# Assignment 1 – Algebra 2019

(1) Algebras: Let $k$ be a field. Then a $k$-algebra consists of a vector space $V$ over $k$ along with a multiplication $m : V \times V \longrightarrow V$ satisfying a list of properties. Find this list of properties (either by figuring it out or from the literature) and translate this definition, line by line, into the definition from class (more precisely, the special case of the definition from class where $R = k$ is a field).

(2) Finite fields
  (a) Find all the ways to construct $\mathbb{F}_{16}$ as the quotient of a polynomial ring over $\mathbb{F}_4$ and construct the isomorphisms between them.
  (b) Find all the ways to construct $\mathbb{F}_{27}$ as the quotient of a polynomial ring over $\mathbb{F}_3$ and construct the isomorphisms between them.

(3) Group algebras: Let $G$ be a finite group, and let $k$ be a field. You will show the universal property of the group algebra $kG$ in several steps.
  (a) Given a monoid $M$, we will use the notation $M^\times$ for the group of invertible elements in $M$. Given a group $G$ and a monoid $M$, show that the set of monoid homomorphisms from $G$ to $M$ can be identified with the set of group homomorphisms from $G$ to $M^\times$.
  (b) Let $R$ be a commutative ring, and let $M$ be a monoid. Show that the monoid algebra $RM$ along with the map $i : M \longrightarrow RM$ defined in class satisfies the following universal property: for any $R$-algebra $A$ and any monoid map $\phi$ from $M$ to $(A, \cdot)$, there exists exactly one $R$-algebra homomorphism $f : RM \longrightarrow A$ satisfying $f \circ i = \phi$.
  (c) Combine the two previous points to formulate and prove the universal property of $kG$.

(4) Representations: A representation of $G$ on a $k$-vector space $V$ is an action of $G$ on $V$ via $k$-linear maps. In other words, a representation of $G$ on $V$ is a $G$-action

$$\begin{aligned} \varrho : G \times V &\longrightarrow V \\ (g, v) &\longmapsto \varrho(g)(v) \end{aligned}$$

such that for each $g \in G$ the map $\varrho(g)$ is $k$-linear.
  (a) Prove that a representation of $G$ on $V$ consists of the same data as a $k$-algebra homomorphism $kG \longrightarrow End_k(V)$.
  (b) Prove that a representation of $G$ on $V$ consists of the same data as the structure of a $kG$-module on $V$.
  (c) Describe the defining representations of the dihedral groups as algebra homomorphisms from the group algebra to the Endomorphism ring of $\mathbb{R}^2$.

(5) Constructions with straight edge and compas. Let $\alpha = \cos(15°)$.
  (a) Using the addition formulas for sine and cosine, find the irreducible polynomial of $\alpha$ over $\mathbb{Q}$.
  (b) Show that the irreducible polynomial for $\alpha^2$ over $\mathbb{Q}$ has degree two.
  (c) From here, argue that $\cos(15°)$ is constructible.
  (d) Assume now that $\cos(45°)$ has already been constructed. (How?) Find the irreducible polynomial of $\cos(15°)$ over $\mathbb{Q}[\cos(45°)]$.
  (e) Translate (d) into a step by step construction, using straight-edge and compass, of a $15°$ angle from an already constructed $45°$ angle.

(f) Translate (a) – (c) into a construction of a 15° angle from scratch (starting with two arbitrary distinct points in the plane).

(g) Write each of the above field extensions

$$\mathbb{Q} \subset \mathbb{Q}[cos(45°)] \quad (1)$$

$$\mathbb{Q} \subset \mathbb{Q}[cos(15°)] \quad (2)$$

$$\mathbb{Q} \subset \mathbb{Q}[cos^2(15°)] \quad (3)$$

$$\mathbb{Q}[cos(45°)] \subset \mathbb{Q}[cos(15°)] \quad (4)$$

$$\mathbb{Q}[cos^2(15°)] \subset \mathbb{Q}[cos(15°)] \quad (5)$$

as quotient of the polynomial ring over the smaller field.