3 Vector spaces and linear transformations

Vector space axioms

Exercise 3.1. Let V be a vector spaces. Prove the following consequences of the vector space axioms:

(a) $0\mathbf{x} = \mathbf{0}$ for all vectors $\mathbf{x} \in V$;

(b) $(-1)\mathbf{x} = -\mathbf{x}$ for all vectors $\mathbf{x} \in V$.

Linear transformations

Exercise 3.2. Show that each of the following functions is a linear transformation:

(a) $S: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ given by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ x + y \end{bmatrix}$;

(b) $T: \mathbf{R}^3 \longrightarrow M_{2\times 2}(\mathbf{R})$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y & z \\ -x & 0 \end{bmatrix}$.

Exercise 3.3. For the linear transformations $\mathbf{R}^2 \longrightarrow \mathbf{R}^2$ given by the following matrices, (i) sketch the image of the rectangle with vertices $(0,0), (2,0), (0,1), (2,1)$; (ii) describe the geometric effect of the linear transformation.

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

(e) $\begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix}$

(f) $\frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

(The constants a, b, c are assumed nonzero.)

Exercise 3.4. Find the matrix of the following linear transformations $\mathbf{R}^2 \longrightarrow \mathbf{R}^2$.

(a) rotation by $\frac{3\pi}{4}$;

(b) rotation by $-\frac{\pi}{2}$;

(c) reflection in the line $y = x$;

(d) reflection in the $x-$axis.

Exercise 3.5. In each part, find a single matrix that performs the indicated succession of operations:

(a) compresses by a factor of $\frac{1}{2}$ in the x-direction, then expands by a factor of 5 in the y-direction;

(b) expands by a factor of 5 in the y-direction, then shears by a factor of 2 in the y-direction;

(c) reflects about $y = x$, then rotates about the origin through an angle of π;

(d) reflects about the y-axis, then expands by a factor of 5 in the x-direction, and then reflects about $y = x$.

(e) rotates through $\frac{\pi}{6}$ or $30°$, then shears by a factor of -2 in the y-direction, and then expands by a factor of 3 in the y-direction.

Exercise 3.6. Determine whether or not $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$ are in the kernel of the linear transformation $T: \mathbf{R}^4 \longrightarrow \mathbf{R}^3$ given by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & -4 & 6 & 2 \end{bmatrix}.$$

Exercise 3.7. Determine whether or not $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}$ are in the image of the linear transformation $T: \mathbf{R}^4 \longrightarrow \mathbf{R}^3$ given by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & -4 & 6 & 2 \end{bmatrix}.$$

Exercise 3.8. Determine whether or not the given linear transformation is invertible. If it is invertible, compute its inverse.

(a) $T: \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ x-y+z \\ y+2z \end{bmatrix}$;

(b) $T: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x+2y \\ -6x-4y \end{bmatrix}$;

(c) $R_\theta: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ an anticlockwise rotation around the origin by an angle of θ;

(d) $S_\theta: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ a reflection in the line through the origin that forms an angle θ with the x-axis.

Exercise 3.9. Show that the transformation $T: \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y+z \\ z+x \end{bmatrix}$$

is invertible and find its inverse.

Exercise 3.10. Consider the matrix

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Evaluate $\det A_\theta$.

(b) Interpret geometrically the effect of multiplying a vector by A_θ.

(c) Show that $A_\theta A_\varphi = A_{\theta+\varphi}$ and interpret this result.

(d) Use the previous part to find the inverse of A_θ. How does this compare this with the transpose A_θ^T?

Subspaces

Exercise 3.11. For each of the following subsets of $\mathbf{R}^2$ sketch the set, then determine whether it is (i) closed under addition, (ii) closed under scalar multiplication, (iii) a subspace of $\mathbf{R}^2$.

(a) $\left\{\begin{bmatrix} x \\ y \end{bmatrix} : y \geqslant 0\right\}$;

(b) $\left\{\begin{bmatrix} x \\ y \end{bmatrix} : x = y\right\}$;

(c) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leqslant 1 \right\}$;

(d) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy = 0 \right\}$.

Exercise 3.12. Decide which of the following are subspaces of $\mathbf{R}^3$. (For the subspaces, see if you can realise them as kernels of linear transformations. For the non-subspaces, find a condition of the Subspace Theorem that is not satisfied.)

(a) $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \in \mathbf{R}^3 : \ a, b \in \mathbf{R} \right\}$;

(b) $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbf{R}^3 : \ 2a - 3b + 5c = 4 \right\}$;

(c) $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbf{R}^3 : \ 2a - 3b + 5c = 0 \right\}$;

(d) $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbf{R}^3 : \ a_1 \geqslant 0 \right\}$;

(e) $\left\{ \begin{bmatrix} a - b \\ a + b \\ 2a \end{bmatrix} \in \mathbf{R}^3 : \ a, b \in \mathbf{R} \right\}$.

Exercise 3.13. Show that the following sets of vectors are subspaces of $\mathbf{R}^m$.

(a) The set of all linear combinations of the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ (of $\mathbf{R}^4$).

(b) The set of all vectors of the form $\begin{bmatrix} a \\ b \\ a - b \\ a + b \end{bmatrix}$ (of $\mathbf{R}^4$).

(c) The set of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x + y + z = 0$ (of $\mathbf{R}^3$).

Exercise 3.14. Show that the following sets of vectors are **not** subspaces of $\mathbf{R}^m$.

(a) The set of all vectors whose first component is 2.

(b) The set of all vectors **except** the vector $\mathbf{0}$.

(c) The set of all vectors the sum of whose components is 1.

Exercise 3.15. Which of the following are real vector spaces with the **usual** operations. You may assume that the set of all real polynomials is a vector space.

(a) The set of real polynomials of degree $\leqslant n$.

(b) The set of real polynomials of degree exactly n.

(c) The set of real polynomials p with $p(0) = 0$.

(d) The set of real polynomials p with $p(0) = 1$.

(e) The set of all differentiable functions.

(f) The set of all solutions of the differential equation $y'' - 3y' + 2y = 0$.

Exercise 3.16. Determine whether or not the given set is a subspace of $M_{2\times 2}$:

(a) The set of all 2×2 matrices, the sum of whose entries is zero.

(b) The set of all 2×2 matrices whose determinant is zero.

Exercise 3.17. Determine whether or not the given set is a subspace of $M_{n\times n}$, the space of all $n \times n$ matrices.

(a) The $n \times n$ diagonal matrices.

(b) The $n \times n$ matrices with trace equal to 0.

Linear combinations and spanning sets

Exercise 3.18. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

(a) Write $\mathbf{w}_1 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(b) Show that $\mathbf{w}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ cannot be written as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(c) For what value of c is the vector $\begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix}$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$?

Exercise 3.19. Is $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ a linear combination of the matrices

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}?$$

Exercise 3.20. Express $q = -9 - 7x - 15x^2$ as a linear combination of $p_1 = 2 + x + 4x^2$, $p_2 = 1 - x + 3x^2$, and $p_3 = 3 + 2x + 5x^2$.

Exercise 3.21. Determine whether the given set spans the given vector space.

(a) In $\mathbf{R}^2$: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$.

(b) In $\mathbf{R}^3$: $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \\ 5 \end{bmatrix} \right\}$.

(c) In $\mathbf{F}_2^3$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Exercise 3.22. Determine which of the following sets span $\mathbf{R}^3$.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

(b) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$.

Exercise 3.23. Find spanning sets for the following subspaces of $\mathbf{R}^3$:

(a) $\left\{ \begin{bmatrix} 2a \\ b \\ 0 \end{bmatrix} \ : \ a, b \in \mathbf{R} \right\}$.

(b) $\left\{ \begin{bmatrix} a+c \\ c-b \\ 3c \end{bmatrix} : a, b, c \in \mathbf{R} \right\}$.

(c) $\left\{ \begin{bmatrix} 4a+d \\ a+2b \\ c-b \end{bmatrix} : a, b, c, d \in \mathbf{R} \right\}$.

Exercise 3.24. Which of the following lie in the complex vector space spanned by $f = e^{ix}$ and $g = e^{-ix}$?

(a) $\cos x$;

(b) $\sin x$;

(c) $\cos x + 3i \sin x$;

(d) $\sin 2x$;

(e) $\cosh x$.

Exercise 3.25. Determine whether the given set of vectors spans the given vector space.

(a) In $\mathcal{P}_2(\mathbf{R})$: $\{1-x, 3-x^2\}$.

(b) In $M_{2\times 2}(\mathbf{R})$: $\left\{ \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix} \right\}$.

(c) In $M_{2\times 2}(\mathbf{F}_2)$: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$.

Linear independence and bases

Exercise 3.26. In this question let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ be vectors in $\mathbf{R}^3$ and let A be the 3×5 matrix with the i-th column given by the vector $\mathbf{v}_i$. Suppose that the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Are the following sets linearly dependent or independent? If linearly dependent, express one vector as a linear combination of the others.

(a) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$;

(b) $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$;

(c) $\{\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5\}$;

(d) $\{\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$.

Exercise 3.27. Determine whether or not the following sets of vectors are linearly independent:

(a) $\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix} \right\}$;

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix} \right\}$.

Exercise 3.28. Determine whether the following sets are linearly dependent or linearly independent.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$;

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$;

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$;

(d) $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \\ -3 \end{bmatrix} \right\}$.

Exercise 3.29. Which of the following sets of vectors in $\mathbf{C}^3$ are linearly independent?

(a) $\left\{ \begin{bmatrix} 1-i \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1+i \\ 0 \end{bmatrix}, \begin{bmatrix} 1+i \\ i \\ 0 \end{bmatrix} \right\}$;

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix}, \begin{bmatrix} 1+i \\ 1 \\ 1-2i \end{bmatrix}, \begin{bmatrix} 0 \\ i \\ 2 \end{bmatrix} \right\}$;

(c) $\left\{ \begin{bmatrix} i \\ 0 \\ 2-i \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}, \begin{bmatrix} -i \\ -1-4i \\ 3 \end{bmatrix} \right\}$.

Exercise 3.30. Which of the following sets of vectors in $\mathbf{F}_2^3$ are linearly independent?

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$;

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$;

(c) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Exercise 3.31. Show that the vectors $\begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}, \begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix}, \begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix}$ are linearly independent if a, b, c are distinct (i.e., $a \neq b$, $a \neq c$ and $b \neq c$). Can you generalise this?

Exercise 3.32. For each of the following sets, determine whether it is (i) linearly independent, (ii) spanning the vector space, (iii) a basis of the vector space. If the set is linearly dependent, write one of its vectors as a linear combination of the others.

(a) $\{1, 1+x, 1+x+x^2\}$ in $\mathcal{P}_2$;

(b) $\{1+x^2, 1+x+2x^2, x+x^2\}$ in $\mathcal{P}_2$;

(c) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ in $M_{2\times 2}$;

(d) $\{1, \sin^2 x, \cos^2 x\}$ in $C(\mathbf{R})$, the vector space of all continuous functions from $\mathbf{R}$ to $\mathbf{R}$;

(e) $\left\{ \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$ in $M_{3\times 3}$;

(f) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ in $M_{2\times 2}(\mathbf{F}_2)$.

Exercise 3.33. In each part determine whether or not the given set forms a basis for the indicated (sub)space.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ for $\mathbf{R}^3$.

(b) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$ for $\mathbf{R}^3$.

(c) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ for the subspace of $\mathbf{R}^3$ consisting of all $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x+y+z = 0$.

(d) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ for the subspace of $\mathbf{R}^3$ consisting of all $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $y = x + z$.

(e) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ for the subspace of $\mathbf{F}_2^3$ consisting of all $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x+y+z = 0$.

Exercise 3.34. Which of the following sets of vectors are bases for $\mathbf{R}^3$?

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right\}$;

(b) $\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -7 \\ 1 \end{bmatrix} \right\}$.

Exercise 3.35. Which of the following sets of vectors are bases for $\mathcal{P}_2(\mathbf{R})$?

(a) $\{1 - 3x + 2x^2, 1 + x + 4x^2, 1 - 7x\}$;

(b) $\{1 + x + x^2, x + x^2, x^2\}$.

Exercise 3.36. Show that the following set of vectors is a basis for $M_{2\times 2}(\mathbf{R})$:

$$\left\{ \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \right\}.$$

Exercise 3.37. Find a basis for and the dimension of the subspace of $\mathbf{R}^n$ spanned by the following sets.

(a) $\left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} \right\}$;

(b) $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \end{bmatrix} \right\}$;

(c) $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -2 \\ 0 \end{bmatrix} \right\}.$

Exercise 3.38. For each of the following sets choose a **subset** that is a basis for the subspace spanned by the set. Then express each vector that is not in the basis as a linear combination of the basis vectors.

(a) $\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -11 \\ 6 \\ 13 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix};$

(b) $\begin{bmatrix} 0 \\ -1 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 1 \end{bmatrix};$

(c) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix};$

(d) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ in $\mathbf{F}_2^4$.

Exercise 3.39. In each part explain why the given statement is true "by inspection."

(a) The set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \right\}$ is linearly dependent.

(b) The set $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ does not span $\mathbf{R}^3$.

(c) If the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ of vectors in $\mathbf{R}^4$ is linearly independent, then it spans $\mathbf{R}^4$.

(d) The set $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is linearly independent, and so it spans the subspace of

$\mathbf{R}^4$ of all vectors of the form $\begin{bmatrix} 0 \\ a \\ b \\ 0 \end{bmatrix}$.

(e) Any four polynomials in $\mathcal{P}_2(\mathbf{R})$ are linearly dependent.

(f) No two polynomials can span $\mathcal{P}_2(\mathbf{R})$.

Exercise 3.40. Prove that if V and W are three-dimensional subspaces of $\mathbf{R}^5$, then V and W must have a nonzero vector in common.
(Hint: Start with bases for the two subspaces, making six vectors in all.)

Exercise 3.41. Find the dimension of the given vector space:

(a) The subspace of $M_{2\times 2}(\mathbf{R})$ consisting of all diagonal 2×2 matrices.

(b) The subspace of $M_{2\times 2}(\mathbf{R})$ consisting of all 2×2 matrices whose diagonal entries are zero.

(c) The subspace of $\mathcal{P}_3(\mathbf{R})$ consisting of all polynomials $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ with $a_2 = 0$.

Exercise 3.42. For each linear transformation below find (i) its standard matrix, (ii) a basis for the kernel, and (iii) a basis for the image.

(a) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 3y \end{bmatrix}$;

(b) $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_1 + x_2 \end{bmatrix}$;

(c) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ -y \\ x-y \end{bmatrix}$;

(d) $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - x_2 - 6x_3 \\ -2x_1 + x_2 + 5x_3 \\ 3x_1 + 3x_2 + 6x_3 \end{bmatrix}$.

Coordinate vectors

Exercise 3.43. Find the coordinate vector of $\mathbf{v}$ with respect to the given ordered basis $\mathcal{B}$ for the vector space V.

(a) $\mathbf{v} = 2 - x + 3x^2$, $\mathcal{B} = (1, x, x^2, x^3)$, $V = \mathcal{P}_3(\mathbf{R})$.

(b) $\mathbf{v} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$, $\mathcal{B} = \left(E^{ij} : \ i = 1,2; j = 1,2,3\right)$, $V = M_{2\times 3}(\mathbf{R})$.

(c) $\mathbf{v} = 2 - 5x$, $\mathcal{B} = (x+1, x-1)$, $V = \mathcal{P}_1(\mathbf{R})$.

(d) $\mathbf{v} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$, $\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$, V is the vector space of all diagonal 2×2 matrices.

Exercise 3.44. (a) Show that the ordered set $\mathcal{B} = \left(\begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right)$ is a basis for $\mathbf{R}^3$.

(b) Find the vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$ whose coordinates with respect to $\mathcal{B}$ are

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad [\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

(c) For each of the following vectors find the coordinates with respect to $\mathcal{B}$:

$$\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix}.$$

Exercise 3.45. Use coordinate vectors to decide whether or not the given set is linearly independent. If it is linearly **dependent**, express one of the vectors as a linear combination of the others.

(a) $\{x^2 + x - 1, x^2 - 2x + 3, x^2 + 4x - 3\} \in \mathcal{P}_2(\mathbf{R})$;

(b) $\left\{\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\right\}$ in $M_{2\times 2}(\mathbf{R})$.

Row space, column space, and kernel

Exercise 3.46. In each part find a basis for and the dimension of the indicated subspace.

(a) The solution space of the homogeneous linear system:

$$\begin{cases} x_1 - 2x_2 + x_3 \phantom{{}+x_4} = 0 \\ \phantom{x_1 - 2{}} x_2 - x_3 + x_4 = 0 \\ x_1 - x_2 \phantom{{}+x_3} + x_4 = 0 \end{cases}$$

(b) The solution space of

$$\begin{cases} x_1 - 3x_2 + x_3 \phantom{{}-x_4} - x_5 = 0 \\ x_1 - 2x_2 + x_3 - x_4 \phantom{{}+x_5} = 0 \\ x_1 - x_2 + x_3 - 2x_4 + x_5 = 0 \end{cases}$$

(c) The subspace of $\mathbf{R}^4$ of all vectors of the form $\begin{bmatrix} x \\ -y \\ x-2y \\ 3y \end{bmatrix}$.

(d) The solution space in $\mathbf{F}_2^5$ of

$$\begin{cases} x_1 + x_2 + x_3 \phantom{{}+x_4} + x_5 = 0 \\ x_1 \phantom{{}+x_2} + x_3 + x_4 \phantom{{}+x_5} = 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 = 0 \end{cases}$$

Exercise 3.47. For each of the following matrices, find (i) the rank, (ii) a basis for the column space, (iii) a basis for the row space, (iv) a basis for the kernel:

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$;

(b) $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$;

(c) $\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$.

Exercise 3.48. Find bases for the following subspaces of $\mathbf{R}^3$.

(a) The set of vectors lying in the plane $2x - y - z = 0$.

(b) The set of vectors on the line $x/2 = y/3 = z/4$.

Matrix representations

Exercise 3.49. Consider the following linear transformations:

$$K\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \\ x+y+z \end{bmatrix}$$
$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x-y \\ x+2y \end{bmatrix}$$
$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z \\ y \\ x \end{bmatrix}$$
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ x+y \\ x-y \\ x-2y \end{bmatrix}.$$

(a) Find the matrix that represents each of these transformations with respect to the standard bases.

(b) Find the indicated linear transformation if it is defined, and give its matrix representation with respect to the standard bases:

i. $L \circ K$;

ii. $T \circ L$;

iii. S^2;

iv. $K + S$;

v. T^2.

Exercise 3.50. Let $T : \mathcal{P}_2(\mathbf{R}) \longrightarrow \mathcal{P}_3(\mathbf{R})$ denote the function defined by multiplication by x: $T(p(x)) = xp(x)$. In other words, $T(a + bx + cx^2) = ax + bx^2 + cx^3$.

(a) Show that T is a linear transformation.

(b) Find the matrix of T with respect to the standard bases $\left(1, x, x^2\right)$ for $\mathcal{P}_2(\mathbf{R})$ and $\left(1, x, x^2, x^3\right)$ for $\mathcal{P}_3(\mathbf{R})$.

Exercise 3.51. For each linear transformation given below: (i) find the matrix that represents the linear transformation with respect to the given bases; (ii) compute the kernel of the transformation, and determine whether it is injective; (iii) compute the image of the linear transformation, and determine whether it is surjective.

(a) $T:\ \mathbf{R}^3 \longrightarrow M_{2\times 2}(\mathbf{R})$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y & z \\ -x & 0 \end{bmatrix},$$

with respect to $\mathcal{B} = \big(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\big)$ and $\mathcal{B}' = \big(E^{ij}:\ i = 1,2; j = 1,2\big)$ (i.e. the standard basis for $M_{2\times 2}(\mathbf{R})$).

(b) $T:\ \mathcal{P}_3(\mathbf{R}) \longrightarrow \mathcal{P}_3(\mathbf{R})$ given by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = (a_0 + a_2) - (a_1 + 2a_3)x^2,$$

with respect to $\mathcal{B} = \mathcal{B}' = \big(1, x, x^2, x^3\big)$.

Exercise 3.52. Let $T:\ M_{2\times 2}(\mathbf{R}) \longrightarrow \mathbf{R}^2$ be the map defined by

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2a_{11} - a_{21} & 2a_{12} - a_{22} \end{bmatrix}.$$

(a) Show that T is a linear transformation.

(b) Find bases for the kernel and image of T. Deduce the rank and nullity of T.

(c) Find the matrix of T with respect to the standard bases of $M_{2\times 2}(\mathbf{R})$ and of $\mathbf{R}^2$.

Exercise 3.53. Let $S:\ \mathcal{P}_2(\mathbf{R}) \longrightarrow \mathcal{P}_3(\mathbf{R})$ be defined as follows. For each $p(x) = a_2x^2 + a_1x + a_0$, define $S(p) = \frac{1}{3}a_2x^3 + \frac{1}{2}a_1x^2 + a_0x$. Find the matrix A that represents S with respect to the bases $\mathcal{B} = \big(1, x, x^2\big)$ and $\mathcal{B}' = \big(1, x, x^2, x^3\big)$.

(The linear transformation S gives the **integral** of $p(x)$, with the constant term equal to zero.)

Use the matrix A to find the integral of $p(x) = 1 - x + 2x^2$.

Change of basis

Exercise 3.54. (a) Find the transition matrix P from $\mathcal{B}$ to $\mathcal{C}$, where

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$$

$$\mathcal{C} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right).$$

(b) Use P to find $[\mathbf{x}]_{\mathcal{B}}$ (the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}$) if

i. $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$;

ii. $\mathbf{x} = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix}$.

Exercise 3.55. Consider the following subset of $\mathbf{R}^2$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}.$$

(a) Check that $\mathcal{B}$ is a basis of $\mathbf{R}^2$.

(b) Find the change of basis matrix from the standard basis $\mathcal{S}$ to $\mathcal{B}$ and use it to find the coordinate vector of $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \in \mathbf{R}^2$ with respect to $\mathcal{B}$.

(c) Let $T: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be given by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Find the matrix representation of T with respect to the basis $\mathcal{B}$.

Exercise 3.56. Consider the following subset of $\mathbf{R}^3$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(a) Check that $\mathcal{B}$ is a basis of $\mathbf{R}^3$.

(b) Find the change of basis matrix from the standard basis $\mathcal{S}$ to $\mathcal{B}$ and use it to find the coordinate vector of $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \in \mathbf{R}^3$ with respect to $\mathcal{B}$.

(c) Let $T: \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be given by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}.$$

Find the matrix representation of T with respect to the basis $\mathcal{B}$.

(d) Let $T' : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be given by $T'\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ x + y + z \\ y - z \end{bmatrix}$. Find the matrix representation of T' with respect to the basis $\mathcal{B}$.

Exercise 3.57. Write down the matrix of T with respect to $\mathcal{B}$, and compute the matrix of T with respect to $\mathcal{B}'$, where $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ -x_2 \end{bmatrix}$ and

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right), \qquad \mathcal{B}' = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix}\right).$$

Exercise 3.58. A linear transformation $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ has matrix

$$\begin{bmatrix} 2 & 3 & 0 \\ -1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

with respect to the standard basis for $\mathbf{R}^3$. Find the matrix of T with respect to the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \right\}.$$

Eigenvectors, eigenvalues, diagonalisation

Exercise 3.59. Find the eigenvalues and linearly independent eigenvectors of the following matrices. If the matrix is diagonalisable, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(a) $\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$;

(b) $\begin{bmatrix} 7 & -2 \\ 15 & -4 \end{bmatrix}$;

(c) $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$;

(d) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Exercise 3.60. Find one-dimensional subspaces of $\mathbf{R}^2$ invariant under left multiplication by the following matrices:

(a) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$;

(b) $\begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}$.

Exercise 3.61. Show that there is no line in the real plane $\mathbf{R}^2$ through the origin that is invariant under left multiplication by the matrix

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

when θ is not an integral multiple of π. Give a geometric interpretation commenting on the case when $\theta = k\pi$ for some $k \in \mathbf{Z}$.

Exercise 3.62. Find the eigenvalues of the given matrix by inspection.

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$;

(b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix}$.

Exercise 3.63. Prove that for an invertible matrix A, λ is an eigenvalue of A if and only if $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. What relationship holds between the eigenvectors of A and A^{-1}?

Exercise 3.64. Find the eigenvalues and linearly independent eigenvectors for the following matrices. If the matrix is diagonalisable, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(a) $\begin{bmatrix} 2 & -3 & 6 \\ 0 & 5 & -6 \\ 0 & 1 & 0 \end{bmatrix}$;

(b) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$;

(c) $\begin{bmatrix} -5 & -8 & -12 \\ -6 & -10 & -12 \\ 6 & 10 & 13 \end{bmatrix}$;

(d) $\begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$.

Exercise 3.65. For each matrix find all eigenvalues and a basis for each eigenspace.

(a) $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$;

(b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Cayley–Hamilton Theorem

Exercise 3.66. Verify the Cayley–Hamilton Theorem for the following matrices.

(a) $\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$;

(b) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$.

Exercise 3.67. Use the Cayley–Hamilton Theorem to calculate the inverse of the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}.$$

Exercise 3.68. For each matrix, find a non-zero polynomial satisfied by the matrix.

(a) $\begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix}$;

(b) $\begin{bmatrix} 1 & 4 & -3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}$.

Applications of diagonalisation

Exercise 3.69. Find A^5, where A is

(a) $\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$;

(b) $\begin{bmatrix} 9 & 18 & -24 \\ 7 & 20 & -24 \\ 7 & 21 & -25 \end{bmatrix}$;

(c) $\frac{1}{8}\begin{bmatrix} 8 & 1 & 27 & 5 \\ 0 & 18 & 14 & -6 \\ 0 & 2 & -18 & -6 \\ 0 & 8 & 8 & -8 \end{bmatrix}$.

Hint: The matrix has eigenvalues $3, 2, -1$ in (b), and $1, 2, -1, -2$ in (c).

Exercise 3.70. The *Fibonacci sequence*

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

by the difference equation $F_{k+2} = F_{k+1} + F_k$ and the initial conditions $F_0 = 0, F_1 = 1$. Writing $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$, show that $u_{k+1} = Au_k$ where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Solve for u_k in terms of $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and show that

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right].$$

Hence find the limit as $k \longrightarrow \infty$ of the ratio F_{k+1}/F_k.

Answers

Solution 3.1. (a) Let $\mathbf{x} \in V$. Then

$$0\mathbf{x} + \mathbf{x} = 0\mathbf{x} + 1\mathbf{x} = (0 + 1)\mathbf{x} = \mathbf{x}.$$

Add $-\mathbf{x}$ to both sides:

$$0\mathbf{x} + \mathbf{x} + (-\mathbf{x}) = \mathbf{x} + (-\mathbf{x}) = \mathbf{0},$$

from which we conclude $0\mathbf{x} + \mathbf{0} = \mathbf{0}$ so $0\mathbf{x} = \mathbf{0}$.

(b) Start with $1 + (-1) = 0$, multiply both sides by $\mathbf{x}$:

$$\mathbf{x} + (-1)\mathbf{x} = 0\mathbf{x} = \mathbf{0}.$$

Now add $-\mathbf{x}$ to both sides:

$$-\mathbf{x} + \mathbf{x} + (-1)\mathbf{x} = -\mathbf{x} + \mathbf{0} = -\mathbf{x},$$

so $\mathbf{0} + (-1)\mathbf{x} = -\mathbf{x}$, hence $(-1)\mathbf{x} = -\mathbf{x}$.

Solution 3.2. (a) We can realise S as left multiplication by the matrix

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

(b) Check the conditions:

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} y_1 + y_2 & z_1 + z_2 \\ -(x_1 + x_2) & 0 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$$

$$T\left(\lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \lambda y & \lambda z \\ -\lambda x & 0 \end{bmatrix} = \lambda T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right).$$

Solution 3.3. (a) Reflection in the line $y = x$.

(b) Orthogonal projection onto the x-axis followed by the reflection in the line $y = x$.

(c) Projection onto the x-axis in the direction of the line $y = -x$.

(d) Shear parallel to the y-axis.

(e) Expansion by factor $|b|$ along the x-axis and by $|c|$ along the y-axis, with a possible change of direction of the axis (if $b < 0$ or $c < 0$).

(f) Rotation about the origin by the angle $\theta = \arctan \frac{4}{3}$.

Solution 3.4. (a) $\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Solution 3.5. (a) $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 5 \end{bmatrix}$;

(b) $\begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}$;

(c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$;

(d) $\begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$;

(e) $\frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ -6\sqrt{3}+3 & 6+3\sqrt{3} \end{bmatrix}$.

Solution 3.6. Both are in the kernel.

Solution 3.7. $\mathbf{w}_1$ is not; $\mathbf{w}_2$ is.

Solution 3.8. (a) $T^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{2}x - \frac{1}{2}y - \frac{1}{2}z \\ x - y \\ -\frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z \end{bmatrix}$;

(b) not invertible;

(c) $R_\theta^{-1} = R_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$;

(d) S_θ is its own inverse.

Solution 3.9.

$$T^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} x - y + z \\ x + y - z \\ -x + y + z \end{bmatrix}.$$

Solution 3.10. (a) 1.

(b) Rotation about the z-axis by θ.

(c) We have

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\varphi - \sin\theta\sin\varphi & -\cos\theta\sin\varphi - \sin\theta\cos\varphi & 0 \\ \sin\theta\cos\varphi + \cos\theta\sin\varphi & -\sin\theta\sin\varphi + \cos\theta\cos\varphi & 0 \\ 0 & 0 & \end{bmatrix}$$
$$= \begin{bmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) & 0 \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where we used the sum-of-angles trigonometric formulas.

(d)

$$A_\theta^{-1} = A_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = A_\theta^T.$$

Solution 3.11. (a) Yes, No, No.

(b) Yes, Yes, Yes.

(c) No, No, No.

(d) No, Yes, No.

Solution 3.12. (a) Yes, take $T:\ \mathbf{R}^3 \longrightarrow \mathbf{R}$ given by $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = c$.

(b) No, as the zero vector is not an element of the subset.

(c) Yes, take $T:\ \mathbf{R}^3 \longrightarrow \mathbf{R}$ given by $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = 2a - 3b + 5c$.

(d) No, as the subset is not closed under scalar multiplication (by -1, for instance).

(e) Yes, take $T:\ \mathbf{R}^3 \longrightarrow \mathbf{R}$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x + y - z$.

Solution 3.13. (a) It is the span of $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(b) It is the span of $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

(c) It is the kernel of the linear transformation $T : \mathbf{R}^3 \longrightarrow \mathbf{R}$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x + y + z$.

Solution 3.14. None of these contain the zero vector of $\mathbf{R}^m$, so they cannot be subspaces.

Solution 3.15. (a) Yes (b) No (c) Yes (d) No (e) Yes (f) Yes

Solution 3.16. (a) Subspace.

(b) Not a subspace.

Solution 3.17. (a) Subspace.

(b) Subspace.

Solution 3.18. (a) $\mathbf{w}_1 = 3\mathbf{u} + 2\mathbf{v}$.

(b) Check that the matrix with columns $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ has rank 3.

(c) $c = -2$.

Solution 3.19. Yes.

Solution 3.20. $q = -2p_1 + p_2 - 2p_3$.

Solution 3.21. (a) Yes.

(b) No.

(c) Yes.

Solution 3.22. (a) Does not span.

(b) Spans.

Solution 3.23. (a) $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$.

(c) $\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Solution 3.24. (a) Yes

(b) Yes

(c) Yes

(d) No

(e) No

Solution 3.25. (a) No

(b) Yes

(c) No

Solution 3.26. (a) Dependent: $\mathbf{v}_2 = 2\mathbf{v}_1$.

(b) Dependent: $\mathbf{v}_4 = -\mathbf{v}_1 + 3\mathbf{v}_3$.

(c) Independent.

(d) Independent.

Solution 3.27. (a) Independent.

(b) Dependent.

Solution 3.28. (a) Dependent.

(b) Independent.

(c) Dependent.

(d) Independent.

Solution 3.29. (a) Dependent.

(b) Independent.

(c) Independent.

Solution 3.30. (a) Dependent.

(b) Independent.

(c) Dependent.

Solution 3.31. Look up the Vandermonde determinant in the self-study problems on matrices and linear systems.

Solution 3.32. (a) Linearly independent, spans, is a basis.

(b) Linearly dependent, does not span, is not a basis. $1 + x + 2x^2 = (1)(1 + x^2) + (1)(x + x^2)$.

(c) Linearly independent, spans, is a basis.

(d) Linearly dependent, does not span, is not a basis. $1 = (1)\sin^2 x + (1)\cos^2 x$.

(e) Linearly independent, does not span, is not a basis.

(f) Linearly dependent, does not span, is not a basis.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Solution 3.33. (a) No.

(b) Yes.

(c) Yes.

(d) No.

(e) Yes.

Solution 3.34. (a) Yes.

(b) No.

Solution 3.35. (a) No

(b) Yes

Solution 3.36. Row reduce the 4×4 matrix

$$\begin{bmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{bmatrix}.$$

(Do you see how it comes from the set in the question?)

Get rank 4 and conclude that the set is a basis.

Solution 3.37. (a) $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$, dimension 2.

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, dimension 2.

(c) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ -1 \\ 14 \end{bmatrix} \right\}$, dimension 2.

Solution 3.38. (a) $\{u_1, u_2, u_3\}, u_4 = 2u_1 + u_2$;

(b) $\{u_1, u_2, u_4\}, u_3 = 2u_1 - 3u_2$;

(c) $\{u_1, u_2, u_4\}, u_3 = 2u_1 - u_2$;

(d) $\{u_1, u_2, u_3\}, u_4 = u_1 + u_2 + u_3$.

Solution 3.39. (a) Have 4 vectors in a 3-dimensional space.

(b) Two vectors cannot span a 3-dimensional subspace.

(c) Four independent vectors span a 4-dimensional space.

(d) Have two linearly independent vectors for a 2-dimensional space.

(e) Any 4 vectors in a 3-dimensional space are linearly dependent.

(f) No 2 vectors can span a 3-dimensional space.

Solution 3.40. As the hint says, start with a basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of V and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of W. Altogether we have 6 vectors in $\mathbf{R}^5$, so they must be linearly dependent. Rewrite the linear dependence relation in the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + b_3\mathbf{w}_3,$$

then the LHS is a vector in V and the RHS is a vector in W. Neither of these can be the zero vector, as that would imply linear dependence among the $\mathbf{v}_i$ or among the $\mathbf{w}_i$, contradicting the fact that they form bases.

Solution 3.41. (a) 2.

(b) 2.

(c) 3.

Solution 3.42. (a) Standard matrix $\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$, basis for kernel $\varnothing$, basis for image $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{vmatrix} 1 \\ 3 \end{vmatrix} \right\}$.

(b) Standard matrix $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$, basis for kernel $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$, basis for image $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

(c) Standard matrix $\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$, basis for kernel $\varnothing$, basis for image $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$.

(d) Standard matrix $\begin{bmatrix} 3 & -1 & -6 \\ -2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$, basis for kernel $\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \right\}$, basis for image $\left\{ \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}$.

Solution 3.43. (a) $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}$;

(b) $\begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}$;

(c) $\begin{bmatrix} -\frac{3}{2} \\ -\frac{7}{2} \end{bmatrix}$;

(d) $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Solution 3.44. (a) Row reduce the matrix with columns the elements of $\mathcal{B}$, get rank 3.

(b) $\begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

Solution 3.45. (a) Independent.

(b) Dependent:

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

Solution 3.46. (a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, dimension 2;

(b) $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, dimension 3;

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 3 \end{bmatrix} \right\}$, dimension 2;

(d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, dimension 2.

Solution 3.47. (a) Rank 2, basis for column space $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$, basis for row space $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$, basis for kernel $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

(b) Rank 1, basis for column space $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, basis for row space $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$, basis for kernel $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(c) Rank 3, basis for column space $\left\{\begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}, \begin{bmatrix}3\\1\\0\\1\end{bmatrix}\right\}$, basis for row space $\left\{\begin{bmatrix}1\\-1\\3\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$, basis for kernel $\varnothing$.

Solution 3.48. (a) $\left\{\begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right\}$;

(b) $\left\{\begin{bmatrix}2\\3\\4\end{bmatrix}\right\}$.

Solution 3.49. (a)

$$\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix} \quad \begin{bmatrix}2&-1&0\\1&2&0\end{bmatrix} \quad \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \quad \begin{bmatrix}2&1\\1&1\\1&-1\\1&-2\end{bmatrix}.$$

(b) i. $LK\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x-y\\2x+y\end{bmatrix}$, matrix representation $\begin{bmatrix}1&-1&0\\3&2&0\end{bmatrix}$;

ii. $TL\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}5x\\3x+y\\x-3y\\-5y\end{bmatrix}$, matrix representation $\begin{bmatrix}5&0&0\\3&1&0\\1&-3&0\\0&-5&0\end{bmatrix}$;

iii. $S^2\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x\\y\\z\end{bmatrix}$, matrix representation $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$;

iv. $(3K-2S)\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x+z\\x+2y\\2x+y+z\end{bmatrix}$, matrix representation $\begin{bmatrix}1&0&1\\1&2&0\\2&1&1\end{bmatrix}$;

v. Not defined (domain $T \neq$ image T).

Solution 3.50. (a) Easy check.

(b) $\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$.

Solution 3.51. (a) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; the kernel is $\{\mathbf{0}\}$, the transformation is injective; the image is $\operatorname{im}(T) = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \ : \ a, b, c \in \mathbf{R} \right\}$, the transformation is not surjective.

(b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; the kernel is $\ker(T) = \{-a - 2bx + ax^2 + bx^3 : \ a, b \in \mathbf{R}\}$, the transformation is not injective; the image is $\operatorname{im}(T) = \{a + bx^2 : \ a, b \in \mathbf{R}\}$, the transformation is not surjective.

Solution 3.52. (a) Easy check.

(b) The kernel of T has basis $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$; the image of T has basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\operatorname{rank}(T) = 2$, $\operatorname{nullity}(T) = 2$.

(c) $\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{bmatrix}$.

Solution 3.53.

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

The integral is $x - x^2/2 + 2x^3/3$.

Solution 3.54. (a)

$$P = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}, \qquad P^{-1} = \frac{1}{10} \begin{bmatrix} 3 & -2 & 3 \\ 2 & 2 & 2 \\ 7 & 2 & -3 \end{bmatrix}.$$

(b) i. $\frac{1}{5} \begin{bmatrix} 14 \\ 6 \\ 1 \end{bmatrix}$;

ii. $\frac{1}{5} \begin{bmatrix} -4 \\ 9 \\ -6 \end{bmatrix}$.

Solution 3.55. (a) The two vectors are not scalar multiples of each other, hence they are linearly independent, therefore they are a basis for the two-dimensional space $\mathbf{R}^2$.

(b) Change of basis matrix $P = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$; coordinate vector $[\mathbf{v}]_{\mathcal{B}} = \frac{1}{4}\begin{bmatrix} 1 \\ -5 \end{bmatrix}$.

(c) $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution 3.56. (a) The 3×3 matrix whose columns are the vectors in $\mathcal{B}$ has determinant $-1 \neq 0$, so the three vectors are linearly independent, hence form a basis for the three-dimensional space $\mathbf{R}^3$.

(b) Change of basis matrix $P = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$; coordinate vector $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

(c) $P^{-1}AP = \begin{bmatrix} 4 & 6 & -4 \\ -1 & -5 & 5 \\ -2 & -3 & 1 \end{bmatrix}$.

(d) The transformation is left multiplication by the matrix

$$A' = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

so we get

$$P^{-1}A'P = \begin{bmatrix} 4 & 5 & -4 \\ -4 & -6 & 5 \\ -1 & -3 & 4 \end{bmatrix}.$$

Solution 3.57.

$$\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}, \qquad \frac{1}{11}\begin{bmatrix} -3 & -56 \\ -2 & 3 \end{bmatrix}.$$

Solution 3.58. $\begin{bmatrix} -28 & -19 & -43 \\ 5 & 4 & 7 \\ 18 & 11 & 28 \end{bmatrix}$.

Solution 3.59. (a) $\lambda = 2,\ u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, not diagonalisable;

(b) $\lambda_1 = 1,\ u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$; $\lambda_2 = 2,\ u_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, diagonalisable with

$$P = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix};$$

(c) $\lambda = 2,\ u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, not diagonalisable;

(d) no real eigenvalues, so not diagonalisable over $\mathbf{R}$; over $\mathbf{C}$, eigenvalues and eigenvectors: $\lambda_1 = i,\ u_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$; $\lambda_2 = -i,\ u_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$, diagonalisable over $\mathbf{C}$ with

$$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}, \quad D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Solution 3.60. (a) Span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and Span $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$;

(b) Span $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$ and Span $\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$.

Solution 3.61. Left multiplication by A_θ is rotation by an angle of θ. If $\theta \neq k\pi$ for some $k \in \mathbf{Z}$, then no lines are invariant under such rotation.

For $\theta = k\pi$, k odd, $A_\theta = -I$ and for $\theta = k\pi$, k even, $A_\theta = I$. In both cases **all** lines are invariant.

Solution 3.62. (a) 1, 3, 4;

(b) 1, 3, 6, 10.

Solution 3.63. A is invertible. This implies that 0 is not an eigenvalue of A.

Suppose λ is an eigenvalue of A, then $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero vector $\mathbf{v}$. Multiply both sides by A^{-1} to get $\mathbf{v} = \lambda A^{-1}\mathbf{v}$, in other words $A^{-1}\mathbf{v} = \frac{1}{\lambda}\mathbf{v}$.

So $1/\lambda$ is an eigenvalue of A^{-1}.

Think about the opposite direction as well.

A and A^{-1} have the same eigenvectors.

Solution 3.64. (a) $\lambda_1 = 2,\ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $\lambda_2 = 2,\ u_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$; $\lambda_3 = 3,\ u_3 = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$,

diagonalisable with

$$P = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & 3 & -6 \\ 0 & -1 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix};$$

(b) $\lambda_1 = 2,\ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};\ \lambda_2 = -3,\ u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, not diagonalisable;

(c) $\lambda_1 = -1,\ u_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix};\ \lambda_2 = 1,\ u_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix};\ \lambda_3 = -2,\ u_3 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$, diagonalisable with

$$P = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 3 \\ 1 & 1 & -2 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 3 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix};$$

(d) $\lambda_1 = 2,\ u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix};\ \lambda_2 = 1,\ u_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix};\ \lambda_3 = 1,\ u_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, diagonalisable with

$$P = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ -1 & -2 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution 3.65. We write $W(\lambda)$ for the eigenspace corresponding to λ. The spanning sets for $W(\lambda)$ listed below are bases.

(a) $\lambda = 2$, $W(2)$ has basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$\lambda = 6$, $W(6)$ has basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(b) $\lambda = 1$, $W(1)$ has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Solution 3.66. Straightforward calculations.

Solution 3.67.

$$\begin{bmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Solution 3.68. (a) $t^2 + t - 11$;

(b) $(t-1)(t^2-2t-5)$.

Solution 3.69. (a) $A^5 = PD^5P^{-1} = \begin{bmatrix} 43 & -22 \\ 22 & -12 \end{bmatrix}$;

(b) $A^5 = PD^5P^{-1} = \begin{bmatrix} 1509 & 198 & -1464 \\ 1477 & 230 & -1464 \\ 1477 & 231 & -1465 \end{bmatrix}$;

(c) $A^5 = PD^5P^{-1} = \frac{1}{8}\begin{bmatrix} 8 & 31 & 357 & 155 \\ 0 & 258 & 254 & -6 \\ 0 & 62 & -318 & -186 \\ 0 & 68 & 188 & 52 \end{bmatrix}$.

Solution 3.70. Limit is the golden ratio $1 + \frac{\sqrt{5}}{2} \approx 1.618$.