# 1 Introduction to rigorous mathematics

## Sets and numbers

**Exercise 1.1.** Show that set union is an *associative* operation, that is for any three sets $A, B, C$ we have

$$\big(A \cup B\big) \cup C = A \cup \big(B \cup C\big).$$

**Exercise 1.2.** We say that two sets $S_1$ and $S_2$ are *disjoint* if $S_1 \cap S_2 = \varnothing$.

Show that given any two sets $A$ and $B$, the sets

$$S_1 := A \setminus B, \qquad S_2 := A \cap B, \qquad S_3 := B \setminus A$$

are pairwise disjoint.

What is their union $S_1 \cup S_2 \cup S_3$?

**Exercise 1.3.** Express the following repeating decimals as rational numbers:

(a) $0.1111\dots$

(b) $2.6666\dots$

(c) $0.9999\dots$

(d) $0.34999\dots$

(e) $0.37373737\dots$

(f) $0.001010101\dots$

**Exercise 1.4.** (Hard) Show that any decimal number that is eventually periodic must represent a rational number.

**Exercise 1.5.** Solve the following equations in $\mathbf{F}_2$:

(a) $x + \hat{1} = \hat{0}$

(b) $x^2 + x = \hat{0}$

(c) $x^2 + x + \hat{1} = \hat{0}$.

**Exercise 1.6.** Let $m$ denote a natural number. Which of the following equations involving only natural numbers have solutions that are also natural numbers? If there is a solution, is it **unique**, in other words, is there **only one** solution?

(a) $m + 1 = 2$

(b) $m + 2 = 1$

(c) $2m = 4$

(d) $2m = 3$

(e) $(0) \,.\, m = 42$

(f) $(0) \,.\, m = 0$

If we replace "natural number" by "integer", which of the preceding equations, which had no solution in the natural numbers, now possess solutions in the integers?

Which now become solvable if "integer" is replaced by "rational number"?

## Complex numbers: prerequisites

**Exercise 1.7.** (a) Using the number $i$, write down an expression for $\sqrt{-25}$.

(b) Simplify $i^7$.

(c) For the complex number $z = 2 - 3i$, write down: (i) $\mathrm{Re}(z)$; (ii) $\mathrm{Im}(z)$; (iii) $\mathrm{Re}(z) - \mathrm{Im}(z)$.

**Exercise 1.8.** Find the solutions of $z^2 - 6z + 10 = 0$. What is the relationship between the solutions? Plot the solutions in the complex plane.

**Exercise 1.9.** Let $z$ and $w$ be complex numbers. Prove the following properties of the complex conjugate.

(a) $z + \bar{z}$ is real;

(b) $z - \bar{z}$ is imaginary;

(c) $z\bar{z}$ is real;

(d) $\overline{z + w} = \bar{z} + \bar{w}$;

(e) $\overline{zw} = \bar{z}\bar{w}$.

**Exercise 1.10.** (a) Express $\dfrac{1 + 2i}{-1 + 3i}$ in cartesian form $x + iy$.

(b) Find $\mathrm{Re}\left(\dfrac{1 + 5i}{2 - 2i}\right)$ and $\mathrm{Im}\left(\dfrac{1 + 5i}{2 - 2i}\right)$.

**Exercise 1.11.** Find the modulus and argument of

(a) $1 + \sqrt{3}i$

(b) $-3 - 3i$

(c) $3 + 4i$

(d) $-7$.

**Exercise 1.12.** Express the following complex numbers in exponential polar form:

(a) $z = \sqrt{3} + i$

(b) $z = -1 - i$.

**Exercise 1.13.** (a) Express $z = -2 + 2\sqrt{3}i$ in exponential polar form $re^{i\theta}$.

(b) Express $z = 5e^{i\frac{3\pi}{4}}$ in cartesian form $x + iy$.

**Exercise 1.14.** (a) Describe geometrically what happens when a complex number $z$ is multiplied by $w = i$. By $u = -i$?

(b) Describe geometrically what happens when a complex number $z$ is divided by $w = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

**Exercise 1.15.** If $z = \sqrt{3} + i$ and $w = 1 - \sqrt{3}i$, use exponential polar form to find $\frac{1}{z}$ and $zw$.

**Exercise 1.16.** Using properties of the modulus, evaluate

$$\left| \frac{-2(3-i)(5+2i)}{(1+3i)(7-i)} \right| .$$

**Exercise 1.17.** Using properties of the argument, evaluate:

(a) $\arg\big((1+i)(-1+\sqrt{3}i)\big)$;

(b) $\arg\left(\frac{-i}{-2+2i}\right)$.

**Exercise 1.18.** What is the geometric relation between the point $z$ in the complex plane and the following points? Illustrate with a sketch.

(a) $\overline{z}$

(b) $\overline{-z}$

(c) $-z$

(d) $\dfrac{1}{z}$.

**Exercise 1.19.** Sketch the set of points in the complex plane representing the complex numbers $z$ satisfying the following relations:

(a) $|z| = 2$

(b) $|z| < 2$

(c) $|z| > 2$

(d) $|z - 1| = 2$

(e) $|z + 1| < 1$

(f) $|z - 2| = 2|z + 1|$

(g) $|z + i| = |z - 1|$

(h) $|z - a| = |z - b|$

## Polar form and exponential form

**Exercise 1.20.** Express each of the following complex numbers in the polar form $re^{i\theta}$. Illustrate with a sketch.

(a) $2 + i2\sqrt{3}$

(b) $-5 + 5i$

(c) $-\sqrt{6} - i\sqrt{2}$

(d) $3 + 4i$

(e) $1 - i\sqrt{3}$

(f) $(1 + i\sqrt{3})^2$

(g) $\dfrac{1 + i}{1 - i}$

(h) $\frac{1+i\sqrt{3}}{1-i\sqrt{3}}$

(i) $(2 + 3i)(1 - 2i)$

**Exercise 1.21.** Express the following numbers in Cartesian form:

(a) $e^{2+i\frac{\pi}{4}}$

(b) $e^{i\pi}$

(c) $e^{i2\pi}$

**Exercise 1.22.** Express in exponential form $e^z$:

(a) $1 - i\sqrt{3}$

(b) $-3$

(c) $1 + i$

Is exponential form unique?

**Exercise 1.23.** Compute $(5-i)^4(1+i)$ and use the result to prove that

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).$$

The latter expression has frequently been used to calculate $\pi$. The first term on the right hand side is easy to calculate and the second term converges very quickly.

## Applications of exponential form

**Exercise 1.24.** Let $z$ be a complex variable and write $z = a + ib$. Make sense of the derivative

$$\frac{d}{dt}\, e^{zt},$$

where $t$ is a real variable.

Find a general formula for the indefinite integral

$$\int e^{zt}\, dt.$$

**Exercise 1.25.** Find the fourth derivative with respect to $t$ of $e^{at}\cos(bt)$ using that of $e^{(a+ib)t}$. Show that

$$\frac{d^4}{dt^4}\big(e^{-t}\cos(2t)\big) = -25e^{-t}\sin(2t+\alpha),$$

where $\alpha = \arctan\left(\frac{7}{24}\right)$.

**Exercise 1.26.** Calculate $\int 3^t \cos(t)\, dt$ using the complex exponential.

**Exercise 1.27.** Using the complex exponential, calculate

(a) $\dfrac{d^{51}}{dt^{51}}\, e^t \cos\left(\sqrt{3}t\right)$

(b) $\dfrac{d^{75}}{dt^{75}}\, e^t \sin(t)$.

**Exercise 1.28.** Use de Moivre's formula to express the following functions as polynomials in $\sin(\theta)$ and $\cos(\theta)$. Which of them can be expressed as polynomials in (i) $\cos(\theta)$ alone, (ii) $\sin(\theta)$ alone?

(a) $\cos(4\theta)$

(b) $\sin(3\theta)$

(c) $\cos(5\theta)$

(d) $\sin(4\theta)$ .

**Exercise 1.29.** Express the following as linear combinations of trigonometric functions of integral multiples of $\theta$:

(a) $\cos^5(\theta)$

(b) $\sin^5(\theta)$

(c) $\cos^6(\theta)$

(d) $\cos^3(\theta)\sin^2(\theta)$.

**Exercise 1.30.** (Hard) Give an argument to show that, for any natural number $n$, $\cos(n\theta)$ can be expressed in the form

$$\cos(n\theta) = P_n(\cos(\theta)),$$

where $P_n$ is a polynomial of degree $n$ with real coefficients. (For example, $P_1(x) = x$ and $P_2(x) = 2x^2 - 1$.)

(These are called *Chebyshev polynomials*.)

## Finding roots and solving equations

**Exercise 1.31.** Find all cube roots of unity and locate them graphically.

**Exercise 1.32.** Find the square roots of $i$.

**Exercise 1.33.** Find all complex numbers $z$ for which $z^3 = -8i$ and sketch them in the complex plane.

**Exercise 1.34.** Find the sixth roots of 64 and illustrate with a diagram.

**Exercise 1.35.** Solve the following equations and locate the zeros graphically:

(a) $z^5 = -32$

(b) $z^3 = -1 + i$

(c) $z^4 = -2\sqrt{3} - 2i$.

**Exercise 1.36.** Find all zeros of the following equations:

(a) $z^4 - 2z^2 + 4 = 0$

(b) $z^6 + 2z^3 + 2 = 0$

(c) $z^4 + 4z^2 + 16 = 0$

(d) $z^4 + 1 = 0$

(e) $z^2 + (1+i)z - \frac{1}{2\sqrt{3}} + i = 0$

(f) $z^3 - (1+2i)z^2 + iz + i = 0$

**Hint.** In (f) $z = 1$ is a zero. Divide by $z - 1$.

## Proofs

**Exercise 1.37.** Prove carefully the following results about integers. (You will need to assume some facts about the integers but you should state clearly what they are and you should make them as simple as possible.)

(a) The square of an even integer is even.

(b) The product of two odd integers is odd.

(c) The sum of two odd integers is even.

(d) The cube of an odd integer is odd.

(e) If $k$ is an odd integer, $k^2 - 1$ is divisible by 4.

**Exercise 1.38.** (Hard) Prove that if $a_0, a_1, \ldots, a_{n-1}$ are integers and $x$ is a rational number such that

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0,$$

then $x$ has to be an integer which is a factor of $a_0$. This theorem shows in particular that for $n \in \mathbf{N}$ and $n > 1$, the $n$-th root of any natural number $k$ is irrational, unless $k$ is of the form $m^n$ for some natural number $m$. The theorem can also be used in some cases to deduce that solutions of algebraic equations are not rational numbers.

**Exercise 1.39.** One of the following statements is true and the other is false. For the one that is true, give a careful proof. For the one that is false, give an example to show that it is false.

(a) If a natural number $n$ is divisible by 12 then its square is also divisible by 12.

(b) If the square $n^2$ of a natural number $n$ is divisible by 12 then $n$ is also divisible by 12.

**Exercise 1.40.** Show by example that

(a) the sum of two irrational numbers need not be irrational;

(b) the product of two irrational numbers need not be irrational.

**Exercise 1.41.** Prove that there is no rational number whose square is six.

**Exercise 1.42.** Prove that $\sqrt{2}+\sqrt{3}$ is irrational.

**Exercise 1.43.** Prove that $\log_{10} 2$ is irrational.

**Exercise 1.44.** (Hard) Given 5 points in a square of side 1, show that there are two points of the five for which the distance apart is no more than $\frac{\sqrt{2}}{2}$.

**Exercise 1.45.** (Hard) Suppose that the points of the plane are each coloured either red, yellow or blue. Prove that there are two points at distance one apart which have the same colour.

**Exercise 1.46.** (Hard) Prove that, for any set of 2026 integers, there is a subset whose sum is divisible by 2026.

## Inequalities

**Exercise 1.47.** Transform each of the following inequalities into an equivalent inequality free of the modulus sign, such as $a < x < b$. Simplify as much as possible.

(a) $|x| < 3$

(b) $|x-2| < 5$

(c) $|3-2x| < 1$

(d) $|1+2x| \leqslant 3$

(e) $|x+2| \geqslant 5$

(f) $|5-x^{-1}| < 1$

(g) $|x-5| < |x+1|$

(h) $|x^2-2| \leqslant 1$

**Exercise 1.48.** Rewrite each of the following inequalities in terms of intervals:

(a) $|x+3| \geqslant 1$

(b) $|x-2| < 3$

(c) $|x-2| < 3$ or $|x+1| < 1$

(d) $|x-2| < 3$ and $|x+1| < 1$

(e) $|x+2| \leqslant 2$ and $|x| > 1$

(f) $|x+2| \leqslant 2$ or $|x| > 1$

## Mathematical induction

**Exercise 1.49.** Rewrite each equality using summation notation, and prove that the equality holds for every positive integer $n$.

(a) $2+4+6+\cdots+2n = n(n+1)$

(b) $1+4+7+\cdots+(3n-2) = \frac{1}{2}n(3n-1)$

(c) $2+7+12+\cdots+(5n-3) = \frac{1}{2}n(5n-1)$

(d) $1+2\cdot 2+3\cdot 2^2+4\cdot 2^3+\cdots+n\cdot 2^{n-1} = 1+(n-1)2^n$

(e) $1^2+2^2+3^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$

(f) $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

(g) $3+3^2+3^3+\cdots+3^n = \frac{3}{2}(3^n-1)$

(h) $(1+2^5+\cdots+n^5)+(1+2^7+\cdots+n^7) = 2\left\{\frac{n(n+1)}{2}\right\}^4$

(i) $1+r+r^2+\cdots+r^{n-1} = \dfrac{1-r^n}{1-r}$ if $r \neq 1$

**Exercise 1.50.** Prove by induction that the following statements are true for every positive integer $n$.

(a) 3 is a factor of $n^3-n+3$

(b) 9 is a factor of $10^{n+1}+3\cdot 10^n+5$

(c) 4 is a factor of $5^n-1$

(d) $x-y$ is a factor of $x^n-y^n$

(e) $7^{2n}-48n-1$ is divisible by 2304.

**Exercise 1.51.** Prove that the following inequalities hold for all $n \in \mathbf{N}$.

(a) $(1+x)^n \geqslant 1 + nx$ if $x \geqslant -1$ (the so-called **Bernoulli's inequality**)

(b) $1^3 + 2^3 + \cdots + (n-1)^3 < \frac{1}{4}n^4 < 1^3 + 2^3 + \cdots + n^3$

(c) $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}} \geqslant \sqrt{n}$

(d) $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \ldots \dfrac{1}{\sqrt{n}} \leqslant 2\sqrt{n} - 1.$

**Exercise 1.52.** In each case find $n_0 \in \mathbf{N}$ such that the inequality holds for all $n \geqslant n_0$, and give a proof by induction.

(a) $n! > 2^n$

(b) $n! > 3n^2$

(c) $n! > 2n^3$

(d) $2^n > n^2$

(e) $2^n > 2n^3$

(f) $2^{2n} > 20 \cdot 3^n$

**Exercise 1.53.** Derive the following formulas (without using mathematical induction):

(a) $\displaystyle\sum_{k=1}^{n} (2k-1) = n^2.$
(**Hint:** $2k - 1 = k^2 - (k-1)^2$.)

(b) $\displaystyle\sum_{k=1}^{n} k = \frac{1}{2}n^2 + \frac{1}{2}n.$
(**Hint:** Use the preceding formula.)

(c) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$
(**Hint:** $k^3 - (k-1)^3 = 3k^2 - 3k + 1$.)

(d) $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$
(**Hint:** $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$.)

**Exercise 1.54.** Consider the triangle in the complex plane that has vertices at the origin, at $z_1$ and at $z_1 + z_2$. By considering the sides of this triangle, show that

$$|z_1 + z_2| \leqslant |z_1| + |z_2|.$$

Now use mathematical induction to show that, for any complex numbers $z_1, \ldots, z_n$,

$$|z_1 + \cdots + z_n| \leqslant |z_1| + \cdots + |z_n|.$$

## Functions

**Exercise 1.55.** Let $X$ be a nonempty set and let $y$ be an element that may or may not be in $X$.

(a) How many functions $X \longrightarrow \varnothing$ are there?

(b) How many functions $\varnothing \longrightarrow X$ are there?

(c) How many functions $X \longrightarrow \{y\}$ are there?

(d) How many functions $\{y\} \longrightarrow X$ are there?

**Exercise 1.56.** Consider two functions $f: A \longrightarrow B$ and $g: B \longrightarrow C$.

(a) Suppose $g \circ f$ is injective. Does it follow that $f$ must be injective? That $g$ must be injective?

(b) Suppose $g \circ f$ is surjective. Does it follow that $f$ must be surjective? That $g$ must be surjective?

(c) Suppose $g \circ f$ is bijective. What can you conclude about $f$ and $g$?

**Exercise 1.57.** Let $f: A \longrightarrow C$ and $g: B \longrightarrow D$ be bijective functions. Give a bijective function $h: A \times B \longrightarrow C \times D$ (and prove that it is bijective).

**Exercise 1.58.** Let $f: A \longrightarrow B$ be a function and let $X \subseteq A$. The **image of $X$ under $f$** is the subset of $B$ defined by

$$f(X) = \{b \in B : b = f(x) \text{ for some } x \in X\}.$$

Suppose we have another subset $Y \subseteq A$.

(a) Prove that
$$f(X \cup Y) = f(X) \cup f(Y).$$

(b) Prove that
$$f(X \cap Y) \subseteq f(X) \cap f(Y).$$

Does the opposite inclusion always hold?

**Exercise 1.59.** Let $f: A \longrightarrow B$ be a function and let $X \subseteq B$. The **inverse image of $X$ under $f$** is the subset of $A$ defined by

$$f^{-1}(X) = \{a \in A: f(a) \in X\}.$$

Suppose we have another subset $Y \subseteq B$.

(a) Prove that
$$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y).$$

(b) Prove that
$$f^{-1}(X \cap Y) \subseteq f^{-1}(X) \cap f^{-1}(Y).$$
Does the opposite inclusion always hold?

**Exercise 1.60.** Let $f: A \longrightarrow B$ be a function.

(a) Prove that
$$f^{-1}(f(X)) = X \qquad \text{for every } X \subseteq A$$
if and only if $f$ is injective.

(b) Prove that
$$f(f^{-1}(X)) = X \qquad \text{for every } X \subseteq B$$
if and only if $f$ is surjective.

**Exercise 1.61.** Let $A, B$ be two nonempty sets. Consider the functions

$$\begin{aligned} \pi_A: A \times B \longrightarrow A \quad & \text{given by } \pi_A(a,b) = a \text{ for all } (a,b) \in A \times B \\ \pi_B: A \times B \longrightarrow B \quad & \text{given by } \pi_B(a,b) = b \text{ for all } (a,b) \in A \times B. \end{aligned}$$

(a) Prove that $\pi_A$ and $\pi_B$ are surjective.

(b) Prove that given any set $X$ and any functions $f: X \longrightarrow A$ and $g: X \longrightarrow B$, there exists a unique function $h: X \longrightarrow A \times B$ such that
$$\pi_A \circ h = f \qquad \text{and} \qquad \pi_B \circ h = g.$$
(This is called the **universal mapping property** of the Cartesian product $A \times B$.)

**Exercise 1.62.** Let $A$ be a set of cardinality $n$, where $n \geqslant 1$.

(a) Prove that $A$ is not the empty set.

(b) Prove that for any $a \in A$, the subset $A \setminus \{a\}$ has cardinality $n - 1$.

**Exercise 1.63.** Let $A$ be a set of cardinality $n \in \mathbf{N}$. Prove that for any $m \neq n$, $A$ cannot have cardinality $m$.

(Hint: use induction on $n$, and Exercise 1.62.)

**Exercise 1.64.** Let $A$ be a finite set and let $B \subseteq A$. Prove that $B$ is finite and $\#B \leqslant \#A$. (Hint: try induction on the cardinality of $A$.)

**Exercise 1.65.** Let $X$ be an arbitrary nonempty set, $A$ a finite subset of $X$, and $x \in X \setminus A$.

(a) Prove that if $x \in X \setminus A$ then $A \cup \{x\}$ is finite and

$$\#\big(A \cup \{x\}\big) = \#A + 1.$$

(b) Prove that for any $n \in \mathbf{N}$, if $x_1, \dots, x_n$ are distinct elements of $X \setminus A$, then $A \cup \{x_1, \dots, x_n\}$ is finite and

$$\#\big(A \cup \{x_1, \dots, x_n\}\big) = \#A + n.$$

Therefore if $A$ and $B$ are finite subsets of $X$ that are disjoint, then

$$\#\big(A \cup B\big) = \#A + \#B.$$

**Exercise 1.66.** (Inclusion-exclusion principle). Let $A$ and $B$ be two finite sets.

(a) Prove that the union $A \cup B$ and the intersection $A \cap B$ are finite sets.

(b) Prove that

$$\#\big(A \cup B\big) = \#A + \#B - \#\big(A \cap B\big).$$

**Exercise 1.67.** (Pigeonhole principle). Let $A_1, \dots, A_n$ be finite sets, let $A$ be their union, and suppose the cardinality of $A$ is $> n$. Prove that there exists $j$ such that the cardinality of $A_j$ is $> 1$.

## Equivalence relations

**Exercise 1.68.** An equivalence relation $\sim$ is a way of identifying elements of a set. More precisely, given a set $A$ and $x, y \in A$, we will write $x \sim y$ to signify that "$x$ is equivalent to $y$", and we require this to satisfy three properties:

- $x \sim x$ for all $x \in A$ ("reflexivity");
- if $x \sim y$ then $y \sim x$ ("symmetry");
- if $x \sim y$ and $y \sim z$ then $x \sim z$ ("transitivity").

(a) Let $A$, $B$ be sets and $f\colon A \longrightarrow B$ a function. For $x, y \in A$, define $x \sim y$ if $f(x) = f(y)$. Show that this satisfies the properties of an equivalence relation on $A$.

(b) Fix a natural number $n$. For $k, m \in \mathbf{Z}$, define $k \sim m$ if $m - k$ is divisible by $n$. Show that this satisfies the properties of an equivalence relation on $\mathbf{Z}$.

(c) Fix a set $\Omega$ and let $A$ denote the set of all subsets of $\Omega$. For $X, Y \in A$ define $X \sim Y$ if there exists a bijective function $X \longrightarrow Y$. Show that this satisfies the properties of an equivalence relation on $A$.

(d) Suppose we are given an equivalence relation on a set $A$. For any element $x \in A$, we define the equivalence class of $x$ as:

$$[x] = \{y \in A \mid x \sim y\}.$$

Show that, for any elements $x, z \in A$, their equivalence classes are either identical or disjoint, in other words:

$$\text{either } [x] = [z] \quad \text{or } [x] \cap [z] = \varnothing.$$

(e) How many distinct equivalence classes are there for the equivalence relation on $\mathbf{Z}$ defined in part (b)? Make sure to prove that the number you give (and which will probably depend on $n$) is the **exact** number of classes (not just an upper bound, for instance).

(f) Suppose we are given an equivalence relation on a set $A$, and consider the set $B$ of equivalence classes[1]:

$$B = \{[x] \mid x \in A\}.$$

There is an obvious surjective function $\pi\colon A \longrightarrow B$ defined by $[x]$. Under what circumstances (if any) is $\pi$ a bijection?

(g) Let $A = \mathbf{N} \times \mathbf{N}$ and define $(a, b) \sim (c, d)$ if $a + d = b + c$.

  i. Show that this satisfies the conditions of an equivalence relation on $A$.

  ii. Construct a bijective function $B \longrightarrow \mathbf{Z}$, where $B$ is the set of equivalence classes. (Don't forget to prove that your function is well-defined, and that it is bijective.)

[1]A frequently used notation for the set of equivalence classes is $A/\sim$, read as "A mod tilde", and it is referred to as the quotient of $A$ by the relation $\sim$

## Answers

**Solution 1.1.** We have to show that

$$\big(A \cup B\big) \cup C \subseteq A \cup \big(B \cup C\big) \qquad \text{and} \qquad A \cup \big(B \cup C\big) \subseteq \big(A \cup B\big) \cup C.$$

We work out the first of these inclusions; the other is very similar.

Let $x \in \big(A \cup B\big) \cup C$. Then there are two (non-mutually exclusive) possibilities:

- $x \in A \cup B$, so $x \in A$ or $x \in B$. If $x \in A$, then certainly $x \in A \cup \big(B \cup C\big)$. Otherwise, $x \in B$ so $x \in B \cup C$, so again $x \in A \cup \big(B \cup C\big)$.
- $x \in C$, in which case $x \in B \cup C$, so $x \in A \cup \big(B \cup C\big)$.

In all cases we concluded that $x \in A \cup \big(B \cup C\big)$.

**Solution 1.2.** If $x \in S_1 \cap S_2$, then we have simultaneously that $x \in A$, $x \notin B$, $x \in A$, and $x \in B$, contradiction. So $S_1 \cap S_2 = \varnothing$. The same argument shows that $S_3 \cap S_2 = \varnothing$.

If $x \in S_1 \cap S_2$, then $x \in A$, $x \notin B$, $x \in B$, and $x \notin A$, contradiction.

The union is

$$S_1 \cup S_2 \cup S_3 = A \cup B.$$

To show this, first note that if $C \subseteq X$ and $D \subseteq Y$ then $C \cup D \subseteq X \cup Y$. Since $S_1 = A \setminus B \subseteq A$ and $S_2 = A \cap B \subseteq A$, we have that $S_1 \cup S_2 \subseteq A$. Now $S_3 = B \setminus A \subseteq B$, so $(S_1 \cup S_2) \cup S_3 \subseteq A \cup B$.

For the inclusion in the other direction, let $x \in A \cup B$. Suppose first that $x \in A$. Either $x \in B$ or $x \notin B$, leading to $x \in A \setminus B = S_1$ or $x \in A \cap B = S_2$, so that $x \in S_1 \cup S_2 \subseteq S_1 \cup S_2 \cup S_3$.

Similarly, if $x \in B$ we conclude that $x \in S_2 \cup S_3$, so in either case $x \in S_1 \cup S_2 \cup S_3$.

**Solution 1.3.** (a) $\frac{1}{9}$

(b) $\frac{8}{3}$

(c) 1

(d) $0.35 = \frac{7}{20}$

(e) $\frac{37}{99}$

(f) $\frac{1}{990}$

**Solution 1.4.** Without loss of generality we may assume that the number is between 0 and 1 (otherwise subtract the integer part of the number).

So we have something of the form

$$x = 0.a_1 a_2 a_3 \ldots a_n \overline{a_{n+1} a_{n+2} \ldots a_{n+r}}$$

where the bar indicates the repeating part. Then

$$10^n x = a_1 a_2 \ldots a_n.\overline{a_{n+1} a_{n+2} \ldots a_{n+r}}$$

Set $A = a_1 a_2 \ldots a_n \in \mathbf{N}$ and $B = a_{n+1} \ldots a_{n+r} \in \mathbf{N}$, then

$$10^n x - A = 0.\overline{a_{n+1} a_{n+2} \ldots a_{n+r}}$$

We finally get to the cool part: because of the periodicity, we have

$$10^r(10^n x - A) - B = 0.\overline{a_{n+1} a_{n+2} \ldots a_{n+r}} = 10^n x - A,$$

so

$$10^n x - A = \frac{B}{10^r - 1}$$

and finally

$$x = \frac{\frac{B}{10^r - 1} + A}{10^n},$$

which is manifestly a rational number.

**Solution 1.5.** (a) $x = \hat{1}$

(b) $x = \hat{0}$ or $x = \hat{1}$

(c) no solution

**Solution 1.6.** (a) Yes, unique

(b) No

(c) Yes, unique

(d) No

(e) No

(f) Yes, not unique
Equation (b) has a unique solution in the set of integers.
Equation (d) has a unique solution in the set of rational numbers.

**Solution 1.7.** (a) $5i$; (b) $-i$; (c) i. 2, ii. $-3$, iii. 5.

**Solution 1.8.** From the quadratic formula, $z = 3 \pm i$. The two solutions are complex conjugates.

**Solution 1.9.** (a) $z + \overline{z} = (x + iy) + (x - iy) = 2x \in \mathbf{R}$.

(b) $z - \overline{z} = (x + iy) - (x - iy) = 2yi \in i\mathbf{R}$.

(c) $z\overline{z} = (x+iy)(x-iy) = x^2 + y^2 \in \mathbf{R}$.

(d) $\overline{z+w} = \overline{(x+iy)+(a+ib)} = (x+a) - i(y+b) = \overline{z} + \overline{w}$.

(e) You do it.

**Solution 1.10.** (a) $\frac{1}{2} - \frac{1}{2}\, i$.

(b) $-1$, resp. $3/2$.

**Solution 1.11.** (a) 2 and $\pi/3$;

(b) $3\sqrt{2}$ and $5\pi/4$;

(c) 5 and $\arctan(4/3)$;

(d) 7 and $\pi$.

**Solution 1.12.** (a) $2e^{i\pi/6}$;

(b) $\sqrt{2}\, e^{i5\pi/4}$.

**Solution 1.13.** (a) $4e^{i2\pi/3}$;

(b) $-\frac{5\sqrt{2}}{2} + i\, \frac{5\sqrt{2}}{2}$.

**Solution 1.14.** (a) for $i$: anti-clockwise rotation by $\pi/2$, for $-i$: clockwise rotation by $\pi/2$;

(b) clockwise rotation by $\pi/3$.

**Solution 1.15.** $1/z = (1/2)\, e^{-i\pi/6}$, $zw = 4e^{-i\pi/6}$.

**Solution 1.16.** $\sqrt{58}/5$.

**Solution 1.17.** (a) $11\pi/12$;

(b) $3\pi/4$.

**Solution 1.18.** (a) Reflection in the real axis.

(b) Reflection in the imaginary axis.

(c) Symmetry in the origin.

(d) Reflection in the real axis followed by scaling by the coefficient $|z|^{-2}$.

**Solution 1.19.** (a) Circle, centre $(0,0)$, radius 2.

(b) Interior of the circle with centre $(0,0)$ and radius 2.

(c) Exterior of the circle with centre $(0, 0)$ and radius 2.

(d) Circle, centre $(1, 0)$, radius 2.

(e) Interior of circle with centre $(-1, 0)$, radius 1.

(f) Circle, centre $(-2, 0)$, radius 2.

(g) The line $y = -x$.

(h) Bisector of line segment with end points $a, b$.

**Solution 1.20.** (a) $4e^{i\frac{\pi}{3}}$

(b) $5\sqrt{2}e^{i\frac{3\pi}{4}}$

(c) $2\sqrt{2}e^{i\frac{7\pi}{6}}$

(d) $5e^{i\arctan\frac{4}{3}}$

(e) $2e^{-i\frac{\pi}{3}}$

(f) $4e^{i\frac{2\pi}{3}}$

(g) $e^{i\frac{\pi}{2}}$

(h) $e^{i\frac{2\pi}{3}}$

(i) $\sqrt{65}e^{-i\arctan\frac{1}{8}}$

**Solution 1.21.** (a) $\frac{1}{\sqrt{2}}e^2(1 + i)$

(b) $-1$

(c) $1$

**Solution 1.22.** (a) $e^{\log 2 + i(2k\pi - \frac{\pi}{3})}$

(b) $e^{\log 3 + i(2k+1)\pi}$

(c) $e^{\frac{1}{2}\log 2 + i(\frac{\pi}{4} + 2k\pi)}$

**Solution 1.23.** In Cartesian form, we have

$$\begin{aligned} z &= (5-i)^4(1+i) \\ &= \big(5^4 - 4\cdot 5^3 i + 6\cdot 5^2 i^2 - 4\cdot 5\cdot i^3 + i^4\big)(1+i) \\ &= (625 - 500i - 150 + 20i + 1)\,(1+i) \\ &= (476 - 480i)(1+i) \\ &= 956 - 4i \\ &= 4(239 - i). \end{aligned}$$

Therefore

$$\arg(z) = -\arctan(1/239).$$

On the other hand:

$$\arg(z) = 4\arg(5-i) + \arg(1+i) = -4\arctan(1/5) + \frac{\pi}{4}.$$

Equate the two expressions and you're done.

(I'm being careless with argument versus principal argument here, but if you stare at the numbers you notice that it's not a problem.)

**Solution 1.24.**

$$\begin{aligned} \frac{d}{dt}\, e^{zt} &= z e^{zt} \\ \int e^{zt}\, dt &= \frac{1}{z}\, e^{zt} + C. \end{aligned}$$

**Solution 1.25.**

$$\frac{d^4}{dt^4}\, e^{at}\cos bt = \operatorname{Re}\big((a+ib)^4\, e^{(a+ib)t}\big) = e^{at}\,(A\cos bt - B\sin bt),$$

where

$$A = a^4 - 6a^2b^2 + b^4 \quad \text{and} \quad B = 4ab(a^2 - b^2).$$

**Solution 1.26.**

$$\int 3^t \cos t\, dt = \alpha 3^t(\log 3\cdot\cos t + \sin t) + C,$$

where $\alpha = 1/((\log 3)^2 + 1)$.

**Solution 1.27.** (a) $-2^{51}e^t\cos\sqrt{3}t$

(b) $2^{37}e^t(\cos t - \sin t)$.

**Solution 1.28.** (a) $8\cos^4\theta - 8\cos^2\theta + 1$

(b) $3\sin\theta - 4\sin^3\theta$

(c) $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(d) $4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ – cannot be expressed in one alone.

**Solution 1.29.** (a) $2^{-4}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

(b) $2^{-4}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

(c) $2^{-5}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

(d) $-2^{-4}(\cos 5\theta + \cos 3\theta - 2\cos\theta)$.

**Solution 1.30.** Use the complex exponential and the binomial theorem.

**Solution 1.31.** $1, \quad -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

**Solution 1.32.** $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \quad -\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$.

**Solution 1.33.** $2i, \quad -\sqrt{3} - i, \quad \sqrt{3} - i$.

**Solution 1.34.** $2, \quad 1 + i\sqrt{3}, \quad -1 + i\sqrt{3}, \quad -2, \quad -1 - i\sqrt{3}, \quad 1 - i\sqrt{3}$.

**Solution 1.35.** (a) $2e^{i\frac{1}{5}\pi}, \quad 2e^{i\frac{3}{5}\pi}, \quad -2, \quad 2e^{-i\frac{1}{5}\pi}, \quad 2e^{-i\frac{3}{5}\pi}$

(b) $\sqrt[6]{2}e^{i\frac{1}{4}\pi} = 2^{-\frac{1}{3}}(1+i), \quad \sqrt[6]{2}e^{i\frac{11}{12}\pi}, \quad \sqrt[6]{2}e^{-i\frac{5}{12}\pi}$

(c) $\sqrt{2}e^{-i\frac{5}{24}\pi}, \quad \sqrt{2}e^{i\frac{7}{24}\pi}, \quad \sqrt{2}e^{i\frac{19}{24}\pi}, \quad \sqrt{2}e^{-i\frac{17}{24}\pi}$.

**Solution 1.36.** (a) $\frac{1}{\sqrt{2}}(\sqrt{3}+i), \quad -\frac{1}{\sqrt{2}}(\sqrt{3}+i), \quad \frac{1}{\sqrt{2}}(\sqrt{3}-i), \quad \frac{1}{\sqrt{2}}(-\sqrt{3}+i)$

(b) $\sqrt[6]{2}e^{i\frac{1}{4}\pi}, \quad \sqrt[6]{2}e^{i\frac{11}{12}\pi}, \quad \sqrt[6]{2}e^{-i\frac{5}{12}\pi}, \quad \sqrt[6]{2}e^{-i\frac{1}{4}\pi}, \quad \sqrt[6]{2}e^{i\frac{5}{12}\pi}, \quad \sqrt[6]{2}e^{-i\frac{11}{12}\pi}$

(c) $1 + i\sqrt{3}, \quad -(1 + i\sqrt{3}), \quad 1 - i\sqrt{3}, \quad -1 + i\sqrt{3}$

(d) $\frac{1}{\sqrt{2}}(1+i), \quad \frac{1}{\sqrt{2}}(1-i), \quad -\frac{1}{\sqrt{2}}(1+i), \quad \frac{1}{\sqrt{2}}(-1+i)$

(e) $\frac{1}{2}(-1-i) + \frac{1}{2\sqrt[4]{3}}(\sqrt{3}-i), \quad \frac{1}{2}(-1-i) - \frac{1}{2\sqrt[4]{3}}(\sqrt{3}-i)$

(f) $1, \quad \sqrt[4]{2}\cos\frac{3}{8}\pi + (1 + \sqrt[4]{2}\sin\frac{3}{8}\pi)i, \quad -\sqrt[4]{2}\cos\frac{3}{8}\pi + (1 - \sqrt[4]{2}\sin\frac{3}{8}\pi)i$

**Solution 1.37.** (a) Let $n = 2k$ be an even integer, with $k \in \mathbf{Z}$. Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$ is an even integer, since $2k^2 \in \mathbf{Z}$.

(b) Let $n = 2k + 1$ and $m = 2h + 1$ be odd integers, with $k, h \in \mathbf{Z}$. Then $nm = (2k+1)(2h+1) = 4kh+2k+2h+1 = 2(2kh+k+h)+1$ is odd, since $2kh+k+h \in \mathbf{Z}$.

(c) Let $n = 2k+1$ and $m = 2h+1$ be odd integers, with $k, h \in \mathbf{Z}$. Then $n + m = (2k+1) + (2h+1) = 2k + 2h + 2 = 2(k+h+1)$ is even, since $k + h + 1 \in \mathbf{Z}$.

(d) Let $n = 2k+1$ be an odd integer, with $k \in \mathbf{Z}$. Then $n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$ is odd, since $4k^3 + 6k^2 + 3k \in \mathbf{Z}$.

(e) Let $k = 2n+1$ be an odd integer, with $n \in \mathbf{Z}$. Then $k^2 - 1 = (2n+1)^2 - 1 = 4n^2 + 4n + 1 - 1 = 4(n^2 + n)$ is divisible by 4, since $n^2 + n \in \mathbf{Z}$.

(In fact, $k^2 - 1$ is divisible by 8. Can you see why?)

**Solution 1.38.** Write $x = \frac{p}{q}$ with $p$ and $q$ relatively prime. The relation becomes

$$\frac{p^n}{q^n} + a_{n-1}\frac{p^{n-1}}{q^{n-1}} + \cdots + a_1\frac{p}{q} + a_0 = 0,$$

which we can rewrite as

$$p\left(p^{n-1} + a_{n-1}p^{n-2}q + \cdots + a_1 q^{n-1}\right) = -a_0 q^n.$$

Since $q$ divides the right hand side, it must divide the left hand side. But it is relatively prime to $p$, so it must divide the parenthesis. This implies that it must divide $p^{n-1}$, which in turn (because $p$ and $q$ are relatively prime) forces $q = 1$. In other words, $x = p$ is an integer.

The last equation becomes

$$p\left(p^{n-1} + a_{n-1}p^{n-2} + \cdots + a_1\right) = -a_0,$$

so $p$ must divide $a_0$.

**Solution 1.39.** The first statement is true. If $n = 12k$ with $k \in \mathbf{Z}$ then $n^2 = 144k^2 = 12(12k^2)$ is divisible by 12 since $12k^2 \in \mathbf{Z}$.

The second statement is false. For example, take $n = 6$. Then $n$ is not divisible by 12, but its square $n^2 = 36$ is divisible by 12.

**Solution 1.40.** (a) $\pi + (-\pi) = 0$;

(b) $\sqrt{2} \cdot \sqrt{2} = 2$.

**Solution 1.41.** We prove the statement by contradiction. Suppose there exists $x \in \mathbf{Q}$ with $x^2 = 6$. Write $x = \frac{p}{q}$ where $p$ and $q$ are relatively prime integers.

Then $\frac{p^2}{q^2} = 6$ so $p^2 = 6q^2$. This implies that $p^2$ is even, therefore (as we have seen) $p$ must be even, say $p = 2r$ for $r \in \mathbf{Z}$. We get $(2r)^2 = 6q^2$, so $4r^2 = 6q^2$, so $2r^2 = 3q^2$. This means that $3q^2$ is even, so $q^2$ is even (since 3 is odd), so $q$ is even.

But then both $p$ and $q$ are even, contradicting the fact that they are relatively prime. So our original assumption is false and $\sqrt{6}$ is irrational.

**Solution 1.42.** Let $x = \sqrt{2} + \sqrt{3}$. Suppose $x$ is rational, then so is its square $x^2 = (\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$. Rewrite this as $\sqrt{6} = \frac{1}{2}(x^2 - 5)$, which must then also be rational. But this contradicts the fact that $\sqrt{6}$ is irrational that we proved in Exercise 1.41.

**Solution 1.43.** We proceed by contradiction.

Suppose $\log_{10}(2)$ is rational, and write it as $\frac{p}{q}$ with $p, q \in \mathbf{Z}$ relatively prime.

Then $10^{p/q} = 2$. Raising to the $q$-th power we get $10^p = 2^q$. Clearly this forces $q > p$. Multiply by $2^{-p}$ on both sides of the equation to get $5^p = 2^{q-p}$. The right hand side is an even integer, so the left hand side is also an even integer, which is a contradiction since 5 is odd hence $5^p$ is also odd.

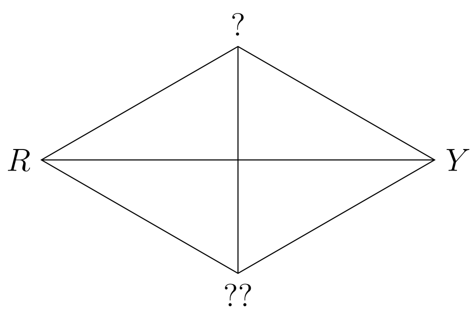
**Solution 1.44.** We will use the pigeonhole principle: if you have $n$ pigeonholes and you place $n + 1$ objects in them, then there is at least one pigeonhole containing more than one object.

Take the square and partition it into the 4 squares of side length $1/2$. We are distributing the 5 points among these 4 squares, so at least one square contains 2 points. But then the distance between these 2 points is at most the length of the diagonal of the small square, that is $\sqrt{2}/2$.

**Solution 1.45.** We prove this by contradiction. Assume there are no two points of the same colour at a distance of 1 apart.

There must be at least one red point $R$. Otherwise all points are either yellow or blue so any equilateral triangle of side length 1 will have two vertices of the same colour.

Take this red point $R$ and consider a circle centred at it, of radius $\sqrt{3}$. This circle must have some yellow or blue point, because if all the points on it were red then some of them would be at distance 1 apart. Let's say this point $Y$ is yellow (if it's blue, the argument is similar). Divide the segment $RY$ in two halves and construct the two equilateral triangles centred along this axis:




These equilateral triangles have side length 1 (since the length of $RY$ is $\sqrt{3}$). Consider the two new vertices. They are at distance 1 from $R$, so they cannot be red. They are also at distance 1 from $Y$, so they cannot be yellow. Therefore they must be blue, but this is a contradiction since they are at distance 1 from each other.

**Solution 1.46.** This is another application of the pigeonhole principle: if you have $n$ pigeonholes and you place $n+1$ objects in them, there is at least one pigeonhole that contains at least two objects.

In the statement of the question, there is nothing special about the number 2026, it can be replaced by any natural number $N$.

Enumerate the set of integers

$$\{a_1, a_2, \dots, a_N\}$$

and consider the sums

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_N &= a_1 + a_2 + \cdots + a_N. \end{aligned}$$

Let $r_i$ denote the remainder of the division of $s_i$ by $N$. If any of the $r_i$ is zero, then $s_i$ is divisible by $N$ and we are done.

If none of the $r_i$'s are zero, it means that the $N$ numbers $r_1, r_2, \dots, r_N$ are placed in the $N-1$ pigeonholes $1, 2, \dots, N-1$. So by the pigeonhole principle, at least two of the $r_i$ must be equal, let's say $r_j = r_k$ with $j < k$. Consider then the sum

$$s_k - s_j = a_{j+1} + a_{j+2} + \cdots + a_k.$$

The remainder of its division by $N$ is $r_k - r_j = 0$, so the sum is divisible by $N$.

**Solution 1.47.** (a) $-3 < x < 3$

(b) $-3 < x < 7$

(c) $1 < x < 2$

(d) $-2 \leqslant x \leqslant 1$

(e) $x \leqslant -7$ or $x \geqslant 3$

(f) $\frac{1}{6} < x < \frac{1}{4}$

(g) $x > 2$

(h) $1 \leqslant x \leqslant \sqrt{3}$ or $-\sqrt{3} \leqslant x \leqslant -1$

**Solution 1.48.** (a) $(-\infty, -4] \cup [-2, \infty)$

(b) $(-1, 5)$

(c) $(-1,5)\cup(-2,0)=(-2,5)$

(d) $(-1,5)\cap(-2,0)=(-1,0)$

(e) $[-4,-1)$

(f) $(-\infty,0]\cup(1,\infty)$

**Solution 1.49.** For all of these we let $P(n)$ denote the desired statement corresponding to $n\in\mathbf{N}$.

(a)
$$\sum_{k=1}^{n} 2k = n(n+1).$$

Base case: $P(1)$ says $2 = 1\cdot 2$, true.

Induction step: Let $n\in\mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

The left hand side of $P(n+1)$ is

$$2+4+\cdots+2n+2(n+1) = n(n+1)+2(n+1) = (n+1)(n+2),$$

so $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n\in\mathbf{N}$.

(b)
$$\sum_{k=1}^{n}(3k-2) = \frac{1}{2}n(3n-1).$$

Base case: $P(1)$ says $1=\frac{1}{2}\,1\cdot 2$, true.

Induction step: Let $n\in\mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

The left hand side of $P(n+1)$ is

$$1+4+\cdots+(3n-2)+(3(n+1)-2) = \frac{1}{2}\,n(3n-1)+(3n+1) = \frac{3n^2-n+6n+2}{2} = \frac{(n+1)(3n+2)}{2},$$

so $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n\in\mathbf{N}$.

(c)
$$\sum_{k=1}^{n}(5k-3) = \frac{1}{2}n(5n-1).$$

Base case: $P(1)$ says $2 = \frac{1}{2}\, 1 \cdot 4$, true.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

The left hand side of $P(n+1)$ is

$$2+7+\cdots+(5n-3)+(5(n+1)-3) = \frac{1}{2}\,n(5n-1)+(5n+2) = \frac{(n+1)(5n+4)}{2},$$

so $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

(d)

$$\sum_{k=1}^{n} k \cdot 2^{k-1} = 1 + (n-1)2^n.$$

Base case: $P(1)$ says $1 = 1$, true.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

The left hand side of $P(n+1)$ is

$$1+2\cdot 2+\cdots+n\cdot 2^{n-1}+(n+1)\cdot 2^n = 1+(n-1)2^n+(n+1)2^n = 1+(2n)2^n = 1+n\cdot 2^{n+1},$$

so $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

(e)

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1).$$

Base case: $P(1)$ says $1^2 = \frac{1}{6}\, 1 \cdot 2 \cdot 3$, true.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

The left hand side of $P(n+1)$ is

$$1^2+2^2+\cdots+n^2+(n+1)^2 = \frac{n(n+1)(2n+1)}{6}+(n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6},$$

so $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

(f)

$$\sum_{k=1}^{n} 1/k(k+1) = \frac{n}{n+1}.$$

Base case: $P(1)$ says $\frac{1}{2} = \frac{1}{2}$, true.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

The left hand side of $P(n+1)$ is

$$\frac{1}{1\cdot 2} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2},$$

so $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

(g)

$$\sum_{k=1}^{n} 3^k = \frac{3}{2}(3^n - 1).$$

Base case: $P(1)$ says $3 = \frac{3}{2}\,2$, true.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

The left hand side of $P(n+1)$ is

$$3 + 3^2 + \cdots + 3^n + 3^{n+1} = \frac{3(3^n - 1)}{2} + 3^{n+1} = \frac{3^{n+2} - 3}{2},$$

so $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

(h)

$$\sum_{k=1}^{n} k^5 + \sum_{k=1}^{n} k^7 = 2\left(\frac{n(n+1)}{2}\right)^4.$$

Base case: $P(1)$ says $1 + 1 = 2$, true.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

The left hand side of $P(n+1)$ is

$$\begin{aligned}(1 + 2^5 + \cdots + n^5 + (n+1)^5) + (1 + 2^7 + \cdots + n^7 + (n+1)^7) &= \frac{n^4(n+1)^4}{8} + (n+1)^5 + (n+1) \\ &= \frac{(n+1)^4\,(n^4 + 8n^3 + 24n^2 + 32r}{8} \\ &= \frac{(n+1)^4(n+2)^4}{8},\end{aligned}$$

so $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

(i)
$$\sum_{k=1}^{n} r^{k-1} = \begin{cases} \frac{1-r^n}{1-r} & \text{if } r \neq 1 \\ n & \text{if } r = 1. \end{cases}$$

Base case: $P(1)$ says $1 = \frac{1-r}{1-r}$, true.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

The left hand side of $P(n+1)$ is

$$1 + r + r^2 + \cdots + r^{n-1} + r^n = \frac{1-r^n}{1-r} + r^n = \frac{1-r^{n+1}}{1-r},$$

so $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

**Solution 1.50.** Let $P(n)$ denote the desired statement for $n \in \mathbf{N}$.

(a) Base case: $P(1)$ says 3 divides $1 - 1 + 3$, true.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

We have
$$(n+1)^3 - (n+1) + 3 = (n^3 - n + 3) + 3(n^2 + n),$$
and on the right hand side the first term is divisible by 3 by the induction hypothesis, while the second term is clearly divisible by 3. So $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

(b) Base case: $P(1)$ says 9 divides $100 + 30 + 5$, true as $135 = 9 \cdot 15$.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

We have
$$10^{n+2} + 3 \cdot 10^{n+1} + 5 = 10\left(10^{n+1} + 3 \cdot 10^n + 5\right) - 45,$$
which is divisible by 9 by the induction hypothesis and the fact that 45 is divisible by 9. So $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

(c) Base case: $P(1)$ says 4 divides $5-1$, true.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

We have
$$5^{n+1}-1 = 5(5^n-1)+4,$$
which is divisible by 4 by the induction hypothesis and the fact that 4 is divisible by 4. So $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

(d) Base case: $P(1)$ says $x-y$ divides $x-y$, true.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

We have
$$x^{n+1}-y^{n+1} = x(x^n-y^n)+xy^n-y^{n+1} = x(x^n-y^n)+y^n(x-y),$$
which is divisible by $x-y$ by the induction hypothesis and the fact that $x-y$ is divisible by $x-y$. So $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

(e) Base case: $P(1)$ says 2304 divides $7^2-48-1=0$, true.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

We have
$$7^{2(n+1)}-48(n+1)-1 = 49(7^{2n}-48n-1)+48\cdot 48n,$$
which is divisible by 2304 by the induction hypothesis and the fact that $48\cdot 48 = 2304$. So $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

**Solution 1.51.** Let $P(n)$ denote the desired statement for $n \in \mathbf{N}$.

(a) Base case: $P(1)$ says that $1+x \geqslant 1+x$, true.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

We have
$$(1+x)^{n+1} = (1+x)^n(1+x) \geqslant (1+nx)(1+x),$$
by the induction hypothesis together with the fact that $x \geqslant -1$ (so that $1+x \geqslant 0$). We continue then
$$(1+x)^{n+1} = (1+x)^n(1+x) \geqslant (1+nx)(1+x) = 1+(n+1)x+nx^2 \geqslant 1+(n+1)x,$$

so $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

(b) To preserve sanity, we do the two inequalities separately.

Base case: $P(1)$ says that $0 < \frac{1}{4}$, true.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

We have

$$1^3+2^3+\cdots+(n-1)^3+n^3 < \frac{1}{4}\,n^4+n^3 < \frac{1}{4}\,\left(n^4+4n^3+6n^2+4n+1\right) = \frac{1}{4}\,(n+1)^4,$$

so $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that the first inequality holds for all $n \in \mathbf{N}$.

Now we do the other inequality.

Base case: $P(1)$ says that $\frac{1}{4} < 1$, true.

Induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

We have

$$1^3+2^3+\cdots+n^3+(n+1)^3 > \frac{1}{4}\,n^4+(n+1)^3 = \frac{1}{4}\,\left(n^4+4n^3+12n^2+12n+4\right) > \frac{1}{4}\,(n+1)^4,$$

so $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that the second inequality holds for all $n \in \mathbf{N}$.

(c) Base case: $P(1)$ says that $1 \geqslant 1$, true.

Before going further, I claim that $\sqrt{n}+\frac{1}{\sqrt{n+1}} \geqslant \sqrt{n+1}$ for all $n \in \mathbf{N}$. To see this, we start with the obviously true inequality

$$\sqrt{n(n+1)} \geqslant n.$$

Add 1 on both sides:

$$\sqrt{n(n+1)}+1 \geqslant n+1,$$

then divide both sides by $\sqrt{n+1}$:

$$\frac{\sqrt{n(n+1)}+1}{\sqrt{n+1}} \geqslant \sqrt{n+1},$$

and finally notice that the left hand side can be rewritten

$$\sqrt{n} + \frac{1}{\sqrt{n+1}} \geqslant \sqrt{n+1}.$$

Now we do the induction step: Let $n \in \mathbf{N}$ be arbitrary but fixed. Suppose $P(n)$ is true.

We have

$$1 + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \geqslant \sqrt{n} + \frac{1}{\sqrt{n+1}} \geqslant \sqrt{n+1},$$

first by using the induction hypothesis, and then by using the fact we proved above. So $P(n+1)$ is true.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

(d) We use the same approach as in part (c). I will just show you the auxiliary inequality needed here, and let you do the induction yourselves.

I claim that for all $n \in \mathbf{N}$ we have

$$2\sqrt{n} - 1 + \frac{1}{\sqrt{n+1}} \leqslant 2\sqrt{n+1} - 1.$$

To see this, start with

$$4n^2 + 4n \leqslant 4n^2 + 4n + 1,$$

take square root on both sides (the inequality survives because both sides are positive):

$$2\sqrt{n(n+1)} \leqslant 2n + 1,$$

add 1 on both sides:

$$2\sqrt{n(n+1)} + 1 \leqslant 2(n+1),$$

divide by $\sqrt{n+1}$ on both sides:

$$2\sqrt{n} + \frac{1}{\sqrt{n+1}} \leqslant 2\sqrt{n+1},$$

then subtract 1 on both sides to get the claim.

**Solution 1.52.** (a) $n_0 = 4$

(b) $n_0 = 5$

(c) $n_0 = 6$

(d) $n_0 = 5$

(e) $n_0 = 12$

(f) $n_0 = 11$

**Solution 1.53.** (a)

$$\begin{aligned}\sum_{k=1}^{n}(2k-1) &= \sum_{k=1}^{n}\left(k^2-(k-1)^2\right)\\ &= (1^2-0^2)+(2^2-1^2)+(3^2-2^2)+\cdots+\left((n-1)^2-(n-2)^2\right)+\left(n^2-(n-1)^2\right)\\ &= n^2.\end{aligned}$$

(b) Note that

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n}2k+\sum_{k=1}^{n}(-1) = 2\sum_{k=1}^{n}k-n,$$

so we conclude

$$\sum_{k=1}^{n}k = \frac{1}{2}\left(\sum_{k=1}^{n}(2k-1)+n\right) = \frac{1}{2}n^2+\frac{1}{2}n.$$

(c) By the same approach as in part (a), we have

$$\sum_{k=1}^{n}\left(k^3-(k-1)^3\right) = n^3.$$

On the other hand, the hint suggests

$$n^3 = \sum_{k=1}^{n}\left(k^3-(k-1)^3\right) = \sum_{k=1}^{n}\left(3k^2-3k+1\right) = 3\sum_{k=1}^{n}k^2-3\sum_{k=1}^{n}k+n = 3\sum_{k=1}^{n}k^2-\frac{3}{2}\left(n^2+n\right)+n.$$

It remains to solve for the last sum.

(d) Same idea as part (a), just follow the hint.

**Solution 1.54.** The sides of the triangle have lengths $|z_1+z_2|$, $|z_1|$, and $|z_2|$, and we know that any side is at most equal to the sum of the other two.

**Solution 1.55.** (a) There are no such functions: suppose $f : X \longrightarrow \varnothing$ is a function and let $x \in X$ (such $x$ exists since $X$ is nonempty). Then $f(x) \in \varnothing$, contradicting the fact that the empty set does not have any elements.

(b) There is a unique function $f : \varnothing \longrightarrow X$. It doesn't do anything.

(c) There is a unique function $f: X \longrightarrow \{y\}$: for any $x \in X$ we are forced to let $f(x) = y$, which gives a uniquely determined function.

(d) There are as many functions $f: \{y\} \longrightarrow X$ as there are elements in $X$. Given $x \in X$, setting $f_x(y) = x$ gives a function $f_x: \{y\} \longrightarrow X$. Clearly if $x_1 \neq x_2$ then $f_{x_1} \neq f_{x_2}$.

Conversely, if $f: \{y\} \longrightarrow X$ is a function, then $f(y) \in X$. If we let $x = f(y)$, then $f = f_x$ as defined above.

If I wrote this more carefully, the argument would make it clear that there is a bijection between the set $X$ and the set of all functions $\{y\} \longrightarrow X$.

**Solution 1.56.** (a) I claim that $f$ must be injective. Suppose $a, a' \in A$ such that $f(a) = f(a')$. Then

$$(g \circ f)(a) = g(f(a)) = g(f(a')) = (g \circ f)(a').$$

Since $g \circ f$ is injective, we must have $a = a'$.

There is nothing we can say about the function $g$. For a counterexample showing that $g$ need not be injective, we can take $f: \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = e^x$ and $g: \mathbf{R} \longrightarrow \mathbf{R}$ given by $g(x) = x^2$.

(b) I claim that $g$ must be surjective. Suppose $c \in C$. Since $g \circ f: A \longrightarrow C$ is surjective, there exists $a \in A$ such that $g(f(a)) = c$. Let $b = f(a) \in B$. Then $g(b) = g(f(a)) = c$.

There is nothing we can say about the function $f$. For a counterexample showing that $f$ need not be surjective, we can take $f: \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = 0$ and $g: \mathbf{R} \longrightarrow \{1\}$ given by $g(x) = 1$.

(c) We can only conclude what parts (a) and (b) tell us, namely that $f$ is injective and $g$ is surjective.

**Solution 1.57.** Let $h(a, b) = \big(f(a), g(b)\big)$.

We check that $h$ is injective: suppose $h(a_1, b_1) = h(a_2, b_2)$, so that $f(a_1) = f(a_2)$ and $g(b_1) = g(b_2)$. Since $f$ and $g$ are injective, we get $a_1 = a_2$ and $b_1 = b_2$, so that $(a_1, b_1) = (a_2, b_2)$.

To see that $h$ is surjective, consider an arbitrary element $(c, d) \in C \times D$. Then $c \in C$; since $f$ is surjective, there exists $a \in A$ such that $f(a) = c$. Also $d \in D$; since $g$ is surjective, there exists $b \in B$ such that $g(b) = d$. Then $(a, b) \in A \times B$ and $h(a, b) = \big(f(a), g(b)\big) = (c, d)$.

**Solution 1.58.** (a) First we show that $f(X \cup Y) \subseteq f(X) \cup f(Y)$. If $b \in f(X \cup Y)$, then there exists $a \in X \cup Y$ such that $b = f(a)$. So we have $a \in X$ with $b = f(a)$, so that $b \in f(X)$, or $a \in Y$ with $b = f(a)$, so that $b \in f(Y)$. In any case, $b \in f(X) \cup f(Y)$.

Next we show that $f(X) \cup f(Y) \subseteq f(X \cup Y)$. If $b \in f(X) \cup f(Y)$, then $b \in f(X)$ or $b \in f(Y)$. In the first case, we have that $b = f(a)$ for some $a \in X$; in the second case, we have that $b = f(a)$ for some $a \in Y$. In any of the cases, we have $b = f(a)$ for some $a \in X \cup Y$, so that $b \in f(X \cup Y)$.

(b) If $b \in f(X \cap Y)$ then $b = f(a)$ for some $a \in X \cap Y$. Since $a \in X$, we see that $b \in f(X)$; and since $a \in Y$, we see that $b \in f(Y)$, so we conclude that $b \in f(X) \cap f(Y)$.

The opposite inclusion does not always hold. (Follow-up exercise for you: try to prove the opposite inclusion, and identify where the "proof" fails to be correct.) For a counterexample, take $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x^2$, $X = [1, 2]$ and $Y = [-2, -1]$. Then $f(X) = [1, 4] = f(Y)$, so that $f(X) \cap f(Y) = [1, 4]$, but $X \cap Y = \varnothing$ so that $f(X \cap Y) = \varnothing$.

**Solution 1.59.** (a) If $a \in f^{-1}(X \cup Y)$ then $f(a) \in X \cup Y$, so $f(a) \in X$ or $f(a) \in Y$, which means that $a \in f^{-1}(X)$ or $a \in f^{-1}(Y)$, so that $a \in f^{-1}(X) \cup f^{-1}(Y)$.

If $a \in f^{-1}(X) \cup f^{-1}(Y)$, then $a \in f^{-1}(X)$ or $a \in f^{-1}(Y)$, so $f(a) \in X$ or $f(a) \in Y$, which means that $f(a) \in X \cup Y$, so that $a \in f^{-1}(X \cup Y)$.

(b) If $a \in f^{-1}(X \cap Y)$ then $f(a) \in X \cap Y$, so $f(a) \in X$ and $f(a) \in Y$, which means that $a \in f^{-1}(X)$ and $a \in f^{-1}(Y)$, so that $a \in f^{-1}(X) \cap f^{-1}(Y)$.

The opposite inclusion always holds: if $a \in f^{-1}(X) \cap f^{-1}(Y)$, then $a \in f^{-1}(X)$ and $a \in f^{-1}(Y)$, so $f(a) \in X$ and $f(a) \in Y$, hence $f(a) \in X \cap Y$, so that $a \in f^{-1}(X \cap Y)$. (Follow-up exercise: why doesn't the counterexample from the proof of Exercise 1.58(b) translate into a counterexample here?)

**Solution 1.60.** (a) Suppose $f$ is injective and let $X \subseteq A$.

If $a \in f^{-1}(f(X))$ then $f(a) \in f(X)$ so there exists $x \in X$ such that $f(a) = f(x)$; since $f$ is injective, this implies that $a = x$, so in particular $a \in X$. Therefore $f^{-1}(f(X)) \subseteq X$.

If $x \in X$ then $f(x) \in f(X)$ so $x \in f^{-1}(f(X))$; in other words $X \subseteq f^{-1}(f(X))$.

Conversely, suppose $f^{-1}(f(X)) = X$ for all $X \subseteq A$ but $f$ is not injective. Then there exist $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ but $a_1 \neq a_2$. Let $X = \{a_1\}$ and $b = f(a_1) = f(a_2)$, then $f(X) = \{f(a_1)\} = \{b\}$, but $\{a_1, a_2\} \subseteq f^{-1}(\{b\}) = f^{-1}(f(X))$, so certainly $f^{-1}(f(X)) \neq X$, contradiction.

(b) Suppose $f$ is surjective and let $X \subseteq B$.

If $b \in f(f^{-1}(X))$ then there exists $a \in f^{-1}(X)$ such that $f(a) = b$, but then $f(a) \in X$, so $b \in X$. We conclude that $f(f^{-1}(X)) \subseteq X$.

Now suppose $x \in X$. Since $f$ is surjective, there exists $a \in A$ such that $f(a) = x$. This means that $a \in f^{-1}(X)$, and then that $x \in f(f^{-1}(X))$, so we conclude that $X \subseteq f(f^{-1}(X))$.

Conversely, suppose $f(f^{-1}(X)) = X$ for all $X \subseteq B$. Take an arbitrary $b \in B$ and define $X = \{b\}$. If $f^{-1}(\{b\}) = \varnothing$, then $f(f^{-1}(\{b\})) = f(\varnothing) = \varnothing \neq \{b\}$, contradiction. So we must have that $f^{-1}(\{b\}) \neq \varnothing$ for all $b \in B$, which means that $f$ is surjective.

**Solution 1.61.** (a) Since $B$ is nonempty, it contains at least one element $b_0 \in B$. For any $a \in A$, we have $\pi_A(a, b_0) = a$, so $\pi_A$ is surjective.

The proof for $\pi_B$ is similar.

(b) We are asked to look for $h : X \longrightarrow A \times B$ such that for all $x \in X$ we have

$$\pi_A(h(x)) = f(x),$$
$$\pi_B(h(x)) = g(x).$$

By the definition of $\pi_A$ and $\pi_B$, this forces us to take

$$h(x) = \big(f(x), g(x)\big).$$

This function $h$ satisfies the required property, and is unique.

**Solution 1.62.** (a) Suppose the empty set $\varnothing$ has cardinality $n \geqslant 1$. Then there exists a bijection $f : \varnothing \longrightarrow \{1, 2, \ldots, n\}$. This has an inverse function $f^{-1} : \{1, 2, \ldots, n\} \longrightarrow \varnothing$, but this contradicts the fact that there are no such functions, see 1.1.55(a).

(b) Since $A$ has cardinality $n$, there exists a bijection $f : A \longrightarrow \{1, 2, \ldots, n\}$. Given $a \in A$, define

$$g : A \setminus \{a\} \longrightarrow \{1, 2, \ldots, n-1\}$$

by the rule

$$g(x) = \begin{cases} f(x) & \text{if } f(x) < f(a) \\ f(x) - 1 & \text{if } f(x) > f(a). \end{cases}$$

(Since $f$ is bijective, we cannot have $f(x) = f(a)$ with $x \in A \setminus \{a\}$.)

I claim that $g$ is a bijection.

Suppose $g(x_1) = g(x_2)$ with $x_1, x_2 \in A \setminus \{a\}$. There are three possibilities:

- if $f(x_1) < f(a)$ and $f(x_2) < f(a)$, then $f(x_1) = g(x_1) = g(x_2) = f(x_2)$, so $x_1 = x_2$ by the injectivity of $f$;
- if $f(x_1) > f(a)$ and $f(x_2) > f(a)$, then $f(x_1) - 1 = g(x_1) = g(x_2) = f(x_2) - 1$, so $f(x_1) = f(x_2)$ so $x_1 = x_2$ by the injectivity of $f$;
- if $f(x_1) < f(a)$ and $f(x_2) > f(a)$, then, since $f(a), f(x_1), f(x_2) \in \mathbf{N}$, we have $f(x_2) \geqslant f(a) + 1 \geqslant f(x_1) + 2$; but $f(x_1) = g(x_1) = g(x_2) = f(x_2) - 1$, contradiction. So this case is impossible.

In all possible cases we concluded that $x_1 = x_2$, so $g$ is injective.

To show that $g$ is surjective, let $k \in \{1, 2, \ldots, n-1\}$.

If $k < f(a)$, let $x = f^{-1}(k)$, then $f(x) = k < f(a)$ so $g(x) = k$.

If $k \geqslant f(a)$, let $x = f^{-1}(k+1)$, then $f(x) = k+1 > f(a)$, so $g(x) = f(x) - 1 = k$.

**Solution 1.63.** We use induction on $n$.

For the base case: if $n = 0$ then $A$ is the empty set, so by Exercise 1.62(a) it cannot have another cardinality $m \geqslant 1$.

For the induction step: take $n \in \mathbf{N}$ arbitrary but fixed, and suppose the statement holds for all sets of cardinality $n$. Let $A$ be a set of cardinality $n+1$. By Exercise 1.62(a), $A$ is not empty, so let $a \in A$. Suppose that $A$ also has another cardinality $m \neq n+1$. By Exercise 1.62(b), $A \setminus \{a\}$ has cardinality $n$ and cardinality $m - 1 \neq n$, contradicting the induction hypothesis.

**Solution 1.64.** We proceed by induction on the cardinality $n$ of $A$.

For the base case $n = 0$: $A$ is the empty set, which forces $B$ to also be the empty set, hence $B = A$ and we are done.

For the induction step: let $n \in \mathbf{N}$ be arbitrary but fixed and suppose that the statement holds for all sets $A$ of cardinality $n$. Let $A'$ be a set of cardinality $n+1$ and let $B \subseteq A'$. If $B = A'$ then the claim is obvious. If $B \subsetneq A'$ then there exists $a \in A'$ such that $a \notin B$. Let $A = A' \setminus \{a\}$, then by 1.1.62(b) we know that $A$ has cardinality $n$. Since $a \notin B$, we know that $B \subseteq A$, so by the induction hypothesis we conclude that $B$ is finite and $\#B \leqslant \#A \leqslant \#A'$.

**Solution 1.65.** (a) Let $m$ be the cardinality of $A$. If $m = 0$ then $A = \varnothing$ and $A \cup \{x\} = \{x\}$ clearly has cardinality 1, so we are done.

If $m \geqslant 1$, then there exists a bijection $f : A \longrightarrow \{1, 2, \ldots, m\}$. We construct a function $g : A \cup \{x\} \longrightarrow \{1, 2, \ldots, m, m+1\}$ by the rule:

$$g(y) = \begin{cases} f(y) & \text{if } y \in A, \\ m+1 & \text{if } y = x. \end{cases}$$

You should check that $g$ is indeed bijective.

(b) Induction on $n$.

For the base case $n = 0$, the statement is trivial as $A \cup \{x_1, \ldots, x_n\} = A$ is clearly finite of cardinality $\#A + 0$.

For the induction step: let $n \in \mathbf{N}$ be arbitrary but fixed and suppose the statement holds for this value of $n$. Given $x_1, \ldots, x_n, x_{n+1}$ distinct elements of $X \setminus A$, let $B = A \cup \{x_1, \ldots, x_n\}$. By the induction hypothesis $B$ is finite and

$$\#B = \#A + n.$$

Now apply part (a) to $B \cup \{x_{n+1}\}$ to conclude that it is finite and

$$\#\big(A \cup \{x_1, \ldots, x_n, x_{n+1}\}\big) = \#\big(B \cup \{x_{n+1}\}\big) = \#B + 1 = \#A + (n+1).$$

**Solution 1.66.** (a) Since $A \cap B \subseteq A$ and $A$ is finite, $A \cap B$ is also finite by 1.1.64(a). For the union, note that

$$A \cup B = A \cup \big(B \setminus A\big),$$

and since $B \setminus A \subseteq B$, it is finite, say of cardinality $n$. Write $B \setminus A = \{x_1, \ldots, x_n\}$, then we conclude by applying 1.1.65(b).

(b) Continuing with the notation of the previous part, we have

(1) $$\#(A \cup B) = \#(A \cup (B \setminus A)) = \#A + \#(B \setminus A).$$

However

$$B = \big(B \setminus A\big) \cup \big(B \cap A\big)$$

and the two subsets $B \setminus A$ and $B \cap A$ are disjoint, so by 1.1.65(b) we have

$$\#B = \#\big(B \setminus A\big) + \#\big(B \cap A\big).$$

Combining this with Equation (1) we get

$$\#(A \cup B) = \#A + \#B - \#(A \cap B).$$

**Solution 1.67.** First, I claim that

$$\#A \leqslant \sum_{j=1}^{n} \#A_j.$$

We prove this by induction on $n$. If $n = 0$, there are no sets so $A = \varnothing$ and the claimed inequality is $0 = 0$.

For the induction step: let $n \in \mathbf{N}$ be arbitrary but fixed and suppose the statement holds for this value of $n$. Given finite sets $A_1, \ldots, A_n, A_{n+1}$, we let $A$ denote their union and $B$ the union of $A_1, \ldots, A_n$. Then

$$\#A = \#\big(B \cup A_{n+1}\big) \leqslant \#B + \#A_{n+1} \leqslant \left(\sum_{j=1}^{n} A_j\right) + \#A_{n+1} = \sum_{j=1}^{n+1} \#A_j,$$

where we first used the definitions of $A$ and $B$, then the inclusion-exclusion principle 1.1.66(b) for $B$ and $A_{n+1}$, and then the induction hypothesis.

Now to prove the pigeonhole principle, we proceed by contradiction, assuming that for all $j$ we have $\#A_j \leqslant 1$.

We have then

$$n < \#A \leqslant \sum_{j=1}^{n} \#A_j \leqslant \sum_{j=1}^{n} 1 = n,$$

contradiction.

**Solution 1.68.**

(a)
- Given $x \in A$, we have $f(x) = f(x)$ so $x \sim x$.
- If $x \sim y$, then $f(x) = f(y)$, so $f(y) = f(x)$, that is $y \sim x$.
- If $x \sim y$ and $y \sim z$ then $f(x) = f(y)$ and $f(y) = f(z)$, so that $f(x) = f(z)$, that is $x \sim z$.

(b)
- Given $k \in \mathbf{Z}$, $k - k = 0$ is divisible by $n$.
- If $k \sim m$, then $m - k = na$ for some $a \in \mathbf{Z}$, therefore $k - m = -na$, so $m \sim k$.
- If $k \sim m$ and $m \sim \ell$ then $m - k = na$ and $\ell - m = nb$ for some $a, b \in \mathbf{Z}$. Therefore $\ell - k = n(a + b)$ so $k \sim \ell$.

(c)
- Given $X \in A$, the identity function $\mathrm{id}_X \colon X \longrightarrow X$ is bijective, so $X \sim X$.
- If $X \sim Y$ then there is a bijective function $f \colon X \longrightarrow Y$, so there's a bijective inverse function $f^{-1} \colon Y \longrightarrow X$, that is $Y \sim X$.
- If $X \sim Y$ and $Y \sim Z$, then there are bijective functions $f \colon X \longrightarrow Y$ and $g \colon Y \longrightarrow Z$. The composition $g \circ f \colon X \longrightarrow Z$ is bijective, so $X \sim Z$.

(d) Let $x, z \in A$. There are two possibilities:
- $x \sim z$: given $y \in [x]$, we have $x \sim y$, so $y \sim x$, so $y \sim z$, so $y \in [z]$. This tells us that $[x] \subseteq [z]$, and the other inclusion follows the same way from $z \sim x$. Therefore $[x] = [z]$.
- $x \not\sim z$: suppose $[x] \cap [z]$ is not empty, and pick some element $y$ in there. Then $y \in [x]$ so $y \sim x$, and $y \in [z]$ so $y \sim z$, implying that $x \sim z$, contradiction. Therefore $[x] \cap [z] = \varnothing$.

(e) Given $m \in \mathbf{Z}$, let $0 \leqslant r \leqslant n - 1$ be the remainder of the division of $m$ by $n$: $m = qn + r$. Then $m - r$ is divisible by $n$, hence $m \sim r$. From the previous part, we know that there are at most $n$ equivalence classes, one for each possible value of $r$. To show that we have exactly $n$ equivalence classes, we need to prove that $[r_1] \neq [r_2]$ for any $r_1 \neq r_2$ with $0 \leqslant r_1, r_2 \leqslant n - 1$. We do this by contradiction: if $[r_1] = [r_2]$ then $r_1 \sim r_2$, so $r_2 - r_1$ is a multiple of $n$. But $-(n - 1) \leqslant r_2 - r_1 \leqslant (n - 1)$, and the only multiple of $n$ in that interval is 0, in other words $r_2 = r_1$, contradiction.

(f) Suppose $\pi$ is bijective. I claim that the only way $x \sim y$ can happen is if $x = y$: if $x \sim y$ then $\pi(x) = \pi(y)$, but $\pi$ is bijective so $x = y$.
We conclude that the equivalence relation on $A$ must be given by: $x \sim y$ if and only if $x = y$.

(g) i.
- Given $(a, b) \in \mathbf{N} \times \mathbf{N}$, we have $a + b = b + a$ so $(a, b) \sim (a, b)$.
- If $(a, b) \sim (c, d)$ then $a + d = b + c$, so $c + b = d + a$, that is $(c, d) \sim (a, b)$.

- If $(a,b) \sim (c,d)$ and $(c,d) \sim (x,y)$ then $a+d=b+c$ and $c+y=d+x$. Adding these two equalities gives $a+d+c+y=b+c+d+x$, and cancelling out $c+d$ on both sides we get $a+y=b+x$, that is $(a,b) \sim (x,y)$.

ii. Define $g\colon B \longrightarrow \mathbf{Z}$ by $g\big([(a,b)]\big) = b-a$. We first need to make sure that this is a well-defined function, in other words that the value does not depend on the chosen representative $(a,b)$ of $[(a,b)]$: suppose $(a',b') \in [(a,b)]$, then $(a',b') \sim (a,b)$ so $a'+b=b'+a$, hence $a'-b'=a-b$.

Let's show that $g$ is injective: if $g([(a,b)]) = g([(c,d)])$ then $a-b=c-d$, so $a+d=b+c$, so $(a,b) \sim (c,d)$, so $[(a,b)] = [(c,d)]$.

Finally, to see that $g$ is surjective, let $n \in \mathbf{Z}$. If $n \geqslant 0$ then $n = g([(n+1,1)])$; if $n < 0$ then $n = g([(1,1-n)])$.