MAST10008 Written Assignment 1

Due 31 March 2026 at 18:00 on Canvas

Name:

Student ID:

Some guidelines:

- Submit your assignment through Gradescope before the deadline following the instructions on the Assignment Page. Submissions by email will not be accepted; late submissions require special consideration.
- There is no need to include your preparatory scratch work (do this on separate paper) but make sure that the solution you write in the box is a complete proof/explanation.

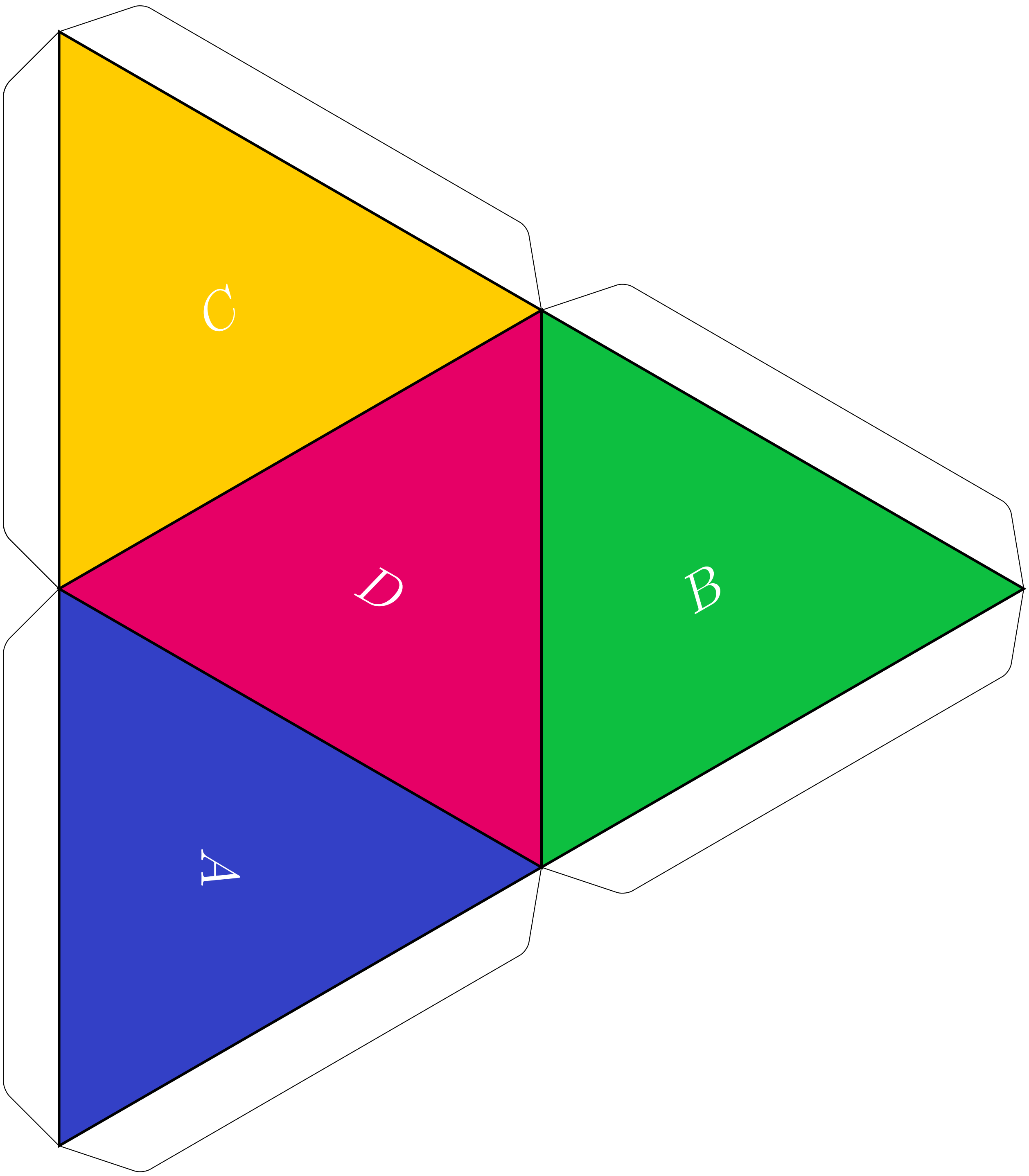
  The quality of the exposition will be assessed alongside the correctness of the approach.
- For technical reasons (since you will be scanning your solutions to upload to GradeScope), please write legibly with a very readable writing implement.
- Results stated in the lectures can be used (without having to re-prove them); make sure you say clearly what result you are using, though.
- It is acceptable for students to discuss the questions on the assignments and strategies for solving them. However, each student must write down their solutions in their own words and notation (and make sure that they understand what they are writing).
- Assignments are a valuable learning tool in this subject, so strive to maximise their impact on your understanding of the material.
- You may assume that not all questions will have the same weight in the assessment (shorter/easier questions will probably have fewer allocated marks than the longer/more involved ones).
- No Chegg or anything similar. At all. Please.

Please scan your answer pages and upload them to GradeScope in the correct order.

Question 1

For this question, you will be working with the group of rigid symmetries of the regular tetrahedron. Your first task is to craft this thing (include a photo of your work in your submission).

(a) Crafting Instructions for a regular tetrahedron



It is recommended to reinforce the sides, for instance by cutting up your breakfast cereal carton.

Do not throw away your tetrahedron when you are done with your homework! You may well find that you need it again in the future.

(b) Composition table: Next you are to fill in the composition table. Here symmetries are treated as functions and denoted by the way they permute the faces. The function in the top row is to be carried out first, followed by that in the left-most column. So, in the example entry, you first flip A with B and C with D and follow this by the rotation about the axis passing perpendicularly through the midpoint of D and moving face A to face B, face B to face C and face C back to face A.

$\circ$	Id	$\begin{smallmatrix} A \leftrightarrow B \\ C \leftrightarrow D \end{smallmatrix}$	$\begin{smallmatrix} A \leftrightarrow C \\ B \leftrightarrow D \end{smallmatrix}$	$\begin{smallmatrix} A \leftrightarrow D \\ B \leftrightarrow C \end{smallmatrix}$	$\begin{smallmatrix} & A & \\ \swarrow & & \nwarrow \\ B & \rightarrow & C \end{smallmatrix}$	$\begin{smallmatrix} & A & \\ \nearrow & & \searrow \\ B & \leftarrow & C \end{smallmatrix}$	$\begin{smallmatrix} & A & \\ \swarrow & & \nwarrow \\ D & \rightarrow & B \end{smallmatrix}$	$\begin{smallmatrix} & A & \\ \nearrow & & \searrow \\ D & \leftarrow & B \end{smallmatrix}$	$\begin{smallmatrix} & A & \\ \swarrow & & \nwarrow \\ C & \rightarrow & D \end{smallmatrix}$	$\begin{smallmatrix} & A & \\ \nearrow & & \searrow \\ C & \leftarrow & D \end{smallmatrix}$	$\begin{smallmatrix} & B & \\ \swarrow & & \nwarrow \\ D & \rightarrow & C \end{smallmatrix}$	$\begin{smallmatrix} & B & \\ \nearrow & & \searrow \\ D & \leftarrow & C \end{smallmatrix}$
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We will write $(\boldsymbol{T}, \circ)$ for the group (inform yourself what that is) consisting of the twelve rotation symmetries of the tetrahedron and their composition.

(c) What type of 3D rotations turn up in $\boldsymbol{T}$ and how many of each kind?

(d) Give an example of two rotations R and S of the tetrahedron such that $R \circ S \neq S \circ R$. Give an example of two distinct no-trivial rotations P and Q of the tetrahedron such that $P \circ Q = Q \circ P$.

(e) Familiarise yourself with the notion of equivalence relation in the problem sheet collection. Let $\boldsymbol{K}$ be the set consisting of the first four entries in the table above, i.e.,

$$\boldsymbol{K} = \left\{ Id, \ {}^{A \leftrightarrow B}_{C \leftrightarrow D}, \ {}^{A \leftrightarrow C}_{B \leftrightarrow D}, \ {}^{A \leftrightarrow D}_{B \leftrightarrow C} \right\},$$

and show that

$$R \sim S \quad \Longleftrightarrow \quad (\exists Q \in \boldsymbol{K})\,(R = SQ)$$

defines an equivalence relation on $\boldsymbol{T}$.

(f) List all the equivalence classes with respect to $\sim$. How many are there?

(g) Now go through the same discussion but with

$$\boldsymbol{C} = \{Id, A \longrightarrow B \longrightarrow C \longrightarrow A, A \longrightarrow C \longrightarrow B \longrightarrow A\}$$

in the place of $\boldsymbol{K}$.

(h) Only one of the equivalence relations above is compatible with the composition $\circ$ in that it allows you to declare a well defined operation $\circ'$ on the quotient set, such that the quotient map respects composition. Which one is it?

(k) Discuss your findings in (h).

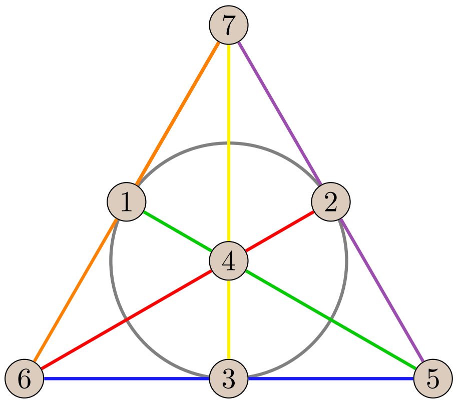
Question 2

Ernie and Bert are both intelligent and honest. They play a game: each of them thinks of a natural number and whispers it into Big Bird's ear. Big Bird then writes two natural numbers on the board (in random order). One of these numbers is the sum of the two numbers that were whispered to him, the other is a randomly chosen natural number. He then asks Ernie: "Do you know Bert's number?" If Ernie answers "Yes" then the game is over. If the answer is "No", Big Bird asks Bert "Do you know Ernie's number?" If Bert answers "Yes" then the game is over. If the answer is "No", Big Bird asks Earnie "Do you know Bert's number now?", and so on. Prove that the game is over after a finite number of steps.

Note: This question is not just about figuring out the correct reason, your challenge is bringing it to paper in a formal and mathematically accurate manner. A good start is to remember to use full sentences and to declare the objects you are speaking about before you say something about them, give them names and refer to them by their names (pronouns quickly get confusing).

Question 3

The Fano plane $\mathcal{F}$, depicted on the left, is the smallest example of a projective geometry. It consists of seven points and seven lines, each line has three points, and there are three lines passing through any given point. The grey circle counts as a line. Before we start to formalise this, your first task is to fill in the incidence table on the right: in every given row, colour in the boxes whose column number indicates a point on that line.



The Fano plane

	1	2	3	4	5	6	7
Line 1		■			■		■
Line 2			■	■			■
Line 3	■			■	■		
Line 4	■					■	■
Line 5		■		■		■	
Line 6	■	■	■				
Line 7			■		■	■	

Formally, $\mathcal{F}$ is given by the following data:

- A set $P = \{1, 2, 3, 4, 5, 6, 7\}$, whose elements we will call the points of $\mathcal{F}$.
- A set $L = \{\text{Line 1}, \text{ Line 2}, \text{ Line 3}, \text{ Line 4}, \text{ Line 5}, \text{ Line 6}, \text{ Line 7}\}$, whose elements we will call the lines of $\mathcal{F}$.
- A subset of their Cartesian product, $R \subseteq L \times P$, indicating which points lie on which lines
$$R = \{(l, p) \mid \text{ point } p \text{ lies on line } l\}$$
(This is called an incidence relation.)

In what follows, we will identify lines in $\mathcal{F}$ with subsets of P: for $1 \leqslant j \leqslant 7$, let

$$\ell_j = \{p \in P \mid (\text{Line j}, p) \in R\}.$$

Let $\mathcal{P}(P)$ be the power set of P, and consider the set $\mathcal{H} \subseteq \mathcal{P}(P)$ defined by

$$\mathcal{H} = \mathcal{L} \cup C \cup \{\varnothing, P\},$$

where

$$\mathcal{L} = \{\ell_j \mid 1 \leqslant j \leqslant 7\}$$

and

$$C = \{X \subseteq P \mid (\exists j)(X = P \setminus \ell_j)\}$$

(b) How many elements does $\mathcal{P}(P)$ have? How many elements does $\mathcal{H}$ have?

(c) For each element $X \in \mathcal{H}$, draw its incidence vector. This is a column of numbers with seven entries that are either zero or one, a one in position k indicating that point k is an element of X. For instance, the incidence vector of Line 1 is given by

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

(d) The parallel axiom is a property that a given geometry may or may not satisfy. It goes as follows:

For every given line ℓ and every given point p not on ℓ, there exists one and only one line ℓ' through p such that ℓ and ℓ' have no point in common.

What is the negation of the parallel axiom?

(e) Give a formal proof that the Fano plane does not satisfy the parallel axiom.

(f) Show that in fact any two distinct lines in $\mathcal{F}$ have exactly one point in common.

(g) Using the addition rules in $\mathbb{F}_2$ component-wise, show that the sum of any choice of two incidence vectors of elements of $\mathcal{H}$ is again the incidence vector of an element of $\mathcal{H}$.

Question 4

Let

$$\mathcal{P}_2 = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid (\exists a, b, c \in \mathbb{R})(f(x) = ax^2 + bx + c)\}$$

be the set of all real quadratic functions (where linear and constant functions are viewed as a degenerate version of quadratics), and let

$$\mathcal{P}_1 = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid (\exists a, b \in \mathbb{R})(f(x) = ax + b)\}$$

be the set of all linear functions on $\mathbb{R}$. Consider the map

$$\begin{aligned} D : \mathcal{P}_2 &\longrightarrow \mathcal{P}_1 \\ f &\longmapsto f' \end{aligned}$$

sending a function f to its derivative.

(a) Is D injective? Prove your answer!

(b) Is D surjective? Prove your answer!

(c) Is D bijective? Prove your answer!

(d) **(Challenge)** Give an example of a surjective map $S : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ or prove that no such map exists.