The University of Melbourne

Masters Thesis

The Homotopy Theory of 2-Groups

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Abstract

In this thesis, we develop the homotopy theory of 2-groups, using the Garzon-Miranda model structure on the category of strict 2-groups. We explore the extent to which homotopy theoretic methods recover the 2-category theory of 2-groups. In particular, we apply this theory to study the construction of free 2-groups and kernels, with a view to developing 2-dimensional analogues of classical constructions.

Contents

1 Introduction

A 2-group is a monoidal groupoid in which every object is weakly invertible. We call a 2-group strict if its monoidal structure is strict and every object is strictly invertible. There is a well-known result that any 2-group is equivalent to a strict 2-group; this strictification theorem greatly simplifies calculations involving 2-groups. Moreover, the category of strict 2-groups admits a model structure, which allows us to study their homotopy theory. This thesis will discuss the extent to which the homotopy theory of strict 2-groups recovers the 2-category theory of 2-groups.

Our philosophy is thus to work with 2-groups in as strict a way as possible. However, 2-groups naturally form a 2-category, so we must consider 2-categorical and bicategorical constructions on 2-groups. We briefly recall the definitions of 2-categories and internal categories in Section 2.

Historically, 2-groups have arisen in a range of contexts, in forms which are not obviously equivalent. Of these formulations, crossed modules are simplest to do computations with. They were introduced by Whitehead in [39] to capture the algebraic properties of relative homotopy groups, and have since become a central part of the machinery of algebraic homotopy advanced by Baues and Brown. We define crossed modules in Section 3 and discuss some examples.

In Section 4 we introduce monoidal categories and sketch a proof of a strictification theorem, which is the basis of the strictification theorem for 2-groups. In Section 5 we review a number of different formulations of 2-groups and sketch proofs of their equivalence. We also develop aspects of the category theory of 2-groups, culminating in a statement of the strictification theorem.

In Section 6 we define model categories and develop some aspects of their theory. In Section 7 we recall the Garzon-Miranda model structure on the category of strict 2-groups and prove the following result, which gives a precise connection between the homotopy theory of 2-groups and their 2-category theory. Given a cofibrant strict 2-group $\mathcal{A}$ and any 2-group $\mathcal{G}$ we have the following equivalence of groupoids:

$$Hom_{\mathbf{Str2Grp}}\left(\mathcal{A},\bar{\mathcal{G}}\right) \simeq Hom_{\mathbf{2Grp}}\left(\mathcal{A},\mathcal{G}\right)$$

Here $\bar{\mathcal{G}}$ is the strict 2-group equivalent to $\mathcal{G}$, defined via the strictification theorem. The objects of $Hom_{\mathbf{Str2Grp}}\left(\mathcal{A},\bar{\mathcal{G}}\right)$ are strict monoidal functors, while the objects of $Hom_{\mathbf{2Grp}}\left(\mathcal{A},\mathcal{G}\right)$ are monoidal functors. The arrows in both groupoids are monoidal natural isomorphisms.

In Section 8, we describe some elements of the 2-dimensional algebra of 2-groups. In particular, we recall Vitale's definition of the kernel for a morphism of 2-groups. This is defined via a 2-categorical universal property. Following a remark in [21], we note that Vitale's construction of kernels satisfies the stronger universal property of a strict homotopy limit. For strict 2-groups, we show that Vitale's construction coincides with that of the homotopy kernel coming from the model structure.

In Section 9, we use the model structure on strict 2-groups to construct the free 2-group $\mathcal{F}(G)$ on a groupoid G. The strict 2-group $\mathcal{F}(G)$ is free in the sense that, given any groupoid G and any 2-group $\mathcal{G}$, we have a 2-natural isomorphism of groupoids as below:

$$Hom_{\mathbf{Str2Grp}}\left(\mathcal{F}\left(G\right),\mathcal{G}\right)\cong Hom_{\mathbf{Grpd}}\left(G,U\left(\mathcal{G}\right)\right)$$

Here $U\left(\mathcal{G}\right)$ is the underlying groupoid of $\mathcal{G}$. This construction agrees with the free crossed module on a groupoid constructed in [5], although we give the construction in terms of truncated simplicial groups rather than crossed modules. The proof also makes more explicit use of the model structures on groupoids and strict 2-groups than the proof in [5].

Sections 7, 8 and 9 illustrate the facility of working with 2-groups via their homotopy theory. The topics we have chosen to develop are 2-dimensional analogues of elementary concepts in group theoy. We conclude the thesis with a brief overview of possible future work in this direction, and a discussion of the difficulties encountered in the 2-dimensional setting.

1.1 Notation and Assumed Background

We assume the reader is familiar with elementary category theory, as presented in [29].

We will use the same notation for 2-categories as their underlying category. For example, we denote the category of groupoids (that is, categories in which every morphism is invertible) and functors between them by **Grpd**. We also denote the 2-category of groupoids, functors between groupoids and natural transformations between functors by **Grpd**. It will be clear from the context whether we are referring to the 2-dimensional structure or the 1-dimensional.

2 2-Dimensional Structures

In this section we introduce 2-categories and internal categories. Both concepts may be seen as forms of categorification, in the sense used in [2]. In later sections, we will be interested in studying stuctures on 2-groups that are stable under equivalence rather than isomorphism. For this reason, the appropriate notions of limits, colimits and adjunctions for 2-groups are 2-categorical.

The 2-category of strict 2-groups, **Str2Grp**, defined in Definition 5.16, is equivalent to the 2-category **CatGrp** of internal categories in **Grp**. Thus, we define internal categories in this section. We will also use the language introduced in this section to discuss groupoids in Section 9, thinking of groupoids as internal to **Set**.

2.1 2-Categories

Definition 2.1. A 2-category is a category enriched over **Cat**, in the same way that small categories are enriched over **Set** and preadditive categories (precursors to abelian categories) are enriched over **Ab**. Explicitly, a 2-category $\mathcal{C}$ consists of a set (or a class) of objects, and for each pair of objects x and y, a category $Hom_{\mathcal{C}}(x,y)$. We will often denote this by $\mathcal{C}(x,y)$. For any objects x, y and z we have the following functors, defining identities and composition:

$$\begin{array}{rcl} i : 1 & \longrightarrow & \mathcal{C}(x,x) \\ \circ : \mathcal{C}(y,z) \times \mathcal{C}(x,y) & \longrightarrow & \mathcal{C}(x,z) \end{array}$$

Here 1 is the terminal category with one object and a single identity morphism. We require that the following diagrams commute, expressing associativity and unit constraints:

$$\begin{array}{ccccc} (\mathcal{C}(z,w) \times \mathcal{C}(y,z)) \times \mathcal{C}(x,y) & & \longrightarrow & & \mathcal{C}(z,w) \times (\mathcal{C}(y,z) \times \mathcal{C}(x,y)) \\ \Big\downarrow{\scriptstyle \circ \times \mathcal{C}(x,y)} & & & & \Big\downarrow{\scriptstyle \mathcal{C}(z,w) \times \circ} \\ \mathcal{C} \times \mathcal{C}(x,y) & \xrightarrow{\circ} & \mathcal{C}(x,w) & \xleftarrow{\circ} & \mathcal{C}(z,w) \times \mathcal{C}(x,z) \end{array}$$

$$\begin{array}{ccc} 1 \times \mathcal{C}(x,y) & \xrightarrow{i \times \mathcal{C}(x,y)} & \mathcal{C}(y,y) \times \mathcal{C}(x,y) \\ & {\scriptstyle p_2} \searrow & \Big\downarrow{\scriptstyle \circ} \\ & & \mathcal{C}(x,y) \end{array}$$

$$\begin{array}{ccc} \mathcal{C}(x,y) \times \mathcal{C}(x,x) & \xleftarrow{\mathcal{C}(x,y) \times i} & \mathcal{C}(x,y) \times 1 \\ \Big\downarrow{\scriptstyle \circ} & \swarrow {\scriptstyle p_1} & \\ \mathcal{C}(x,y) & & \end{array}$$

We call the objects of $\mathcal{C}(x,y)$ morphisms from x to y, and denote them by $f : x \longrightarrow y$. They are also often called 1-morphisms or 1-cells. Given two objects $f, g \in \mathcal{C}(x,y)$ a morphism α from f to g is known as a 2-morphism or 2-cell in $\mathcal{C}$. We denote this by $\alpha : f \Longrightarrow g$. Note that we have two different notions of composition for 2-cells. Given $\alpha : f \Longrightarrow g$ and $\beta : g \Longrightarrow h$ in $\mathcal{C}(x,y)$, their vertical composite is defined to be their composite in $\mathcal{C}(x,y)$, denoted $\beta\alpha : f \Longrightarrow h$. Moreover, given $\alpha : f \Longrightarrow g$ in $\mathcal{C}(x,y)$ and $\beta : k \Longrightarrow h$ in $\mathcal{C}(y,z)$ the functor $\circ$ defines their horizontal composite, shown below:

$$\beta \circ \alpha : k \circ f \Longrightarrow h \circ g$$

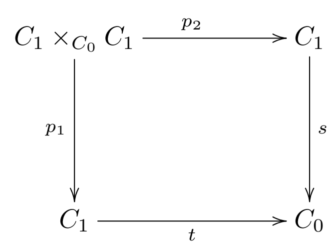
If all of the categories $\mathcal{C}(x,y)$ are groupoids, we will sometimes call the 2-category $\mathcal{C}$ a **Grpd**-enriched category. The 2-categories **Grpd**, **2Grp**, **Str2Grp** and **Cross** defined in Sections 3 and 5 are all **Grpd**-enriched.

We will often make use of structures related to 2-categories, which are known as bicategories. Roughly, bicategories are to 2-categories as the strict monoidal categories of Definition 4.18 are to the monoidal categories of Definition 4.1. With this philosophy, bicategories are often called weak 2-categories. Although we make use of bicategories and their theory, we will omit the defintion. For background on bicategories and 2-categories, in particular the appropriate notions of limits and adjunctions, see [24, 22, 23, 25, 27].

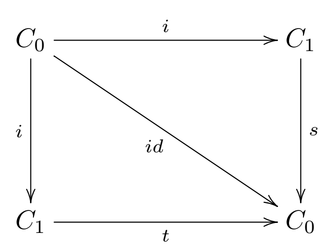
Example 2.2. We may form a 2-category **Cat** with categories as objects, functors as 1-cells, and natural transformations as 2-cells. In the same way that **Set** may be considered the archetypal example of a category, **Cat** is the archetypal example of a 2-category. The 2-category **Grpd** is a full sub-2-category of **Cat**, in that the inclusion form **Grpd** to **Cat** is full on both 1-morphisms and 2-morphisms.

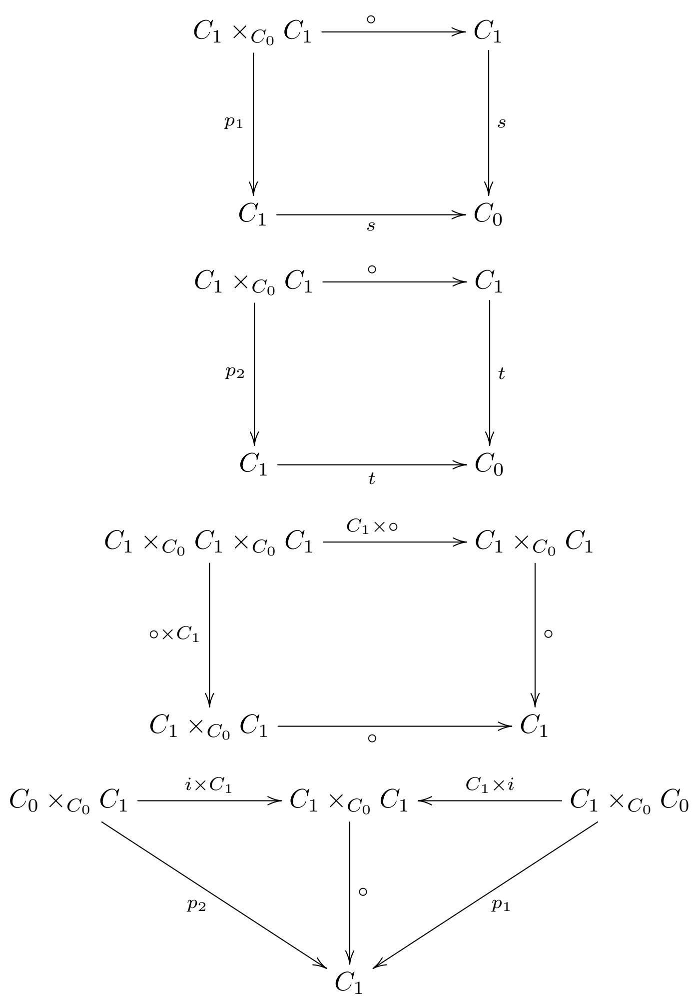
2.2 Internal Categories

Definition 2.3. Let **C** be a category with finite limits. An internal category in **C** is given by a pair of objects $C_0, C_1 \in \mathbf{C}$ called the object of objects and the object of arrows. In addition, we must specify source, target and identity morphsims $s, t : C_1 \longrightarrow C_0$ and $i : C_0 \longrightarrow C_1$, and a composition morphism $\circ : C_1 \times_{C_0} C_1 \longrightarrow C_1$. The object $C_1 \times_{C_0} C_1$ is defined via the pullback diagram below:



We require the following diagrams to commute, expressing the source and target for identity arrows and composites, and the associative and unit laws for composition:





We will often denote an internal category by $C_1 \rightrightarrows C_0$. Note that the assumption that **C** has finite limits is not really necessary; we need only guarantee the existence of the pullbacks used to define composition. Note also that the definition of a category internal to **Set** recovers the usual definition of a small category.

Definition 2.4. Let **C** be a category and let C and D be internal categories in **C**. An internal functor $F : C \longrightarrow D$ is given by a pair of morphisms in **C**:

$$F_0 : C_0 \longrightarrow D_0$$

$$F_1 : C_1 \longrightarrow D_1$$

We require the following diagrams to commute. These say that F preserves source, target, identities and composition:

$$
\begin{array}{ccc}
C_1 & \xrightarrow{s} & C_0 \\
{\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_0} \\
D_1 & \xrightarrow[s]{} & D_0
\end{array}
$$

$$
\begin{array}{ccc}
C_1 & \xrightarrow{t} & C_0 \\
{\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_0} \\
D_1 & \xrightarrow[t]{} & D_0
\end{array}
$$

$$
\begin{array}{ccc}
C_0 & \xrightarrow{i} & C_1 \\
{\scriptstyle F_0}\downarrow & & \downarrow{\scriptstyle F_1} \\
D_0 & \xrightarrow[i]{} & D_1
\end{array}
$$

$$
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{F_1 \times_{C_0} F_1} & D_1 \times_{D_0} D_1 \\
{\scriptstyle \circ}\downarrow & & \downarrow{\scriptstyle \circ} \\
C_1 & \xrightarrow[F_1]{} & D_1
\end{array}
$$

If we take $\mathcal{X} = \mathbf{Set}$ then once again we get the familiar definition of a functor between categories.

Definition 2.5. Let $\mathbf{C}$ be a category and let $F, G : C \longrightarrow D$ be internal functors in $\mathbf{C}$. An internal natural transformation $\theta : F \Longrightarrow G$ is given by a morphism in $\mathbf{C}$:

$$\theta : C_0 \longrightarrow D_1$$

We require that the following diagrams commute:

$$\begin{array}{ccc} C_0 & & \\ {\scriptstyle\theta}\downarrow & \searrow^{F_0} & \\ D_1 & \xrightarrow[s]{} & D_0 \end{array}$$

$$\begin{array}{ccc} C_0 & & \\ {\scriptstyle\theta}\downarrow & \searrow^{G_0} & \\ D_1 & \xrightarrow[t]{} & D_0 \end{array}$$

$$\begin{array}{ccc} C_1 & \xrightarrow{(\theta\circ s, G_1)} & D_1 \times_{D_0} D_1 \\ {\scriptstyle (F_1, \theta\circ t)}\downarrow & & \downarrow{\scriptstyle\circ} \\ D_1 \times_{D_0} D_1 & \xrightarrow[\circ]{} & D_1 \end{array}$$

Definition 2.6. Internal categories, functors and natural transformations in **C** form a 2-category. We denote this 2-category by **CatC**.

Example 2.7. The category **Ab** of abelian groups has all small limits, so we may consider internal categories in **Ab**. We call internal categories in **Ab** abelian 2-groups. The 2-category **CatAb** is equivalent to the 2-category $\mathbf{Ch}^2_{\mathbb{Z}}$, with objects 2-term chain complexes of abelian groups:

$$\delta : H \longrightarrow G$$

A chain map between two such chain complexes $\delta_1 : H_1 \longrightarrow G_1$ and $\delta_2 : H_2 \longrightarrow G_2$ is a pair of group homomorphisms $u : G_1 \longrightarrow G_2$ and $v : H_1 \longrightarrow H_2$ such that the diagram below commutes:

$$\begin{array}{ccc} H_1 & \xrightarrow{\delta_1} & G_1 \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ H_2 & \xrightarrow[\delta_2]{} & G_2 \end{array}$$

Let $(u_1, v_1), (u_2, v_2) : \delta_1 \longrightarrow \delta_2$ be chain maps. A chain homotopy $\phi : (u_1, v_1) \Longrightarrow (u_2, v_2)$ is a group homomorphism $\phi : G_1 \longrightarrow H_2$ such that for any $g \in G_1$ and $h \in H_1$ we have the following equalities:

$$u_2(g) - u_1(g) = \delta_2(\phi(g))$$

$$v_2(h) - v_1(h) = \phi(\delta_1(h))$$

The proof of the equivalence is a special case of the equivalence in Section 5.3.

Remark 2.8. Any internal category in **Ab** is in fact an internal groupoid. This is also true in **Grp**. See [7] for a proof of this fact.

Example 2.9. The category $\mathbf{Vect}_{\mathbb{C}}$ of complex vector spaces and linear transformations has all small limits, so we may consider internal categories in $\mathbf{Vect}_{\mathbb{C}}$. These are known as Baez-Crans 2-vector spaces, and were introduced in [1]. The 2-category of Baez-Crans 2-vector spaces is equivalent to the 2-category $\mathbf{Ch}^2_{\mathbb{C}}$ of 2-term chain complexes of vector spaces, chain maps and chain homotopies. The representation theory of strict 2-groups on Baez-Crans 2-vector spaces is studied in [14].

Remark 2.10. An internal category $C = (C_1 \rightrightarrows C_0)$ in **Cat** is known as a double category. Double categories offer an approach to higher category theory which is distinct from that given by 2-categories. We may think of a double category as having objects given by the objects of C_0, horizontal arrows the arrows of C_0, vertical arrows the objects of C_1 and 2-morphisms the morphisms of C_1. Given any 2-category $\mathcal{C}$, we may think of $\mathcal{C}$ as a double category either by taking all vertical morphisms or all horizontal morphisms to be identities. In this way, we may think of 2-categories as examples of double categories.

3 Crossed Modules and Quadratic Modules

Crossed modules were first introduced in [39] in 1949. In [39], Whitehead defines a crossed module associated to any CW-complex, which we call the fundamental crossed module in Definition 3.16. In [30] Mac Lane and Whitehead show that any pointed, connected 2-type may be reconstructed up to homotopy from its fundamental crossed module. In this way, crossed modules are algebraic models for connected, pointed 2-types. There are a number of ways to make this precise; in particular, we may use the model structure on crossed modules given in Section 7.11 to express this as an equivalence of homotopy categories. This is described in Example 7.14.

There is a large body of literature following on from Whitehead's work, relating 2-group theory to homotopy theory. In particular, in [5], the authors use crossed modules to model two-stage spaces - that is, spaces with non-trivial homotopy groups only in two consecutive dimensions n and $n+1$. The authors show that for $n = 2$ the associated crossed module forms a reduced quadratic module, and for $n \geq 3$ the crossed module is a stable quadratic module. In the language of 2-groups, introduced in Section 5, reduced quadratic modules are equivalent to braided strict 2-groups and stable quadratic modules are equivalent to symmetric strict 2-groups. Symmetric 2-groups are the 2-dimensional analogue of abelian groups. Thus, the result in [5] the 2-dimensional analogue of the classical fact that for a topological space X, the homotopy groups $\pi_n(X)$ are abelian for $n \geq 2$.

In this section we introduce crossed modules and quadratic modules. To fix notation, we begin by recalling the definition of a group action.

3.1 Group Actions

Definition 3.1. Let G and H be groups. An action of G on H is a group homomorphism as below:

$$\rho : G \longrightarrow Aut_{\mathbf{Grp}}(H)$$

We will call a group H equipped with a G-action a G-group. If H is abelian we will say that H is a G-module. For any $g \in G$ and $h \in H$ we will denote $\rho(g)(h) := {}^{g}h$. With this notation, a morphism of G-groups $\rho_1 : G \longrightarrow Aut(H_1)$ and $\rho_2 : G \longrightarrow Aut(H_2)$ is a group homomorphism $\delta : H_1 \longrightarrow H_2$ such that for any $g \in G$ and $h \in H_1$

$$\delta\left({}^{g}h\right) = {}^{g}\delta(h).$$

We say that such a morphism $\delta : H_1 \longrightarrow H_2$ is G-equivariant.

Remark 3.2. Any group action $\rho : G \longrightarrow Aut_{\mathbf{Grp}}(H)$ gives a function:

$$\begin{array}{rcl} \alpha : G \times H & \longrightarrow & H \\ (g, h) & \longmapsto & {}^{g}h \end{array}$$

The fact that ρ is a group homomorphism and that any element $g \in G$ acts is mapped to a group homomorphism on H corresponds to the following conditions:

$$\begin{array}{rcl} {}^{g}(h_1 h_2) & = & {}^{g}h_1 {}^{g}h_2 \\ {}^{g_1}\left({}^{g_2}h\right) & = & {}^{g_1 g_2}h \\ {}^{1}h & = & h \end{array}$$

A morphism of G-groups is then a group homomorphism $\delta : H_1 \longrightarrow H_2$ making the following diagram commute:

$$\begin{array}{ccc} G \times H_1 & \xrightarrow{G \times \delta} & G \times H_2 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} \\ H_1 & \xrightarrow[\delta]{} & H_2 \end{array}$$

We will make one final equivalent definition that generalises easily to other categories. This will description will be useful when we consider actions of monoidal categories in Section 4.2.

Definition 3.3. Let G be a group. The delooping of G, denoted $\boldsymbol{B}G$, is a groupoid with a single object $\bullet$ and morphisms given by

$$Hom_{\mathbf{B}G}(\bullet, \bullet) = G.$$

Composition in $\boldsymbol{B}G$ is given by $g \circ h = gh$.

Remark 3.4. The groupoid $\boldsymbol{B}G$ is called the delooping of G because its loop space $\Omega\boldsymbol{B}G$ may be identified with G. See Definition 6.73 for the definition of loop space, and Example 6.74 for a construction of $\Omega\boldsymbol{B}G$.

Definition 3.5. An action of G on H is given by a functor $\rho : \boldsymbol{B}G \longrightarrow \mathbf{Grp}$ such that $\rho(\bullet) = H$. The category of G-groups is then the functor category

$$\mathbf{Grp}_G := [\boldsymbol{B}G, \mathbf{Grp}].$$

Example 3.6. Any group G acts on itself by conjugation. The action is defined via the following homomorphism:

$$\begin{array}{rcl} \varphi : G & \longrightarrow & Aut(G) \\ g & \longmapsto & \varphi_g \end{array}$$

Here $\varphi_g(h) = ghg^{-1}$ for any $h \in G$. Note that the kernel of this homomorphism is the centre of G:

$$Z(G) = \left\{g \in G \mid ghg^{-1} = h, \forall h \in G\right\} = \{g \in G \mid \varphi_g = id_G\}$$

We call this action the adjoint action of G on itself, and we will often denote the induced function as follows:

$$\begin{array}{rcl} Ad : G \times G & \longrightarrow & G \\ (g, h) & \longmapsto & ghg^{-1} \end{array}$$

We will describe a 2-dimensional analogue of this action in Example 8.7 and the corresponding notion of the centre of a 2-group, which is known as the Drinfeld centre.

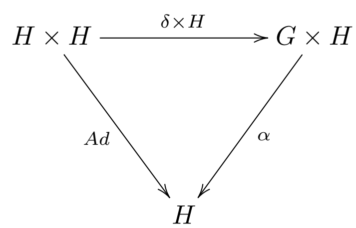
3.2 Crossed Modules

Informally, we may think of a crossed module as a group homomorphism $\delta : H \longrightarrow G$ which behaves as though the target group is abelian. To make sense of this, H must be equipped with G-action, which is related by δ to the action of G on itself by conjugation. Whitehead's motivation for introducing crossed modules in [39] was to capture the algebraic properties of the relative homotopy groups of a CW-complex. This is expanded upon in Section 3.3.

Definition 3.7. Let H be a G-group. Recall that we may consider G itself as a G-group, via the adjoint action of Example 3.6. A G-precrossed module is a G-group morphism $\delta : H \longrightarrow G$. The category of G-precrossed modules is the comma category below:

$$\mathbf{Precross}_G = (\mathbf{Grp}_G \downarrow G)$$

Definition 3.8. Let $\delta : H \longrightarrow G$ be a G-precrossed module. We call $\delta : H \longrightarrow G$ a G-crossed module if the diagram below commutes:



This condition is known as the Peiffer identity. G-crossed modules form a full subcategory $\mathbf{Cross}_G$ of $\mathbf{Precross}_G$.

Unwinding Definitions 3.7 and 3.8, a G-crossed module consists of a group H equipped with an action $\alpha : G \times H \longrightarrow H$ and a group homomorphism $\delta : H \longrightarrow G$ such that for all $g \in G$ and $h_1, h_2 \in H$ we have the following equalities:

$$\begin{aligned} h_1 h_2 h_1^{-1} &= {}^{\delta(h_1)}h_2 \\ \delta\left({}^{g}h\right) &= g\delta\left(h\right)g^{-1} \end{aligned}$$

Example 3.9. Let $H \subseteq G$ be a normal subgroup. Then G acts on H by conjugation, and the inclusion map $i : H \longrightarrow G$ defines a G-crossed module. It is easy to show that $Im\left(\delta\right) \subseteq G$ is a normal subgroup for any G-crossed module $\delta : H \longrightarrow G$, so normal subgroups $H \subseteq G$ correspond exactly to G-crossed modules $\delta : H \longrightarrow G$ with δ injective.

Example 3.10. Suppose we have a central extension as below:

$$1 \longrightarrow N \xrightarrow{i} H \xrightarrow{\delta} G \longrightarrow 1$$

That is, $N \subseteq Z\left(H\right)$ and $G \cong H/N$. Define an action of G on H via the function below:

$$\begin{aligned} \alpha : G \times H &\longrightarrow H \\ (g, h) &\longmapsto khk^{-1} \end{aligned}$$

Here $k \in H$ is an element chosen such that $\delta\left(k\right) = g$. Such an element always exists since $\delta : H \longrightarrow G$ is surjective. To see that this action is well-defined, suppose $\delta\left(k_1\right) = \delta\left(k_2\right)$ for some $k_1, k_2 \in H$. Then $h_1^{-1}h_2 \in Ker\left(\delta\right)$. However,

$Ker\,(\delta) = N$ is in the centre of H so for any $h \in H$ we have the following equality:

$$k_1^{-1} k_2 h = h k_1^{-1} k_2$$

Thus, we have the required equality below:

$${}^{\delta(k_1)}h = k_1 h k_1^{-1} = k_2 h k_2^{-1} = {}^{\delta(k_2)}h$$

With this action it is easy to check that $\delta : H \longrightarrow G$ defines a G-crossed module. Conversely, suppose $\delta : H \longrightarrow G$ is a G-crossed module with δ surjective. Then

$$1 \longrightarrow Ker\,(\delta) \longrightarrow H \xrightarrow{\delta} G \longrightarrow 1$$

is a central extension by Remark 3.11, so central extensions correspond to surjective G-crossed modules.

Remark 3.11. For any crossed module $\delta : H \longrightarrow G$, we have $Ker\,(\delta) \subseteq Z\,(H)$. To see this, let $z \in Ker\,(\delta)$. Then by the Peiffer identity, for any $h \in H$ we have the following equality:

$$zhz^{-1} = {}^{\delta(z)}h = {}^{1}h = h$$

Note that this argument also shows that $Ker\,(\delta)$ is an abelian group.

Definition 3.12. We may generalise the notion of G-crossed module to define a category **Cross** in which we vary the group G. We call the objects of **Cross** crossed modules. Given two crossed modules $\delta_1 : H_1 \longrightarrow G_1$ and $\delta_2 : H_2 \longrightarrow G_2$, a morphism

$$(u, v) : (\delta_1 : H_1 \longrightarrow G_1) \longrightarrow (\delta_2 : H_2 \longrightarrow G_2)$$

is given by a pair of group homomorphisms $u : G_1 \longrightarrow G_2$ and $v : H_1 \longrightarrow H_2$ such that the diagrams below commute:

$$\begin{array}{ccc} G_1 \times H_1 & \xrightarrow{u \times v} & G_2 \times H_2 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} \\ H_1 & \xrightarrow[v]{} & H_2 \end{array}$$

$$\begin{array}{ccc} H_1 & \xrightarrow{v} & H_2 \\ \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} \\ G_1 & \xrightarrow[u]{} & G_2 \end{array}$$

Thus, from the first diagram, for any $g \in G_1$ and $h \in H_1$ we have ${}^{u(g)}v\,(h) = v\,({}^{g}h)$.

We will often denote a crossed module $\delta : H \longrightarrow G$ simply by δ.

We now give a definition which we will make a great deal of use of in subsequent sections. The functors introduced in Definition 3.13 may be thought of as the analogue of the first and second homotopy groups of a topological space. We will make this precise in Section 3.3 and Example 7.14.

Definition 3.13. We may define the following two functors:

$$\begin{array}{rcl} h_0 : \mathbf{Cross} & \longrightarrow & \mathbf{Grp} \\ \delta & \longmapsto & Coker\,(\delta) \\ ((u,v) : \delta_1 \longrightarrow \delta_2) & \longmapsto & (\overline{u} : Coker\,(\delta_1) \longrightarrow Coker\,(\delta_2)) \end{array}$$

$$\begin{array}{rcl} h_1 : \mathbf{Cross} & \longrightarrow & \mathbf{Ab} \\ \delta & \longmapsto & Ker\,(\delta) \\ ((u,v) : \delta_1 \longrightarrow \delta_2) & \longmapsto & \left(v\,|_{Ker(\delta_1)}: Ker\,(\delta_1) \longrightarrow Ker\,(\delta_2)\right) \end{array}$$

Here $\overline{u} : Coker\,(\delta_1) \longrightarrow Coker\,(\delta_2)$ is given by $\overline{u}\,(g) = u\,(g)$ for any $g \in Coker\,(\delta_1)$. Note that $Ker\,(\delta)$ is always abelian by Remark 3.11. We will call a morphism of crossed modules $(u,v) : \delta_1 \longrightarrow \delta_2$ an equivalence if it induces isomorphisms $h_0\,(\delta_1) \cong h_0\,(\delta_2)$ and $h_1\,(\delta_1) \cong h_1\,(\delta_2)$.

Example 3.14. Let G be a group with presentation $F/H \cong G$, where F is a free group and $H \subseteq F$ is a normal subgroup. By Example 3.9, the normal subgroup inclusions below define crossed modules:

$$\begin{array}{rcl} \delta_1 : H & \longrightarrow & F \\ \delta_2 : 1 & \longrightarrow & G \end{array}$$

Let $p : F \longrightarrow F/H$ be projection. Then $(0,p) : \delta_1 \longrightarrow \delta_2$ is a morphism of crossed modules. It is not hard to check that this morphism is an equivalence.

3.3 Fundamental Crossed Modules

We now introduce the fundamental crossed module studied in [39]. For background on CW-complexes and relative homotopy see [17].

Let X be a connected CW-complex. Consider the pair of topological spaces $\left(X, X^{(1)}\right)$, where $X^{(1)} \subseteq X$ is the 1-skeleton of X. For any pair of pointed spaces (X, Y, x_o) we have the following long exact sequence induced by the inclusions $i : (Y, x_o) \longrightarrow (X, x_o)$ and $j : (X, x_o) \longrightarrow (X, Y)$

$$\ldots \longrightarrow \pi_n\,(Y, x_o) \xrightarrow{i_\star} \pi_n\,(X, x_o) \xrightarrow{j_\star} \pi_n\,(X, Y, x_o) \xrightarrow{\delta} \pi_{n-1}\,(Y, x_o) \longrightarrow \ldots$$

If X is a connected CW-complex then in particular its 1-skeleton is path connected, so we may suppress the basepoint from our notation. Furthermore, $\pi_n\left(X^{(1)}\right) = 0$ for $n \neq 1$. Thus we have a short exact sequence as below:

$$1 \longrightarrow \pi_2\,(X) \xrightarrow{\alpha} \pi_2\left(X, X^{(1)}\right) \xrightarrow{\delta} \pi_1\left(X^{(1)}\right) \xrightarrow{\beta} \pi_1\,(X) \longrightarrow 1$$

Lemma 3.15. *For any connected CW-complex X*

$$\delta : \pi_2\left(X, X^{(1)}\right) \longrightarrow \pi_1\left(X^{(1)}\right)$$

defines a crossed module.

See [4] for a proof of Lemma 3.15. The action of $\pi_1\left(X^{(1)}\right)$ on $\pi_2\left(X, X^{(1)}\right)$ is induced by travelling along a path in X corresponding to an element of $\pi_1\left(X^{(1)}\right)$. In this way, the crossed module encodes a great deal of the geometric information of X.

Definition 3.16. Let X be a connected CW-complex. Then we call

$$\delta : \pi_2\left(X, X^{(1)}\right) \longrightarrow \pi_1\left(X^{(1)}\right)$$

the fundamental crossed module of X. We will denote this by $\Pi_1(X)$.

Remark 3.17. Note that $\pi_1\left(X^{(1)}\right)$ is a free group since $X^{(1)}$ is a connected graph. We call such crossed modules 0-free. The 0-free objects of **Cross** are the cofibrant objects in the model structure on **Cross** of Definition 7.11.

Remark 3.18. By the exactness of the sequence above we have the following isomorphisms:

$$\begin{aligned}
h_0(\Pi_1(X)) &= Coker(\delta) \\
&= \pi_1\left(X^{(1)}\right)/Im(\delta) \\
&= \pi_1\left(X^{(1)}\right)/Ker(\beta) \\
&= Im(\beta) \\
&= \pi_1(X) \\
h_1(\Pi_1(X)) &= Ker(\delta) \\
&= Im(\alpha) \\
&= \pi_2(X)
\end{aligned}$$

3.4 Reduced and Stable Quadratic Modules

Definition 3.19. A reduced quadratic module is a sequence of group homomorphisms as below:

$$G^{ab} \otimes G^{ab} \xrightarrow{\omega} H \xrightarrow{\delta} G$$

Here G^{ab} is the abelianisation of G. We require that the following equalities hold for any $g_1, g_2 \in G$ and $h_1, h_2 \in H$:

$$\begin{aligned}
(\delta \circ \omega)(g_1 \otimes g_2) &= g_2 g_1 g_2^{-1} g_1^{-1} \\
\omega(\delta(h_1) \otimes \delta(h_2)) &= h_2 h_1 h_2^{-1} h_1^{-1} \\
\omega(\delta(h) \otimes g + g \otimes \delta(h)) &= 0
\end{aligned}$$

We will often denote a reduced quadratic module by (ω, δ).

Definition 3.20. Let (ω_1, δ_1) and (ω_2, δ_2) be reduced quadratic modules. A morphism of reduced quadratic modules $(u, v) : (\omega_1, \delta_1) \longrightarrow (\omega_2, \delta_2)$ is given by a pair of group homomorphisms $u : G_1 \longrightarrow G_2$ and $v : H_1 \longrightarrow H_2$ such that the diagram below commutes:

$$\begin{array}{ccccc} G_1^{ab} \otimes G_1^{ab} & \xrightarrow{\omega_1} & H_1 & \xrightarrow{\delta_1} & G_1 \\ {\scriptstyle u^{ab}\otimes u^{ab}}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\ G_2^{ab} \otimes G_2^{ab} & \xrightarrow{\omega_2} & H_2 & \xrightarrow[\delta_2]{} & G_2 \end{array}$$

Definition 3.21. Let (ω, δ) be a reduced quadratic module. If the following condition is satisfied for any $g_1, g_2 \in G$, we call (ω, δ) a stable quadratic module:

$$\omega\left(g_1 \otimes g_2 + g_2 \otimes g_1\right) = 0$$

Note that this is a stronger condition than the final equality of Definition 3.19. Let **SQuad** denote the full subcategory of **RQuad** generated by the stable quadratic modules.

Remark 3.22. There is an obvious forgetful functor U_1 : **SQuad** $\longrightarrow$ **RQuad**. In addition, there is a forgetful functor U_2 : **RQuad** $\longrightarrow$ **Cross** defined as follows. For any reduced quadratic module

$$G^{ab} \otimes G^{ab} \xrightarrow{\quad\omega\quad} H \xrightarrow{\delta} G$$

define its image in **Cross** to be the crossed module $\delta : H \longrightarrow G$ with the action given by:

$${}^{g}h = \omega\left(\delta\left(h\right) \otimes g\right) h$$

Any morphism of reduced quardatic modules $(u, v) : (\omega_1, \delta_1) \longrightarrow (\omega_2, \delta_2)$ is mapped to the corresponding morphism of crossed modules $(u, v) : \delta_1 \longrightarrow \delta_2$.

3.5 The 2-Category of Crossed Modules

We now define 2-morphisms of crossed modules, making the category **Cross** into a 2-category.

Definition 3.23. Let $(u_1, v_1), (u_2, v_2) : \delta_1 \longrightarrow \delta_2$ be morphisms in **Cross**. A 2-morphism $\phi : (u_1, v_1) \Longrightarrow (u_2, v_2)$ is a function $\phi : G_1 \longrightarrow H_2$ as in the diagram below:

$$\begin{array}{ccc} H_1 & \xrightarrow{v_1, v_2} & H_2 \\ {\scriptstyle \delta_1}\downarrow & \overset{\phi}{\nearrow} & \downarrow{\scriptstyle \delta_2} \\ G_1 & \xrightarrow[u_1, u_2]{} & G_2 \end{array}$$

For any $g, g_1, g_2 \in G_1$ and $h \in H_1$ we require the following equalities:

$$\phi(g_1 g_2) = {}^{u_1(g_2)}\phi(g_1)\,\phi(g_2)$$

$$u_2(g) = u_1(g)\,\delta_2(\phi(g))$$

$$v_2(h) = v_1(h)\,\phi(\delta_1(h))$$

We call a function $\phi : G_1 \longrightarrow H_2$ safisfying the first condition a u_1-crossed homomorphism. Vertical composition of 2-morphism is given by pointwise multiplication. Therefore, any 2-morphism ϕ has an inverse, given by $\phi^{-1}(g) = (\phi(g))^{-1}$ for every $g \in G_1$. Thus, the 2-category **Cross** is a **Grpd**-enriched category.

Example 3.24. Let H and G be abelian groups. We may consider H a G-module via the trivial action ${}^{g}h := h$ for all $g \in G$ and $h \in H$. With respect to this action any homomorphism $\delta : H \longrightarrow G$ between abelian groups defines a crossed module. Such crossed modules are 2-term chain complexes of abelian groups, as in Example 2.7. In this case, the definitions of crossed module morphisms and 2-morphisms reduce to those of chain maps and chain homotopies.

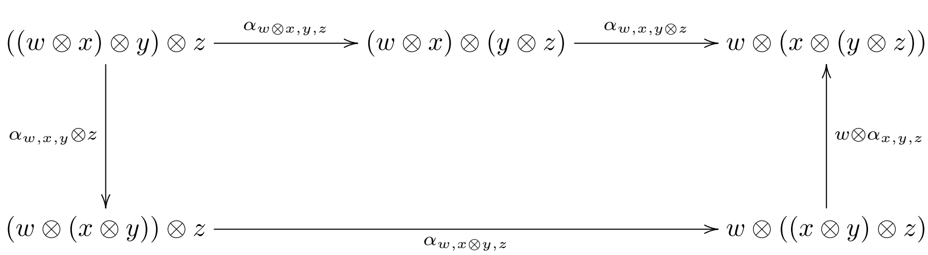
4 Monoidal Categories

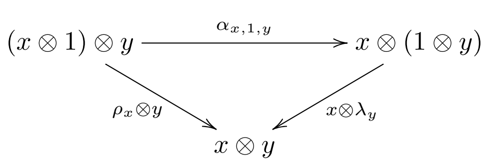
Definition 4.1. A monoidal category is a category $\mathcal{C}$ equipped with:

- A functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
- A distinguished object $1 \in \mathcal{C}$ called the unit object
- Natural isomorphisms called, respectively, the associator, and the left and right unitors:

$$\begin{aligned} \alpha_{x,y,z} : (x \otimes y) \otimes z &\longrightarrow x \otimes (y \otimes z) \\ \lambda_x : 1 \otimes x &\longrightarrow x \\ \rho_x : x \otimes 1 &\longrightarrow x \end{aligned}$$

such that the following diagrams commute for any objects $w, x, y, z \in \mathcal{C}$





These diagrams express the pentagon axiom and the triangle axiom for monoidal categories. Note that we will usually suppress the objects from our notation when writing α, λ and ρ.

Example 4.2. Consider the category $\mathbf{Vect}_{\mathbb{C}}$ of complex vector spaces and linear transformations. $\mathbf{Vect}_{\mathbb{C}}$ forms is a monoidal category with respect to both tensor product and direct sum. The unit object with respect to the tensor product is $\mathbb{C}$ and the unit with respect to direct sum is the trivial vector space 0. In both cases the isomorphisms α, λ and ρ are canonical.

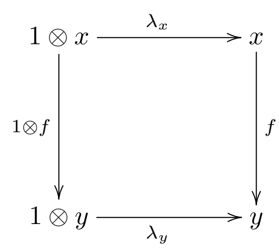
Remark 4.3. For any object $x \in \mathcal{C}$ we may define the following functors. The first, $\mathcal{Y}_x : \mathcal{C} \longrightarrow \mathcal{C}$ is analogous to right multiplication:

$$\begin{array}{rcl} z & \longmapsto & z \otimes x \\ (f : z \longrightarrow w) & \longmapsto & (f \otimes x : z \otimes x \longrightarrow w \otimes x) \end{array}$$

The second, $\mathcal{W}_x : \mathcal{C} \longrightarrow \mathcal{C}$ is analogous to left multiplication:

$$\begin{array}{rcl} z & \longmapsto & x \otimes z \\ (f : z \longrightarrow w) & \longmapsto & (x \otimes f : x \otimes z \longrightarrow x \otimes w) \end{array}$$

The unitors are the components of natural isomorphisms $\rho : \mathcal{Y}_1 \Longrightarrow id_{\mathcal{C}}$ and $\lambda : \mathcal{W}_1 \Longrightarrow id_{\mathcal{C}}$ from the functors $\mathcal{Y}_1$ and $\mathcal{W}_1$ to the identity functor. Since they are naturally isomorphic to the identity, both functors $\mathcal{Y}_1$ and $\mathcal{W}_1$ are equivalences. The naturality condition for λ implies that for any arrow $f : x \longrightarrow y$ in $\mathcal{C}$ the diargam below commutes:



In particular, taking $f = \lambda_x : 1 \otimes x \longrightarrow x$, we have the following identity:

$$\lambda_{1 \otimes x} = 1 \otimes \lambda_x$$

Similarly, the naturality of the right unitor gives the following equality for any $x \in \mathcal{C}$:

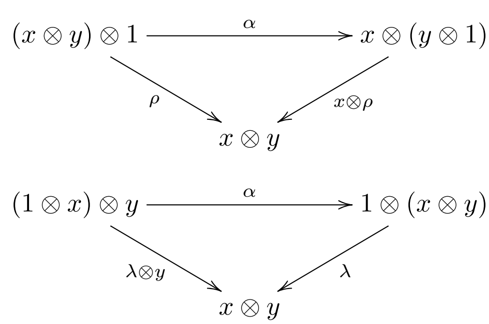
$$\rho_{x \otimes 1} = \rho_x \otimes 1$$

Example 4.4. Suppose $\mathcal{C}$ is a category with finite coproducts. In particular, $\mathcal{C}$ has an empty coproduct, which is simply an initial object $\emptyset \in \mathcal{C}$. The universal property of the copoduct yields isomorphsims, natural in $a, b, c \in \mathcal{C}$:

$$\begin{array}{rcl} \alpha_{a,b,c} : (a \amalg b) \amalg c & \longrightarrow & a \amalg (b \amalg c) \\ \lambda_a : \emptyset \amalg a & \longrightarrow & a \\ \rho_a : a \amalg \emptyset & \longrightarrow & a \end{array}$$

With these canonical isomorphisms, $\mathcal{C}$ forms a monoidal category with respect to the coproduct. Dually, if $\mathcal{C}$ has finite products then $\mathcal{C}$ is monoidal with respect to the product, with unit given by the terminal object. These constructions make many familiar categories monoidal; for example, the category **Set** has coproduct given by disjoint union with initial object the empty set $\emptyset$, and product given by the cartesian product with terminal object given by any singleton. **Set** has all finite products and coproducts, so **Set** is monoidal with respect to both cartesian product and disjoint union.

Lemma 4.5. *Let $\mathcal{C}$ be a monoidal category. Then for any $x, y \in \mathcal{C}$ the diagrams below commute:*

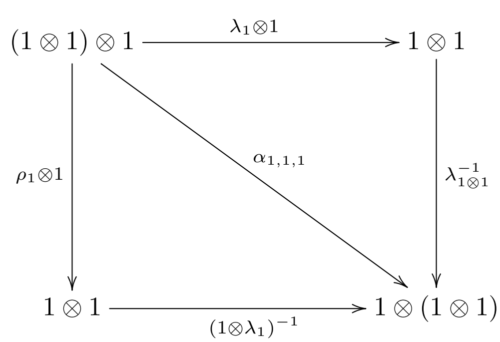


See [20] for a proof of Lemma 4.5.

Lemma 4.6. *In any monoidal category*

$$\lambda_1 = \rho_1 : 1 \otimes 1 \longrightarrow 1$$

Proof. Setting every object to 1 in the triangle axiom and in the second diagram of Lemma 4.5 we obtain the following commutative square:



Now, by Remark 4.3, we have $\lambda_{1\otimes 1} = 1 \otimes \lambda_1$. Thus, the diagram above reduces to the following equality:

$$\rho_1 \otimes 1 = \lambda_1 \otimes 1$$

That is, we have $\mathcal{Y}_1(\lambda_1) = \mathcal{Y}_1(\rho_1)$. By Remark 4.3, $\mathcal{Y}_1 : \mathcal{C} \longrightarrow \mathcal{C}$ is an equivalence, so in particular it is faithful. Thus, $\mathcal{Y}_1(\lambda_1) = \mathcal{Y}_1(\rho_1)$ implies that $\lambda_1 = \rho_1$. □

Definition 4.7. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A monoidal functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor F between the underlying categories equipped with the following natural isomorphisms:

$$\begin{array}{rcl} \mu_{x,y} : F(x) \otimes F(y) & \longrightarrow & F(x \otimes y) \\ \mu_1 : 1 & \longrightarrow & F(1) \end{array}$$

We require that the following diagrams commute for any $x, y, z \in \mathcal{C}$:

$$\begin{array}{ccccc} (F(x) \otimes F(y)) \otimes F(z) & \xrightarrow{\mu_{x,y} \otimes F(z)} & F(x \otimes y) \otimes F(z) & \xrightarrow{\mu_{x \otimes y, z}} & F((x \otimes y) \otimes z) \\ \Big\downarrow {\scriptstyle \alpha_{F(x),F(y),F(z)}} & & & & \Big\downarrow {\scriptstyle F(\alpha_{x,y,z})} \\ F(x) \otimes (F(y) \otimes F(z)) & \xrightarrow[F(x) \otimes \mu_{y,z}]{} & F(x) \otimes F(y \otimes z) & \xrightarrow[\mu_{x, y \otimes z}]{} & F(x \otimes (y \otimes z)) \end{array}$$

$$\begin{array}{ccc} F(x) \otimes F(1) & \xrightarrow{\mu_{x,1}} & F(x \otimes 1) \\ \Big\uparrow {\scriptstyle F(x) \otimes \mu_1} & & \Big\downarrow {\scriptstyle F(\rho)} \\ F(x) \otimes 1 & \xrightarrow[\rho]{} & F(x) \end{array}$$

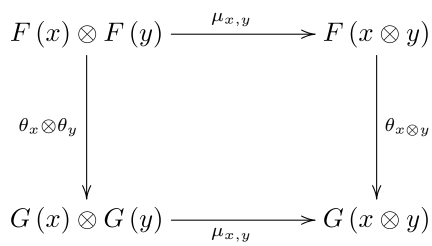
$$\begin{array}{ccc} F(1) \otimes F(x) & \xrightarrow{\mu_{1,x}} & F(1 \otimes x) \\ \Big\uparrow {\scriptstyle \mu_1 \otimes F(x)} & & \Big\downarrow {\scriptstyle F(\lambda)} \\ 1 \otimes F(x) & \xrightarrow[\lambda]{} & F(x) \end{array}$$

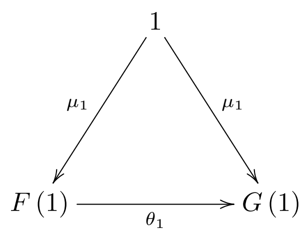
As with the associator and the unitors, we will usually suppress the objects from our notation when writing μ. We call a monoidal functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ an equivalence if the underlying functor is an equivalence of categories.

Remark 4.8. If $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{E}$ are monoidal functors then their composite $G \circ F : \mathcal{C} \longrightarrow \mathcal{E}$ has the following structure isomorphisms in $\mathcal{E}$:

$$\begin{array}{rcl} \mu^{G \circ F}_{x,y} & = & G\left(\mu^F_{x,y}\right) \circ \mu^G_{F(x),F(y)} \\ \mu^{G \circ F}_1 & = & G\left(\mu^F_1\right) \circ \mu^G_1 \end{array}$$

Definition 4.9. Let $G, F : \mathcal{C} \longrightarrow \mathcal{D}$ be monoidal functors between monoidal categories. Then a monoidal natural transformation $\theta : F \Longrightarrow G$ is a natural transformation such that the diagrams below commute, for any $x, y \in \mathcal{C}$:





Lemma 4.10. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a monoidal functor. Then F is an equivalence if and only if there is a monoidal functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ and monoidal natural isomorphisms $\eta : id_{\mathcal{C}} \Longrightarrow G \circ F$ and $\varepsilon : F \circ G \Longrightarrow id_{\mathcal{D}}$.*

A proof of Lemma 4.10 may be found in [12].

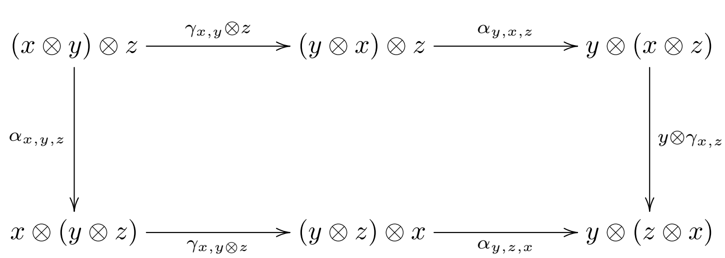
Definition 4.11. There is a 2-category with objects monoidal categories, 1-cells monoidal functors and 2-cells monoidal natural transformations. We will denote this 2-category by **MonCat**.

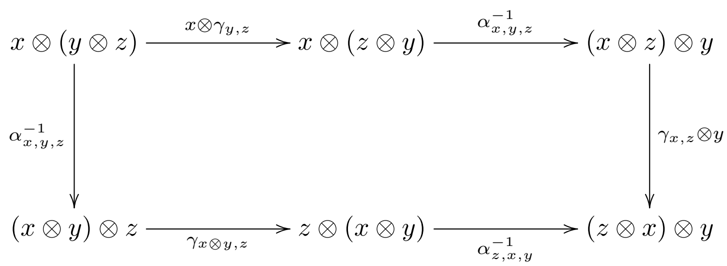
4.1 Braided and Symmetric Monoidal Categories

Definition 4.12. A braided monoidal category is a monoidal category $\mathcal{C}$ equipped with a family of natural isomorphisms

$$\gamma_{x,y} : x \otimes y \longrightarrow y \otimes x$$

making the diagrams below commute for all $x, y, z \in \mathcal{C}$:



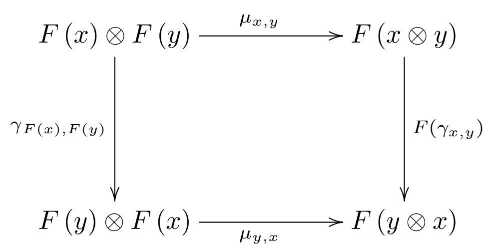


Note that, given a braiding $\gamma_{x,y} : x \otimes y \longrightarrow y \otimes x$, the natural isomorphisms $\gamma^{-1}_{x,y} : y \otimes x \longrightarrow x \otimes y$ also constitute a braiding on $\mathcal{C}$. In general these braidings are not the same; we call a braided monoidal category symmetric if for any $x, y \in \mathcal{C}$ we have the following identity:

$$\gamma^{-1}_{x,y} = \gamma_{y,x} : y \otimes x \longrightarrow x \otimes y$$

Thus, we have two notions of commutativity for the tensor product on a monoidal category. The morphisms between braided monoidal categories are those which preserve the braiding. Note that this implies a condition on both the underlying functor and on its structure isomorphisms. Unlike monoid morphisms, which automatically preserve commutativity, not every monoidal functor between braided monoidal categories is a braided monoidal functor.

Definition 4.13. Let $\mathcal{C}$ and $\mathcal{D}$ be braided monoidal categories. A braided monoidal functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a monoidal functor F such that the diagram below commutes:



Definition 4.14. There is a 2-category with objects braided monoidal categories, 1-cells braided monoidal functors and 2-cells monoidal natural transformations. We will denote this 2-category **BraidMonCat**. The full (on 1-cells and on 2-cells) sub-2-category of **BraidMonCat** generated by symmetric monoidal categories will be denoted **SymMonCat**.

4.2 Actions of Monoidal Categories

Definition 4.15. Let $\mathcal{C}$ be a monoidal category. We may associate $\mathcal{C}$ with the bicategory $\boldsymbol{B}\mathcal{C}$ with one object $\bullet$ and morphism category

$$Hom_{\boldsymbol{B}\mathcal{C}}(\bullet, \bullet) = \mathcal{C}.$$

Composition of 1-cells in $\boldsymbol{B}\mathcal{C}$ is induced by the tensor product, and vertical composition of 2-cells is induced by composition in $\mathcal{C}$. Note that, since the tensor product is in general neither strictly associative nor strictly unital, $\boldsymbol{B}\mathcal{C}$ does not form a 2-category.

Definition 4.16. Let $\mathcal{C}$ be a monoidal category. A left action of $\mathcal{C}$ on a category A is a pseudofunctor $\rho : \boldsymbol{B}\mathcal{C} \longrightarrow \mathbf{Cat}$ into the 2-category $\mathbf{Cat}$, such that $\rho(\bullet) = A$. We call A a left $\mathcal{C}$-category. The strict 2-category of left $\mathcal{C}$-categories is the 2-category $[\boldsymbol{B}\mathcal{C}, \mathbf{Cat}]$ with objects given by pseudofunctors, 1-morphisms given by pseudonatural transformations and 2-morphisms given by modifications. The 2-category of right $\mathcal{C}$-categories is the functor 2-category $[\boldsymbol{B}\mathcal{C}^{op}, \mathbf{Cat}]$. Note that the bicategory $\boldsymbol{B}\mathcal{C}^{op}$ has 1-morphisms reversed, but not 2-morphisms.

We will not define bicategories, pseudofunctors, pseudonatural transformations and modifications in general. However, in the case that the source bicategory has only one object, pseudofunctors amonut to monoidal functors. Note, however, that pseudonatural transformations do not recover monoidal natural transformations. We will unwind the definitions of pseudonatural transformations and modifications in the particular case of Theorem 4.20. See [27] for the definitions in general.

Example 4.17. Any monoidal category $\mathcal{C}$ acts on itself by right multiplication via the pseudofunctor $\mathcal{Y} : \boldsymbol{B}\mathcal{C}^{op} \longrightarrow \mathbf{Cat}$. This can be described as follows:

$$\begin{array}{rcl} \bullet & \longmapsto & \mathcal{C} \\ (x : \bullet \longrightarrow \bullet) & \longmapsto & (\mathcal{Y}_x : \mathcal{C} \longrightarrow \mathcal{C}) \\ (\phi : x \Longrightarrow y) & \longmapsto & (\mathcal{Y}_\phi : \mathcal{Y}_x \Longrightarrow \mathcal{Y}_y) \end{array}$$

As in Remark 4.3, the functor $\mathcal{Y}_x : \mathcal{C} \longrightarrow \mathcal{C}$ is given by:

$$\begin{array}{rcl} z & \longmapsto & z \otimes x \\ (f : z \longrightarrow w) & \longmapsto & (f \otimes x : z \otimes x \longrightarrow w \otimes x) \end{array}$$

The natural transformation $\mathcal{Y}_\phi : \mathcal{Y}_x \Longrightarrow \mathcal{Y}_y$ has components

$$(\mathcal{Y}_\phi)_z = z \otimes \phi : z \otimes x \longrightarrow z \otimes y$$

The coherence isomorphisms for $\mathcal{Y}$ are induced by the associator α and the right unitor ρ in $\mathcal{C}$. Explicitly, these are natural transformations as below:

$$\mu_{x,y} : \mathcal{Y}_x \circ \mathcal{Y}_y \Longrightarrow \mathcal{Y}_{y \otimes x}$$

The component of this natural transformation at $z \in \mathcal{C}$ is given by $\alpha_{z,y,x} : (z \otimes y) \otimes x \longrightarrow z \otimes (y \otimes x)$. The other structure isomorphism is given by

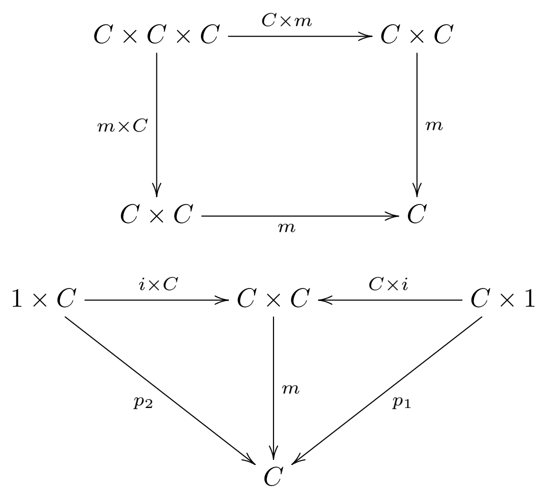
$$\mu_1 = \rho^{-1} : id_{\mathcal{C}} \Longrightarrow \mathcal{Y}_1$$

Note that we are thinking of $\mathcal{C}$ as an element of $\mathbf{Cat}$ rather than $\mathbf{MonCat}$. That is, we are associating the monoidal category $\mathcal{C}$ with its underlying category.

4.3 Strictification for Monoidal Categories

Definition 4.18. We say a monoidal category $\mathcal{C}$ is strict if the isomorphisms α, λ and ρ are all identities. Similarly, a monoidal functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is strict if μ_1 and μ are identities. We will denote the 2-category of strict monoidal categories, strict monoidal functors, and monoidal natural transformations by **StrMonCat**.

Remark 4.19. Strict monoidal categories are equivalent to monoid objects in **Cat**. Explicitly, a monoid is a set C equipped with a multiplication function $m : C \times C \longrightarrow C$ and a function $i : 1 \longrightarrow C$ from the terminal set into C, which picks out the identity element in C. We require the following diagrams to commute, expressing associativity and unit laws for multiplication:



In the same way, a strict monoidal category $\mathcal{C}$ is a category equipped with functors $m : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ and $i : 1 \longrightarrow \mathcal{C}$ from the terminal category, satisfying the same diagrams as above. Monoid morphisms are arrows between monoids which preserve both the multiplication m and the unit i. Internal monoid morphisms in **Cat** amount to strict monoidal functors.

Note that a monoid object in **Cat** encodes the same information as an internal category in **Mon**, the category of monoids. This is a general principle, referred to as the commutativity of internalisation in [2]. For a proof of this in the case of group objects in **Cat** and internal categories in **Grp**, see [13].

Theorem 4.20. *For any monoidal category $\mathcal{C}$ there is a monoidal equivalence $[\boldsymbol{B}\mathcal{C}^{op}, \mathbf{Cat}](\mathcal{Y}, \mathcal{Y}) \simeq \mathcal{C}$.*

Here $\mathcal{Y} : \boldsymbol{B}\mathcal{C}^{op} \longrightarrow \mathbf{Cat}$ is the pseudofunctor of Example 4.17. The category $[\boldsymbol{B}\mathcal{C}^{op}, \mathbf{Cat}](\mathcal{Y}, \mathcal{Y})$ has objects pseudonatural transformations from $\mathcal{Y}$ to itself and arrows the modifications between these. This can be thought of as the category of $\mathcal{C}$-equivariant functors, where $\mathcal{C}$ acts on itself by right multiplication.

Explicitly, a pseudonatural transformation from $\mathcal{Y}$ to itself amounts to a functor $\sigma : \mathcal{C} \longrightarrow \mathcal{C}$ and a family of natural isomorphisms:

$$\sigma_x : \sigma \circ \mathcal{Y}_x \Longrightarrow \mathcal{Y}_x \circ \sigma$$

such that the following diagrams commute:

$$\begin{array}{ccc}
\sigma \circ \mathcal{Y}_x \circ \mathcal{Y}_y & \xrightarrow{\sigma_x \circ \mathcal{Y}_y} & \mathcal{Y}_x \circ \sigma \circ \mathcal{Y}_y \\
\downarrow{\scriptstyle \sigma \circ \mu_{x,y}} & & \\
\sigma \circ \mathcal{Y}_{y \otimes x} & & \downarrow{\scriptstyle \mathcal{Y}_x \circ \sigma_y} \\
\downarrow{\scriptstyle \sigma_{y \otimes x}} & & \\
\mathcal{Y}_{y \otimes x} \circ \sigma & \xleftarrow[\mu_{x,y} \circ \sigma]{} & \mathcal{Y}_x \circ \mathcal{Y}_y \circ \sigma
\end{array}$$

$$\begin{array}{ccccc}
id_{\mathcal{C}} \circ \sigma & = & \sigma & = & \sigma \circ id_{\mathcal{C}} \\
\downarrow{\scriptstyle \mu_1 \circ \sigma} & & & & \downarrow{\scriptstyle \sigma \circ \mu_1} \\
\mathcal{Y}_1 \circ \sigma & & \xleftarrow[\sigma_1]{} & & \sigma \circ \mathcal{Y}_1
\end{array}$$

Note that the functor $\sigma : \mathcal{C} \longrightarrow \mathcal{C}$ need not be monoidal. Given two $\mathcal{C}$-equivariant functors $\sigma, \tau : \mathcal{C} \longrightarrow \mathcal{C}$, a modification between the corresponding pseudonatural transformations amounts to a $\mathcal{C}$-equivariant natural transformation. This is a natural transformation $\mathscr{M} : \sigma \Longrightarrow \tau$ such that the diagram below commutes:

$$\begin{array}{ccc}
\sigma \circ \mathcal{Y}_x & \xrightarrow{\mathscr{M} \circ \mathcal{Y}_x} & \tau \circ \mathcal{Y}_x \\
\downarrow{\scriptstyle \sigma_x} & & \downarrow{\scriptstyle \tau_x} \\
\mathcal{Y}_x \circ \sigma & \xrightarrow[\mathcal{Y}_x \circ \mathscr{M}]{} & \mathcal{Y}_x \circ \tau
\end{array}$$

In components, this diagram gives the commutative square below:

$$\begin{array}{ccc} \sigma(z \otimes x) & \xrightarrow{\mathscr{M}_{z \otimes x}} & \tau(z \otimes x) \\ \Big\downarrow{\scriptstyle \sigma_{x,z}} & & \Big\downarrow{\scriptstyle \tau_{x,z}} \\ \sigma(z) \otimes x & \xrightarrow[\mathscr{M}_z \otimes x]{} & \tau(z) \otimes x \end{array}$$

The tensor product on $[\boldsymbol{B}\mathcal{C}^{op}, \mathbf{Cat}](\mathcal{Y}, \mathcal{Y})$ is given by composition of functors.

Proof. We will give a only description of the monoidal functors $\omega : [\boldsymbol{B}\mathcal{C}^{op}, \mathbf{Cat}](\mathcal{Y}, \mathcal{Y}) \longrightarrow \mathcal{C}$ and $\bar{\omega} : \mathcal{C} \longrightarrow [\boldsymbol{B}\mathcal{C}^{op}, \mathbf{Cat}](\mathcal{Y}, \mathcal{Y})$. See [20] for the complete proof. The first functor is as follows:

$$\begin{array}{rcl} \omega : [\boldsymbol{B}\mathcal{C}^{op}, \mathbf{Cat}](\mathcal{Y}, \mathcal{Y}) & \longrightarrow & \mathcal{C} \\ (\sigma : \mathcal{C} \longrightarrow \mathcal{C}) & \longmapsto & \sigma(1) \\ (\mathscr{M} : \sigma \Longrightarrow \tau) & \longmapsto & (\mathscr{M}_1 : \sigma(1) \longrightarrow \tau(1)) \end{array}$$

Its weak inverse is the following functor:

$$\begin{array}{rcl} \bar{\omega} : \mathcal{C} & \longrightarrow & [\boldsymbol{B}\mathcal{C}^{op}, \mathbf{Cat}](\mathcal{Y}, \mathcal{Y}) \\ x & \longmapsto & (\mathcal{W}_x : \mathcal{C} \longrightarrow \mathcal{C}) \\ (\phi : x \longrightarrow y) & \longmapsto & (\mathcal{W}_\phi : \mathcal{W}_x \Longrightarrow \mathcal{W}_y) \end{array}$$

Here $\mathcal{W}_x : \mathcal{C} \longrightarrow \mathcal{C}$ is the functor of Remark 4.3, given by

$$\begin{array}{rcl} z & \longmapsto & x \otimes z \\ (f : z \longrightarrow w) & \longmapsto & (x \otimes f : x \otimes z \longrightarrow x \otimes w) \end{array}$$

The natural transformation $\mathcal{W}_\phi : \mathcal{W}_x \Longrightarrow \mathcal{W}_y$ has components

$$(\mathcal{W}_\phi)_z = \phi \otimes z : x \otimes z \longrightarrow y \otimes z$$

□

The first proof of a coherence theorem for monoidal categories was given by Mac Lane. This proof appears in [29]. The proof given in [20] and outlined above is very different to Mac Lane's proof. In fact, as stated above, this theorem is a 2-dimensional version of Cayley's theorem, which states that any monoid acts faithfully on its underlying set by right multiplication. From a categorical perspective, Cayley's theorem is a special case of the Yoneda lemma applied to the delooping of a monoid. In exactly the same way, Theorem 4.20 is a special case of the Yoneda lemma for bicategories. For a proof of this result in full generality see [3].

Theorem 4.20 implies that any monoidal category $\mathcal{C}$ is equivalent to a strict monoidal category via a monoidal functor. Note, however, that in general this monoidal functor is not strict.

Furthermore, although any braided (or symmetric) monoidal category is equivalent to a braided (symmetric) monoidal category in which α, λ and ρ are identites, we cannot in general make γ an identity. The corresponding coherence theorem, proved in [20], states that any braided (symmetric) monoidal category is equivalent to a braided (symmetric) strict monoidal category in which two composites of the commutativity isomorphism are equal if and only if their action on the objects corresponds to the same element of the braid group (respectively the symmetric group). In the case of symmetric monoidal categories, we call such a monoidal category permutative.

One statement of Mac Lane's coherence theorem for monoidal categories is that all diagrams made up of α, λ and ρ commute. Note that Lemma 4.5 and Lemma 4.6 both follow from this theorem.

5 2-Groups

We have now developed the required machinery to give a concise definition of 2-groups. We begin this section by developing some elementary facts about 2-groups. Many of these results may be found in the expository article [2], although there are more concise accounts in the early sections of [11] and [15]. For a modern account from a higher categorical point of view, see the relevant section of [37].

Definition 5.1. Let $\mathcal{G}$ be a monoidal groupoid. We call $\mathcal{G}$ a 2-group if, for every object $g \in \mathcal{G}$, the functor $\mathcal{Y}_g : \mathcal{G} \longrightarrow \mathcal{G}$ of Remark 4.3 is an equivalence. The 2-category **2Grp** of 2-groups is the full sub-2-category of **MonCat** whose objects are 2-groups. That is, the arrows in **2Grp** are monoidal functors and the 2-arrows are monoidal natural transformations.

If the underlying monoidal category of a 2-group $\mathcal{G}$ is braided (or symmetric) we call $\mathcal{G}$ a braided (respectively symmetric) 2-group. We may thus define the 2-category **Braid2Grp** of braided 2-groups, braided monoidal functors and monoidal natural transformations, and the full sub-2-category **Sym**2**Grp**.

By Definition 5.1, for any object g in a 2-group $\mathcal{G}$, the functor $\mathcal{Y}_g : \mathcal{G} \longrightarrow \mathcal{G}$ is full, faithful and essentially surjective. In particular, since $\mathcal{Y}_g$ is essentially surjective, there is an object $\bar{g} \in \mathcal{G}$ and an isomorphism as below:

$$e_g : \mathcal{Y}_g(\bar{g}) = \bar{g} \otimes g \longrightarrow 1$$

We say that such an object $\bar{g}$ is a weak left inverse to g.

Remark 5.2. Suppose $g \in \mathcal{G}$ has two weak left inverses, $e_g : \bar{g} \otimes g \longrightarrow 1$ and $\hat{e}_g : \hat{g} \otimes g \longrightarrow 1$. Then, since $\mathcal{Y}_g$ is full and faithful, there is a unique isomorphism

$\gamma : \hat{g} \longrightarrow \bar{g}$ making the diagram below commute

$$
\begin{array}{ccc}
\hat{g} \otimes g & \xrightarrow{\gamma \otimes g} & \bar{g} \otimes g \\
& {\scriptstyle \hat{e}_g} \searrow \quad \swarrow {\scriptstyle e_g} & \\
& 1 &
\end{array}
$$

Remark 5.3. If we choose a system of weak left inverses for each object in a 2-group $\mathcal{G}$, then for each $g \in \mathcal{G}$ the isomorphism $e_g : \bar{g} \otimes g \longrightarrow 1$ determines a unique isomorphism $i_g : 1 \longrightarrow g \otimes \bar{g}$ making the diagram below commute:

$$
\begin{array}{ccccc}
(g \otimes \bar{g}) \otimes g & & \xrightarrow{\alpha} & & g \otimes (\bar{g} \otimes g) \\
\uparrow {\scriptstyle i_g \otimes g} & & & & \downarrow {\scriptstyle g \otimes e_g} \\
1 \otimes g & \xrightarrow[\lambda]{} & g & \xrightarrow[\rho^{-1}]{} & g \otimes 1
\end{array}
$$

Thus, the weak left inverse $\bar{g}$ is also weak right inverse to g. As in Remark 5.2, this diagram commutes since $\mathcal{Y}_g$ is full and faithful. See [2] for a proof that the commutativity of the diagram above also implies that the diagram below commutes:

$$
\begin{array}{ccccc}
\bar{g} \otimes (g \otimes \bar{g}) & & \xrightarrow{\alpha^{-1}} & & (\bar{g} \otimes g) \otimes \bar{g} \\
\uparrow {\scriptstyle \bar{g} \otimes i_g} & & & & \downarrow {\scriptstyle e_g \otimes \bar{g}} \\
\bar{g} \otimes 1 & \xrightarrow[\rho]{} & \bar{g} & \xrightarrow[\lambda^{-1}]{} & 1 \otimes \bar{g}
\end{array}
$$

Definition 5.4. We call the 4-tuple $(g, \bar{g}, i_g, e_g)$ an adjoint equivalence if the two diagrams of Remark 5.3 commute.

Thus, in a 2-group $\mathcal{G}$ we may choose an adjoint equivalence $(g, \bar{g}, i_g, e_g)$ for any $g \in \mathcal{G}$. This implies that, as well as the functor $\mathcal{Y}_g : \mathcal{G} \longrightarrow \mathcal{G}$, the functor $\mathcal{W}_g : \mathcal{G} \longrightarrow \mathcal{G}$ of Remark 4.3 is an autoequivalence.

Remark 5.5. A choice of adjoint equivalence $(g, \bar{g}, i_g, e_g)$ for each $g \in \mathcal{G}$ defines a functor $\bar{*} : \mathcal{G} \longrightarrow \mathcal{G}$ given as follows:

$$
\begin{array}{rcl}
g & \longmapsto & \bar{g} \\
(f : g \longrightarrow h) & \longmapsto & \left(\bar{f} : \bar{g} \longrightarrow \bar{h}\right)
\end{array}
$$

The morphism $\bar{f} : \bar{g} \longrightarrow \bar{h}$ is given by the composite below:

$$\bar{g} \longrightarrow \bar{g} \otimes 1 \xrightarrow{\bar{g}\otimes i_h} \bar{g} \otimes \left(h \otimes \bar{h}\right) \xrightarrow{\bar{g}\otimes f^{-1}\otimes \bar{h}} \bar{g} \otimes \left(g \otimes \bar{h}\right)$$

$$\longrightarrow (\bar{g} \otimes g) \otimes \bar{h} \xrightarrow[e_g\otimes \bar{h}]{} 1 \otimes \bar{h} \longrightarrow \bar{h}$$

Definition 5.6. Let $\mathcal{G}$ and $\mathcal{H}$ be 2-groups. The zero morphism $0 : \mathcal{G} \longrightarrow \mathcal{H}$ in **2Grp** is defined as follows:

$$\begin{array}{rcl} g & \longmapsto & 1 \\ (\phi : g \longrightarrow h) & \longmapsto & (id : 1 \longrightarrow 1) \end{array}$$

Note that for any monoidal functor $F : \mathcal{H} \longrightarrow \mathcal{K}$ between 2-groups there is a canonical monoidal natural isomorphism $\imath : F \circ 0 \Longrightarrow 0$ with components given, for each $h \in \mathcal{H}$, by the structure isomorphism of F:

$$\imath_h = \mu_1^{-1} : F(1) \longrightarrow 1$$

Similarly, for any $F : \mathcal{K} \longrightarrow \mathcal{G}$ we have $0 \circ F = 0$.

Remark 5.7. Given a choice of adjoint equivalence for each object of $\mathcal{G}$, the isomorphisms $e_g : \bar{g} \otimes g \longrightarrow 1$ and $i_g : 1 \longrightarrow g \otimes \bar{g}$ are the comopnents of natural isomorphisms to the zero morphism. That is, for any arrow $f : g \longrightarrow h$ in $\mathcal{G}$ the diagrams below commute:

$$\begin{array}{ccc} \bar{g} \otimes g & \xrightarrow{\bar{f}\otimes f} & \bar{h} \otimes h \\ & {\scriptstyle e_g} \searrow \quad \swarrow {\scriptstyle e_h} & \\ & 1 & \end{array}$$

$$\begin{array}{ccc} g \otimes \bar{g} & \xrightarrow{f\otimes \bar{f}} & h \otimes \bar{h} \\ & {\scriptstyle i_g} \nwarrow \quad \nearrow {\scriptstyle i_h} & \\ & 1 & \end{array}$$

We will assume from now on that any 2-group $\mathcal{G}$ comes equipped with a choice of adjoint equivalence $(g, \bar{g}, i_g, e_g)$ for each $g \in \mathcal{G}$. Note that we then also have canonical natural isomorphisms as below:

$$\overline{(\bar{g})} \cong g$$

$$\overline{(g \otimes h)} \cong \bar{h} \otimes \bar{g}$$

Remark 5.8. Let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be a morphism of 2-groups. Then there are unique isomorphisms $\xi_g : \overline{F(g)} \longrightarrow F(\bar{g})$ for each $g \in \mathcal{G}$ such that the following diagrams in $\mathcal{H}$ commute, expressing compatibility of ξ with e and i:

$$\begin{array}{ccccc}
F(g) \otimes \overline{F(g)} & \xrightarrow{F(g)\otimes\xi} & F(g) \otimes F(\bar{g}) & \xrightarrow{\mu} & F(g \otimes \bar{g}) \\
{\scriptstyle i}\uparrow & & & & \uparrow{\scriptstyle F(i)} \\
1 & & \xrightarrow[\mu_1]{\qquad\qquad} & & F(1)
\end{array}$$

$$\begin{array}{ccccc}
\overline{F(g)} \otimes F(g) & \xrightarrow{\xi\otimes F(g)} & F(\bar{g}) \otimes F(g) & \xrightarrow{\mu} & F(\bar{g} \otimes g) \\
{\scriptstyle e}\downarrow & & & & \downarrow{\scriptstyle F(e)} \\
1 & & \xrightarrow[\mu_1]{\qquad\qquad} & & F(1)
\end{array}$$

See [2] for a proof of this fact.

Example 5.9. Let $\mathcal{C}$ be a category. The autoequivalence 2-group of $\mathcal{C}$, denoted $\mathcal{A}ut(\mathcal{C})$, is the category whose objects are autoequivalences of $\mathcal{C}$, and whose arrows are the natural isomorphisms between them. The monoidal product on $\mathcal{A}ut(\mathcal{C})$ is given by composition of functors:

$$\circ : \mathcal{A}ut(\mathcal{C}) \times \mathcal{A}ut(\mathcal{C}) \longrightarrow \mathcal{A}ut(\mathcal{C})$$

The unit object is the identity functor. Since composition of functors is strictly associative and unital, $\mathcal{A}ut(\mathcal{C})$ forms a strict monoidal category. Note, however, that its objects are not strictly invertible; by definition they need only be invertible up to isomorphism.

5.1 Homotopy Invariants of 2-Groups

We now introduce the analogues for 2-groups of the functors h_0 and h_1 introduced in 3.13. We make the connection explicit in Lemma 5.22.

Let $\mathcal{G}$ be a 2-group. Let $h_0(\mathcal{G})$ be the set of isomorphism classes of objects in $\mathcal{G}$. Let $h_1(\mathcal{G}) = Aut_{\mathcal{G}}(1)$ be the automorphisms of the unit object $1 \in \mathcal{G}$. The tensor product on $\mathcal{G}$ induces a group operation on both $h_0(\mathcal{G})$ and $h_1(\mathcal{G})$.

Explicitly, for $a, b \in h_1(\mathcal{G})$ their product $a \star b$ is given by the composite morphism below:

$$1 \xrightarrow{\lambda_1^{-1}} 1 \otimes 1 \xrightarrow{a\otimes b} 1 \otimes 1 \xrightarrow{\lambda_1} 1$$

Note that $\lambda_1 = \rho_1$ by Lemma 4.6, so we have the following equalities:

$$\begin{aligned} a \star b &= \lambda_1 \circ (a \otimes b) \circ \lambda_1^{-1} \\ &= \rho_1 \circ (a \otimes b) \circ \rho_1^{-1} \end{aligned}$$

Composition of morphisms induces a second group operation on $h_1(\mathcal{G})$, which is a homomorphism for the operation induced by the tensor product. To see this, note that the functoriality of the tensor product gives, for any $a, b, c, d \in h_1(\mathcal{G})$, the following equality:

$$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$$

This gives the following:

$$\begin{aligned}(a \star b) \circ (c \star d) &= \lambda_1 \circ (a \otimes b) \circ \lambda_1^{-1} \circ \lambda_1 \circ (c \otimes d) \circ \lambda_1^{-1} \\ &= \lambda_1 \circ (a \otimes b) \circ (c \otimes d) \circ \lambda_1^{-1} \\ &= \lambda_1 \circ (a \circ c) \otimes (b \circ d) \circ \lambda_1^{-1} \\ &= (a \circ c) \star (b \circ d)\end{aligned}$$

Furthermore, by Remark 4.3, we have the two identities below:

$$a \star id = \rho_1 \circ (a \otimes id) \circ \rho_1^{-1} = a$$

$$id \star a = \lambda_1 \circ (id \otimes a) \circ \lambda_1^{-1} = a$$

Thus, $id_1 : 1 \longrightarrow 1$ is the identity for both operations. We have the following equalities:

$$\begin{aligned}a \star b &= (a \circ id) \star (id \circ b) \\ &= (a \star id) \circ (id \star b) \\ &= a \circ b \\ &= (id \star a) \circ (b \star id) \\ &= (id \circ b) \star (a \circ id) \\ &= b \star a\end{aligned}$$

Thus, the two operations on $h_1(\mathcal{G})$ coincide, and $h_1(\mathcal{G})$ is abelian. This argument, known as the Eckmann-Hilton argument, is described in [2].

Now, let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be a morphism of 2-groups. The functor F induces a group homomorphism on $h_0(\mathcal{G})$ as follows:

$$\begin{aligned}h_0(F) : h_0(\mathcal{G}) &\longrightarrow h_0(\mathcal{H}) \\ [g] &\longmapsto [F(g)]\end{aligned}$$

Here $[g]$ denotes the isomorphism class of an object $g \in \mathcal{G}$. Similarly, via its structure isomorphism μ_1, F induces a group homomorphism on $h_1(\mathcal{G})$:

$$\begin{aligned}h_1(F) : h_1(\mathcal{G}) &\longrightarrow h_1(\mathcal{H}) \\ a &\longmapsto \mu_1^{-1} \circ F(a) \circ \mu_1\end{aligned}$$

Definition 5.10. Let **2Grp** denote the 1-category obtained from the 2-category **2Grp** by forgetting the 2-morphsims. The constructions given above define the following functors:

$$h_0 : \mathbf{2Grp} \longrightarrow \mathbf{Grp}$$

$$h_1 : \mathbf{2Grp} \longrightarrow \mathbf{Ab}$$

Lemma 5.11. *Let $F, G : \mathcal{G} \longrightarrow \mathcal{H}$ be morphisms of 2-groups. Suppose we have a 2-morphism $\phi : F \Longrightarrow G$. Then $h_1(F) = h_1(G)$ and $h_0(F) = h_0(G)$.*

Proof. Suppose we have a monoidal natural transformation $\phi : F \Longrightarrow G$. Let $g \in \mathcal{G}$. Then the component of ϕ at g gives us an arrow in $\mathcal{H}$:

$$\phi_g : F(g) \longrightarrow G(g)$$

Since all arrows are invertible, this implies that $F(g)$ and $G(g)$ are isomorphic. Thus, $h_0(F) = h_0(G)$.

Now, let $a : 1 \longrightarrow 1$ be an element of $h_1(\mathcal{G})$. Then we have the following equality:

$$\begin{aligned} h_1(G)(a) &= \left(\mu_1^G\right)^{-1} \circ G(a) \circ \mu_1^G \\ &= \left(\phi_1 \circ \mu_1^F\right)^{-1} \circ G(a) \circ \left(\phi_1 \circ \mu_1^F\right) \end{aligned}$$

This follows from the triangle diagram of Definition 4.9. This gives the equality below:

$$\begin{aligned} h_1(G)(a) &= \left(\mu_1^F\right)^{-1} \circ \phi_1^{-1} \circ G(a) \circ \phi_1 \circ \mu_1^F \\ &= \left(\mu_1^F\right)^{-1} \circ F(a) \circ \mu_1^F \\ &= h_1(F)(a) \end{aligned}$$

This follows from the naturality of $\phi : F \Longrightarrow G$ at $a : 1 \longrightarrow 1$. Thus we have $h_1(F) = h_1(G)$. □

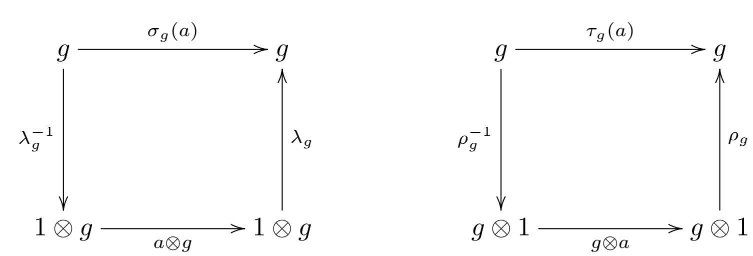
Lemma 5.12. *In any 2-group $\mathcal{G}$, for any object $g \in \mathcal{G}$, we have the following group isomorphisms:*

$$\begin{aligned} \sigma_g : h_1(\mathcal{G}) &\longrightarrow Hom_{\mathcal{G}}(g, g) \\ a &\longmapsto \lambda_g \circ (a \otimes g) \circ \lambda_g^{-1} \end{aligned}$$

$$\begin{aligned} \tau_g : h_1(\mathcal{G}) &\longrightarrow Hom_{\mathcal{G}}(g, g) \\ a &\longmapsto \rho_g \circ (g \otimes a) \circ \rho_g^{-1} \end{aligned}$$

Here the set $Hom_{\mathcal{G}}(g, g)$ has group structure given by composition.

Proof. For any $a : 1 \longrightarrow 1$, $\sigma_g(a)$ and $\tau_g(a)$ are defined by the following commutative diagrams:



Thus, $\sigma_g(a) = \lambda_g \circ \mathcal{Y}_g(a) \circ \lambda_g^{-1}$ and $\tau_g(a) = \rho_g \circ \mathcal{W}_g(a) \circ \rho_g^{-1}$ where $\mathcal{Y}_g$ and $\mathcal{W}_g$ are the functors of Remark 4.3. Since $\mathcal{Y}_g$ and $\mathcal{W}_g$ are functors they preserve composition, so it is easy to see that σ_g and τ_g define group homomorphisms.

Since $\mathcal{Y}_g$ and $\mathcal{W}_g$ are equivalences the induced maps below are bijections:

$$\begin{array}{rcl} Hom_{\mathcal{G}}(1,1) & \longrightarrow & Hom_{\mathcal{G}}(1 \otimes g, 1 \otimes g) \\ a & \longmapsto & a \otimes g \\ Hom_{\mathcal{G}}(1,1) & \longrightarrow & Hom_{\mathcal{G}}(g \otimes 1, g \otimes 1) \\ a & \longmapsto & g \otimes a \end{array}$$

It follows that σ_g and τ_g are invertible. □

Lemma 5.13. *Let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be a morphism of 2-groups. Then we have the following:*

1. F is essentially surjective if and only if $h_0(F)$ is surjective.
2. F is faithful if and only if $h_1(F)$ is injective.
3. F is full if and only if $h_0(F)$ is injective and $h_1(F)$ is surjective.
4. F an equivalence if and only if both $h_0(F)$ and $h_1(F)$ are isomorphisms.

Proof. We will outline the proofs of Statement 2 and Statement 3, since Statement 1 is not difficult, and Statement 4 follows from the first three. The proofs of Statements 2 and 3 make use of the isomorphisms of Lemma 5.12.

If F is faithful then it follows immediately that $h_1(F)$ is injective. To see the converse, suppose $h_1(F)$ is injective and let $i, j : g \longrightarrow h$ be morphisms in $\mathcal{G}$ such that $F(i) = F(j)$. We wish to show that $i = j$. Consider the morphism below:

$$\sigma_g^{-1}\left(j^{-1} \circ i\right) : 1 \longrightarrow 1$$

We have the following equalities in $\mathcal{H}$:

$$\begin{array}{rcl} F\left(j^{-1} \circ i\right) & = & id_{F(g)} \\ & = & F(\lambda_g) \circ F\left(\sigma_g^{-1}\left(j^{-1} \circ i\right) \otimes g\right) \circ F\left(\lambda_g^{-1}\right) \end{array}$$

Thus, $F\left(\sigma_g^{-1}\left(j^{-1} \circ i\right) \otimes g\right) = id_{F(1 \otimes g)}$. This implies the equality below:

$$F\left(\sigma_g^{-1}\left(j^{-1} \circ i\right)\right) = id_{F(1)}$$

Since $h_1(F)$ is injective, this implies $\sigma_g^{-1}\left(j^{-1} \circ i\right) = id_1$. Thus, since σ_g is an isomorphism, we have $i = j$. Therefore F is faithful.

Now, suppose F is full. Then it is immediate that $h_1(F)$ is surjective. To see that $h_0(F)$ is injective, suppose we have $g, h \in \mathcal{G}$ such that $F(g) \cong F(h)$. Then we have a morphism $f : F(g) \longrightarrow F(h)$ in $\mathcal{H}$. Since F is full, there is a morphism $f' : g \longrightarrow h$ in $\mathcal{G}$ such that $F(f') = f$. Thus, $g \cong h$ in $\mathcal{G}$. Therefore, $h_0(F)$ is injective.

To see the converse, suppose that $h_0(F)$ is injective and $h_1(F)$ is surjective, and let $f : F(g) \longrightarrow F(h)$ be a morphism in $\mathcal{H}$. Then we have $F(g) \cong F(h)$. Thus, since $h_0(F)$ is injective, there is a morphism $u : g \longrightarrow h$ in $\mathcal{G}$. Consider the morphism below:

$$f^{-1} \circ F(u) : F(g) \longrightarrow F(g)$$

We may apply the isomorphism $\sigma^{-1}_{F(g)}$ to obtain $\sigma^{-1}_{F(g)}\left(f^{-1} \circ F(u)\right)$ in $h_1(\mathcal{H})$. Since $h_1(F)$ is surjective, there is a morphism $a \in h_1(\mathcal{G})$ satisfying the following:

$$\begin{aligned}(h_1(F))(a) &= \mu_1^{-1} \circ F(a) \circ \mu_1 \\ &= \sigma^{-1}_{F(g)}\left(f^{-1} \circ F(u)\right)\end{aligned}$$

That is, we have the following equality:

$$\begin{aligned}f^{-1} \circ F(u) &= \sigma_{F(g)}\left(\mu_1^{-1} \circ F(a) \circ \mu_1\right) \\ &= \lambda_{F(g)} \circ \left(\mu_1^{-1} \otimes F(g)\right) \circ (F(a) \otimes F(g)) \circ (\mu_1 \otimes F(g)) \circ \lambda^{-1}_{F(g)}\end{aligned}$$

Now, consider the morphism $\sigma_g(a) : g \longrightarrow g$, and make the following definition:

$$f' := u \circ (\sigma_g(a))^{-1} : g \longrightarrow h$$

We claim that $F(f') = f$.

To see this, consider the following equalities:

$$\begin{aligned}f^{-1} \circ F(f') &= f^{-1} \circ F(u) \circ F(\sigma_g(a))^{-1} \\ &= id_{F(g)}\end{aligned}$$

This follows from the expression above for $f^{-1} \circ F(u)$ and the third commutative diagram of Definition 4.7. □

Remark 5.14. Consider the category $\mathcal{H}(\mathbf{2Grp})$, obtained from the 2-category **2Grp** by identifying 2-isomorphic morphisms. We have the following full but non-faithful functor:

$$\mathcal{H} : \mathbf{2Grp} \longrightarrow \mathcal{H}(\mathbf{2Grp})$$

This functor takes each morphism in the 1-category **2Grp** to its 2-isomorphism class. Lemma 5.11 implies that both $h_0 : \mathbf{2Grp} \longrightarrow \mathbf{Grp}$ and $h_1 : \mathbf{2Grp} \longrightarrow \mathbf{Ab}$ factor through $\mathcal{H}$. That is, we have induced functors

$$\tilde{h_0} : \mathcal{H}(\mathbf{2Grp}) \longrightarrow \mathbf{Grp}$$

$$\tilde{h_1} : \mathcal{H}(\mathbf{2Grp}) \longrightarrow \mathbf{Ab}$$

such that $h_0 = \tilde{h_0} \circ \mathcal{H}$ and $h_1 = \tilde{h_1} \circ \mathcal{H}$. In Section 7 we will show that this category $\mathcal{H}(\mathbf{2Grp})$ is equivalent to $Ho(\mathbf{Str2Grp})$, the homotopy category of **Str2Grp**.

Definition 5.15. For any group $G \in \mathbf{Grp}$ we may define a 2-group $G[0]$ with object set G, only identity morphisms and tensor product induced by multiplication on G. Similarly, for any abelian group $A \in \mathbf{Ab}$ we may define a 2-group $A[1]$ with a single object whose automorphism set is the set A, with composition induced by multiplication on A. These are the object functions of the following full and faithful functors:

$$[0] : \mathbf{Grp} \longrightarrow \mathbf{2Grp}$$

$$[1] : \mathbf{Ab} \longrightarrow \mathbf{2Grp}$$

By composing [0] and [1] with $\mathcal{H} : \mathbf{2Grp} \longrightarrow \mathcal{H}(\mathbf{2Grp})$, we may define functors into $\mathcal{H}(\mathbf{2Grp})$:

$$\tilde{[0]} : \mathbf{Grp} \longrightarrow \mathcal{H}(\mathbf{2Grp})$$

$$\tilde{[1]} : \mathbf{Ab} \longrightarrow \mathcal{H}(\mathbf{2Grp})$$

In [21], the authors describe the following adjunctions:

$$\tilde{[1]} : \mathbf{Ab} \Longleftrightarrow \mathcal{H}(\mathbf{2Grp}) : \tilde{h_1}$$

$$\tilde{[0]} : \mathbf{Grp} \Longleftrightarrow \mathcal{H}(\mathbf{2Grp}) : \tilde{h}_0$$

$$\tilde{h}_0 : \mathcal{H}(\mathbf{2Grp}) \Longleftrightarrow \mathbf{Grp} : \tilde{[0]}$$

5.2 Strictification for 2-Groups

Definition 5.16. We call a 2-group $\mathcal{G}$ strict if $\mathcal{G}$ is a strict monoidal category, and both e and i are identities. We will denote the 2-category of strict 2-groups, strict monoidal functors and monoidal natural transformations by **Str2Grp**.

A braided 2-group is called strict if the underlying 2-group is strict. We may thus define a 2-category **BraidStr2Grp** of braided strict 2-groups, braided strict monoidal functors and monoidal natural transformations. We may similarly define the 2-category **SymStr2Grp**. Note that the definition of strict braided 2-group does not impose any strictness conditions on the braiding.

Remark 5.17. Strict 2-groups are equivalent to group objects in **Cat**, just as strict monoidal categories are monoid objects in **Cat**. As indicated in Remark 4.19, this allows us to think of strict 2-groups as internal categories in **Grp**.

To be precise, there is an isomorphism of 2-categories $\mathbf{Str2Grp} \cong \mathbf{CatGrp}$, where **CatGrp** is the 2-category of internal categories of Definition 2.6. Thus, to describe a strict 2-group we may give a group of objects and a group of arrows, and group homomorphisms s, t, i and $\circ$ satisfying the axioms of Definition 2.3. This reflects the fact that, for a strict 2-group, the monoidal product induces group structures on both the set of arrows and the set of objects. We will denote both **Str2Grp** and **CatGrp** by **Str2Grp**.

Theorem 5.18. *Any* 2*-group* $\mathcal{G}$ *is equivalent in* **2Grp** *to a strict* 2*-group.*

Proof. We know from Theorem 4.20 that any 2-group $\mathcal{G}$ is equivalent in **MonCat** to a monoidal category in which α, ρ and λ are identities. We may transport the 2-group structure along this equivalence, and thus define a 2-group $\tilde{\mathcal{G}}$, which is equivalent to $\mathcal{G}$ in **2Grp** and whose underlying monoidal category is strict. In general, the natural isomorphisms i and e in $\tilde{\mathcal{G}}$ are not identities. In [2], the authors sketch a construction from $\tilde{\mathcal{G}}$ of an equivalent 2-group which is strict. □

For a more direct approach to coherence for 2-groups, see [26].

Remark 5.19. As noted after the proof of Theorem 4.20 for braided and symmetric monoidal categories, there are similar results to Theorem 5.18 for braided and symmetric 2-groups. In particular, the strictification theorem for symmetric 2-groups states that any symmetric 2-group $\mathcal{G}$ is equivalent to a strict permutative 2-group $\bar{\mathcal{G}}$. That is, $\bar{\mathcal{G}}$ is a permutative monoidal category in which every object has a strict inverse. See [19] for a proof of this result.

5.3 2-Groups and Crossed Modules

We now prove a well-known result relating strict 2-groups and crossed modules.

Theorem 5.20. *There is an equivalence of categories* **Cross** $\simeq$ **Str2Grp**.

Proof. We will describe the functors Υ : **Cross** $\longrightarrow$ **Str2Grp** and Ψ : **Str2Grp** $\longrightarrow$ **Cross**.

Let $\delta : H \longrightarrow G$ be a crossed module. We will construct $\Upsilon(\delta) = (C_1 \rightrightarrows C_0)$, an internal category in **Grp**. Let the group of objects be given by $C_0 = G$. Define the group of arrows to be given by the semidirect product $C_1 = H \rtimes G$, with multiplication defined as follows:

$$(h_1, g_1)(h_2, g_2) = (h_1{}^{g_1}h_2, g_1 g_2)$$

We define source and target maps as follows:

$$s(h, g) = g$$

$$t(h, g) = \delta(h)\, g$$

Finally, the identity map is given as follows:

$$i(g) = (1, g)$$

Composition in $\Upsilon(\delta)$ is induced by multiplication in H. That is, given morphisms as below

$$\begin{array}{rcl} (h_1, g) : g & \longrightarrow & \delta(h_1)\, g \\ (h_2, \delta(h_1)\, g) : \delta(h_1)\, g & \longrightarrow & \delta(h_2)\, \delta(h_1)\, g \end{array}$$

their composite is given as follows:

$$(h_1 h_2, g) : g \longrightarrow \delta(h_1 h_1)\, g$$

Now, suppose $(u, v) : \delta_1 \longrightarrow \delta_2$ is a morphism of crossed modules. To define the internal functor $\Upsilon(u, v) : \Upsilon(\delta_1) \longrightarrow \Upsilon(\delta_2)$ we must give two group homomorphisms, one between the groups of objects and the other between the groups of arrows. On the group of objects we take the group homomorphism $u : G_1 \longrightarrow G_2$. On the group of arrows we take the group homomorphism below:

$$u \rtimes v : H_1 \rtimes G_1 \longrightarrow H_2 \rtimes G_2$$

$$(h, g) \longmapsto (u(h), v(g))$$

That this is a group homomorphism follows easily from the first diagram of Definition 3.12, which states that for any $h \in H_1$ and $g \in G_1$ we have ${}^{u(g)}v(h) = v({}^{g}h)$.

Now, let $C = (C_1 \rightrightarrows C_0)$ be a strict 2-group. We can construct a crossed module $\Psi(C) = (\delta : H \longrightarrow G)$ as follows. Let $G = C_0$. Let $H = Ker(s) \subseteq C_1$ and define $\delta = t|_H : H \longrightarrow G$. The action of G on H is then given as follows:

$${}^{g}h = i(g)\, hi(g)^{-1}$$

To see that this is well-defined, note that for any $h \in Ker(s)$ and for any $g \in G$ we have the following:

$$s\left(i(g)\, hi(g)^{-1}\right) = gs(h)\, g^{-1} = gg^{-1} = 1$$

Now, suppose $(F_0, F_1) : C \longrightarrow C'$ is a morphism of strict 2-groups. To define a morphism of crossed modules $\Psi(F_0, F_1) : \Psi(C) \longrightarrow \Psi(C')$ we must give two group homomorphisms. These are defined as follows:

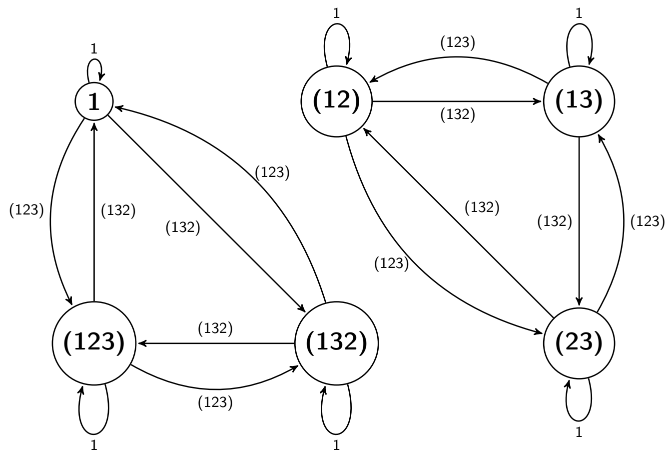
$$F_0 : G \longrightarrow G'$$

$$F_1|_H : H \longrightarrow H'$$

□

We may use the model structures on the categories **Str2Grp** and **Cross**, defined in Section 7, to extend the equivalence of categories in Theorem 5.20 to an equivalence between the 2-categories **Cross** and **Str2Grp**. This is described in Remark 7.15. Note that the induced 2-functors are isomorphisms on the hom-groupoids.

Example 5.21. Consider the normal subgroup inclusion $\delta : A_3 \longrightarrow S_3$ of the alternating group A_3 into the symmetric group S_3. By Example 3.9, this is a crossed module. We can picture the construction of $\Upsilon(\delta)$ by constructing a directed graph whose vertices are labelled by the elements of S_3 and whose edges are given by pairs $(h, g) : g \longrightarrow hg$, for $g \in S_3$ labelling a vertex, and $h \in A_3$. We label such edges by h in the picture below:



It is easy to see that this directed graph can be given the structure of a category. The category corresponds to the action of A_3 on S_3 by right multiplication; moving to isomorphism classes of objects identifies elements of S_3 in the same orbit. Thus, as we can see by the picture, $h_0(\Upsilon(\delta)) = S_3/A_3 = \mathbb{Z}_2$. We can also see that the only endomorphism of the unit object is the arrow $(1,1) : 1 \longrightarrow 1$. Thus, $h_1(\Upsilon(\delta)) = 1$.

Lemma 5.22. *The functor Υ :* **Cross** $\longrightarrow$ **Str2Grp** *preserves both h_0 and h_1 of Definitions 5.10 and 3.13. That is, we have $h_0 \circ \Upsilon = h_0$, and a natural isomorphism $h_1 \circ \Upsilon \cong h_1$.*

Proof. Let $\delta : H \longrightarrow G$ be a crossed module. We need to describe $h_0(\Upsilon(\delta))$, the isomorphism classes of objects in $\Upsilon(\delta)$, and $h_1(\Upsilon(\delta))$, the endomorphisms of the unit object. The objects of $\Upsilon(\delta)$ are given by the group G. Let $g_1, g_2 \in G$. By the description of arrows in the proof of Theorem 5.20, there is an arrow from g_1 to g_2 if and only if $g_1 = \delta(h)g_2$ for some $h \in H$. Thus, objects g_1 and g_2 are isomorphic if and only if $g_1 g_2^{-1} \in Im(\delta)$. So the group of isomorphism classes of objects in $\Upsilon(\delta)$ is given by the following quotient:

$$\begin{aligned} h_0(\Upsilon(\delta)) &= G/Im(\delta) \\ &= Coker(\delta) \\ &= h_0(\delta) \end{aligned}$$

Let $(u,v) : \delta_1 \longrightarrow \delta_2$ be a morphism of crossed modules. The homomorphism induced by the functor $\Upsilon(u,v) : \Upsilon(\delta_1) \longrightarrow \Upsilon(\delta_2)$ on isomorphism classes of objects is as follows:

$$h_0(\Upsilon(u,v)) = h_0(u,v) = \overline{u} : Coker(\delta_1) \longrightarrow Coker(\delta_2)$$

Here $\overline{u}$ is the group homomorphism defined in Definition 3.13. Thus, we have $h_0 \circ \Upsilon = h_0$.

Now, the group of morphisms in $\Upsilon(\delta)$ is given by the semidirect product $H \rtimes G$. The endomorphisms of the unit object in $\Upsilon(\delta)$ are the elements $(h,g) \in H \rtimes G$ for which the following holds:

$$s(h,g) = t(h,g) = 1$$

Thus, if $(h,g) \in h_1(\Upsilon(\delta))$, we have the following equalities:

$$s(h,g) = g = 1$$

$$t(h,g) = \delta(h)\, g = \delta(h) = 1$$

Thus, $h_1(\Upsilon(\delta))$ is the following subgroup:

$$h_1(\Upsilon(\delta)) = \{(h,1) \in H \rtimes G \mid h \in Ker(\delta)\} := K_\delta$$

Clearly we have an isomorphism taking $(h,1) \in K_\delta$ to $h \in Ker(\delta)$:

$$K_\delta \cong Ker(\delta) = h_1(\delta)$$

Now, if $(u,v) : \delta_1 \longrightarrow \delta_2$ is a morphism of crossed modules then we have the following:

$$h_1(\Upsilon(u,v)) = (u \rtimes v)\,|_{K_{\delta_1}} : K_{\delta_1} \longrightarrow K_{\delta_2}$$

The canonical isomorphisms $K_\delta \cong Ker(\delta)$ make the diagram below commute:

$$\begin{array}{ccc} K_{\delta_1} & \xrightarrow{(u \rtimes v)|_{K_{\delta_1}}} & K_{\delta_2} \\ \downarrow & & \downarrow \\ Ker(\delta_1) & \xrightarrow[u_{Ker(\delta_1)}]{} & Ker(\delta_2) \end{array}$$

Thus they give the components of a natural isomorphism $h_1 \circ \Upsilon \cong h_1$. $\square$

Remark 5.23. In general, for a strict 2-group $C = (C_1 \rightrightarrows C_0)$, h_0 is given by the coequaliser $h_0(C) = Coequ(s,t)$, and h_1 is given by $h_1(C) = Ker(s) \cap Ker(t)$.

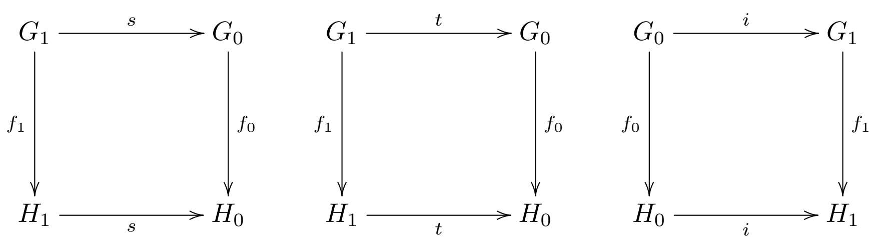
Remark 5.24. As in Theorem 5.20, there are equivalences of categories **RQuad** $\simeq$ **BraidStr2Grp** and **SQuad** $\simeq$ **SymStr2Grp**. These equivalence are described in [33].

5.4 2-Groups and Simplicial Groups

We give one more equivalent formulation of strict 2-groups, first described in [28], which is closely related to the description of strict 2-groups as internal categories in **Grp**. We will use this formulation of strict 2-groups in the construction of the free 2-group on a groupoid in Section 9.

Definition 5.25. A 1-truncated simplicial group is a pair of groups G_1 and G_0 and group homomorphisms $s, t : G_1 \longrightarrow G_0$ and $i : G_0 \longrightarrow G_1$ such that $s \circ i = t \circ i = id_{G_0}$. We will denote a 1-truncated simplicial group by $G = (G_1 \rightrightarrows G_0)$.

Given two 1-truncated simplicial groups, $G = (G_1 \rightrightarrows G_0)$ and $H = (H_1 \rightrightarrows H_0)$, a morphism of 1-truncated simplicial groups $f : G \longrightarrow H$ is given by a pair of group homomorphisms $f_0 : G_0 \longrightarrow H_0$ and $f_1 : G_1 \longrightarrow H_1$ such that the diagrams below commute:



We denote a morphism by $f = (f_0, f_1)$. We may define a category $\mathbf{Tr^1}\left(\mathbf{SimpGrp}\right)$ of 1-truncated simplicial groups and their morphisms.

Theorem 5.26. *We may associate* **Str2Grp** *with the full category of* $\mathbf{Tr^1}\left(\mathbf{SimpGrp}\right)$ *on* 1*-truncated simplicial groups* $G_1 \rightrightarrows G_0$ *such that* $[Ker\,(s)\,, Ker\,(t)] = 1$. *The inclusion functor* $J : \mathbf{Str2Grp} \longrightarrow \mathbf{Tr^1}\left(\mathbf{SimpGrp}\right)$ *has a left adjoint* $P : \mathbf{Tr^1}\left(\mathbf{SimpGrp}\right) \longrightarrow \mathbf{Str2Grp}$.

Proof. Let $G = (G_1 \rightrightarrows G_0)$ be a 1-truncated simplicial group with structure morphisms s, t and i. To give G the structure of an internal category in **Grp** we need only define composition. Given $g, h \in G_1$ with $s\,(h) = t\,(g)$, define their composite as follows:

$$h \circ g := hi\,(s\,(h))^{-1}\,g$$

It is easy to check that composition is a group homomorphism if and only if $[Ker\,(s)\,, Ker\,(t)] = 1$, where $[Ker\,(s)\,, Ker\,(t)] \subseteq G_1$ is the subgroup generated by elements of the form $xyx^{-1}y^{-1}$ with $x \in Ker\,(s)$ and $y \in Ker\,(t)$.

Now, the functor $P : \mathbf{Tr^1}\left(\mathbf{SimpGrp}\right) \longrightarrow \mathbf{Str2Grp}$ is given on objects as follows. Let $G = (G_1 \rightrightarrows G_0)$ be a 1-truncated simplicial group. Define $P\,(G) = (C_1 \rightrightarrows C_0)$, where $C_0 = G_0$ and $C_1 = G_1/\,[Ker\,(s)\,, Ker\,(t)]$. The structural morphisms for $P\,(G)$ are induced by those of G. Explicitly, denote the projection as follows:

$$p : G_1 \longrightarrow C_1 = G_1/\,[Ker\,(s)\,, Ker\,(t)]$$

Then the source and target morphisms of $P\,(G)$ are given by $\tilde{s}, \tilde{t} : C_1 \longrightarrow C_0$ where $\tilde{s}$ and $\tilde{t}$ are the unique morphisms such that $\tilde{s} \circ p = s$ and $\tilde{t} \circ p = t$, and the identity morphism is given by $p \circ i : C_0 \longrightarrow C_1$.

Similarly, for any morphism of 1-truncated simplicial groups $f = (f_1, f_0) : G \longrightarrow H$, define

$$P\,(f) = \left(\tilde{f}_1, f_0\right) : P\,(G) \longrightarrow P\,(H)$$

Here we denote the morphism induced by f_1 on the quotient by

$$\tilde{f}_1 : G_1/\left[Ker\left(s\right), Ker\left(t\right)\right] \longrightarrow H_1/\left[Ker\left(s\right), Ker\left(t\right)\right].$$

By Definition 5.25 we have the following inclusion:

$$f_1\left(\left[Ker\left(s\right), Ker\left(t\right)\right]\right) \subseteq \left[Ker\left(s\right), Ker\left(t\right)\right]$$

Thus, $\left[Ker\left(s\right), Ker\left(t\right)\right] \subseteq Ker\left(p \circ f_1\right)$. We take $\tilde{f}_1$ to be the unique map such that $\tilde{f}_1 \circ p = p \circ f_1$.

We wish to show that we have the following adjunction:

$$P : \mathbf{Tr^1}\left(\mathbf{SimpGrp}\right) \Longleftrightarrow \mathbf{Str2Grp} : J$$

The components of the unit are as follows. For any 1-truncated simplicial group G, take $\eta_G : G \longrightarrow \left(J \circ P\right)\left(G\right)$ to be given by $\eta_G = \left(p, id\right)$. Given any morphism $f : G \longrightarrow \mathcal{G}$, where $\mathcal{G} = \left(\mathcal{G}_1 \rightrightarrows \mathcal{G}_0\right)$ is a strict 2-group, we know that $\left[Ker\left(s\right), Ker\left(t\right)\right] = 1$ in $\mathcal{G}_1$. We have

$$f_1\left(\left[Ker\left(s\right), Ker\left(t\right)\right]\right) \subseteq \left[Ker\left(s\right), Ker\left(t\right)\right] = 1$$

so the homomorphism $f_1 : G_1 \longrightarrow \mathcal{G}_1$ induces a unique morphism

$$\tilde{f}_1 : G_1/\left[Ker\left(s\right), Ker\left(t\right)\right] \longrightarrow \mathcal{G}_1$$

such that $\tilde{f}_1 \circ p = f_1$. The pair $\tilde{f} = \left(\tilde{f}_1, f_0\right) : P\left(G\right) \longrightarrow \mathcal{G}$ is then the unique morphism such that $\tilde{f} \circ \eta_G = f$ in $\mathbf{Tr^1}\left(\mathbf{SimpGrp}\right)$. □

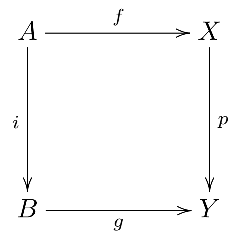
Remark 5.27. Limits in **Str2Grp** may be computed on the underlying groupoids. Colimits in $\mathbf{Tr^1}$ (**SimpGrp**) are computed dimension-wise; we may then compute colimits in **Str2Grp** by applying the functor P of Theorem 5.26. Thus, the category **Str2Grp** has all small limits and colimits.

6 Model Categories

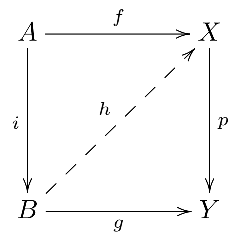
In this section we recall the definition of a Quillen model category and develop some model category theory. Model categories provide an axiomatic approach to homotopy theory. A model category has three classes of morphisms, known as weak equivalences, fibrations and cofibrations. Roughly, the weak equivalences in a model category are morphisms which we want to regard as being invertible. For example, in the model structure on **Cat** given in Example 6.13, the weak equivalences are the equivalences of categories, and in the model structure on **Top** of Example 6.11 the weak equivalences are weak homotopy equivalences. Thus, the morphisms of primary interest in a model category are the weak equivalences; the other two classes may be seen as tools to facilitate working with the weak equivalences.

The account of model categories given in this section largely follows [10]. We will apply the results of this section to study the homotopy theory of 2-groups in the subsequent sections. Therefore, we have included proofs for a number of the results so that we may refer back to the details in later sections.

Definition 6.1. In any category C, suppose we have a commutative square as below:

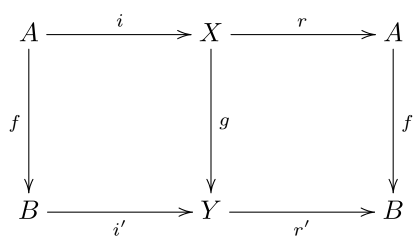


We say this square has a lift if there exists a morphism $h : B \longrightarrow X$ such that $h \circ i = f$ and $p \circ h = g$. That is, h makes the diagram below commute:



A morphism $i : A \longrightarrow B$ is said to have the left lifting property (LLP) with respect to a morphism $p : X \longrightarrow Y$, if for any $f : A \longrightarrow X$ and $g : B \longrightarrow Y$ making the top diagram commute, there is a lift $h : B \longrightarrow X$ as in the bottom diagram. If i has the LLP with respect to p, then we say that p has the right lifting property (RLP) with respect to i.

Definition 6.2. Let $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ be morphisms in some category C. We say that f is a retract of g if there are morphisms i, r, i' and r' making the diagram below commute, such that $r \circ i = id_A$ and $r' \circ i' = id_B$.



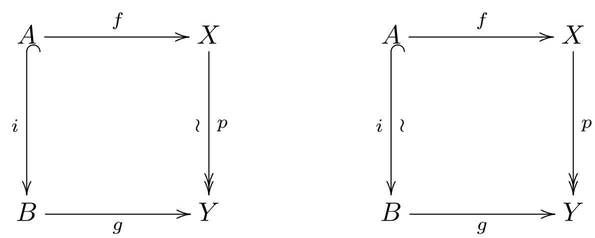
Definition 6.3. A closed Quillen model category C is a category with three distinguished classes of morphisms, each closed under composition and containing all identity maps:

- Weak equivalences, which we denote $A \xrightarrow{\sim} B$.
- Fibrations, which we denote $A \twoheadrightarrow B$.

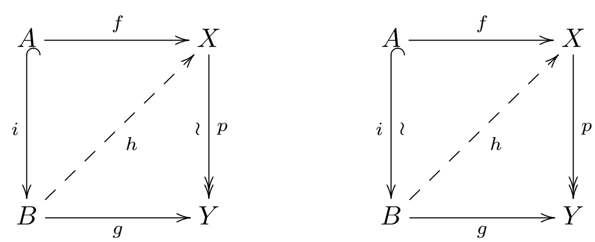
- Cofibrations, which we denote $A \hookrightarrow B$.

If a morphism is both a fibration and a weak equivalence we call it an acyclic fibration. Morphisms which are both cofibrations and weak equivalences are called acyclic cofibrations. We require that the following five axioms hold:

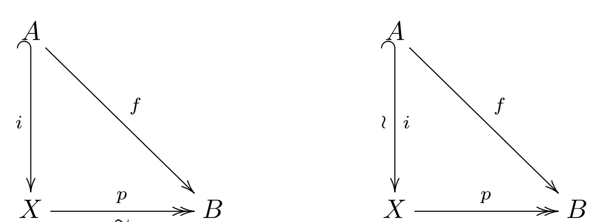
1. C has all finite limits and colimits.
2. Suppose f and g are composable arrows in C. If two of the three maps f, g and $g \circ f$ are weak equivalences, then so is the third.
3. Suppose f is a retract of g. Then if g is a fibration, f is a fibration. If g is a cofibration, f is a cofibration. If g is a weak equivalence, f is weak equivalence.
4. Suppose we have one of the two diagrams below:



   Then we have a lift $h : B \longrightarrow X$ as in the diagrams below:



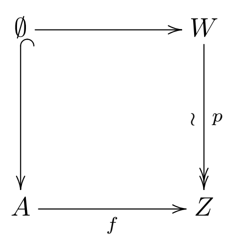
5. Given any map $f : A \longrightarrow B$ in C, we may factor $f = p \circ i$, both as a cofibration followed by an acyclic fibration and as an acyclic cofibration followed by a fibration, as shown below:



Since they are the only type of model category we will consider, we will refer to closed Quillen model categories simply as model categories. Axiom 4 of Definition 6.3 states that any cofibration in C has the LLP with respect to all acyclic fibrations, and that any acyclic cofibration has the LLP with respect to all fibrations. In fact, by Lemma 6.7, this property determines the cofibrations and acyclic cofibrations.

Definition 6.4. Let C be a model category. Since C has all finite limits and colimits, C has both an initial object $\emptyset$ and a terminal object $*$. We call an object $A \in C$ cofibrant if the unique map $\emptyset \hookrightarrow A$ is a cofibration. We call $X \in C$ fibrant if the unique map $X \twoheadrightarrow *$ is a fibration.

Remark 6.5. Suppose $A \in C$ is cofibrant, and let $f : A \longrightarrow Z$ be a morphism. Given any acyclic fibration $p : W \overset{\sim}{\twoheadrightarrow} Z$ there is a morphism $h : A \longrightarrow W$ with $p \circ h = f$. This follows since, by the uniqueness of arrows from the initial object, the diagram below must commute:



Thus we have a lift $h : A \longrightarrow W$ in this diagram. Similarly, if $X \in C$ is fibrant then for any morphism $g : W \longrightarrow X$ and any acyclic cofibration $i : W \overset{\sim}{\hookrightarrow} Z$, there is a morphism $k : Z \longrightarrow X$ such that $k \circ i = g$.

Example 6.6. For any model category C, we may define a model structure on C^{op} as follows:

- An arrow $f : A \longrightarrow B$ in C^{op} is a weak equivalence if the corresponding arrow $f : B \longrightarrow A$ in C is a weak equivalence.
- $f : A \longrightarrow B$ in C^{op} is a cofibration if $f : B \longrightarrow A$ in C is a fibration.
- $f : A \longrightarrow B$ in C^{op} is a fibration if $f : B \longrightarrow A$ in C is a cofibration.

Example 6.6 allows us to translate statements about cofibrations into the dual statement about fibrations, and vice versa. This reflects the duality built into the axioms of Definition 6.3.

Lemma 6.7. *Let $f : A \longrightarrow B$ be morphism in a model category C.*

1. f is a cofibration if and only if f has the LLP with respect to all acyclic fibrations.

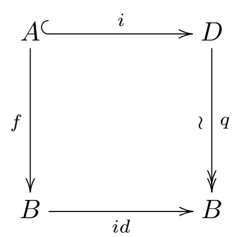
2. f is an acyclic cofibration if and only if f has the LLP with respect to all fibrations.

3. f is a fibration if and only if f has the RLP with respect to all acyclic cofibrations.

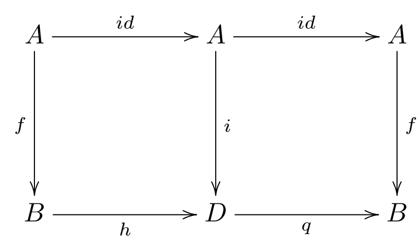
4. f is an acyclic fibration if and only if f has the RLP with respect to all cofibrations.

Proof. We will only prove Statement 1. The proof of Statement 2 is similar, and 3 and 4 are dual. We know by Axiom 4 of Definition 6.3 that if $f : A \longrightarrow B$ is a cofibration, f has the LLP with respect to all acyclic fibrations. We must prove the converse.

Suppose $f : A \longrightarrow B$ is a morphism in C with the LLP with respect to all acyclic fibrations. By Axiom 5 of Definition 6.3, we may factor f as $f = q \circ i$, where i is a cofibration and q is an acyclic fibration, as below:



Since f has the LLP with respect to all acyclic fibrations, f has the LLP with respect to q. Thus, we have a lift in the diagram above. That is, we have a map $h : B \longrightarrow D$ such that $q \circ h = id$ and $f \circ h = i$. This morphism h makes the diagram below commute, realising f as a retract of i:

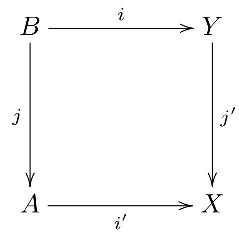


By Axiom 3 of Definition 6.3, since i is a cofibration, this implies that f is a cofibration. ☐

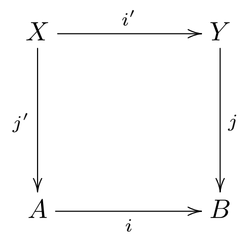
Remark 6.8. Any two of the three classes of morphisms in a model category determines the third. If we know the weak equivalences, then by Lemma 6.7 we need only specify one other class of morphism and the last is determined. Furthermore, if we know both the fibrations and the cofibrations then, by Lemma 6.7, we may characterise acyclic cofibrations and acyclic fibrations. By Axiom 2 and 5 of Definition 6.3 the weak equivalences are exactly those morphisms that factorise as an acyclic cofibration followed by an acyclic fibration.

Thus, when defining a model structure on a category, we need only describe two of the three classes of morphisms.

Definition 6.9. Suppose the diagram below is a pushout:



We call $i' : A \longrightarrow X$ the cobase change of i along j, and $j' : Y \longrightarrow X$ the cobase change of j along i. Similarly, suppose the diagram below is a pullback:

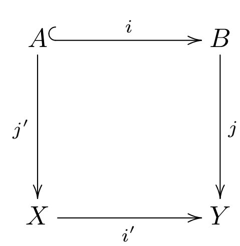


We call $i' : X \longrightarrow Y$ the base change of i along j, and $j' : X \longrightarrow A$ the base change of j along i.

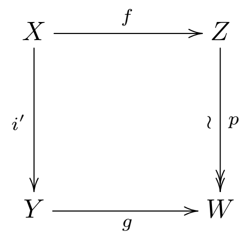
Lemma 6.10. *Let C be a model category. Then we have the following:*

1. The class of cofibrations is closed under cobase change.
2. The class of acyclic cofibrations is closed under cobase change.
3. The class of fibrations is closed under base change.
4. The class of acyclic fibrations is closed under base change.

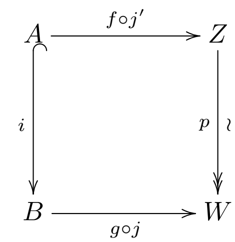
Proof. We will only prove Statement 1. The proof of Statement 2 is similar, and 3 and 4 are dual. Let $i : A \hookrightarrow B$ be a cofibration, and suppose the diagram below is a pushout:



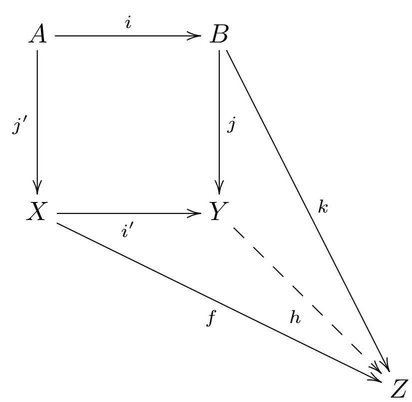
We wish to show that $i' : X \longrightarrow Y$ is a cofibration. By Lemma 6.7, this is equivalent to showing that i' has the LLP with respect to all acyclic fibrations. Suppose we have an acyclic fibration $p : Z \overset{\sim}{\twoheadrightarrow} W$ and a commutative diagram as below:



We wish to find a lift for this diagram. Consider the commutative diagram below:

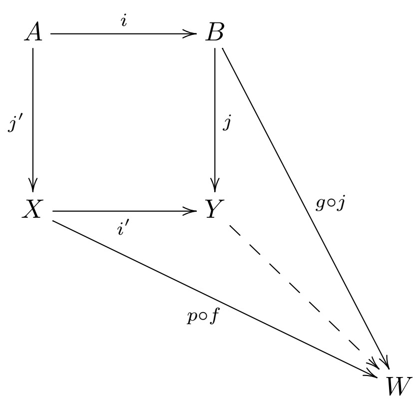


By Axiom 4 of Definition 6.3, since i is a cofibration and p is an acyclic fibration, there is a lift $k : B \longrightarrow Z$ in the diagram above. Thus, we have $k \circ i = f \circ j'$. However, by the universal property of the pushout, this implies that there is a unique morphism $h : Y \longrightarrow Z$ such that $h \circ j = k$ and $h \circ i' = f$, as in the diagram below:



We know that $h \circ i' = f$. Furthermore, since both g and $p \circ h$ fill in the pushout diagram below, by the uniqueness property, we have $p \circ h = g$. Thus, $h : Y \longrightarrow Z$

is the desired lift.



Therefore, i' has the LLP with respect to any acyclic fibration. Thus, i' is a cofibration. □

Example 6.11. Consider the category **Top**. We may define a model structure on **Top** as follows:

- An arrow $f : A \longrightarrow B$ is a weak equivalence if it is a weak homotopy equivalence. That is, f is a weak equivalence if f induces isomorphisms

$$f_* : \pi_n (A, a_o) \longrightarrow \pi_n (B, f (a_o))$$

  for any basepoint $a_0 \in A$, and for all $n \geq 0$.

- An arrow $f : A \longrightarrow B$ is a fibration if it has the RLP with respect to the inclusions

$$D^n \times \{0\} \longrightarrow D^n \times [0, 1]$$

  for all $n \geq 0$. Here D^n denotes the n-disc.

- An arrow $f : A \longrightarrow B$ is a cofibration if it has the LLP with respect to all fibrations.

This is known as the Quillen model structure on **Top**. With respect to this model structure, every object of **Top** is fibrant and CW complexes are cofibrant. Note that **Top** admits other model structures. For example, the Strøm model structure has weak equivalences given by homotopy equivalences and all objects are both fibrant and cofibrant.

Definition 6.12. Let $F : A \longrightarrow B$ be a functor. We call F an isofibration if, for any object $a \in A$ and any isomorphism $f : F (a) \longrightarrow b$ in B, there is an isomorphism $g : a \longrightarrow a'$ in A with $F (g) = f$. Thus we also have $F (a') = b$. Note that any functor which is both full and surjective on objects is an isofibration.

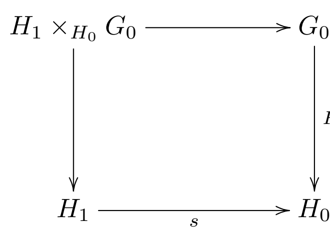
Example 6.13. Consider the category **Cat**. We may define a model structure on **Cat** as follows:

- An arrow $F : A \longrightarrow B$ is a weak equivalence if it is an equivalence of categories.
- An arrow $F : A \longrightarrow B$ is a cofibration if it is injective on objects.
- An arrow $F : A \longrightarrow B$ is a fibration if it is an isofibration.

This model structure is known as the canonical model structure on **Cat**. It is the only model structure on **Cat** for which the weak equivalences are the equivalences of categories.

Example 6.14. The model structure of Example 6.13 restricts to a model structure on **Grpd**, the category of groupoids.

Remark 6.15. Let $F : G \longrightarrow H$ be a functor between groupoids. Thinking of $G = (G_1 \rightrightarrows G_0)$ and $H = (H_1 \rightrightarrows H_0)$ as internal groupoids in **Set**, as in Definition 2.3, F determines functions $F_0 : G_0 \longrightarrow H_0$ and $F_1 : G_1 \longrightarrow H_1$. Consider the pullback below:

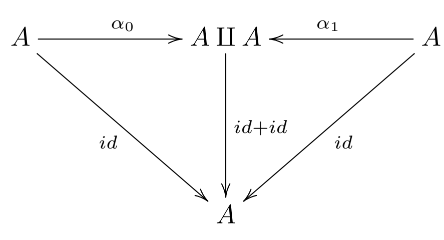


Since $F_0 \circ s = s \circ F_1$, the functions $s : G_1 \longrightarrow G_0$ and $F_1 : G_1 \longrightarrow H_1$ induce a canonical map $L : G_1 \longrightarrow H_1 \times_{H_1} G_0$. The functor F is an isofibration if and only if L is surjective.

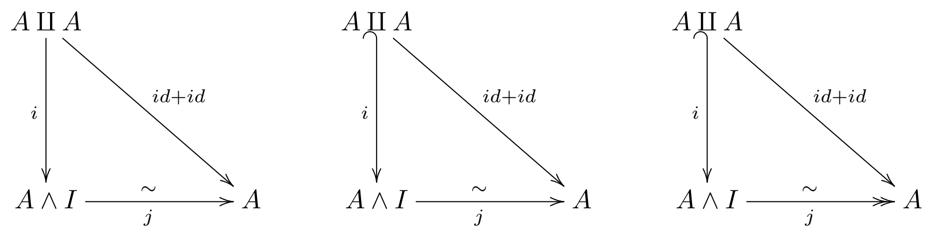
6.1 Homotopies in a Model Cateogry

6.1.1 Cylinder Objects and Left Homotopies

Definition 6.16. Let C be a model category and let $A \in C$. Consider the coproduct $A \amalg A$ and the folding map $id + id : A \amalg A \longrightarrow A$. This map makes the diagram below commute, where $\alpha_i : A \longrightarrow A \amalg A$ are the canonical inclusions:



A cylinder object for A is an object $A \wedge I \in C$ with a morphism $i : A \amalg A \longrightarrow A \wedge I$ and a weak equivalence $j : A \wedge I \xrightarrow{\sim} A$ such that $j \circ i = id + id$. We call a cylinder object $A \wedge I$ a good cylinder object if $i : A \amalg A \hookrightarrow A \wedge I$ is a cofibration and a very good cylinder object if in addition $j : A \wedge I \overset{\sim}{\twoheadrightarrow} A$ is an acyclic fibration. A cylinder object, good cylinder object and very good cylinder object for an object $A \in C$ are shown below:



We will denote $i \circ \alpha_0 = i_0$ and $i \circ \alpha_1 = i_1$. Thus, $i = i_0 + i_1 : A \amalg A \longrightarrow A \wedge I$.

Remark 6.17. By Axiom 5 of Definition 6.3, the folding map for any object $A \in C$ must factorise as $id + id = j \circ i$, where i is a cofibration and j is an acyclic fibration. Thus, every object $A \in C$ has at least one very good cylinder object.

Lemma 6.18. *Let A be a cofibrant object and let $A \wedge I$ be a good cylinder object for A. Then the morphisms $i_0 : A \amalg A \overset{\sim}{\hookrightarrow} A \wedge I$ and $i_1 : A \amalg A \overset{\sim}{\hookrightarrow} A \wedge I$ are acyclic cofibrations.*

Definition 6.19. Let $f, g : A \longrightarrow X$ be morphisms in a model category C. Let $A \wedge I$ be a cylinder object for A. Then a left homotopy from f to g via $A \wedge I$ is a morphism $H : A \wedge I \longrightarrow X$ such that $H \circ i = f + g$. That is, $H \circ i_0 = f$ and $H \circ i_1 = g$. If $A \wedge I$ is a good cylinder object then we call H a good left homotopy. If $A \wedge I$ is a very good cylinder object we call H a very good left homotopy. If we have a left homotopy from f to g then we write $f \sim_l g$. We say that f is left homotopic to g.

Remark 6.20. Let $f, g : A \longrightarrow X$ be morphisms in a model category. If $f \sim_l g$ then we may use Axiom 5 of Definition 6.3 to find a good left homotopy from f to g. Furthermore, if X is fibrant, then there is a very good left homotopy from f to g.

Lemma 6.21. *Let A and X be two objects in a model category. Then for any morphism $f : A \longrightarrow X$ we have $f \sim_l f$. For any two morphisms $f, g : A \longrightarrow X$, if $f \sim_l g$ then $g \sim_l f$. Furthermore, if A is cofibrant, then given any three morphisms $f, g, h : A \longrightarrow X$, if $f \sim_l g$ and $g \sim_l h$, then $f \sim_l h$.*

Remark 6.22. By Lemma 6.21, for any cofibrant object $A \in C$ and any object $X \in C$, the relation $\sim_l$ is an equivalence relation on $Hom_C(A, X)$. Even if A is not cofibrant, so that $\sim_l$ is not necessarily an equivalence realtion, we may consider the equivalence relation generated by $\sim_l$ on $Hom_C(A, X)$. We denote the set of equivalence classes under this equivalence relation by $\pi^l(A, X)$.

Example 6.23. For any space $A \in \mathbf{Top}$, one choice of cylinder object is the cylinder $A \times [0,1]$. The maps i and j are given as follows:

$$\begin{array}{rcl} i_0 : A & \longrightarrow & A \times [0,1] \\ a & \longmapsto & (a,0) \end{array}$$

$$\begin{array}{rcl} i_1 : A & \longrightarrow & A \times [0,1] \\ a & \longmapsto & (a,1) \end{array}$$

$$\begin{array}{rcl} j : A \times [0,1] & \longrightarrow & A \\ (a,t) & \longmapsto & a \end{array}$$

For any space A the map j is a homotopy equivalence and thus a weak homotopy equivalence. If A is a CW complex, so A is cofibrant, then the map $i = i_0 + i_1$ is a cofibration, so $A \times [0,1]$ is a good cylinder object. Note that for a general space $A \in \mathbf{Top}$ this need not be the case. Given continuous functions $f, g : A \longrightarrow X$ a left homotopy from f to g via $A \times [0,1]$ is a continuous function $H : A \times [0,1] \longrightarrow X$ such that $H(a,0) = f(a)$ and $H(a,1) = g(a)$ for all points $a \in A$. Thus, left homotopies from f to g via this choice of cylinder object $A \times [0,1]$ coincide with the classical notion of homotopy.

Example 6.24. Let $\mathbf{I}$ denote the interval groupoid. That is, $\mathbf{I}$ is the category with two objects $0, 1 \in \mathbf{I}$ and an invertible arrow $\alpha : 0 \longrightarrow 1$. For any category $A \in \mathbf{Cat}$, one choice of cylinder object is the product category $A \times \mathbf{I}$. The functors i and j are given as follows:

$$\begin{array}{rcl} i_0 : A & \longrightarrow & A \times \mathbf{I} \\ a & \longmapsto & (a,0) \\ (f : a \longrightarrow b) & \longmapsto & ((f, id) : (a,0) \longrightarrow (b,0)) \end{array}$$

$$\begin{array}{rcl} i_1 : A & \longrightarrow & A \times \mathbf{I} \\ a & \longmapsto & (a,1) \\ (f : a \longrightarrow b) & \longmapsto & ((f, id) : (a,1) \longrightarrow (b,1)) \end{array}$$

The functor $j = p_1 : A \times \mathbf{I} \longrightarrow A$ is given by projection onto the first factor. This functor is full, faithful and surjective on objects, so it is both an isofibration and an equivalence of categories. Thus, $j : A \times \mathbf{I} \xrightarrow{\sim} A$ is an acyclic fibration. The functor $i = i_0 + i_1$ is injective on objects, so i is a cofibration. Thus for any $A \in \mathbf{Cat}$, $A \times \mathbf{I}$ is a very good cylinder object. Given functors $F, G : A \longrightarrow X$, a left homotopy from F to G via $A \times \mathbf{I}$ is a functor $H : A \times \mathbf{I} \longrightarrow X$ such that $H \circ i_0 = F$ and $H \circ i_1 = G$. For any arrow $f : a \longrightarrow a'$ in A, consider the commutative diagram below in $A \times \mathbf{I}$:

$$\begin{array}{ccc}
(a,0) & \xrightarrow{(id,\alpha)} & (a,1) \\
\Big\downarrow{\scriptstyle (f,id)} & \searrow^{(f,\alpha)} & \Big\downarrow{\scriptstyle (f,id)} \\
(a',0) & \xrightarrow[(id,\alpha)]{} & (a',1)
\end{array}$$

Applying H to this diagram gives the following commutative diagram in X:

$$\begin{array}{ccc}
F(a) & \xrightarrow{H(id,\alpha)} & G(a) \\
\Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} \\
F(a') & \xrightarrow[H(id,\alpha)]{} & G(a')
\end{array}$$

Thus, the arrows $H(id,\alpha) : F(a) \longrightarrow G(a)$ are the components of a natural transformation from F to G. Since the arrow $(id,\alpha) : (a,0) \longrightarrow (a,1)$ in $A \times \mathbf{I}$ is an isomorphism, every component $H(id,\alpha)$ is an isomorphism. Thus, $H : A \times \mathbf{I} \longrightarrow X$ gives a natural isomorphism from F to G. Conversely, any natural isomorphism from F to G defines a functor from $A \times \mathbf{I}$ to X. Thus, left homotopies from F to G via this choice of cylinder object $A \times \mathbf{I}$ coincide with the natural isomorphisms from F to G. Note that this choice of cylinder object is functorial. (For an explicit description of the functor see Remark 6.34.)

Remark 6.25. Consider the category $\mathbf{2}$, with two objects $0, 1 \in \mathbf{2}$ and a single noninvertible arrow $\alpha : 0 \longrightarrow 1$. For any two functors $F, G : A \longrightarrow X$, functors $H : A \times \mathbf{2} \longrightarrow X$ with $H(-,0) = F$ and $H(-,1) = G$ are exactly natural transformations from F to G. Note, however, that $A \times \mathbf{2}$ is not a cylinder object for A since A is not equivalent to $A \times \mathbf{2}$. Thus, only natural isomorphisms define left homotopies.

Lemma 6.26. *Let $A \in C$ be a cofibrant object and let $p : Y \xrightarrow{\sim} X$ be an acyclic fibration. Then composition with p induces the following bijection on left homotopy classes of morphisms:*

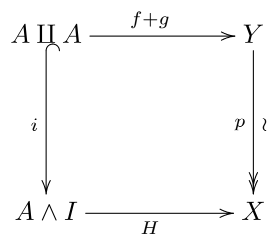
$$\begin{array}{rcl}
p_* : \pi^l(A,Y) & \longrightarrow & \pi^l(A,X) \\
{[f]} & \longmapsto & [p \circ f]
\end{array}$$

Proof. To see that p_* is well-defined, let $f, g : A \longrightarrow Y$ be morphisms, and let $H : A \wedge I \longrightarrow Y$ be a left homotopy from f to g, via some cylinder object

$A \wedge I$. Then $p \circ H : A \wedge I \longrightarrow X$ is a left homotopy from $p \circ f$ to $p \circ g$, so p_* is well-defined.

To see that p_* is surjective, consider any morphism $f : A \longrightarrow X$. By Remark 6.5, since A is cofibrant and p is an acyclic fibration, there is a map $h : A \longrightarrow Y$ with $p \circ h = f$. Thus, $p_* ([h]) = [p \circ h] = [f]$.

To see that p_* is injective, suppose we have maps $f, g : A \longrightarrow Y$ such that $[p \circ f] = [p \circ g]$. By Remark 6.20, we may find a good left homotopy $H : A \wedge I \longrightarrow X$ from $p \circ f$ to $p \circ g$. Thus we have $H \circ i = (p \circ f) + (p \circ g)$, where $i : A \amalg A \hookrightarrow A \wedge I$ is the canonical map into the good cylinder object $A \wedge I$. That is, the diagram below commutes:



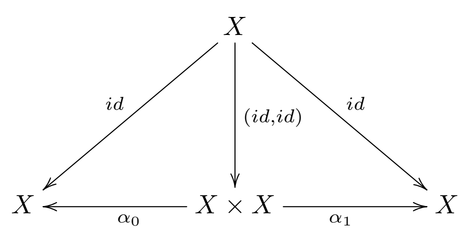
By Axiom 4 of Definition 6.3, we have a lift $K : A \wedge I \longrightarrow Y$ in this diagram. In particular, $K \circ i = f + g$, so K is a (good) left homotopy from f to g. Thus $[f] = [g]$. □

Lemma 6.27. *Let $X \in C$ be fibrant. Then, for any $A, B \in C$, the function below is well-defined:*

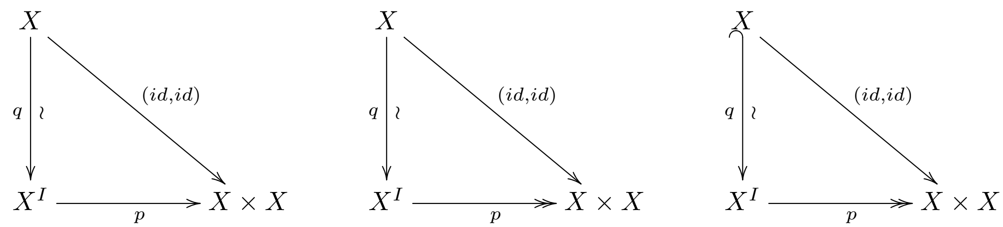
$$\begin{array}{rcl} \pi^l (A, B) \times \pi^l (B, X) & \longrightarrow & \pi^l (A, X) \\ ([f], [g]) & \longmapsto & [g \circ f] \end{array}$$

6.1.2 Path Objects and Right Homotopies

Definition 6.28. Let C be a model category and let $X \in C$. Consider the product of X with itself and the diagonal map $(id, id) : X \longrightarrow X \times X$, making the diagram below commute, where $\alpha_i : X \times X \longrightarrow X$ are the projection morphisms:



A path object for X is an object $X^I \in C$ with a weak equivalence $q : X \xrightarrow{\sim} X^I$ and a morphism $p : X^I \longrightarrow X \times X$ such that $p \circ q = (id, id)$. We call a path object X^I a good path object if $p : X \twoheadrightarrow X$ is a fibration and a very good cylinder object if in addition $q : X \overset{\sim}{\hookrightarrow} X^I$ is an acyclic cofibration. A path object, good path object and very good path object for an object $X \in C$ are shown below:



We will denote $\alpha_0 \circ p = p_0$ and $\alpha_1 \circ p = p_1$. Thus, $p = (p_0, p_1) : X^I \longrightarrow X \times X$.

Definition 6.29. Let $f, g : A \longrightarrow X$ be morphisms in a model category C. Let X^I be a path object for X. Then a right homotopy from f to g via X^I is a morphism $H : A \longrightarrow X^I$ such that $p \circ H = (f, g)$. That is, $p_0 \circ H = f$ and $p_1 \circ H = g$. If X^I is a good path object then we call H a good right homotopy. If X^I is a very good path object we call H a very good right homotopy. If we have a right homotopy from f to g then we write $f \sim_r g$. We say that f is right homotopic to g.

Remark 6.30. Any path object for $X \in C$ corresponds to a cylinder object for $X \in C^{op}$, and left homotopies correspond to right homotopies. This gives us a series of statements about path objects and right homotopies, dual to Remark 6.20, and to Lemmas 6.18, 6.21, 6.26 and 6.27. For example, the dual to Lemma 6.21 states that if $X \in C$ is fibrant then $\sim_r$ is an equivalence relation on $Hom_C(A, X)$ for any $A \in C$. For any $X \in C$, we will denote the set of equivalence classes under the equivalence relation generated by $\sim_r$ on $Hom_C(A, X)$ by $\pi^r(A, X)$.

Lemma 6.31. *Let $f, g : A \longrightarrow X$ be morphisms in a model category.*

- If A is cofibrant, then $f \sim_l g \Longrightarrow f \sim_r g$.
- If X is fibrant, then $f \sim_r g \Longrightarrow f \sim_l g$.

Thus, if A is cofibrant and X is fibrant, two morphisms $f, g : A \longrightarrow X$ are left homotopic if and only if they are right homotopic. We write $f \sim g$, and we denote the equivalence classes under this relation as follows:

$$\pi(A, X) = \pi^l(A, X) = \pi^r(A, X)$$

Note that this result is independant of the choice of (good) path object and (good) cylinder object in the following sense: If A is cofibrant and X is fibrant then for any fixed good cylinder object $A \wedge I$, and for any fixed good path object X^I, two morphisms $f, g : A \longrightarrow X$ are left homotopic via $A \wedge I$ if and only if they are right homotopic via X^I.

Example 6.32. For any space $X \in \mathbf{Top}$, one choice of path object is the mapping space $Map([0,1], X)$. The maps p and q are given as follows:

$$\begin{array}{rcl} q : X & \longrightarrow & Map([0,1], X) \\ x & \longmapsto & \gamma_x \end{array}$$

$$\begin{array}{rcl} p : Map([0,1], X) & \longrightarrow & X \times X \\ \gamma & \longmapsto & (\gamma(0), \gamma(1)) \end{array}$$

Here $\gamma_x : [0,1] \longrightarrow X$ is the constant path $\gamma_x(t) = x$ for any $t \in [0,1]$. Given continuous functions $f, g : A \longrightarrow X$ a right homotopy from f to g via $Map([0,1], X)$ is a continuous function $H : A \longrightarrow Map([0,1], X)$ such that, for any $a \in A$, $H(a,0) = f(a)$ and $H(a,1) = g(a)$, where $H(a,t)$ denotes the value of the path $H(a) \in Map([0,1], X)$ at $t \in [0,1]$. Thus, right homotopies from f to g via this choice of path object $Map([0,1], X)$ coincide with the classical notion of homotopy.

Example 6.33. For any category $X \in \mathbf{Cat}$, one choice of path object is the functor category $X^{\mathbf{I}}$, where $\mathbf{I}$ is the interval groupoid of Example 6.24. We may think of $X^{\mathbf{I}}$ as the category with objects the invertible arrows $\alpha : x \longrightarrow x'$ in X and morphisms given by commutative squares as below:

$$\begin{array}{ccc} x & \xrightarrow{f_0} & y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ x' & \xrightarrow[f_1]{} & y' \end{array}$$

The functor $q : X \longrightarrow X^{\mathbf{I}}$ takes any object $x \in X$ to arrow $id_x : x \longrightarrow x$, and any morphism $f : x \longrightarrow x'$ to the commutative square below:

$$\begin{array}{ccc} x & \xrightarrow{id_x} & x \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ x' & \xrightarrow[id_{x'}]{} & x' \end{array}$$

The functor $p : X^{\mathbf{I}} \longrightarrow X \times X$ takes any object $\alpha : x \longrightarrow x'$ to the pair (x, x') and a commutative square with horizontal sides (f_0, f_1) to the arrow (f_0, f_1) in $X \times X$. Note that for any category X, the functor category $X^{\mathbf{I}}$ is a very good path object.

Now, given two functors $F, G : A \longrightarrow X$, a right homotopy from F to G via $X^{\mathbf{I}}$ is a functor $H : A \longrightarrow X^{\mathbf{I}}$ such that $p \circ H = (F, G)$. Thus, for any $a \in A$ we have an invertible arrow $H_a : F(a) \longrightarrow G(a)$ in X and for any arrow $f : a \longrightarrow a'$ we have the commutative diagram below:

$$\begin{array}{ccc} F(a) & \xrightarrow{H_a} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(a') & \xrightarrow[H_{a'}]{} & G(a') \end{array}$$

Thus, the morphisms $H_a : F(a) \longrightarrow G(a)$ are the components of a natural isomorphism from F to G. Conversely, any natural isomorphism from F to G defines a functor from A to $X^{\mathbf{I}}$. Therefore, right homotopies from F to G via this choice of path object $X^{\mathbf{I}}$ coincide with the natural isomorphisms from F to G.

Remark 6.34. The choices of cylinder object in Example 6.24 and path object in Example 6.33 define the two functors below:

$$\begin{array}{rcl} - \times \mathbf{I} : \mathbf{Cat} & \longrightarrow & \mathbf{Cat} \\ A & \longmapsto & A \times \mathbf{I} \\ (F : A \longrightarrow B) & \longmapsto & (F \times \mathbf{I} : A \times \mathbf{I} \longrightarrow B \times \mathbf{I}) \end{array}$$

$$\begin{array}{rcl} (-)^{\mathbf{I}} : \mathbf{Cat} & \longrightarrow & \mathbf{Cat} \\ X & \longmapsto & X^{\mathbf{I}} \\ (F : X \longrightarrow Y) & \longmapsto & \left(F \circ - : X^{\mathbf{I}} \longrightarrow Y^{\mathbf{I}}\right) \end{array}$$

These functors form an adjoint pair:

$$- \times \mathbf{I} : \mathbf{Cat} \Longleftrightarrow \mathbf{Cat} : (-)^{\mathbf{I}}$$

Thus for any $A, X \in \mathbf{Cat}$ we have the following bijection, natural in A and X:

$$\varphi : Hom_{\mathbf{Cat}}(A \times \mathbf{I}, X) \longrightarrow Hom_{\mathbf{Cat}}\left(A, X^{\mathbf{I}}\right)$$

Given $H : A \times \mathbf{I} \longrightarrow X$, the functor $\varphi(H) : A \longrightarrow X^{\mathbf{I}}$ is defined as follows:

$$a \longmapsto (H(id, \alpha) : H(a, 0) \longrightarrow H(a, 1))$$

$$(f : a \longrightarrow a') \longmapsto \begin{array}{ccc} H(a, 0) & \xrightarrow{H(id_a, \alpha)} & H(a, 1) \\ {\scriptstyle H(f, id)}\downarrow & & \downarrow{\scriptstyle H(f, id)} \\ H(a', 0) & \xrightarrow[H(id_{a'}, \alpha)]{} & H(a', 1) \end{array}$$

Let $F, G : A \longrightarrow X$ be functors, and let $H : A \times \mathbf{I} \longrightarrow X$ be a functor with $H \circ i = F + G$. Then we can see that $\varphi(H) : A \longrightarrow X^{\mathbf{I}}$ satisfies $p \circ \varphi(H) = (F, G)$. Similarly, if $K : A \longrightarrow X^{\mathbf{I}}$ is a functor such that $p \circ K = (F, G)$, then $\varphi^{-1}(H) \circ i = F + G$. In this way, φ gives a bijection between left and right homotopies from F to G. Note that this is stronger than Lemma 6.31, which does not associate a unique right homotopy to a given left homotopy. (The proof of Lemma 6.31 uses a lifting property, and so does not give uniqueness for an arbitrary choice of cylinder object and path object.)

6.2 The Homotopy Category of a Model Category

In this section we define the homotopy category of a model category C. This is the category obtained from C by formally inverting the weak equivalences. Given any category A and any class of morphisms in A, it is always possible to formally invert the desired morphisms using a construction given in [18]. However, in general, it is almost impossible to work with the resulting category. In fact, starting with a small category A, the resulting category may no longer even be locally small. Using the tools available in model category it is possible to get a much more explicit description of its homotopy category than in the general case. This description is given in Corollary 6.48.

6.2.1 Homotopies and Weak Equivalences

Lemma 6.35. *Let $f, g : A \longrightarrow X$ be morphisms in a model category.*

- Suppose $f \sim_l g$. Then f is a weak equivalence if and only if g is a weak equivalence.
- Suppose $f \sim_r g$. Then f is a weak equivalence if and only if g is a weak equivalence.

Definition 6.36. Suppose $A, X \in C$ are both fibrant and cofibrant, and let $f : A \longrightarrow X$ be a morphism. We call a map $g : X \longrightarrow A$ a homotopy inverse for f if we have $f \circ g \sim id_X$ and $g \circ f \sim id_A$.

Lemma 6.37. *Let $f : A \longrightarrow X$ be a morphism between objects which are both fibrant and cofibrant. Then f is a weak equivalence if and only if f has a homotopy inverse $g : X \longrightarrow A$.*

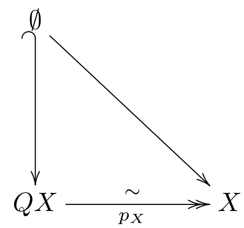
6.2.2 Fibrant and Cofibrant Replacement

Definition 6.38. For any model category C, we have the following associated categories:

- The objects of $Ho(C_c)$ are the cofibrant objects of C, and the morphisms are right homotopy classes of morphisms in C.
- The objects of $Ho(C_f)$ are the fibrant objects of C, and the morphisms are left homotopy classes of morphisms in C.

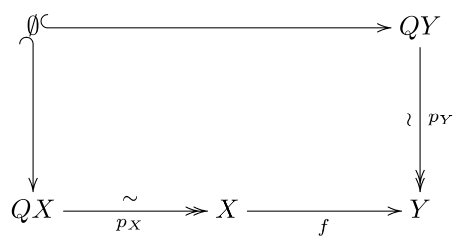
- The objects of $Ho(C_{cf})$ are the objects in C which are both fibrant and cofibrant, and the morphisms are homotopy classes of morphisms in C.

Definition 6.39. Let $X \in C$, and consider the unique map $\emptyset \longrightarrow X$. By Axiom 5 of Definition 6.3, we may factor this map as follows:

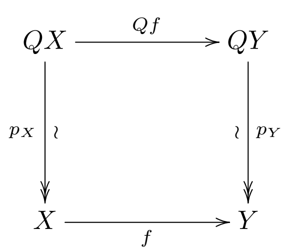


This gives an object QX, which is cofibrant, and an acyclic fibration $p_X : QX \overset{\sim}{\twoheadrightarrow} X$. We call the object QX a cofibrant replacement for X. If X is itself cofibrant then we take $QX = X$.

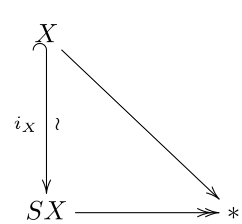
Given a morphism $f : X \longrightarrow Y$, consider the commutative diagram below:



By Axiom 4 of Definition 6.3, we have a lift in this diagram. Thus we have a morphism $Qf : QX \longrightarrow QY$ making the diagram below commute:



Similarly, we may factor the unique map $X \longrightarrow *$ as follows:



So we have an object SX, which is fibrant, and an acyclic cofibration $i_X : X \overset{\sim}{\hookrightarrow} SX$. We call the object SX a fibrant replacement for X. If X is itself fibrant then we take $SX = X$. Given a morphism $f : X \longrightarrow Y$, there is a morphism $Sf : SX \longrightarrow SY$ making the diagram below commute:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle i_X}\ \big\downarrow\wr & & \wr\big\downarrow\ {\scriptstyle i_Y} \\ SX & \xrightarrow[\quad Sf \quad]{} & SY \end{array}$$

Note that the choices for Qf and Sf are not uniquely determined. However, Lemma 6.40 and its dual imply that they are determined up to left and right homotopy.

Lemma 6.40. *Let $f : X \longrightarrow Y$ be a morphism in C. Consider any morphism $Qf : QX \longrightarrow QY$, as in Definition 6.39, such that $p_Y \circ Qf = f \circ p_X$. Then we have the following:*

- If $g : Qf \longrightarrow QY$ is any morphism such that $p_Y \circ g = f \circ p_X$, then $g \sim_l Qf$ and $g \sim_r Qf$.
- If Y is fibrant, then for any morphism $h : X \longrightarrow Y$ such that $h \sim_l f$ we have $Qh \sim_l Qf$ and $Qh \sim_r Qf$.
- $f : X \longrightarrow Y$ is a weak equivalence if and only if $Qf : QX \longrightarrow QY$ is a weak equivalence.

There is a dual statement for the morphism $Sf : SX \longrightarrow SY$.

Theorem 6.41. *Fibrant replacement and cofibrant replacement induce the functors below:*

$$Q : C \longrightarrow Ho(C_c)$$
$$S : C \longrightarrow Ho(C_f)$$
$$Q : Ho(C_f) \longrightarrow Ho(C_{cf})$$
$$S : Ho(C_c) \longrightarrow Ho(C_{cf})$$

Proof. We will only show that taking cofibrant replacement preserves identities and composition up to right (and left) homotopy. Let $X \in C$. Then both Qid_X and id_{QX} fill in the diagram below:

$$\begin{array}{ccc} QX & \dashrightarrow & QX \\ {\scriptstyle p_X}\ \big\downarrow\wr & & \wr\big\downarrow\ {\scriptstyle p_Y} \\ X & \xrightarrow[\quad id_X \quad]{} & X \end{array}$$

Thus, by Lemma 6.40, we have $id_{QX} \sim_l Qid_X$ and $id_{QX} \sim_r Qid_X$. Similarly, given $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, we have the following equalities:

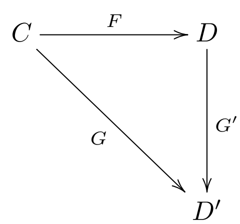
$$p_Z \circ Qg \circ Qf = g \circ p_Y \circ Qf = g \circ f \circ p_X$$

This is the defining property of $Q\,(gof)$. That is, $p_Z \circ Q\,(g \circ f) = g \circ f \circ p_X$. Thus, by Lemma 6.40, we have $Qg \circ Qf \sim_l Q\,(g \circ f)$ and $Qg \circ Qf \sim_r Q\,(g \circ f)$. $\square$

6.2.3 Localisation and the Homotopy Category

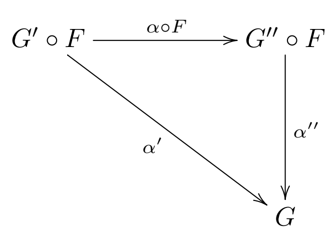
Definition 6.42. Let C be a category and let W be a class of morphisms in C. A functor $F : C \longrightarrow D$ is called a localisation with respect to W if it satisfies the following conditions:

- For any $f : A \longrightarrow B$ in W, $F\,(f) : F\,(A) \longrightarrow F\,(B)$ is an isomorphism in D.
- Given any functor $G : C \longrightarrow D'$ such that $G\,(f)$ is an isomorphism for any f in W there is a unique functor $G' : D \longrightarrow D'$ making the diagram below commute:



Remark 6.43. The universal property of a localisation implies that any two localisations are canonically isomorphic. Note also that for any localisation $F : C \longrightarrow D$ the functor F is a bijection on objects. This follows from the general construction given in [18].

Remark 6.44. Let C be a category, W be a class of morphisms in C, and $F : C \longrightarrow D$ be the localisation of C with respect to W. Then $F : C \longrightarrow D$ satisfies the following weak universal property, which characterises D up to equivalence. Given any $G : C \longrightarrow D'$ taking morphisms in W to isomorphisms, there is a functor $G' : D \longrightarrow D'$ and a natural isomorphism $\alpha' : G' \circ F \Longrightarrow G$. Furthermore, given any other functor $G'' : D \longrightarrow D'$ and natural isomorphism $\alpha'' : G'' \circ F \Longrightarrow G$ there is a unique natural isomorphism $\alpha : G' \Longrightarrow G''$ such that the diagram below commutes:



Example 6.45. For any model category C, denote by C_c the full subcategory of C generated by the cofibrant objects. Then the canonical functor $\gamma_c : C_c \longrightarrow Ho(C_c)$ is the localisation of C_c with respect to weak equivalences. Similarly, $Ho(C_f)$ and $Ho(C_{cf})$ are localisations of the full subcategories C_f, generated by fibrant objects, and C_{cf}, generated by objects which are both fibrant and cofibrant.

Explicitly, suppose $F : C_c \longrightarrow D$ is a functor such that for any weak equivalence $f : A \xrightarrow{\sim} B$ in C_c, $F(f)$ is an isomorphism in D. We claim that F identifies right homotopic maps. To see this, let $f, g : A \longrightarrow B$ be maps in C_c with $f \sim_r g$. Since A is cofibrant, by the dual of Remark 6.20, we may find a very good right homotopy $H : A \longrightarrow B^I$ from f to g. Note that since B is cofibrant and the canonical morphism $q : B \overset{\sim}{\hookrightarrow} B^I$ is a cofibration, B^I is also cofibrant, so H and q are both morphisms in C_c, as are the canonical maps $p_0, p_1 : B^I \longrightarrow B$. Since F maps weak equivalences to isomorphisms, $F(q)$ is an isomorphism in D.

Now, $p_0 \circ q = p_1 \circ q = id_B$, so we have $F(p_0) \circ F(q) = F(p_1) \circ F(q)$. Since $F(q)$ is an isomorphism, this implies that $F(p_0) = F(p_1)$. However, since H is a right homotopy from f to g, we have $p_0 \circ H = f$ and $p_1 \circ H = g$. Thus we have the following equalities in D:

$$\begin{aligned} F(f) &= F(p_0) \circ F(H) \\ &= F(p_1) \circ F(H) \\ &= F(g) \end{aligned}$$

From this observation, it is apparent that F induces a unique functor $F' : Ho(C_c) \longrightarrow D$ with $F' \circ \gamma_c = F$. Thus, $\gamma_c : C_c \longrightarrow Ho(C_c)$ is the localisation of C_c with respect to weak equivalences.

Definition 6.46. Let C be a model category and let W be the class of weak equivalences in C. The homotopy category of C is the localisation of C with respect to W. We denote this by $\gamma : C \longrightarrow Ho(C)$.

Theorem 6.47. *Let C be a model category. Then we have an equivalence $Ho(C_{cf}) \simeq Ho(C)$.*

Proof. Consider the embedding $i : C_{cf} \longrightarrow C$. The composite $\gamma \circ i : C_{cf} \longrightarrow Ho(C)$ takes weak equivalences in C_{cf} to isomorphisms. Thus, there is a unique functor $i' : Ho(C_{cf}) \longrightarrow Ho(C)$ such that $i' \circ \gamma_{cf} = \gamma \circ i$. We claim that i' is an equivalence of categories with weak inverse induced by cofibrant-fibrant replacement. Consider the functor $S \circ Q : C \longrightarrow Ho(C_{cf})$ of Theorem 6.41. By Lemma 6.40, $S \circ Q$ takes weak equivalences to isomorphisms, so $S \circ Q$ induces a functor $F : Ho(C) \longrightarrow Ho(C_{cf})$ with $F \circ \gamma = S \circ Q$.

Now, $F \circ i' = id_{\pi C_{cf}}$. To see this, we have the following equality:

$$\begin{aligned} F \circ i' \circ \gamma_{cf} &= F \circ \gamma \circ i \\ &= S \circ Q \circ i \\ &= \gamma_{cf} \end{aligned}$$

This follows since we choose $QX = X$ for cofibrant objects and $SX = X$ for fibrant objects, so $S \circ Q \circ i$ maps any object $X \in C_{cf}$ to the object $X \in Ho(C_{cf})$ and any morphism f in C_{cf} to its homotopy class. Thus, by the universal property of localisation, we have $F \circ i' = id$.

Now, consider $i' \circ F : Ho(C) \longrightarrow Ho(C)$. For each object $X \in C$, the morphism $p_X : QX \overset{\sim}{\twoheadrightarrow} X$ is mapped under γ to an isomorphism $\gamma(p_X) : \gamma(QX) \longrightarrow \gamma(X)$ in $Ho(C)$. Similarly, the morphism $i_{QX} : QX \overset{\sim}{\hookrightarrow} SQX$ is mapped to an isomorphism $\gamma(i_{QX}) : \gamma(QX) \longrightarrow \gamma(SQX)$. By construction, for any $f : X \longrightarrow Y$ in C, the diagram below commutes in $Ho(C)$:

$$\begin{array}{ccc} \gamma(SQX) & \xrightarrow{\gamma(SQf)} & \gamma(SQY) \\ \Big\downarrow{\scriptstyle \gamma(p_X)\circ\gamma(i_{QX})^{-1}} & & \Big\downarrow{\scriptstyle \gamma(p_Y)\circ\gamma(i_{QY})^{-1}} \\ \gamma(X) & \xrightarrow[\gamma(f)]{} & \gamma(Y) \end{array}$$

Now, we have $i' \circ F \circ \gamma = i' \circ S \circ Q$. On objects this functor is given by $(i' \circ S \circ Q)(X) = i'(SQX)$. Note, however, that we have $i' \circ \gamma_{cf} = \gamma \circ i$, so for the object $SQX \in Ho(C_{cf})$ we must have the following:

$$i'(SQX) = (\gamma \circ i)(SQX) = \gamma(SQX)$$

Given an arrow $f : X \longrightarrow Y$, we have $(i' \circ S \circ Q)(f) = i'([SQf])$. Now, applying $i' \circ \gamma_{cf}$ to the morphism $SQf : SQX \longrightarrow SQY$ gives the following:

$$\begin{aligned} (i' \circ \gamma_{cf})(SQf) &= i'([SQf]) \\ &= (\gamma \circ i)(SQf) \\ &= \gamma(SQf) \end{aligned}$$

Thus, $i' \circ F \circ \gamma$ takes $X \in C$ to the object $\gamma(SQX) \in Ho(C)$, and any morphism f in C to $\gamma(SQf)$. Therefore, the maps $\gamma(p_X) \circ \gamma(i_{QX})^{-1}$ are the components of a natural isomorphism $i' \circ F \circ \gamma \Longrightarrow \gamma$. Thus, by Remark 6.44, there is a natural isomorphism $i' \circ F \Longrightarrow id_{Ho(C)}$.

Therefore, F is a weak inverse for i', so these functors give the desired equivalence $Ho(C_{cf}) \simeq Ho(C)$. Note that a similar argument implies that the functor induced by $Q \circ S : C \longrightarrow Ho(C_{cf})$ also yields a weak inverse to i'. □

Corollary 6.48. *Let C be a model category. We may construct the homotopy category $Ho(C)$ as follows. The objects of $Ho(C)$ are the objects of C. The morphisms are given by:*

$$Hom_{Ho(C)}(X, Y) = Hom_{\pi C_{cf}}(SQX, SQY) = \pi(SQX, SQY)$$

The functor $\gamma : C \longrightarrow Ho(C)$ is defined as follows:

$$\begin{array}{rcl} X & \longmapsto & X \\ (f : X \longrightarrow Y) & \longmapsto & ([SQf] : X \longrightarrow Y) \end{array}$$

Furthermore, any morphism $f : X \longrightarrow Y$ in $Ho(C)$ may be represented as a composite

$$f = \gamma(p_Y) \circ \gamma(i_{QY})^{-1} \circ \gamma(f') \circ \gamma(i_{QX}) \circ \gamma(p_X)^{-1}$$

for some map $f' : SQX \longrightarrow SQY$ in C.

Remark 6.49. In the construction of the homotopy category in Corollary 6.48 we may take $Hom_{Ho(C)}(X, Y) = \pi(QSX, QSY)$ rather than $Hom_{Ho(C)}(X, Y) = \pi(SQX, SQY)$. Since both constructions yield localisations, the resulting categories are canonically isomorphic. In fact, if A is cofibrant and X is fibrant, then γ induces a bijection between $\pi(A, X)$ and $Hom_{Ho(C)}(A, X)$, so we may take $Hom_{Ho(C)}(X, Y) = \pi(QX, SY)$.

Note that the description of morphisms in the homotopy category given in Corollary 6.48 implies that $Ho(C)$ is locally small if C is locally small.

Remark 6.50. Given $f, g : X \longrightarrow Y$ in with $f \sim_l g$ or $f \sim_r g$, we have $\gamma(f) = \gamma(g)$. Thus, by the universal property of $\gamma : C \longrightarrow Ho(C)$, if $F : C \longrightarrow D$ is any functor taking weak equivalences to isomorphisms, then for any f, g with $f \sim_l g$ or $f \sim_r g$, we have $F(f) = F(g)$.

Lemma 6.51. *Let $f : X \longrightarrow Y$ be a morphism in C. Then $\gamma(f) : X \longrightarrow Y$ is an isomorphism if and only if $f : X \longrightarrow Y$ is a weak equivalence.*

Proof. By definition, if $f : X \xrightarrow{\sim} Y$ is a weak equivalence in C then $\gamma(f)$ is an isomorphism. Conversely, if $\gamma(f) : X \longrightarrow Y$ is an isomorphism then $\gamma(f)$ has an inverse $g : Y \longrightarrow X$ in $Ho(C)$. Choose a representative $g' : SQY \longrightarrow SQX$ in C such that

$$g = \gamma(p_X) \circ \gamma(i_{QX})^{-1} \circ \gamma(g') \circ \gamma(i_{QY}) \circ \gamma(p_Y)^{-1}$$

Then g' is a homotopy inverse for SQf. By Lemma 6.37, this implies that SQf is a weak equivalence. By Lemma 6.40 and its dual, this implies that $f : X \longrightarrow Y$ is a weak equivalence. □

Example 6.52. We call a CW-complex X an n-type if $\pi_i(X, x_0) = 0$ for any choice of basepoint $x_0 \in X$, and any $i > n$. We will denote the full subcategory of **Top** generated by n-types by **nType**, and the full category of $\mathbf{Top}^*$ generated by pointed connected n-types by $\mathbf{nType}^*_{\mathbf{c}}$. Consider the homotopy categories $Ho(\mathbf{nType})$ and $Ho(\mathbf{nType}^*_{\mathbf{c}})$, obtained by passing to homotopy classes of maps. Note that, although neither **nType** nor $\mathbf{nType}^*_{\mathbf{c}}$ are model categories, we may form these localisations. It is well-known that the fundamental group induces an equivalence of categories:

$$\pi_1 : Ho(\mathbf{1Type}^*_{\mathbf{c}}) \longrightarrow \mathbf{Grp}$$

Similarly, for any topological space X we may define its fundamental groupoid $\Pi_0(X)$. The objects of $\Pi_0(X)$ are the points $x \in X$ and the morphisms are homotopy classes of continuous paths. That is, given two points $x, y \in X$, we have

$$Hom_{\Pi_0(X)}(x, y) = \{[\gamma] \mid \gamma : [0,1] \longrightarrow X, \gamma(0) = x, \gamma(1) = y\}$$

where $[\gamma]$ denotes the homotopy class of γ. Note that for any $x \in X$ the automorphisms of x in $\Pi_0(X)$ give the fundamental group of X at x:

$$Hom_{\Pi_0(X)}(x, x) = \pi_1(X, x)$$

In [6], it is shown that the fundamental groupoid construction induces an equivalence of categories as below:

$$\Pi_0 : Ho(\mathbf{1Type}) \longrightarrow Ho(\mathbf{Grpd})$$

Here $Ho(\mathbf{Grpd})$ is the homotopy category of **Grpd** with respect to the canonical model structure of Example 6.14. Since every groupoid is both fibrant and cofibrant, and homotopies in **Grpd** correspond to natural isomorphisms, the category $Ho(\mathbf{Grpd})$ has groupoids as objects and natural isomorphism classes of functors as morphisms:

$$\begin{aligned} Hom_{Ho(\mathbf{Grpd})}(G, H) &= \pi(G, H) \\ &= \{[F] \mid F : G \longrightarrow H\} \end{aligned}$$

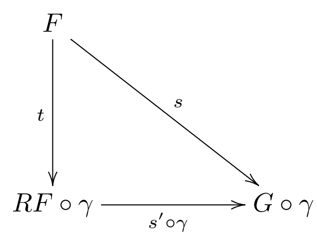
6.3 Quillen Adjunctions

6.3.1 Derived Functors

Definition 6.53. Let C be a model category and let $F : C \longrightarrow D$ be a functor into any category D. A left derived functor for F is a pair (LF, t), where $LF : Ho(C) \longrightarrow D$ is a functor and $t : LF \circ \gamma \Longrightarrow F$ is a natural transformation such that, given any other $G : Ho(C) \longrightarrow D$ and $s : G \circ \gamma \Longrightarrow F$, there is a unique natural transformation $s' : G \Longrightarrow LF$ such that the diagram below commutes:

$$\begin{array}{ccc} G \circ \gamma & \xrightarrow{s' \circ \gamma} & LF \circ \gamma \\ & \searrow^{s} & \downarrow^{t} \\ & & F \end{array}$$

A right derived functor for F is a pair (RF, t), where $RF : Ho(C) \longrightarrow D$ is a functor and $t : F \Longrightarrow RF \circ \gamma$ is a natural transformation such that, given any other $G : Ho(C) \longrightarrow D$ and $s : F \Longrightarrow G \circ \gamma$, there is a unique natural transformation $s' : RF \Longrightarrow G$ such that the diagram below commutes:



Note that the universal property of a left derived functor implies that, if it exists it, is unique up to canonical natural isomorphism. Similarly, if a right derived functor exists, it is unique up to canonical natural isomorphism.

Example 6.54. If $F : C \longrightarrow D$ is a functor taking weak equivalences to isomorphisms, then by the universal property of $Ho(C)$ there is a unique functor $F' : Ho(C) \longrightarrow D$ such that $F' \circ \gamma = F$. This functor F' is both a left and right derived functor for F.

Theorem 6.55. *Let C be a model category and let $F : C \longrightarrow D$ be a functor. Assume that for any cofibrant objects $A, B \in C$ and any weak equivalence $f : A \xrightarrow{\sim} B$, the image $F(f) : F(A) \longrightarrow F(B)$ is an isomorphism in D. Then the left derived functor $LF : Ho(C) \longrightarrow D$ exists. Moreover, for any cofibrant object $A \in C$, the component $t_A : LF(A) \longrightarrow F(A)$ is an isomorphism.*

Proof. The functor F restricts to a functor $F : C_c \longrightarrow D$. Since F maps weak equivalences between cofibrant objects to isomorphisms, by the universal property of the localisation, there is a unique functor $F' : Ho(C_c) \longrightarrow D$ such that $F' \circ \gamma_c = F$. Precomposing with $Q : C \longrightarrow Ho(C_c)$ gives the functor $F' \circ Q : C \longrightarrow D$, defined as follows:

$$\begin{array}{rcl} X & \longmapsto & F(QX) \\ (f : X \longrightarrow Y) & \longmapsto & (F'([Qf]) = F(Qf) : F(QX) \longrightarrow F(QY)) \end{array}$$

This takes weak equivalences in C to isomorphisms in D, so by the universal property of $Ho(C)$ we have a functor $LF : Ho(C) \longrightarrow D$ such that $LF \circ \gamma = F' \circ Q$. By Example 6.54, the pair (LF, id) is both left and right derived functor for $F' \circ Q : C \longrightarrow D$.

Now, the morphisms $t_X := F(p_X) : F(QX) \longrightarrow F(X)$ give the components of a natural transformation $t : LF \circ \gamma = F' \circ Q \Longrightarrow F$. Although we will not prove it, (LF, t) is the left derived functor for $F : C \longrightarrow D$. See [9] for the remainder of the proof. □

Definition 6.56. Let $F : C \longrightarrow D$ be a functor between model categories. Consider $\gamma \circ F : C \longrightarrow Ho(D)$. We may form the left derived functor for $\gamma \circ F$, which we will call the total left derived functor for F. We will denote this by $LF : Ho(C) \longrightarrow Ho(D)$. Dually, the total right derived functor for F is the right derived functor for $\gamma \circ F$. We will denote this by $RF : Ho(C) \longrightarrow Ho(D)$.

Lemma 6.57. *Let C and D be model categories, and let $F : C \longrightarrow D$ be a functor taking any acyclic cofibration between cofibrant objects to a weak equivalence. Then if $A, B \in C$ are cofibrant and* $f : A \xrightarrow{\sim} B$ *is a weak equivalence,* $F(f) : F(A) \xrightarrow{\sim} F(B)$ *is a weak equivalence.*

For a proof of Lemma 6.57 see [9].

Theorem 6.58. *Let C and D be model categories. Suppose we have an adjoint pair:*

$$F : C \Longleftrightarrow D : G$$

If any one of the equivalent conditons below is satisfied then the total derived functors $LF : Ho(C) \longrightarrow Ho(D)$ and $RG : Ho(D) \longrightarrow Ho(C)$ exist and they form an adjoint pair:

$$LF : Ho(C) \Longleftrightarrow Ho(D) : RG$$

1. F preserves cofibrations and G preserves fibrations.
2. F preserves cofibrations and acyclic cofibrations.
3. G preserves fibrations and acyclic fibrations.

We call such an adjunction $F : C \Longleftrightarrow D : G$ a Quillen adjunction.

Proof. For the proof that Statements 1, 2 and 3 are equivalent see [9].

Now, since F is a left adjoint, F preserves colimits, so in particular $F(\emptyset)$ is initial in D. Since F also preserves cofibrations, F maps any cofibrant object in C to a cofibrant object in D. Dually, G maps fibrant objects in D to fibrant objects in C.

By Lemma 6.57, F preserves weak equivalences between cofibrant objects of C. Thus, $\gamma \circ F : C \longrightarrow Ho(D)$ takes weak equivalences between cofibrant objects to isomorphisms, so by Lemma 6.55, the left derived functor of $\gamma \circ F$ exists. This is the total left derived functor of F, denoted $LF : Ho(C) \longrightarrow Ho(D)$. Lemma 6.55 gives us the following construction for $LF \circ \gamma : C \longrightarrow Ho(D)$:

$$\begin{array}{rcl} X & \longmapsto & F(QX) \\ (f : X \longrightarrow Y) & \longmapsto & ([SF(Qf)] : F(QX) \longrightarrow F(QY)) \end{array}$$

By the duals of Lemma 6.57 and Lemma 6.55, the total right derived functor of G, $RG : Ho(D) \longrightarrow Ho(C)$, exists, and $RG \circ \gamma : D \longrightarrow Ho(C)$ may be described as follows:

$$\begin{array}{rcl} X & \longmapsto & G(SX) \\ (f : X \longrightarrow Y) & \longmapsto & ([QG(Sf)] : G(SX) \longrightarrow G(SY)) \end{array}$$

Now, let $A \in C$ be cofibrant and let $X \in D$ be fibrant. We will show that the bijection

$$\varphi : Hom_{\mathbf{D}}(F(A), X) \longrightarrow Hom_{\mathbf{C}}(A, G(X))$$

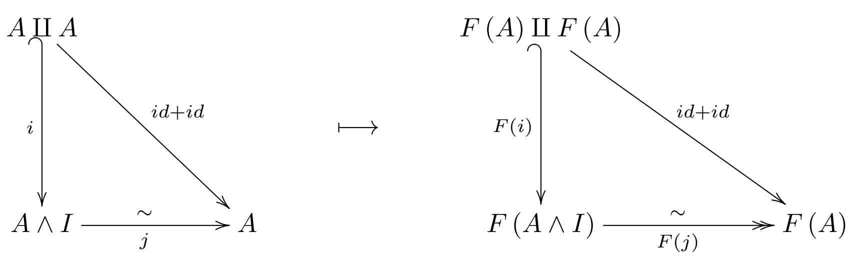
induces a bijection between homotopy classes of maps:

$$\pi(F(A), X) \cong \pi(A, G(X))$$

By Remark 6.49 these sets may be identified with the morphisms in the homotopy categories. Explicitly, we claim that this bijection gives us the following bijection, natural in Z and W, where the first map and the last both follow from Remark 6.49:

$$\begin{aligned} Hom_{Ho(\mathbf{D})}(LF(Z), W) &\cong \pi(F(QZ), SW) \\ &\cong \pi(QZ, G(SW)) \\ &\cong Hom_{Ho(\mathbf{C})}(Z, SG(W)) \end{aligned}$$

Now, let $f, g : A \longrightarrow G(X)$ be morphisms in C, with $f \sim g$. By Lemma 6.31, we may choose a good left homotopy $H : A \wedge I \longrightarrow G(X)$ with $H \circ i = f + g$, where $i : A \amalg A \hookrightarrow A$ is the canonical morphism. Since $A \wedge I$ is a good cylinder object, by Lemma 6.18, the maps $i_0, i_1 : A \overset{\sim}{\hookrightarrow} A \wedge I$ are acyclic cofibrations. This implies that $A \wedge I$ is cofibrant. Now, since F preserves colimits, cofibrations and acyclic cofibrations (and thus, by Lemma 6.57, weak equivalences between cofibrant objects), $F(A \wedge I)$ is a good cylinder object for $F(A)$. This is illustrated below:



Now, consider the morphisms $\varphi^{-1}(H) : F(A \wedge I) \longrightarrow X$ and $\varphi^{-1}(f), \varphi^{-1}(g) : F(A) \longrightarrow X$. Let $\varepsilon_X : (F \circ G)(X) \longrightarrow X$ be the component of the counit at X. Then, since F preserves colimits we have the following equalities:

$$\begin{aligned} \varphi^{-1}(H) \circ F(i) &= \varepsilon_X \circ F(H \circ i) \\ &= \varepsilon_X \circ F(f + g) \\ &= \varepsilon_X \circ (F(f) + F(g)) \\ &= (\varepsilon_X \circ F(f)) + (\varepsilon_X \circ F(g)) \\ &= \varphi^{-1}(f) + \varphi^{-1}(g) \end{aligned}$$

Thus, $\varphi^{-1}(H)$ is a good left homotopy from $\varphi^{-1}(f)$ to $\varphi^{-1}(g)$, and so $\varphi^{-1}(f) \sim_l \varphi^{-1}(g)$. Since $F(A)$ is cofibrant and X is fibrant, $\varphi^{-1}(f) \sim_l \varphi^{-1}(g)$ if and only if $\varphi^{-1}(f) \sim_r \varphi^{-1}(g)$, so we have the desired function from $\pi(A, G(X))$ to $\pi(F(A), X)$. We may construct its inverse by a dual argument, using the fact that G preserves limits. See [9] for the proof that the resulting bijection is natural. $\square$

Remark 6.59. If a Quillen adjunction also satisfies the following property then the induced adjunction

$$LF : Ho\,(C) \Longleftrightarrow Ho\,(D) : RG$$

is an equivalence. The property is as follows: For any cofibrant object $A \in C$ and any fibrant object $X \in D$, a map $f : F(A) \longrightarrow X$ is a weak equivalence if and only if $\varphi(f) : A \longrightarrow G(X)$ is a weak equivalence.

6.4 Homotopy Pushouts and Pullbacks

In a model category C, limits and colimits need not be stable under weak equivalence. That is, given two diagrams in C, the objects of which are isomorphic in $Ho\,(C)$, the (co)limits of these diagrams need not be isomorphic in $Ho\,(C)$. For instance, in **Cat**, this is a statement of the well-known fact that two diagrams whose objects are equivalent need not have equivalent (co)limits. This fails even when the equivalences form the components of a natural transformation between the diagrams.

In this section we introduce methods of finding optimal approximations for limits and colimits which are homotopy invariant. We will study examples of these in Section 8 in relation to 2-groups.

6.4.1 Homotopy Pushouts

Definition 6.60. Consider the category D below:

$$a \longleftarrow b \longrightarrow c$$

The diagram category C^D has objects given by diagrams in C of the following form:

$$X(a) \longleftarrow X(b) \longrightarrow X(c)$$

Arrows are commutative diagrams as below:

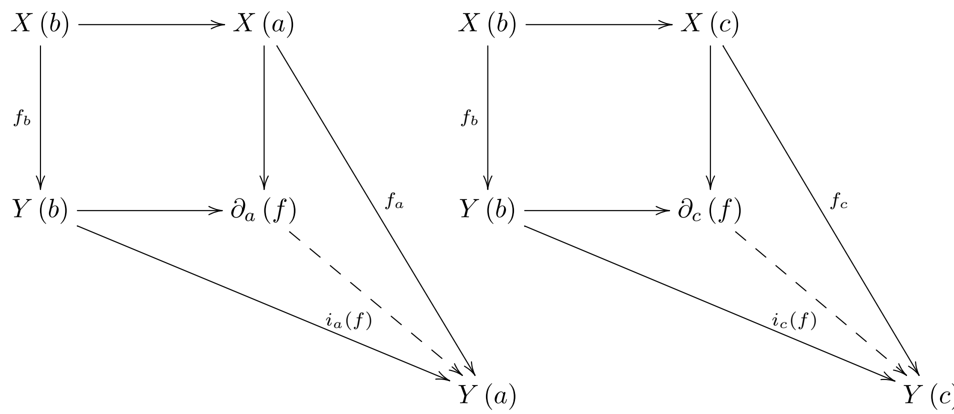
$$\begin{array}{ccccc} X(a) & \longleftarrow & X(b) & \longrightarrow & X(c) \\ \downarrow{\scriptstyle f_a} & & \downarrow{\scriptstyle f_b} & & \downarrow{\scriptstyle f_c} \\ Y(a) & \longleftarrow & Y(b) & \longrightarrow & Y(c) \end{array}$$

We will denote such an arrow by $f : X \longrightarrow Y$. Given any arrow in C^D consider the two diagrams below:

$$Y(b) \xleftarrow{f_b} X(b) \longrightarrow X(a)$$

$$Y(b) \xleftarrow{f_b} X(b) \longrightarrow X(c)$$

Denote the pushout of the first diagram by $\partial_a(f)$ and the pushout of the second diagram by $\partial_c(f)$. The universal property of these pushouts gives us morphisms $i_a(f) : \partial_a(f) \longrightarrow Y(a)$ and $i_c(f) : \partial_c(f) \longrightarrow Y(c)$ as below:



Definition 6.61. Let C be a model category, and consider the category C^D of Definition 6.60. We may define a model structure, known as the projective model structure on C^D as follows:

- An arrow $f : X \longrightarrow Y$ is a weak equivalence if f_a, f_b and f_c are weak equivalences in C.
- An arrow $f : X \longrightarrow Y$ is a cofibration $i_a(f)$, $i_c(f)$ and f_b are cofibrations in C.
- An arrow $f : X \longrightarrow Y$ is a fibration if f_a, f_b and f_c are fibrations in C.

Theorem 6.62. *Now, let C be a model category and let D be the category below*

$$a \longleftarrow b \longrightarrow c$$

Since C is a model category, by Axiom 1 of Definition 6.3, C has all finite colimits. In particular, all pushouts exist, so the diagonal functor $\triangle : C \longrightarrow C^D$ has a left adjoint $colim : C^D \longrightarrow C$. The adjunction

$$colim : C^D \Longleftrightarrow C : \triangle$$

satisfies the equivalent conditions 1, 2 and 3 of Theorem 6.58. Thus, the total derived functors $Lcolim : Ho\left(C^D\right) \longrightarrow Ho(C)$ and $R\triangle : Ho(C) \longrightarrow Ho\left(C^D\right)$ exist, and we have the adjunction below:

$$Lcolim : Ho\left(C^D\right) \Longleftrightarrow Ho(C) : R\triangle$$

Proof. The proof is immediate, since given a fibration $f : A \twoheadrightarrow B$ in C, $\Delta(f)$ has the following components:

$$\Delta(f)_a = \Delta(f)_b = \Delta(f)_c = f$$

Thus, each component of $\Delta(f)$ is a fibration. By the definition of the projective model structure in Definition 6.61, this implies that $\Delta(f)$ is a fibration. Thus, Δ preserves fibrations. By a similar argument, Δ preserves weak equivalences, so Δ satisfies Condition 3 of Theorem 6.58. $\square$

Definition 6.63. For a diagram $X \in C^D$, we call $Lcolim(X)$ the homotopy pushout of X.

Remark 6.64. Given a diagram $X \in C^D$, by the construction of Theorem 6.55, up to isomorphism in $Ho(C)$ we may take $Lcolim(X) = colim(QX)$, where QX is a cofibrant replacement for X. Note that $X \in C^D$ is cofibrant if each of the objects $X(a)$, $X(b)$ and $X(c)$ is cofibrant in C and both maps $X(b) \hookrightarrow X(a)$ and $X(b) \hookrightarrow X(c)$ are cofibrations in C.

Definition 6.65. Let $f : X \longrightarrow Y$ be a map in C. The homotopy pushout of the diagram

$$* \longleftarrow X \xrightarrow{f} Y$$

is called the homotopy cokernel of f, or the mapping cone of f.

Definition 6.66. Let $X \in C$. The suspension object of X, denoted ΣX, is the homotopy pushout of the diagram below:

$$* \longleftarrow X \longrightarrow *$$

6.4.2 Homotopy Pullbacks

Definition 6.67. Consider the category E below:

$$a \longrightarrow b \longleftarrow c$$

Morphisms in the diagram category C^E are commutative diagrams as below:

$$\begin{array}{ccccc} X(a) & \longrightarrow & X(b) & \longleftarrow & X(c) \\ \downarrow{\scriptstyle f_a} & & \downarrow{\scriptstyle f_b} & & \downarrow{\scriptstyle f_c} \\ Y(a) & \longrightarrow & Y(b) & \longleftarrow & Y(c) \end{array}$$

Denote such an arrow by $f : X \longrightarrow Y$. Given any arrow in C^E consider the diagrams two diagrams below:

$$X(b) \xrightarrow{f_b} Y(b) \longleftarrow Y(a)$$

$$X(b) \xrightarrow{f_b} Y(b) \longleftarrow Y(c)$$

Denote the pullback of the first diagram by $\partial_a(f)$ and the pullback of the second diagram by $\partial_c(f)$. The universal property of these pullbacks gives us morphisms $p_a(f) : X(a) \longrightarrow \partial_a(f)$ and $p_c(f) : X(c) \longrightarrow \partial_c(f)$.

Definition 6.68. Let C be a model category, and consider the category C^E of Definition 6.67. We may define a model structure, known as the injective model structure on C^E as follows:

- An arrow $f : X \longrightarrow Y$ is a weak equivalence if f_a, f_b and f_c are weak equivalences in C.
- An arrow $f : X \longrightarrow Y$ is a cofibration if f_a, f_b and f_c are cofibrations in C.
- An arrow $f : X \longrightarrow Y$ is a fibration $p_a(f)$, $p_c(f)$ and f_b are fibrations in C.

Theorem 6.69. *Let C be a model category and let E be the category below:*

$$a \longrightarrow b \longleftarrow c$$

By Axiom 1 of Definition 6.3, C has all pullbacks, so the diagonal functor $\triangle : C \longrightarrow C^E$ has a right adjoint $lim : C^E \longrightarrow C$. The adjunction

$$\triangle : C \Longleftrightarrow C^E : lim$$

satisfies the equivalent conditions 1, 2 and 3 of Theorem 6.58. Thus, the total derived functors $Rlim : Ho\left(C^E\right) \longrightarrow Ho(C)$ and $L\triangle : Ho(C) \longrightarrow Ho\left(C^E\right)$ exist, and we have the adjunction below:

$$L\triangle : Ho(C) \Longleftrightarrow Ho\left(C^E\right) : Rlim$$

Proof. The proof is dual to the proof of Theorem 6.69. The functor $\triangle$ satisfies Condition 2 of Theorem 6.58. $\square$

Definition 6.70. For a diagram $X \in C^E$, we call $Rlim(X)$ the homotopy pullback of X.

Remark 6.71. Given a diagram $X \in C^E$, up to isomorphism in $Ho(C)$, we have $Rlim(X) = lim(SX)$, where SX is a fibrant replacement for X. A diagram $X \in C^E$ is fibrant if each of the objects $X(a)$, $X(b)$ and $X(c)$ is fibrant in C and both maps $X(a) \twoheadrightarrow X(b)$ and $X(a) \twoheadrightarrow X(c)$ are fibrations in C.

Definition 6.72. Suppose the model category C has a zero object $*$. Let $f : X \longrightarrow Y$ be a map in C. The homotopy pullback of the diagram

$$X \xrightarrow{f} Y \longleftarrow *$$

is called the homotopy kernel of f, or the mapping cocone of f.

Definition 6.73. Let C be a model category with a zero object, and let $X \in C$. The loop space object of X, denoted ΩX, is the homotopy pullback of the diagram below:

$$* \longrightarrow X \longleftarrow *$$

Example 6.74. Let G be a group. Consider the delooping groupoid $\mathbf{B}G$ of Definition 3.3. We claim that the loop space $\Omega\mathbf{B}G$ is the discrete groupoid with objects given by the elements of G. (Recall that a category is called discrete if its only morphisms are identities.) We may calculate $\Omega\mathbf{B}G$ up to isomorphism in $Ho\left(\mathbf{Grpd}\right)$ - that is, up to equivalence in $\mathbf{Grpd}$ - by finding a fibrant replacement for the diagram $* \longrightarrow \mathbf{B}G \longleftarrow *$ and then taking a pullback. Note that in $\mathbf{Grpd}$, the zero object $*$ is the discrete groupoid with a single object. By Remark 6.71, since every object of $\mathbf{Grpd}$ is fibrant, a fibrant replacement for this diagram is given by the bottom row of a commutative diagram of the form below:

$$\begin{array}{ccccc}
* & \longrightarrow & \mathbf{B}G & \longleftarrow & * \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
X\left(a\right) & \twoheadrightarrow & X\left(b\right) & \twoheadleftarrow & X\left(c\right)
\end{array}$$

Now, consider the groupoid $\mathbf{E}G$, defined via the pullback below:

$$\begin{array}{ccc}
\mathbf{E}G & \overset{s}{\underset{\sim}{\twoheadrightarrow}} & * \\
{\scriptstyle t}\downarrow & & \downarrow \\
\mathbf{B}G^{\mathbf{I}} & \overset{\sim}{\underset{p_0}{\twoheadrightarrow}} & \mathbf{B}G
\end{array}$$

Since $\mathbf{B}G$ is fibrant, and $\mathbf{B}G^{\mathbf{I}}$ of Example 6.33 is a very good path object, by the dual of Lemma 6.18, $p_0 : \mathbf{B}G^{\mathbf{I}} \overset{\sim}{\twoheadrightarrow} \mathbf{B}G$ is an acyclic fibration. By Lemma 6.10, this implies that $s : \mathbf{E}G \overset{\sim}{\twoheadrightarrow} *$ is also an acyclic fibration. (In this instance this is not hard to verify directly. However, we may use an analogous argument to construct fibrant replacements for morphisms between fibrant objects in any model category.) The objects of $\mathbf{E}G$ are the elements $g \in G$, and for any $g_1, g_2 \in \mathbf{E}G$ we have:

$$Hom_{\mathbf{E}G}\left(g_1, g_2\right) = \left\{g_1^{-1}g_2\right\}$$

The functor $t : \mathbf{E}G \longrightarrow \mathbf{B}G^{\mathbf{I}}$ is given as follows:

$$g \longmapsto \left(\bullet \overset{g}{\longrightarrow} \bullet\right)$$

$$\left(g_1^{-1}g_2 : g_1 \longrightarrow g_2\right) \longmapsto \quad \begin{array}{ccc} \bullet & \xrightarrow{g_1} & \bullet \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle g_1^{-1}g_2} \\ \bullet & \xrightarrow[g_2]{} & \bullet \end{array}$$

Now, define $F = p_1 \circ t : \mathbf{E}G \longrightarrow \mathbf{B}G$. Explicitly, F is defined as follows:

$$\begin{array}{rcl} g & \longmapsto & \bullet \\ \left(g_1^{-1}g_2 : g_1 \longrightarrow g_2\right) & \longmapsto & \left(g_1^{-1}g_2 : \bullet \longrightarrow \bullet\right) \end{array}$$

For any $g \in \mathbf{E}G$ and any morphism $h : \bullet \longrightarrow \bullet$ in $\mathbf{B}G$, we have $h : g \longrightarrow gh$ in $\mathbf{E}G$ with $F(h) = h$. Thus, $F : \mathbf{E}G \twoheadrightarrow \mathbf{B}G$ is a fibration. Furthermore, since $*$ is initial, the diagram below commutes:

$$\begin{array}{ccccc} * & \longrightarrow & \mathbf{B}G & \longleftarrow & * \\ {\scriptstyle s}\big\downarrow{\scriptstyle \wr} & & \Big\| & & {\scriptstyle \wr}\big\downarrow{\scriptstyle s} \\ \mathbf{E}G & \xrightarrow[F]{} & \mathbf{B}G & \xleftarrow[F]{} & \mathbf{E}G \end{array}$$

Thus, $\mathbf{E}G \twoheadrightarrow \mathbf{B}G \twoheadleftarrow \mathbf{E}G$ is a fibrant replacement for $* \longrightarrow \mathbf{B}G \longleftarrow *$, and so we may calculate $\Omega\mathbf{B}G$ as the pullback of $\mathbf{E}G \twoheadrightarrow \mathbf{B}G \twoheadleftarrow \mathbf{E}G$.

The objects of $\Omega\mathbf{B}G$ are pairs (g, h) with $g, h \in G$. For any object $(g, h) \in \Omega\mathbf{B}G$, there is one morphism $k : (g, h) \longrightarrow (gk, hk)$ for each $k \in G$ and there are no other morphisms in $\Omega\mathbf{B}G$. Now, let G denote the discrete groupoid with objects given by the elements of G. Then the functor $K : G \longrightarrow \Omega\mathbf{B}G$ given by $K(g) = (g, 1)$ on objects is an equivalence. Thus, since the the loop space of a groupoid is only defined up to equivalence, we may take $G = \Omega\mathbf{B}G$.

Remark 6.75. Other than the projective and injective model structures, we may define a number of model structures on functor categories in order to compute homotopy limits and colimits. In particular, if D is the pushout diagram $a \longleftarrow b \longrightarrow c$ of Definition 6.60 and E is the pullback diagram $a \longrightarrow b \longleftarrow c$ of Definition 6.67, then C^D and C^E admit a family model structures known as Reedy model structures. The Reedy model structures have the same weak equivalences as the projective model structure of Definition 6.61 and the injective model structures of Definition 6.68. Thus, these model structures all induce isomorphic homotopy categories, meaning the homotopy colimits we define using the projective model structure will agree with those defined using the Reedy model structures. Dually, homotopy limits constructed using the injective model structure agree with those constructed via the Reedy model structures.

We will not define Reedy categories or Reedy model structures; see [36], from which the following results are taken, for an exposition. The functor category C^D admits a Reedy model structure for which the total left derived functor

$$Lcolim : Ho\left(C^D\right) \longrightarrow Ho\left(C\right)$$

exists and defines a homotopy pushout in C. Moreover, the cofibrant objects of C^D have the form

$$X\left(a\right) \longleftarrow X\left(b\right) \hookrightarrow X\left(c\right)$$

where each of the objects $X\left(a\right)$, $X\left(b\right)$ and $X\left(c\right)$ are cofibrant, but only one of the morphisms is a cofibration. Dually, we may calculate homotopy pullbacks via a Reedy model structure on C^E with fibrant objects given by

$$X\left(a\right) \longrightarrow X\left(b\right) \twoheadleftarrow X\left(c\right)$$

where $X\left(a\right)$, $X\left(b\right)$ and $X\left(c\right)$ are fibrant, but only one of the morphisms is a fibration.

Example 6.76. Let G be a group. By Remark 6.75, we may calculate $\Omega\mathbf{B}G$ by taking the pullback of the diagram below, rather than the pullback of the diagram in Example 6.73:

$$* \longrightarrow \mathbf{B}G \xleftarrow{\;F\;} \mathbf{E}G$$

Here $F = p_1 \circ t$ as in Example 6.73. Thus, $\Omega\mathbf{B}G$ is given by the repeated pullback below:

$$\begin{array}{ccccc}
\Omega\mathbf{B}G & \longrightarrow & \mathbf{E}G & \xrightarrow{\;s\;} & * \\
\big\downarrow & & \big\downarrow{\scriptstyle t} & & \big\downarrow \\
 & & \mathbf{B}G^{\mathbf{I}} & \xrightarrow[p_0]{} & \mathbf{B}G \\
 & & \big\downarrow{\scriptstyle p_1} & & \\
* & \longrightarrow & \mathbf{B}G & &
\end{array}$$

It is not hard to check that this is isomorphic to the following pullback:

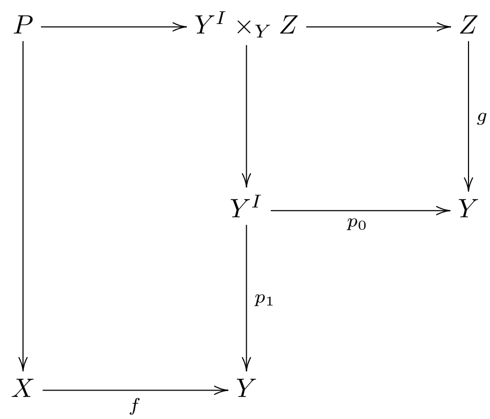
$$\begin{array}{ccc}
\Omega\mathbf{B}G & \longrightarrow & \mathbf{B}G^{\mathbf{I}} \\
\big\downarrow & & \big\downarrow{\scriptstyle p=(p_0,p_1)} \\
* & \longrightarrow & \mathbf{B}G \times \mathbf{B}G
\end{array}$$

Thus, $\Omega\mathbf{B}G = Ker\,(p)$. The objects of $Ker\,(p)$ are the morphisms of $\mathbf{B}G$ - that is the elements $g \in G$ - and $Ker\,(p)$ has only identity morphisms. Thus, as we found in Example 6.74, $\Omega\mathbf{B}G = Ker\,(p) = G$ is the discrete groupoid on the elements of G.

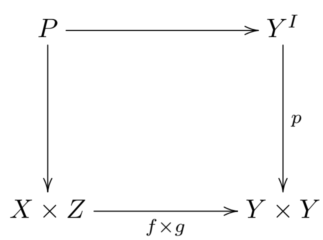
Remark 6.77. The construction of Example 6.76 holds in general. That is, if X, Y and Z are fibrant elements in any model category C, then the homotopy pullback of a diagram

$$X \xrightarrow{f} Y \xleftarrow{g} Z$$

may be calculated as the pullback P below:



It is routine to check that the object P is canonically isomorphic to the pullback below:



Here Y^I is a good path object for Y and $p = (p_0, p_1) : Y^I \longrightarrow Y \times Y$ is the canonical morphism. In particular, if C has a zero object $*$ then the loop space of an object $X \in C$ is presented by the kernel $Ker\,(p)$.

7 Model Structure on Str2Grp

In this section we recall the Garzon-Miranda model structure on the category **Str2Grp**, which was introduced in [31].

Definition 7.1. Consider the category **Str2Grp**, with objects strict 2-groups and morphisms strict monoidal functors. By Remark 5.17, this is isomorphic to the category of internal categories in **Grp**. We may define a model structure on **Str2Grp** as follows:

- An arrow $F : \mathcal{G} \longrightarrow \mathcal{H}$ is a weak equivalence if F is a monoidal equivalence. That is, F is a weak equivalence of strict 2-groups if and only if F is a weak equivalence of the underlying groupoids.

- An arrow $F : \mathcal{G} \longrightarrow \mathcal{H}$ is a fibration if F is a fibration of the underlying groupoids.

- An arrow $F : \mathcal{G} \longrightarrow \mathcal{H}$ is a cofibration if F has the LLP with respect to all acyclic fibrations.

Remark 7.2. The zero object $*$ in **Str2Grp** is the 2-group with one object and one morphism. The underlying groupoid of $*$ is also the zero object in **Grpd**, so, since any groupoid is fibrant and the fibrations in **Str2Grp** are the fibrations of the underlying groupoids, every strict 2-group is fibrant.

Definition 7.3. Let $\mathcal{H} = (\mathcal{H}_1 \rightrightarrows \mathcal{H}_0)$ be a strict 2-group. Define the strict 2-group $\mathcal{H}^I = (C_1 \rightrightarrows C_0)$ as follows. Let $C_0 = H_1$, and let

$$C_1 = \left\{(g_1, g_2, h_1, h_2) \in H_1^4 \mid g_2 \circ h_1 = h_2 \circ g_1\right\}$$

with multiplication defined pointwise. Define the structural morphisms as follows:

$$\begin{aligned} s(g_1, g_2, h_1, h_2) &= g_1 \\ t(g_1, g_2, h_1, h_2) &= g_2 \\ i(g) &= (g, g, i(s(g)), i(t(g))) \end{aligned}$$

Thinking of a strict 2-group as a monoidal category, $\mathcal{H}^I$ has objects given by the morphisms of $\mathcal{H}$ and arrows given by commutative squares. That is, if $U(\mathcal{H})$ is the underlying groupoid of $\mathcal{H}$, then $U\left(\mathcal{H}^I\right) = U(\mathcal{H})^{\mathbf{I}}$, where $U(\mathcal{H})^{\mathbf{I}}$ is the path object for $U(\mathcal{H})$ of Example 6.33.

Lemma 7.4. *For any strict* 2*-group* $\mathcal{H}$*, the* 2*-group* $\mathcal{H}^I$ *is a good path object for* $\mathcal{H}$*.*

Proof. Let $U(\mathcal{H})$ denote the underlying groupoid of $\mathcal{H}$. Then by Definition 7.3, we have $U\left(\mathcal{H}^I\right) = U(\mathcal{H})^{\mathbf{I}}$. Now, $U(\mathcal{H})^{\mathbf{I}}$ is a very good path object for $U(\mathcal{H})$, so in particular $U(\mathcal{H})^{\mathbf{I}}$ is a good path object. The canonical morphisms for the path object, $q : \mathcal{H} \longrightarrow \mathcal{H}^I$ and $p : \mathcal{H}^I \longrightarrow \mathcal{H} \times \mathcal{H}$ are as follows:

$$\begin{array}{rcl} q_0 : \mathcal{H}_0 & \longrightarrow & C_0 = \mathcal{H}_1 \\ x & \longmapsto & i(x) \\ q_1 : \mathcal{H}_1 & \longrightarrow & C_1 \\ g & \longmapsto & (i(s(g)), i(t(g)), g, g) \\ p_0 : C_0 & \longrightarrow & \mathcal{H}_0 \times \mathcal{H}_0 \\ g & \longmapsto & (s(g), t(g)) \\ p_1 : C_1 & \longrightarrow & \mathcal{H}_1 \times \mathcal{H}_1 \\ (g_1, g_2, h_1, h_2) & \longmapsto & (h_1, h_2) \end{array}$$

The underlying functions of q and p recover the functors of Example 6.33, so q is a weak equivalence of groupoids and p is a fibration of groupoids. Therefore, since both weak equivalences and fibrations (but not cofibrations) of strict 2-groups are inherited from **Grpd**, $\mathcal{H}^I$ is a good path object for $\mathcal{H}$ (but not necessarily a very good path object). □

Lemma 7.5. *Let $F, G : \mathcal{G} \longrightarrow \mathcal{H}$ be morphisms in* **Str2Grp**. *A right homotopy $H : \mathcal{G} \longrightarrow \mathcal{H}^I$ from F to G, where $\mathcal{H}^I$ is the path object of Definition 7.3, is precisely a monoidal natural isomorphism from F to G. That is, given 1-morphisms $F, G : \mathcal{G} \longrightarrow \mathcal{H}$ in the 2-category* **Str2Grp**, *the right homotopies from F to G via $\mathcal{H}^I$ are exactly the 2-morphisms from F to G in* **Str2Grp**.

Proof. By Remark 5.17, the data of a monoidal natural isomorphism is precisely that of an internal natural transformation in **Grp**. Given morphisms $F, G : \mathcal{G} \longrightarrow \mathcal{H}$, this amounts to a group homomorphism $\theta : \mathcal{G}_0 \longrightarrow \mathcal{H}_1$ satisfying the naturality condition of Definition 2.5, such that $s \circ \theta = F_0$ and $t \circ \theta = G_0$. Given a right homotopy $H : \mathcal{G} \longrightarrow \mathcal{H}^I$ from F to G, define $\theta = H_0 : \mathcal{G}_0 \longrightarrow C_0 = \mathcal{H}_1$. By the argument of Example 6.33, this satisfies the naturality condition. Thus, since H_0 is a group homomorphism, $H_0 = \theta$ determines a natural isomorphism in **Grp** and thus a monoidal natural isomorphism.

Conversely, given an internal natural isomorphism θ from F to G, define $H : \mathcal{G} \longrightarrow \mathcal{H}^I$ as follows:

$$\begin{array}{rcl} H_0 : \mathcal{G}_0 & \longrightarrow & C_0 \\ x & \longmapsto & \theta(x) \\ H_1 : \mathcal{G}_1 & \longrightarrow & C_1 \\ f & \longmapsto & (\theta(s(f)), \theta(t(f)), F_1(f), G_1(f)) \end{array}$$

□

Lemma 7.6. *A morphism $G : \mathcal{G} \longrightarrow \mathcal{H}$ in* **Str2Grp** *is a cofibration if and only if G is a retract of a morphism $H : \mathcal{G} \longrightarrow \mathcal{L}$ with $\mathcal{L}_0 = \mathcal{G}_0 * F(X)$, where $F(X)$ is the free group on some set X.*

Proof. We will only prove the reverse implication. Suppose $H : \mathcal{G} \longrightarrow \mathcal{L}$ is a morphism between strict 2-groups, where $\mathcal{L}_0 = \mathcal{G}_0 * F(X)$ for some free group $F(X)$. We wish to show that H has the LLP with respect to all acyclic fibrations. Let $K : \mathcal{A} \overset{\sim}{\twoheadrightarrow} \mathcal{B}$ be an acyclic fibration, and suppose we have morphisms M and N such that the diagram below commutes:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{M} & \mathcal{A} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle \wr\, K} \\ \mathcal{L} & \xrightarrow[N]{} & \mathcal{B} \end{array}$$

Since K is both a weak equivalence and a fibration, K must be surjective on objects. That is, $K_0 : \mathcal{A}_0 \longrightarrow \mathcal{B}_0$ is surjective. Thus, for any generator $x \in X$ we may choose an element $a \in \mathcal{A}_0$ such that $K_0(a) = N_0(x)$. These choices give a function from X to $\mathcal{B}_0$, which induces a group homomorphism $\varphi : F(X) \longrightarrow \mathcal{B}_0$. We may make these choices in such a way that, together with $M_0 : \mathcal{G}_0 \longrightarrow \mathcal{A}_0$, the homomorphism φ determines a morphism $T_0 : \mathcal{L}_0 = \mathcal{G}_0 * F(X) \longrightarrow \mathcal{A}_0$ such that $K_0 \circ T_0 = N_0$ and $T_0 \circ H_0 = M_0$.

Now, let $z \in \mathcal{L}_1$ and consider $N_1(z) \in \mathcal{B}_1$. Since K is an equivalence of categories, we may find $w \in \mathcal{A}_1$, unique such that $s(w) = T_0(s(z))$, $t(w) = T_0(t(z))$, and $K_1(w) = N_1(z)$. Define $T_1(z) = w$. This pair of group homomorphisms defines a morphism $T : \mathcal{L} \longrightarrow \mathcal{A}$ giving a lift in the diagram above. Thus, H has the LLP with respect to an arbitrary acyclic fibration K, so H is a cofibration. By Axiom 3 of Definition 6.3, the class of cofibrations is closed under retracts, so any retract of such a morphism must also be a cofibration.

See [31] for a proof of the reverse direction. □

Corollary 7.7. *A strict* 2*-group* $\mathcal{G}$ *is cofibrant if and only if* $\mathcal{G}_0 = F(X)$ *is a free group.*

Theorem 7.8. *We have an equivalence of categories* $Ho(\mathbf{Str2Grp}) \simeq \mathcal{H}(\mathbf{2Grp})$, *where* $\mathcal{H}(\mathbf{2Grp})$ *is the category defined in Remark 5.14. This makes precise one sense in which the category* **Str2Grp** *recovers the homotopy theory of the* 2*-category* **2Grp**.

Proof. We use the notation of Example 6.45. Consider the category $Ho\left(\mathbf{Str2Grp}_{cf}\right) = Ho\left(\mathbf{Str2Grp}_{c}\right)$, whose objects are the cofibrant 2-groups and whose morphisms, by Lemma 7.5, are isomorphism classes of strict monoidal functors. By Example 6.47, we have an equivalence of categories $Ho\left(\mathbf{Str2Grp}_{cf}\right) \simeq Ho(\mathbf{Str2Grp})$. Thus we need only show that $Ho\left(\mathbf{Str2Grp}_{cf}\right)$ is equivalent to $\mathcal{H}(\mathbf{2Grp})$.

Now, consider the inclusion functor $i : \mathbf{Str2Grp}_{cf} \longrightarrow \mathbf{2Grp}$. Composing i

with the functor $\mathcal{H}$ of Remark 5.14 gives the functor below:

$$\begin{array}{rcl} \mathcal{H}\circ i : \mathbf{Str2Grp}_{cf} & \longrightarrow & \mathcal{H}\left(\mathbf{2Grp}\right) \\ \mathcal{K} & \longmapsto & \mathcal{K} \\ \left(F : \mathcal{J} \longrightarrow \mathcal{K}\right) & \longmapsto & \left([F] : \mathcal{J} \longrightarrow \mathcal{K}\right) \end{array}$$

Here $[F]$ is the 2-isomorphism class of F. By construction, this functor associates 2-isomorphic functors in $\mathbf{Str2Grp}_{cf}$, so it maps equivalences to isomorphisms. Thus, $\mathcal{H}\circ i$ induces a unique functor $\Omega : Ho\left(\mathbf{Str2Grp}_{cf}\right) \longrightarrow \mathcal{H}\left(\mathbf{2Grp}\right)$ such that $\Omega \circ \gamma_{cf} = \mathcal{H}\circ i$. We claim that Ω is an equivalence.

We first show that Ω is faithful. Suppose $F, G : \mathcal{J} \longrightarrow \mathcal{K}$ are morphisms in $\mathbf{Str2Grp}_{cf}$ such that $[F] = [G]$ in $\mathcal{H}\left(\mathbf{2Grp}\right)$. Then there is a monoidal natural isomorphism between F and G. By Lemma 7.5, F and G are 2-isomorphic if and only there is a homotopy from F to G. Thus, in $Ho\left(\mathbf{Str2Grp}_{cf}\right)$ we have $[F] = [G]$. Therefore, Ω is faithful.

To see that Ω is essentially surjective, let $\mathcal{G} \in \mathbf{2Grp}$. By Theorem 5.18, $\mathcal{G}$ is equivalent in $\mathbf{2Grp}$ to a strict 2-group $\bar{\mathcal{G}}$. We may take the cofibrant replacement of definition 6.39, to find a cofibrant strict 2-group $Q\bar{\mathcal{G}}$, which is equivalent to the strict 2-group $\bar{\mathcal{G}}$. Since equivalences in $\mathbf{2Grp}$ are isomorphisms in $\mathcal{H}\left(\mathbf{2Grp}\right)$, this gives an isomorphism $\mathcal{G} \cong Q\bar{\mathcal{G}}$ in $\mathcal{H}\left(\mathbf{2Grp}\right)$. Since $\Omega\left(Q\bar{\mathcal{G}}\right) = Q\bar{\mathcal{G}}$, this implies that Ω is essentially surjective.

Now, suppose $G : \mathcal{J} \longrightarrow \mathcal{K}$ is a monoidal functor in $\mathbf{2Grp}$ between cofibrant strict 2-groups. To conclude that Ω is full we must show that there is a monoidal natural isomorphism $\theta : \hat{G} \Longrightarrow G$ from a strict monoidal functor $\hat{G} : \mathcal{J} \longrightarrow \mathcal{K}$ to G, so that $[G] = \left[\hat{G}\right] = \Omega\left(\left[\hat{G}\right]\right)$. Now, since $\mathcal{J}$ is a cofibrant strict 2-group, we have $\mathcal{J} = \left(\mathcal{J}_1 \rightrightarrows F\left(X\right)\right)$ where $F\left(X\right)$ is the free group on some set X. We may define $\hat{G} : \mathcal{J} \longrightarrow \mathcal{K}$ as follows.

Any object $j \in \mathcal{J}$ has the form $j = x_1^{\varepsilon_1} \otimes x_2^{\varepsilon_2} \otimes \ldots \otimes x_n^{\varepsilon_n}$ for some $x_i \in X$ and $\varepsilon_i = \pm 1$, where x^{-1} denotes the strict inverse of x in $\mathcal{J}$. On objects, we define $\hat{G}$ as follows:

$$\hat{G}\left(x_1^{\varepsilon_1} \otimes x_2^{\varepsilon_2} \otimes \ldots \otimes x_n^{\varepsilon_n}\right) = G\left(x_1\right)^{\varepsilon_1} \otimes G\left(x_2\right)^{\varepsilon_2} \otimes \ldots \otimes G\left(x_n\right)^{\varepsilon_n}$$

Consider the family of isomorphisms below:

$$\begin{array}{c} \hat{G}\left(x_1^{\varepsilon_1} \otimes x_2^{\varepsilon_2} \otimes \ldots \otimes x_n^{\varepsilon_n}\right) = G\left(x_1\right)^{\varepsilon_1} \otimes G\left(x_2\right)^{\varepsilon_2} \otimes \ldots \otimes G\left(x_n\right)^{\varepsilon_n} \\ \downarrow \\ G\left(x_1^{\varepsilon_1}\right) \otimes G\left(x_2^{\varepsilon_2}\right) \otimes \ldots \otimes G\left(x_n^{\varepsilon_n}\right) \\ \downarrow \\ G\left(x_1^{\varepsilon_1} \otimes x_2^{\varepsilon_2} \otimes \ldots \otimes x_n^{\varepsilon_n}\right) \end{array}$$

Here the first map is given by the canonical isomorphisms from $G\left(x_1\right)^{\varepsilon_1}$ to $G\left(x_1^{\varepsilon_1}\right)$ of Remark 5.7. Since $\mathcal{K}$ is strict, these depend only on μ_1 and μ, the structure isomorphisms of G. The second isomorphism is also composed only of the structure isomorphism μ. The commutative diagrams of Definition 4.7 imply that for any $j \in \mathcal{J}$ this gives a unique isomorphism in $\mathcal{K}$:

$$\begin{aligned} \theta_j : \hat{G}(j) &\longrightarrow G(j) \\ \theta_1 = \mu_1 : \hat{G}(1) &\longrightarrow G(1) \end{aligned}$$

Now, for any morphism $f : j \longrightarrow k$ in $\mathcal{J}$, define $\hat{G}(f) = \theta_k^{-1} \circ G(f) \circ \theta_j$. This defines a strict monoidal functor $\hat{G} : \mathcal{J} \longrightarrow \mathcal{K}$. The morphisms $\theta_j : \hat{G}(j) \longrightarrow G(j)$ are the components of the desired monoidal natural isomorphism $\theta : \hat{G} \Longrightarrow G$. Thus, Ω is full. □

Corollary 7.9. *Let $\mathcal{A}$ be a cofibrant object of* **Str2Grp**. *For any 2-group $\mathcal{G}$, let $\bar{\mathcal{G}}$ be the equivalent strict 2-group of Theorem 5.18. We have the following equivalence of categories:*

$$Hom_{\mathbf{2Grp}}(\mathcal{A}, \mathcal{G}) \simeq Hom_{\mathbf{Str2Grp}}(\mathcal{A}, \bar{\mathcal{G}})$$

Proof. The equivalence $\omega : \bar{\mathcal{G}} \longrightarrow \mathcal{G}$ induces the functor below for any $\mathcal{A}$ in **2Grp**:

$$\begin{aligned} \omega \circ - : Hom_{\mathbf{2Grp}}(\mathcal{A}, \bar{\mathcal{G}}) &\longrightarrow Hom_{\mathbf{2Grp}}(\mathcal{A}, \mathcal{G}) \\ F &\longmapsto \omega \circ F \\ (\theta : F \Longrightarrow G) &\longmapsto (\omega \circ \theta : \omega \circ F \Longrightarrow \omega \circ G) \end{aligned}$$

This functor has a weak inverse given by

$$\bar{\omega} \circ - : Hom_{\mathbf{2Grp}}(\mathcal{A}, \mathcal{G}) \longrightarrow Hom_{\mathbf{2Grp}}(\mathcal{A}, \bar{\mathcal{G}})$$

where $\bar{\omega} : \mathcal{G} \longrightarrow \bar{\mathcal{G}}$ is the weak inverse of ω. Thus for any 2-group $\mathcal{A}$ we have the equivalence below:

$$Hom_{\mathbf{2Grp}}(\mathcal{A}, \mathcal{G}) \simeq Hom_{\mathbf{2Grp}}(\mathcal{A}, \bar{\mathcal{G}})$$

Now, suppose $\mathcal{A}$ is a cofibrant object of **Str2Grp**, and consider the inclusion below:

$$j : Hom_{\mathbf{Str2Grp}}(\mathcal{A}, \bar{\mathcal{G}}) \longrightarrow Hom_{\mathbf{2Grp}}(\mathcal{A}, \bar{\mathcal{G}})$$

By the proof of Theorem 7.8, any monoidal functor from a cofibrant strict 2-group to a strict 2-group is naturally isomorphic to a strict monoidal functor. Thus, the inclusion j is essentially surjective. Furthermore, j is clearly full and faithful. Thus, j gives an equivalence as below:

$$Hom_{\mathbf{2Grp}}(\mathcal{A}, \bar{\mathcal{G}}) \simeq Hom_{\mathbf{Str2Grp}}(\mathcal{A}, \bar{\mathcal{G}})$$

Composing these functors gives the desired equivalence:

$$Hom_{\mathbf{2Grp}}(\mathcal{A}, \mathcal{G}) \simeq Hom_{\mathbf{Str2Grp}}(\mathcal{A}, \bar{\mathcal{G}})$$

□

Remark 7.10. Consider the functors of Definition 5.10 and Remark 5.23:

$$\begin{aligned} h_0 : \mathbf{Str2Grp} &\longrightarrow \mathbf{Grp} \\ h_1 : \mathbf{Str2Grp} &\longrightarrow \mathbf{Ab} \end{aligned}$$

By Lemma 5.13, these functors take equivalences in **Str2Grp** to isomorphisms, so they induce functors on the homotopy category:

$$\begin{aligned} \tilde{h_0} : Ho\left(\mathbf{Str2Grp}\right) &\longrightarrow \mathbf{Grp} \\ \tilde{h_1} : Ho\left(\mathbf{Str2Grp}\right) &\longrightarrow \mathbf{Ab} \end{aligned}$$

Using the equivalence of Theorem 7.8, these recover the functors of Remark 5.14.

Definition 7.11. The equivalence of Theorem 5.20 between the categories **Str2Grp** and **Cross** allows us to translate the model structure of Definition 7.1 onto **Cross**. Explicitly, suppose we have an arrow

$$(u, v) : ((\delta_1 : H_1 \longrightarrow G_1) \longrightarrow (\delta_2 : H_2 \longrightarrow G_2))$$

in **Cross**. We make the following definitions:

- The arrow (u, v) is a weak equivalence if (u, v) is an equivalence in the sense of Definition 3.13. That is, (u, v) is a weak equivalence if it induces isomorphisms on h_0 and h_1.
- The arrow (u, v) is a fibration if $v : H_1 \longrightarrow H_2$ is surjective.
- The arrow (u, v) is a cofibration if it is a retract of a morphism

  $$(u', v') : \left(\left(\delta_1^{'} : H_1^{'} \longrightarrow G_1^{'}\right) \longrightarrow \left(\delta_2^{'} : H_2^{'} \longrightarrow G_2^{'}\right)\right)$$

  such that $G_2^{'} = G_1^{'} * F(X)$, where $F(X)$ is the free group on some set X.

The cofibrant objects of **Cross** with this model structure are the 0-free crossed modules - that is, crossed modules of the form $\delta : H \longrightarrow F(X)$, where $F(X)$ is a free group on a set X. Every crossed module is fibrant.

Remark 7.12. In [32], Moerdijk and Svensson introduce a model structure on the category of 2-groupoids, which they use to show that 2-groupoids model all homotopy 2-types. (This is the analogue of the result in Example 6.52, that groupoids model all homotopy 1-types.) Using the delooping of Definition 4.15, we may identify 2-groups with one object 2-groupoids. In [34], Noohi uses this identification to adapt the Moerdijk-Svensson model structure to a model structure on **Cross**. The weak equivalences in this model structure agree with weak equivalences of Definition 7.11. However, a morphism $(u, v) : \delta_1 \longrightarrow \delta_2$ is a fibration in the Moerdijk-Svensson model structure if both u and v are surjective. Thus, although all fibrations in the Moerdijk-Svensson model structure are fibrations in the Garzon-Miranda model structure, the converse is not true.

However, homotopies in the Moerdijk-Svensson model structure agree with homotopies in the Garzon-Miranda model structure, as do the cofibrant and fibrant objects. Thus, in [34], Noohi is able to use the Moerdijk-Svensson model structure to obtain the same characterisation of weak morphisms in terms of strict morphisms as we have in Corollary 7.8.

Example 7.13. Let G be a group with presentation $G \cong F/H$. Consider the morphism below, introduced in Example 3.14:

$$\begin{array}{ccc} H & \xrightarrow{\quad 0 \quad} & 1 \\ {\scriptstyle \delta_1}\big\downarrow & & \big\downarrow{\scriptstyle \delta_2} \\ F & \xrightarrow[\quad p \quad]{} & G \end{array}$$

By Example 3.14 this morphism is a weak equivalence. Moreover, since $0 : H \longrightarrow 1$ is surjective, $(0, p)$ is a fibration. Thus, δ_1 is a cofibrant replacement for δ_2 in **Cross**.

Example 7.14. The fundamental crossed module construction of Definition 3.16 induces an equivalence of categories:

$$\Pi_1 : Ho\left(\mathbf{2Type}^*_{\mathbf{c}}\right) \longrightarrow Ho\left(\mathbf{Cross}\right)$$

As in Example 6.52, the objects of $Ho\left(\mathbf{2Type}^*_{\mathbf{c}}\right)$ are connected pointed 2-types, and the morphisms are homotopy classes of maps. In this way, crossed modules model pointed, connected 2-types.

As in Remark 3.18, we may extract the fundamental group and the second homotopy group of a space from its fundamental crossed module. Explicitly, for any $X \in \mathbf{2Type}^*_{\mathbf{c}}$, we have $h_0\left(\Pi_1\left(X\right)\right) = \pi_1\left(X\right)$ and $h_1\left(\Pi_1\left(X\right)\right) = \pi_2\left(X\right)$. However, given $X, Y \in \mathbf{2Type}^*_{\mathbf{c}}$, the equalities $\pi_1\left(X\right) = \pi_1\left(Y\right)$ and $\pi_2\left(X\right) = \pi_2\left(Y\right)$ alone are not enough to conclude that X is homotopy equivalent to Y. This result tells us that the fundamental crossed module encodes genuinely new information about a CW-complex, which is not captured by homotopy groups.

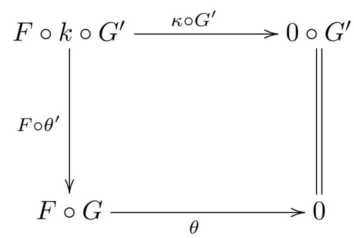
Remark 7.15. The right homotopies in **Cross** recover the 2-morphisms in the 2-category **Cross** of Definition 3.23. Thus, the equivalence of Theorem 5.20 extends to a strong equivalence between the corresponding 2-categories. That is, the functors Υ and Ψ extend to 2-functors which are essentially surjective on objects and full and faithful on both 1-cells and 2-cells.

Moreover, the model structure of Definition 7.1 may be extended to the categories **BraidStr2Grp** and **SymStr2Grp**, and thus to **RQuad** and **SQuad** via the equivalences of Remark 5.24. As in **Cross**, the right homotopies in these model categories recover the 2-cells in the 2-categories. Therefore, the equivalences of Remark 5.24 also induce strong equivalences of 2-categories.

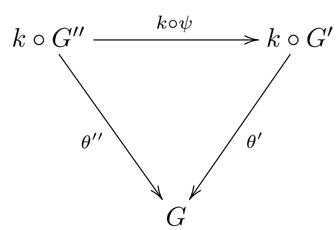
8 Kernels in 2Grp

In this section we discuss the kernel of a morphism in **2Grp**, following a definition given by Vitale in [38].

Definition 8.1. Let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be a morphism in the 2-category **2Grp**. The kernel of F is given by a morphism of 2-groups $k : \mathcal{K}er\,(F) \longrightarrow \mathcal{G}$ together with a monoidal natural isomorphism $\kappa : F \circ k \Longrightarrow 0$, where $0 : \mathcal{K}er\,(F) \longrightarrow \mathcal{H}$ is the zero morphism of Definition 5.6. The kernel satisfies the following universal property, which characterises $\mathcal{K}er\,(F)$ up to equivalence. Given any morphism $G : \mathcal{K} \longrightarrow \mathcal{G}$ and any monoidal natural isomorphism $\theta : F \circ G \Longrightarrow 0$ there exists a morphism $G' : \mathcal{K} \longrightarrow \mathcal{K}er\,(F)$ and a monoidal natural isomorphism $\theta' : k \circ G' \Longrightarrow G$ such that the following diagram commutes:



Furthermore, given $G'' : \mathcal{K} \longrightarrow \mathcal{K}er\,(F)$ and a monoidal natural isomorphism $\theta'' : k \circ G'' \Longrightarrow G$ also making the diagram above commute, there is a unique monoidal natural transformation $\psi : G'' \Longrightarrow G'$ such that the following triangle commutes:



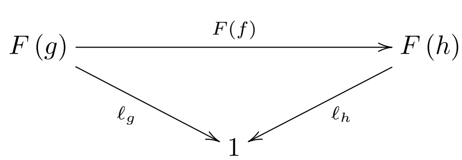
Definition 8.2. Let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be a monoidal functor between 2-groups. We may construct $\mathcal{K}er\,(F)$ as follows:

The objects of $\mathcal{K}er\,(F)$ are pairs (g, ℓ_g) with $g \in \mathcal{G}$ and $\ell_g : F(g) \longrightarrow 1$ a morphism in $\mathcal{H}$.

For any two objects (g, ℓ_g) and (h, ℓ_h) in $\mathcal{K}er\,(F)$, a morphism

$$f : (g, \ell_g) \longrightarrow (h, \ell_h)$$

is given by a morphism $f : g \longrightarrow h$ in $\mathcal{G}$ such that the diagram below commutes:



Composition in $\mathcal{K}er\,(F)$ is induced by composition in $\mathcal{G}$, so $\mathcal{K}er\,(F)$ is a groupoid. In fact, the underlying groupoid of $\mathcal{K}er\,(F)$ agrees with the construction in Remark 6.77 of the homotopy kernel of the underlying functor F in **Grpd**. We will return to this point in Remark 8.3.

The monoidal structure on $\mathcal{K}er\,(F)$ is as follows. The unit object is the pair $\left(1, \mu_1^{-1}\right)$, where $\mu_1 : 1 \longrightarrow F\,(1)$ is the structure isomorphism of F. The tensor product is given on objects by

$$(g, \ell_g) \otimes (h, \ell_h) = (g \otimes h, \ell_{g \otimes h})$$

where $\ell_{g\otimes h} : F\,(g \otimes h) \longrightarrow 1$ is the following composite:

$$F\,(g \otimes h) \xrightarrow{\mu_{g,h}^{-1}} F\,(g) \otimes F\,(h) \xrightarrow{\ell_g \otimes \ell_h} 1 \otimes 1 \longrightarrow 1$$

The functor $k : \mathcal{K}er\,(F) \longrightarrow \mathcal{G}$ is given as follows:

$$\begin{array}{rcl} (g, \ell_g) & \longmapsto & g \\ (f : (g, \ell_g) \longrightarrow (h, \ell_h)) & \longmapsto & (f : g \longrightarrow h) \end{array}$$

The monoidal natural isomorphism $\kappa : F \circ k \Longrightarrow 0$ has components given by $\kappa_{(g,\ell_g)} = \ell_g$ for each object (g, ℓ_g) in $\mathcal{K}er\,(F)$. From now on, when we reference the kernel of a morphism in **2Grp**, we will refer to this construction of the kernel.

Remark 8.3. Let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be a morphism in **Str2Grp**. Since every object of **Str2Grp** is fibrant, by Remark 6.77, the homotopy kernel of F is given by the pullback below:

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{H}^I \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{G} \times * & \xrightarrow[F \times 0]{} & \mathcal{H} \times \mathcal{H} \end{array}$$

By Remark 5.27, pullbacks in **Str2Grp** agree with pullbacks of the underlying groupoids and the underlying groupoid of the path object $\mathcal{H}^I$ is the path object in **Grpd**. This construction of the homotopy kernel P agrees precisely with the construction of $\mathcal{K}er\,(F)$ in Definition 8.2.

Remark 8.4. Definition 8.1 gives $\mathcal{K}er\,(F)$ as a bilimit in the 2-category **2Grp**. Bilimits are one of a number of notions of limit, discussed in [24, 25], that may be considered in a 2-category.

For any morphism $F : \mathcal{G} \longrightarrow \mathcal{H}$ in **2Grp**, the construction of $\mathcal{K}er\,(F)$ given in Definition 8.2 also satisfies the following universal property, which characterises the 2-group up to isomorphism rather than equivalence. Given any morphism

$G : \mathcal{K} \longrightarrow \mathcal{G}$ and any monoidal natural isomorphism $\theta : F \circ G \Longrightarrow 0$ there is a unique morphism $G' : \mathcal{K} \longrightarrow \mathcal{K}er\,(F)$ such that $k \circ G' = G$. We say that limits which satisfy such a universal property are strict homotopy limits.

In a model category C whose objects are fibrant, the construction in Remark 6.77 of the homotopy pullback is an example of a strict homotopy pullback. Note that the universal property of a homotopy pullback in a model category characterises the pullback only up to isomorphism in $Ho\,(C)$, while the a strict homotopy pullback is characterised up to isomorphism in C. Thus, even in a model category (where both concepts are defined), although they often coincide, these concepts are distinct. The fact that the construction of the strict homotopy kernel of a morphism in **2Grp** agrees with the weaker notion of the bilimit of Definition 8.1, and (when they are defined) with the homotopy kernels defined via the model structure on **Str2Grp**, may be seen as an instance in which the homotopy theory of 2-groups recovers their 2-category theory.

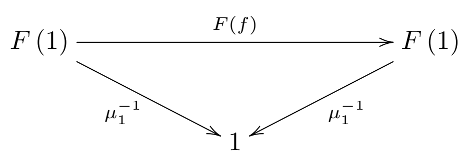
We will now develop some elementary facts about the kernel, which begin to illustrate its central role in the 2-dimensional algebra of 2-groups.

Lemma 8.5. *Let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be a monoidal functor between 2-groups. Then we have a group isomorphism $h_1\,(\mathcal{K}er\,(F)) \cong Ker\,(h_1\,(F))$.*

Proof. Suppose $f : 1 \longrightarrow 1$ is a morphism in $\mathcal{G}$. Recall that by Definition 5.10, we have the following:

$$(h_1\,(F))\,(f) = \mu_1^{-1} \circ F\,(f) \circ \mu_1$$

Now, f is an endomorphism of the unit object $\left(1, \mu_1^{-1}\right)$ in $\mathcal{K}er\,(F)$ if and only if the diagram below commutes:



That is, an arrow f in $h_1\,(\mathcal{G})$ is in $h_1\,(\mathcal{K}er\,(F))$ if and only if $(h_1\,(F))\,(f) = \mu_1^{-1} \circ F\,(f) \circ \mu_1 = 1$. □

Lemma 8.6. *Let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be a morphism of 2-groups. We have the following characterisations of F in terms of $\mathcal{K}er\,(F)$:*

1. F is faithful if and only if $h_1\,(\mathcal{K}er\,(F)) = 1$.
2. F is full if and only if $h_0\,(\mathcal{K}er\,(F)) = 1$.
3. F is full and faithful if and only if $\mathcal{K}er\,(F) \simeq *$.

In this way, $\mathcal{K}er\,(F)$ measures the injectiveness of the functor F, just as $Ker\,(f)$ measures the injectiveness of a group homomorphism $f : G \longrightarrow H$.

Proof. Statement 1 follows from Lemma 8.5 and the fact, form Lemma 5.13, that F is faithful if and only if $h_1(F)$ is injective.

The content of Statement 2 is as follows. We wish to show that $F : \mathcal{G} \longrightarrow \mathcal{H}$ is full if and only if for any two objects $(g, \ell_g), (h, \ell_h) \in \mathcal{K}er(F)$, there is a morphism $f : g \longrightarrow h$ in $\mathcal{G}$ such that $\ell_h \circ F(f) = \ell_g$. To see the forward implication, suppose that F is full. Consider the following morphism in $\mathcal{H}$:

$$\ell_h^{-1} \circ \ell_g : F(g) \longrightarrow F(h)$$

Since F is full, there is a morphism $f : g \longrightarrow h$ in $\mathcal{G}$ such that $F(f) = \ell_h^{-1} \circ \ell_g$. Thus, if F is full, then $h_0(\mathcal{K}er(F)) = 1$.

For the reverse implication, suppose $h_0(\mathcal{K}er(F)) = 1$. By definition, given any object $g \in \mathcal{G}$ for which there exists an isomorphism $\ell_g : F(g) \longrightarrow 1$ in $\mathcal{H}$ (that is, any object $g \in Ker(h_0(F))$), we have an object $(g, \ell_g) \in \mathcal{K}er(F)$. Thus, we have a surjection as below:

$$\begin{array}{rcl} h_0(\mathcal{K}er(F)) & \longrightarrow & Ker(h_0(F)) \\ [(g, \ell_g)] & \longmapsto & [g] \end{array}$$

Here $[x]$ denotes the isomorphism class of the object x. So if $h_0(\mathcal{K}er(F)) = 1$ then $Ker(h_0(F)) = 1$, and so $h_0(F)$ is injective. Now, given any morphism $k : 1 \longrightarrow 1$ in $\mathcal{H}$, we have an associated morphism $k \circ \mu_1^{-1} : F(1) \longrightarrow 1$. Consider the objects $\left(1, k \circ \mu_1^{-1}\right)$ and $\left(1, \mu_1^{-1}\right)$ in $\mathcal{K}er(F)$. Since $h_0(\mathcal{K}er(F)) = 1$, we must have a morphism $f : \left(1, k \circ \mu_1^{-1}\right) \longrightarrow \left(1, \mu_1^{-1}\right)$ in $\mathcal{K}er(F)$. That is, there is a morphism $f : 1 \longrightarrow 1$ such that $\mu_1^{-1} \circ F(f) = k \circ \mu_1^{-1}$. That is, for any $k \in h_1(\mathcal{H})$ there is a morphism $f \in h_1(\mathcal{G})$ with $(h_1(F))(f) = k$. Thus $h_1(F)$ is surjective. By Lemma 5.13, this implies that F is full.

By Lemma 5.13, the unique map $0 : * \longrightarrow \mathcal{K}er(F)$ is an equivalence if and only if both $h_0(0)$ and $h_1(0)$ are isomorphisms. Thus, Statement 3 follows from the first two statements. □

Example 8.7. Let $\mathcal{G}$ be a 2-group and let $\mathbf{B}\mathcal{G}$ be its delooping, as in Definition 4.15. Consider the identity pseudofunctor $id_{\mathbf{B}\mathcal{G}} : \mathbf{B}\mathcal{G} \longrightarrow \mathbf{B}\mathcal{G}$. The Drinfeld centre of $\mathcal{G}$ is the category whose objects are pseudonatural transformations from $id_{\mathbf{B}\mathcal{G}}$ to itself and whose arrows are the modifications between them. We will denote the Drinfeld centre by $\mathcal{Z}(\mathcal{G})$.

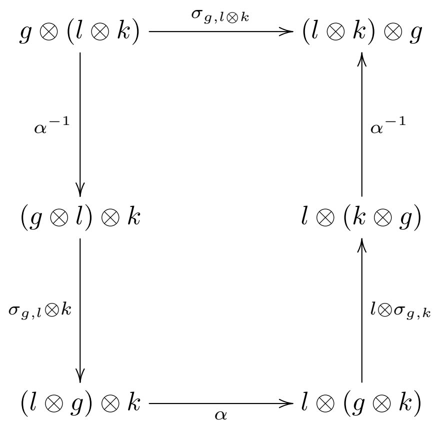
Explicitly, the objects of $\mathcal{Z}(\mathcal{G})$ are pairs (g, σ_g), where $g \in \mathcal{G}$ and σ_g is a natural isomorphism as below:

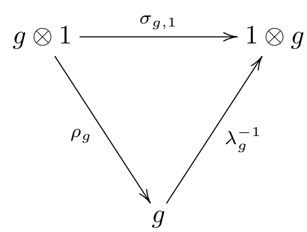
$$\sigma_g : \mathcal{W}_g \Longrightarrow \mathcal{Y}_g$$

Here, $\mathcal{Y}_g$ and $\mathcal{W}_g$ are the functors of Remark 4.3, given on objects by $\mathcal{W}_g(h) = g \otimes h$ and $\mathcal{Y}_g(h) = h \otimes g$. The components of these natural isomorphisms are denoted as follows:

$$\sigma_{g,k} : g \otimes k \longrightarrow k \otimes g$$

The diagrams below must commute:

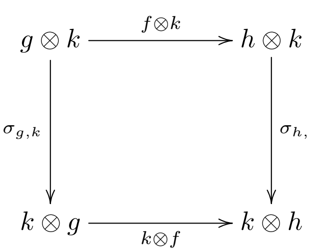




For any two objects (g, σ_g) and (h, σ_h) in $\mathcal{Z}(\mathcal{G})$, a morphism

$$f : (g, \sigma_g) \longrightarrow (h, \sigma_h)$$

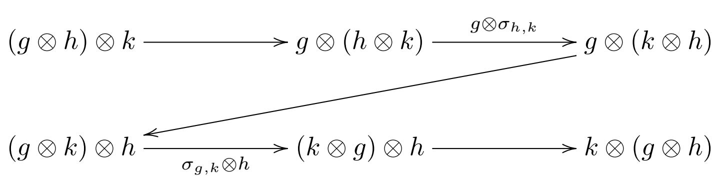
is given by a morphism $f : g \longrightarrow h$ in $\mathcal{G}$ such that the diagram below commutes for any $k \in \mathcal{G}$:



We may define a monoidal structure on $\mathcal{Z}(\mathcal{G})$, given on objects by

$$(g, \sigma_g) \otimes (h, \sigma_h) = (g \otimes h, \sigma_{g\otimes h})$$

where each component $\sigma_{g\otimes h,k} : (g \otimes h) \otimes k \longrightarrow k \otimes (g \otimes h)$ is given by the following composite:



Now, consider the monoidal functor below:

$$\begin{array}{rcl} \varphi : \mathcal{G} & \longrightarrow & \mathcal{A}ut\left(\mathcal{G}\right) \\ g & \longmapsto & \mathcal{W}_g \circ \mathcal{Y}_{\bar{g}} \\ \left(f : g \longrightarrow h\right) & \longmapsto & \left(\varphi_f : \mathcal{W}_g \circ \mathcal{Y}_{\bar{g}} \Longrightarrow \mathcal{W}_h \circ \mathcal{Y}_{\bar{h}}\right) \end{array}$$

Here $\mathcal{A}ut\left(\mathcal{G}\right)$ is the 2-group of monoidal autoequivalences of Example 5.9. The component of the monoidal natural isomorphism φ_f at $k \in \mathcal{G}$ is as follows:

$$f \otimes k \otimes \bar{f} : g \otimes \left(k \otimes \bar{g}\right) \longrightarrow h \otimes \left(k \otimes \bar{h}\right)$$

With these definitions there is an obvious monoidal equivalence $\mathcal{K}er\left(\varphi\right) \simeq \mathcal{Z}\left(\mathcal{G}\right)$. Note that the monoidal functor φ defines an action of $\mathcal{G}$ on itself in the sense of Definition 4.16.

Remark 8.8. The 2-dimensional notions of kernel and centre introduced in Definition 8.1 and Example 8.7 do not embed like their 1-dimensional counterparts. More precisely, given a monoidal functor $F : \mathcal{G} \longrightarrow \mathcal{H}$, although the canonical functor $k : \mathcal{K}er\left(F\right) \longrightarrow \mathcal{G}$ is faithful, in general it is not full. Much of the richness of 2-dimensional algebra derives from this fact, which implies that the kernel of a kernel is in general nontrivial.

To see this, consider the following description of $\mathcal{K}er\left(k\right)$, given by the construction of Definition 8.2. The objects are triples $\left(g, \ell_g, \jmath_g\right)$ where $g \in \mathcal{G}$, and $\ell_g : F\left(g\right) \longrightarrow 1$ and $\jmath_g : g \longrightarrow 1$ are morphisms. Any two objects have at most one morphism between them. We have a morphism

$$\jmath_h^{-1} \circ \jmath_h : \left(g, \ell_g, \jmath_g\right) \longrightarrow \left(h, \ell_h, \jmath_h\right)$$

if and only if the following equality holds:

$$F\left(\jmath_g\right) \circ \ell_g^{-1} = F\left(\jmath_h\right) \circ \ell_h^{-1}$$

The tensor product on $\mathcal{K}er\left(k\right)$ is given on objects as follows:

$$\left(g, \ell_g, \jmath_g\right) \otimes \left(h, \ell_h, \jmath_h\right) = \left(g \otimes h, \ell_{g\otimes h}, \jmath_{g\otimes h}\right)$$

The morphisms $\ell_{g\otimes h}$ and $\jmath_{g\otimes h}$ are defined as in Definition 8.2.

It is easy to see that $h_1\left(\mathcal{K}er\left(k\right)\right) = 1$, so by Lemma 8.6 k is faithful.

Now, consider the function below:

$$\begin{array}{rcl} \psi : h_0\left(\mathcal{K}er\left(k\right)\right) & \longrightarrow & h_1\left(\mathcal{H}\right) \\ \left[\left(g, \ell_g, \jmath_g\right)\right] & \longmapsto & \mu_1^{-1} \circ F\left(\jmath_g\right) \circ \ell_g^{-1} \end{array}$$

This is well-defined, since $\left(g, \ell_g, \jmath_g\right)$ is isomorphic to $\left(h, \ell_h, \jmath_h\right)$ if and only if $F\left(\jmath_g\right) \circ \ell_g^{-1} = F\left(\jmath_h\right) \circ \ell_h^{-1}$. This also implies that ψ is injective. To see that ψ is surjective, suppose $a : 1 \longrightarrow 1$ is a morphism in $\mathcal{H}$. The object $\left(1, \left(\mu_1 \circ a\right)^{-1}, id_1\right)$ in $\mathcal{K}er\left(k\right)$ is mapped onto a by ψ. Thus, ψ is surjective. It is also not hard to check that ψ is a group homomorphism. Therefore,

$h_0(\mathcal{K}er(k)) \cong h_1(\mathcal{H})$. By the characterisation of Lemma 8.6, k is full if and only if $h_1(\mathcal{H}) = 1$.

Now, consider the 2-group $(h_1(\mathcal{H}))[0]$, where h_1 and $[0]$ are the functors of Definition 5.10 and Definition 5.15. There is a unique monoidal functor $\Psi : \mathcal{K}er(k) \longrightarrow (h_1(\mathcal{H}))[0]$ such that $h_0(\Psi) = \psi$. Since

$$h_1(\mathcal{K}er(k)) = h_1((h_1(\mathcal{H}))[0]) = 1$$

both $h_1(\Psi) = 0$ and $h_0(\Psi) = \psi$ are isomorphisms. Thus, by the chatacterisation of Lemma 5.13, Ψ is an equivalence.

Therefore, $\mathcal{K}er(k) \simeq (h_1(\mathcal{H}))[0]$. By the description of Example 8.9, this gives us an equivalence $\mathcal{K}er(k) \simeq \Omega(\mathcal{H})$, where $\Omega(\mathcal{H})$ is the loop space of $\mathcal{H}$.

Example 8.9. Let $\mathcal{G}$ be a 2-group. We define the loop space of $\mathcal{G}$ as follows:

$$\Omega(\mathcal{G}) = \mathcal{K}er(0)$$

Here 0 is the unique map $0 : * \longrightarrow \mathcal{G}$. Using the construction of Definition 8.2 it is easy to see that $\Omega(\mathcal{G})$ is equivalent to the discrete 2-group $(h_1(\mathcal{G}))[0]$.

8.1 2-Dimensional Algebra for Symmetric 2-Groups

Let $F : \mathcal{G} \longrightarrow \mathcal{H}$ be a morphism in **2Grp**. We define the cokernel of F to be a morphism of 2-groups $e : \mathcal{H} \longrightarrow \mathcal{C}oker(F)$ together with a monoidal natural isomorphism $\epsilon : e \circ F \Longrightarrow 0$, satisfying a dual universal property to the kernel of Definition 8.1. In [38], Vitale gives a construction for cokernels in **Sym2Grp**. Kernels in **Sym2Grp** may be constructed as they are in **2Grp**.

In general, without the assumption of a symmetric braiding, the construction of the cokernel in [38] does not go through. The difficulty in carrying the construction over to **2Grp** derives from the distinction between morphisms of 2-groups and symmetric 2-groups. There are obvious forgetful 2-functors U_1 : **Sym2Grp** $\longrightarrow$ **Braid2Grp** and U_2 : **Braid2Grp** $\longrightarrow$ **2Grp**, and by the definiton of **Sym2Grp**, U_1 is full on both 1-cells and 2-cells. However, in contrast to the forgetful functor from **Ab** to **Grp**, U_2 (and therefore $U_1 \circ U_2$) is not full on 1-cells.

In [8], the authors introduce the concept of a categorical crossed module, which allows them to define normal sub-2-groups and quotients in **2Grp**. The definition of categorical crossed modules is modelled closely on the definition of crossed modules, with 2-groups taking the place of groups, and the adjoint action of Example 3.6 replaced by its 2-dimensional analogue, which was described in Example 8.7. As in Example 3.9, normal sub-2-groups correspond to categorical crossed modules which are appropriately injective. The relative complexity of the 2-dimensonal case arises in the fact that morphisms of categorical crossed modules are equipped with structure isomorphisms which must satisfy certain coherence conditions. Because of this, the universal property of a quotient as defined in [8] holds only with respect to certain morphisms in **2Grp**. In the case of symmetric 2-groups, these morphisms are exactly the braided monoidal functors of Definition 4.13.

The following examples illustrate the utility of cokernels in **Sym2Grp**.

As in the 1-dimensonal case, the cokernel of a morphism measures its surjectivity. In [38], Vitale gives a dual Statement to Lemma 8.6 for cokernels in **Sym2Grp**. Given a morphism $F : \mathcal{G} \longrightarrow \mathcal{H}$ of symmetric 2-groups, we have the following:

1. F is essentially surjective if and only if $h_0\left(\mathcal{C}oker\left(F\right)\right) = 1$.
2. F is full if and only if $h_1\left(\mathcal{C}oker\left(F\right)\right) = 1$.
3. F is full and essentially surjective if and only if $\mathcal{C}oker\left(F\right) \simeq *$.

Following [21], we may also define suspensions in **Sym2Grp** via cokernels. For any $\mathcal{G} \in$ **Sym2Grp** the suspension of $\mathcal{G}$ is as follows:

$$\Sigma\left(\mathcal{G}\right) = \mathcal{C}oker\left(0\right) \simeq \left(h_0\left(\mathcal{G}\right)\right)[1]$$

If $\mathcal{G}$ is strict then this agrees with the notion of suspension obtained by applying Definition 6.66 to the model structure mentioned in Remark 7.15. In fact, using the cylinder objects defined in [31], we may show that for any $\mathcal{G}$ in **Str2Grp** we have

$$\Sigma\left(\mathcal{G}\right) \simeq \left(h_0\left(\mathcal{G}\right)^{ab}\right)[1]$$

where $h_0\left(\mathcal{G}\right)^{ab}$ denotes the abelianisation.

9 Free 2-Groups

Theorem 9.1. *Consider the forgetful functor* U : **Str2Grp** $\longrightarrow$ **Grpd**. *This functor has a left adjoint* $\mathcal{F}$: **Grpd** $\longrightarrow$ **Str2Grp**.

Proof. Let $G = \left(G_1 \rightrightarrows G_0\right)$ be a groupoid. We define $\mathcal{F}\left(G\right)$ as follows.

Consider the 1-truncated simplicial group $F\left(G\right) = \left(F\left(G_1\right) \rightrightarrows F\left(G_0\right)\right)$ where $F\left(G_i\right)$ is the free group on the set G_i. We will denote the generator inclusions by $j_i : G_i \longrightarrow F\left(G_i\right)$. Given an element $g \in G_i$ we denote the corresponding element of $F\left(G_i\right)$ by $j\left(g\right) = [g]$. Thus, an arbitrary element of $F\left(G_i\right)$ is given by a word of the form $[g_1]^{\varepsilon_1}[g_2]^{\varepsilon_2}\ldots[g_n]^{\varepsilon_n}$ for $g_1, g_2, ...g_n \in G_i$ and $\varepsilon_k = \pm 1$. Note, in particular, that $[g]^{-1}$ denotes the multiplicative inverse of $j_1\left(g\right)$ in $F\left(G_i\right)$, so in general $[g]^{-1} \neq \left[g^{-1}\right]$, where g^{-1} is the inverse morphism to g.

The structure maps for $F\left(G\right)$ are given by the homomorphisms

$$\begin{aligned} F\left(s\right), F\left(t\right) : F\left(G_1\right) &\longrightarrow F\left(G_0\right) \\ F\left(i\right) : F\left(G_0\right) &\longrightarrow F\left(G_1\right) \end{aligned}$$

where s, t and i are the structure morphisms of G. That is, $F\left(s\right)$ is the unique group homomorphism such that $F\left(s\right) \circ j_1 = j_0 \circ s$, $F\left(t\right)$ is unique such that $F\left(t\right) \circ j_1 = j_0 \circ t$, and $F\left(i\right)$ is unique such that $F\left(i\right) \circ j_0 = j_1 \circ i$.

Now, consider the pullback $G_2 := G_1 \times_{G_0} G_1$ of Definition 2.3, whose elements are composable morphisms in G. We have the following functions:

$$\begin{array}{rcl} q_1 : G_2 & \longrightarrow & G_1 \\ (g,h) & \longmapsto & g \\ \circ : G_2 & \longrightarrow & G_1 \\ (g,h) & \longmapsto & h \circ g \\ q_2 : G_2 & \longrightarrow & G_1 \\ (g,h) & \longmapsto & h \end{array}$$

Consider the homomorphisms $F(q_1)$, $F(q_2)$ and $F(\circ)$. Let

$$K = Ker\left(F(q_2)\right) \cap Ker\left(F(\circ)\right) \subseteq F(G_2)$$

and consider the subgroup $H = F(q_1)(K)$ in $F(G_1)$:

$$\begin{array}{rcl} H & = & F(q_1)(K) \\ & = & \left\{ [g_1]^{\varepsilon_1} \ldots [g_n]^{\varepsilon_n} \in F(G_1) \mid \begin{array}{c} [h_1 \circ g_1]^{\varepsilon_1} \ldots [h_n \circ g_n]^{\varepsilon_n} = [h_1]^{\varepsilon_1} \ldots [h_n]^{\varepsilon_n} = 1 \\ [(g_1,h_1)]^{\varepsilon_1} \ldots [(g_n,h_n)]^{\varepsilon_n} \in F(G_2) \end{array} \right\} \end{array}$$

Now, consider the normal subgroup $N \subseteq F(G_1)$ generated by the elements of H and elements of the form $xyx^{-1}y^{-1}$ for $x \in Ker\left(F(s)\right)$ and $y \in Ker\left(F(t)\right)$. Denote the projection onto the quotient as follows:

$$p : F(G_1) \longrightarrow F(G_1)/N$$

For any element $[g_1]^{\varepsilon_1} \ldots [g_n]^{\varepsilon_n} \in H$ corresponding to $[(g_1,h_1)]^{\varepsilon_1} \ldots [(g_n,h_n)]^{\varepsilon_n}$ in $F(G_2)$ we have the following two equalities:

$$\begin{array}{rcl} F(t)\left([g_1]^{\varepsilon_1} \ldots [g_n]^{\varepsilon_n}\right) & = & t(g_1)^{\varepsilon_1} \ldots t(g_n)^{\varepsilon_n} \\ & = & s(h_1)^{\varepsilon_1} \ldots s(h_n)^{\varepsilon_n} \\ & = & F(s)\left([h_1]^{\varepsilon_1} \ldots [h_n]^{\varepsilon_n}\right) \\ & = & F(s)(1) \\ & = & 1 \\ F(s)\left([g_1]^{\varepsilon_1} \ldots [g_n]^{\varepsilon_n}\right) & = & s(g_1)^{\varepsilon_1} \ldots s(g_n)^{\varepsilon_n} \\ & = & s(h_1 \circ g_1)^{\varepsilon_1} \ldots s(h_n \circ g_n)^{\varepsilon_n} \\ & = & F(s)\left([h_1 \circ g_1]^{\varepsilon_1} \ldots [h_n \circ g_n]^{\varepsilon_n}\right) \\ & = & F(s)(1) \\ & = & 1 \end{array}$$

Furthermore, given $x \in Ker\left(F(s)\right)$ and $y \in Ker\left(F(t)\right)$, we have the following equalities:

$$\begin{aligned}
F(t)\left(xyx^{-1}y^{-1}\right) &= (F(t)(x))(F(t)(y))(F(t)(x))^{-1}(F(t)(y))^{-1} \\
&= (F(t)(x))(F(t)(x))^{-1} \\
&= 1 \\
F(s)\left(xyx^{-1}y^{-1}\right) &= (F(s)(x))(F(s)(y))(F(s)(x))^{-1}(F(s)(y))^{-1} \\
&= (F(s)(y))(F(s)(y))^{-1} \\
&= 1
\end{aligned}$$

Thus, the morphisms $F(s)$ and $F(t)$ induce the group homomorphisms below:

$$F\tilde{(s)}, F\tilde{(t)} : F(G_1)/N \longrightarrow F(G_0)$$

These are the unique morphisms such that $F\tilde{(s)} \circ p = F(s)$ and $F\tilde{(t)} \circ p = F(t)$. We also have $p \circ F(i) : F(G_0) \longrightarrow F(G_1)/N$. These form the structure morphisms of a strict 2-group $\mathcal{F}(G) = (F(G_1)/N \rightrightarrows F(G_0))$.

Now, for any groupoid G, we have a canonical morphism $\eta_G : G \longrightarrow \mathcal{F}(G)$ into the underlying groupoid of $\mathcal{F}(G)$, given by $\eta_G = (p \circ j_1, j_0)$. To see that this defines a functor, we require that η_G preserve source, target and identities:

$$\begin{aligned}
j_0 \circ s &= F\tilde{(s)} \circ p \circ j_1 \\
j_0 \circ t &= F\tilde{(t)} \circ p \circ j_1 \\
p \circ j_1 \circ i &= p \circ F(i) \circ j_1
\end{aligned}$$

These equalities hold by definition. Furthermore, we require that η_G preserves composition. To see this, note that by Theorem 5.26, composition in $\mathcal{F}(G)$ is defined as follows. Given $x, y \in F(G_1)/N$ with $F\tilde{(s)}(x) = F\tilde{(t)}(y)$, their composite, which we will denote by $x \bullet y$ to avoid confusion, is given by the following:

$$x \bullet y := x\left(\left(p \circ F(i) \circ F\tilde{(s)}\right)(x)\right)^{-1} y$$

Now, let $g, h \in G_1$ with $s(g) = t(h)$ and consider their composite $g \circ h \in G_1$. We wish to show the following:

$$\begin{aligned}
p([g \circ h]) &= p([g]) \bullet p([h]) \\
&= p([g])\left(\left(p \circ F(i) \circ F\tilde{(s)} \circ p\right)([g])\right)^{-1} p([h]) \\
&= p\left([g]\,[(i \circ s)(g)]^{-1}\,[h]\right)
\end{aligned}$$

That is, we need to show that the element

$$w = [g]\,[(i \circ s)(g)]^{-1}\,[h]\,[g \circ h]^{-1}$$

in $F(G_1)$ is in the subgroup N. Now, consider the element below:

$$v = [(i \circ t)(g)]\,[g]^{-1}\,[g]\,[(i \circ t)(g)]^{-1}$$

Clearly $v = 1$ in $F(G_1)$. Moreover, we have the following:

$$\begin{aligned}&[(i \circ t)(g) \circ g]\,[g \circ (i \circ s)(g)]^{-1}\,[g \circ h]\,[(i \circ t)(g) \circ g \circ h]^{-1}\\ &= [g]\,[g]^{-1}\,[g \circ h]\,[g \circ h]^{-1}\\ &= 1\end{aligned}$$

Thus, by the description of H above, we have $w \in H \subseteq N$. So we have $p(w) = 1$ in $F(G_1)/N$ as desired. Therefore, η_G defines a functor.

Note that a similar calculation to the one above may be used to show that the group homomorphism $p \circ F(\circ) : F(G_2) \longrightarrow F(G_1)/N$ agrees with composition in $\mathcal{F}(G)$. That is, given $[(g_1, h_1)]^{\varepsilon_1} \ldots [(g_n, h_n)]^{\varepsilon_n} \in F(G_2)$, we have the following:

$$\begin{aligned}&(p \circ F(\circ))\left([(g_1, h_1)]^{\varepsilon_1} \ldots [(g_n, h_n)]^{\varepsilon_n}\right)\\ &= p\left([h_1 \circ g_1]^{\varepsilon_1} \ldots [h_n \circ g_n]^{\varepsilon_n}\right)\\ &= p\left([h_1]^{\varepsilon_1} \ldots [h_n]^{\varepsilon_n}\,[(i \circ s)(h_n)]^{-\varepsilon_n} \ldots [(i \circ s)(h_1)]^{-\varepsilon_1}\,[g_1]^{\varepsilon_1} \ldots [g_n]^{\varepsilon_n}\right)\\ &= [h_1]^{\varepsilon_1} \ldots [h_n]^{\varepsilon_n} \bullet [g_1]^{\varepsilon_1} \ldots [g_n]^{\varepsilon_n}\end{aligned}$$

To see that the second equality holds, we must show that the element of $F(G_1)$ below is in H:

$$\begin{aligned}&[h_1]^{\varepsilon_1} \ldots [h_n]^{\varepsilon_n}\,[(i \circ s)(h_n)]^{-\varepsilon_n} \ldots [(i \circ s)(h_1)]^{-\varepsilon_1}\,[g_1]^{\varepsilon_1} \ldots\\ &\qquad \ldots [g_n]^{\varepsilon_n}\,[h_n \circ g_n]^{-\varepsilon_n} \ldots [h_1 \circ g_1]^{-\varepsilon_1}\end{aligned}$$

As above, this can be seen by considering the element below:

$$\begin{aligned}&[(i \circ t)(h_1)]^{\varepsilon_1} \ldots [(i \circ t)(h_n)]^{\varepsilon_n}\,[h_n]^{-\varepsilon_n} \ldots [h_1]^{-\varepsilon_1}\,[h_1]^{\varepsilon_1} \ldots\\ &\qquad \ldots [h_n]^{\varepsilon_n}\,[(i \circ t)(h_n)]^{-\varepsilon_n} \ldots [(i \circ t)(h_1)]^{-\varepsilon_1}\end{aligned}$$

Now, to see that $\mathcal{F}$ is left adjoint to U, let $G = (G_1 \rightrightarrows G_0)$ be a groupoid and let $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ be a strict 2-group, and suppose we have a functor $L : G \longrightarrow \mathcal{G}$ from G to the underlying groupoid of $\mathcal{G}$. We wish to show that there is a unique strict monoidal functor $L' : \mathcal{F}(G) \longrightarrow \mathcal{G}$ such that $L' \circ \eta_G = L$, so that η_G is a universal arrow from G to the functor U. Consider the morphism of 1-truncated simplicial groups

$$\tilde{L} = \left(\tilde{L_1}, L_0^{'}\right) : F(G) \longrightarrow \mathcal{G}$$

where $\tilde{L_1}$ is the unique group homomorphism such that $\tilde{L_1} \circ j_1 = L_1$, and $L_0^{'}$ is unique such that $L_0^{'} \circ j_0 = L_0$. Since $\tilde{L}$ is a morphism of simplicial groups, and $\mathcal{G}$ is a strict 2-group, for any $x \in Ker(F(s))$ and $y \in Ker(F(t))$ we have the following:

$$\tilde{L_1}\left(xyx^{-1}y^{-1}\right) \in [Ker(s), Ker(t)] = 1$$

This follows by the description of strict 2-groups established in Theorem 5.26. This gives us the inclusion below:

$$[Ker\left(F\left(s\right)\right), Ker\left(F\left(t\right)\right)] \subseteq Ker\left(\tilde{L_1}\right)$$

Furthermore, since L is a functor, we know that L preserves compositon. If we denote composition in $\mathcal{G}$ by $\star$ then, using the description of composition in $\mathcal{G}$ from Theorem 5.26, for composable morphisms $g, h \in G_1$ we have the following:

$$\begin{aligned} L_1\left(h \circ g\right) &= L_1\left(h\right) \star L_1\left(g\right) \\ &= L_1\left(h\right)\left(\left(i \circ s \circ L_1\right)\left(h\right)\right)^{-1} L_1\left(g\right) \\ &= L_1\left(h\right)\left(\left(i \circ L_0 \circ s\right)\left(h\right)\right)^{-1} L_1\left(g\right) \\ &= L_1\left(h\right)\left(\left(L_1 \circ i \circ s\right)\left(h\right)\right)^{-1} L_1\left(g\right) \\ &= \tilde{L_1}\left([h]\left[\left(i \circ s\right)\left(h\right)\right]^{-1}[g]\right) \end{aligned}$$

Now, let $[(g_1, h_1)]^{\varepsilon_1} \dots [(g_n, h_n)]^{\varepsilon_n} \in F\left(G_2\right)$. We have the following expression:

$$\begin{aligned} &\tilde{L_1}\left([h_1 \circ g_1]^{\varepsilon_1} \dots [h_n \circ g_n]^{\varepsilon_n}\right) \\ &= \tilde{L_1}\left([h_1]^{\varepsilon_1}\left[\left(i \circ s\right)\left(h_1\right)\right]^{-\varepsilon_1}[g_1]^{\varepsilon_1} \dots [h_n]^{\varepsilon_n}\left[\left(i \circ s\right)\left(h_n\right)\right]^{-\varepsilon_n}[g_n]^{\varepsilon_n}\right) \end{aligned}$$

We may expand the right hand side above to obtain the expression below:

$$\begin{aligned} &L_1\left(h_1\right)^{\varepsilon_1}\tilde{L_1}\left(\left[\left(i \circ s\right)\left(h_1\right)\right]^{-\varepsilon_1}[g_1]^{\varepsilon_1}\right)\tilde{L_1}\left([h_2]^{\varepsilon_2}\left[\left(i \circ s\right)\left(h_2\right)\right]^{-\varepsilon_2}\right) \\ &\quad L_1\left(g_2\right)^{\varepsilon_2}\tilde{L_1}\left([h_3]^{\varepsilon_3}\left[\left(i \circ s\right)\left(h_3\right)\right]^{-\varepsilon_3}\right)\dots \end{aligned}$$

Now, since $s\left(h_k\right) = t\left(g_k\right)$ for each k, we have the following:

$$\begin{aligned} F\left(t\right)\left(\left[\left(i \circ s\right)\left(h_k\right)\right]^{-\varepsilon_k}[g_k]^{\varepsilon_k}\right) &= F\left(t\right)\left(\left[\left(i \circ t\right)\left(g_k\right)\right]^{-\varepsilon_k}[g_k]^{\varepsilon_k}\right) \\ &= t\left(g_k\right)^{-\varepsilon_k} t\left(g_k\right)^{-\varepsilon_k} \\ &= 1 \end{aligned}$$

$$\begin{aligned} F\left(s\right)\left([h_k]^{\varepsilon_k}\left[\left(i \circ s\right)\left(h_k\right)\right]^{-\varepsilon_k}\right) &= s\left(h_k\right)^{\varepsilon_k} s\left(h_k\right)^{-\varepsilon_k} \\ &= 1 \end{aligned}$$

This implies the inclusions below:

$$\left[\left(i \circ s\right)\left(h_k\right)\right]^{-\varepsilon_k}[g_k]^{\varepsilon_k} \in Ker\left(F\left(t\right)\right)$$

$$[h_k]^{\varepsilon_k}\left[\left(i \circ s\right)\left(h_k\right)\right]^{-\varepsilon_k} \in Ker\left(F\left(s\right)\right)$$

Thus, $\tilde{L_1}\left([(i\circ s)(h_k)]^{-\varepsilon_k}[g_k]^{\varepsilon_k}\right)$ and $\tilde{L_1}\left([h_k]^{\varepsilon_k}[(i\circ s)(h_k)]^{-\varepsilon_k}\right)$ commute. So we may rearrange the expression above to obtain the following:

$$\begin{aligned}
&L_1(h_1)^{\varepsilon_1}\tilde{L_1}\left([h_2]^{\varepsilon_2}[(i\circ s)(h_2)]^{-\varepsilon_2}\right)\tilde{L_1}\left([(i\circ s)(h_1)]^{-\varepsilon_1}[g_1]^{\varepsilon_1}\right)\\
&\quad L_1(g_2)^{\varepsilon_2}\tilde{L_1}\left([h_3]^{\varepsilon_3}[(i\circ s)(h_3)]^{-\varepsilon_3}\right)\dots\\
&=L_1(h_1)^{\varepsilon_1}L_1(h_2)^{\varepsilon_2}\tilde{L_1}\left([(i\circ s)(h_2)]^{-\varepsilon_2}[(i\circ s)(h_1)]^{-\varepsilon_1}[g_1]^{\varepsilon_1}[g_2]^{\varepsilon_2}\right)\\
&\quad \tilde{L_1}\left([h_3]^{\varepsilon_3}[(i\circ s)(h_3)]^{-\varepsilon_3}\right)\dots
\end{aligned}$$

By a similar argument to above, we have the following inclusion:

$$[(i\circ s)(h_2)]^{-\varepsilon_2}[(i\circ s)(h_1)]^{-\varepsilon_1}[g_1]^{\varepsilon_1}[g_2]^{\varepsilon_2}\in Ker(F(t))$$

Thus $\tilde{L_1}\left([(i\circ s)(h_2)]^{-\varepsilon_2}[(i\circ s)(h_1)]^{-\varepsilon_1}[g_1]^{\varepsilon_1}[g_2]^{\varepsilon_2}\right)$ and $\tilde{L_1}\left([h_3]^{\varepsilon_3}[(i\circ s)(h_3)]^{-\varepsilon_3}\right)$ commute, so the expression above becomes the following:

$$\begin{aligned}
&L_1(h_1)^{\varepsilon_1}L_1(h_2)^{\varepsilon_2}L_1(h_3)^{\varepsilon_3}\tilde{L_1}\left([(i\circ s)(h_3)]^{-\varepsilon_3}[(i\circ s)(h_2)]^{-\varepsilon_2}[(i\circ s)(h_1)]^{-\varepsilon_1}\right)\\
&\quad L_1(g_1)^{\varepsilon_1}L_1(g_2)^{\varepsilon_2}\dots
\end{aligned}$$

We may continue to argue in this fashion to obtain the equality below:

$$\begin{aligned}
&\tilde{L_1}([h_1\circ g_1]^{\varepsilon_1}\dots[h_n\circ g_n]^{\varepsilon_n})\\
&=L_1(h_1)^{\varepsilon_1}\dots L_1(h_n)^{\varepsilon_n}\tilde{L_1}\left([(i\circ s)(h_n)]^{-\varepsilon_n}\dots[(i\circ s)(h_1)]^{-\varepsilon_1}\right)L_1(g_1)^{\varepsilon_1}\dots L_1(g_n)^{\varepsilon_n}\\
&=\tilde{L_1}([h_1]^{\varepsilon_1}\dots[h_n]^{\varepsilon_n})(i\circ s)\left(\tilde{L_1}([h_1]^{\varepsilon_1}\dots[h_n]^{\varepsilon_n})\right)^{-1}\tilde{L_1}([g_1]^{\varepsilon_1}\dots[g_n]^{\varepsilon_n})\\
&=\tilde{L_1}([h_1]^{\varepsilon_1}\dots[h_n]^{\varepsilon_n})\star\tilde{L_1}([g_1]^{\varepsilon_1}\dots[g_n]^{\varepsilon_n})
\end{aligned}$$

In particular, if $[g_1]^{\varepsilon_1}\dots[g_n]^{\varepsilon_n}\in H$ corresponds to $[(g_1,h_1)]^{\varepsilon_1}\dots[(g_n,h_n)]^{\varepsilon_n}$ in $F(G_2)$ then we have the following equality:

$$[h_1\circ g_1]^{\varepsilon_1}\dots[h_n\circ g_n]^{\varepsilon_n}=[h_1]^{\varepsilon_1}\dots[h_n]^{\varepsilon_n}=1$$

Thus, the equality above becomes the following:

$$\begin{aligned}
1&=\tilde{L_1}([h_1]^{\varepsilon_1}\dots[h_n]^{\varepsilon_n})(i\circ s)\left(\tilde{L_1}([h_1]^{\varepsilon_1}\dots[h_n]^{\varepsilon_n})\right)^{-1}\tilde{L_1}([g_1]^{\varepsilon_1}\dots[g_n]^{\varepsilon_n})\\
&=\tilde{L_1}([g_1]^{\varepsilon_1}\dots[g_n]^{\varepsilon_n})
\end{aligned}$$

Thus, $H\subseteq Ker\left(\tilde{L_1}\right)$. Combining this and the previous observation gives us the inclusion below:

$$N\subseteq Ker\left(\tilde{L_1}\right)$$

Therefore, $\tilde{L_1}$ induces a unique homomorphism $L_1^{'}:F(G_1)/N\longrightarrow\mathcal{G}_1$ such that $L_1^{'}\circ p=\tilde{L_1}$. We make the following definition:

$$L'=\left(L_1^{'},L_0^{'}\right):\mathcal{F}(G)\longrightarrow\mathcal{G}$$

By construction, this is the unique morphism such that $L^{'} \circ \eta_G = L$.

Note that the information above uniquely determines $\mathcal{F}$ on morphisms. Explicitly, suppose we have a morphism in **Grpd** as below:

$$(M_1, M_0) : (G_1 \rightrightarrows G_0) \longrightarrow (H_1 \rightrightarrows H_0)$$

Then $\mathcal{F}(M) = \left(M_1^{'}, M_0^{'}\right)$, where $M_0^{'}$ is unique such that $M_0^{'} \circ j_0 = j_0 \circ M_0$, and $M_1^{'}$ is unique such that $M_1^{'} \circ p = p \circ \tilde{M}_1$, where $\tilde{M}_1$ is the unique group homomorphism such that $\tilde{M}_1 \circ j_1 = j_1 \circ M_1$. $\square$

Remark 9.2. In [5], Baues and Muro give the following construction of the free crossed module on a pointed groupoid $G = (G_1 \rightrightarrows G_0)$. Given G, we may construct a pointed simplicial set NG called the nerve of G. At each dimension $n \in \mathbb{Z}_{\geq 0}$, the set NG_n is the set of strings of n composable morphisms in G. In particular, in the notation of Theorem 9.1, we have $NG_i = G_i$ for $i = 0, 1, 2$.

Now, given this simplicial set we may construct a simplicial group FNG via the Milnor construction. See [16] for a definition of the Milnor construction. Briefly, it is defined by taking the free group at each dimension and quotienting out the basepoint. Given any simplicial group, we may construct its Moore complex, (M_*, ∂), which is a chain complex of nonabelian groups. See [4] for a definition of the Moore complex of a simplicial group. From the Moore complex, again following [4] , we may then extract the crossed module below:

$$\partial_1 : M_1/\partial_2 (M_2) \longrightarrow M_0$$

Using the notation of Theorem 9.1, this amounts to the following construction. We know that $H \subseteq Ker(F(t))$ and $H \subseteq Ker(F(s))$. Thus

$$F(t)\mid_{Ker(F(s))} : Ker(F(s)) \longrightarrow F(G_0)$$

induces a group homomorphism $\delta : Ker(F(s))/H \longrightarrow F(G_0)$. Consider the crossed module below:

$$\mathcal{L}(G) = (\delta : Ker(F(s))/H \longrightarrow F(G_0))$$

This is the free crossed module on G. For any $x \in F(G_0)$ and $g \in Ker(F(s))/H$, the action of $F(G_0)$ on $F(G_1)/H$ is defined as follows:

$${}^{x}g = (F(i))(x)\, g\, (F(i))(x)^{-1}$$

Note that under the equivalence of Theorem 5.20 we have $\Psi(\mathcal{F}(G)) = \mathcal{L}(G)$.

Theorem 9.3. *The adjunction*

$$\mathcal{F} : \mathbf{Grpd} \Longleftrightarrow \mathbf{Str2Grp} : U$$

is a Quillen adjunction. Thus we have an induced adjunction between homotopy categories:

$$L\mathcal{F} : Ho(\mathbf{Grpd}) \Longleftrightarrow Ho(\mathbf{Str2Grp}) : RU$$

Moreover, these adjoint functors induce 2*-functors* $U : \mathbf{Str2Grp} \longrightarrow \mathbf{Grpd}$ *and* $\mathcal{F} : \mathbf{Grpd} \longrightarrow \mathbf{Str2Grp}$ *which are* 2*-adjoint. That is,* $\mathcal{F}$ *and* U *are adjoint as* **Grpd***-enriched functors.*

Proof. The functor $U : \mathbf{Str2Grp} \longrightarrow \mathbf{Grpd}$ preserves both fibrations and weak equivalences. This is immediate by the description of the model structure on **Str2Grp** in Definition 7.1. Thus, U satisfies Condition 3 of Theorem 6.58, so the adjunction

$$\mathcal{F} : \mathbf{Grpd} \Longleftrightarrow \mathbf{Str2Grp} : U$$

is a Quillen adjunction. Note that Condition 3 of Theorem 6.58 is equivalent to both Condition 1 and Condition 2. Thus, we may conclude that $\mathcal{F}$ preserves cofibrations and acyclic cofibrations. By Lemma 6.57, this implies that $\mathcal{F}$ preserves weak equivalences between cofibrant objects. Since every object in **Grpd** is cofibrant and the weak equivalences in **Grpd** and **Str2Grp** are the equivalences of categories, this implies that $\mathcal{F}$ preserves equivalences.

Now, let $K, L : G \longrightarrow H$ be morphisms in **Grpd** and let $\theta : K \Longrightarrow L$ be a natural isomorphism. By Example 6.24, θ uniquely determines a left homotopy $J : G \times \mathbf{I} \longrightarrow H$ via the cylinder object $G \times \mathbf{I}$. The functor $\mathcal{F}$ takes J to a morphism in **Str2Grp**:

$$\mathcal{F}(J) : \mathcal{F}(G \times \mathbf{I}) \longrightarrow \mathcal{F}(H)$$

By the argument in the proof of Theorem 6.58, since $\mathcal{F}$ preserves colimits, cofibrations and equivalences, $\mathcal{F}(G \times \mathbf{I})$ is a good cylinder object for $\mathcal{F}(G)$ and $\mathcal{F}(J)$ is a good left homotopy from $\mathcal{F}(K)$ to $\mathcal{F}(L)$. Now, every object of **Str2Grp** is fibrant, and since $\mathcal{F}$ preserves cofibrant objects, $\mathcal{F}(G)$ is cofibrant. Thus, by Lemma 6.31, the good left homotopy $\mathcal{F}(J)$ determines a good right homotopy $Z : \mathcal{F}(G) \longrightarrow \mathcal{F}(H)^I$ from $\mathcal{F}(K)$ to $\mathcal{F}(L)$ via the good path object $\mathcal{F}(H)^I$ of Definition 7.3. See [31] for a proof that this right homotopy Z is uniquely determined by $\mathcal{F}(J)$. Now, by Lemma 7.5, the right homotopy Z from $\mathcal{F}(K)$ to $\mathcal{F}(L)$ via $\mathcal{F}(H)^I$ uniquely determines a monoidal natural isomorphism $\alpha : \mathcal{F}(K) \Longrightarrow \mathcal{F}(L)$. To define $\mathcal{F}$ on 2-morphisms we take $\mathcal{F}(\theta) = \alpha$.

Now, let $G \in \mathbf{Grpd}$ and let $\mathcal{G} \in \mathbf{Str2Grp}$ and consider morphisms $K, L : G \longrightarrow U(\mathcal{G})$. The proof of Theorem 6.58 describes a bijection between good left homotopies from K to L and good left homotopies from $\varphi^{-1}(K)$ to $\varphi^{-1}(L)$, where φ is the bijection induced by the adjunction:

$$\varphi : Hom_{\mathbf{Str2Grp}}(\mathcal{F}(G), \mathcal{G}) \longrightarrow Hom_{\mathbf{Grpd}}(G, U(\mathcal{G}))$$

Using the definition of $\mathcal{F}$ on natural isomorphisms, this bijection determines the following isomorphism of categories:

$$Hom_{\mathbf{Str2Grp}}(\mathcal{F}(G), \mathcal{G}) \cong Hom_{\mathbf{Grpd}}(G, U(\mathcal{G}))$$

Although we will not prove it, since the bijection of object sets is natural, we may show this isomorphism of categories is strictly 2-natural. See [5] for some justification of this. □

Corollary 9.4. *The 2-functor $\mathcal{F} : \mathbf{Grpd} \longrightarrow \mathbf{Str2Grp}$ of Theorem 9.1 composed with the inclusion $\mathbf{Str2Grp} \longrightarrow \mathbf{2Grp}$ gives a 2-functor from $\mathbf{Grpd}$ to $\mathbf{2Grp}$, which we will also denote by $\mathcal{F} : \mathbf{Grpd} \longrightarrow \mathbf{2Grp}$. This 2-functor defines the free 2-group on any groupoid in the following sense. Given any groupoid G and any 2-group $\mathcal{G}$, we have the following pseudonatural equivalence of categories:*

$$Hom_{\mathbf{2Grp}}\left(\mathcal{F}\left(G\right), \mathcal{G}\right) \simeq Hom_{\mathbf{Grpd}}\left(G, U\left(\mathcal{G}\right)\right)$$

Proof. As noted in the proof of Theorem 9.3, for any groupoid G, $\mathcal{F}\left(G\right)$ is cofibrant. Note that this is also apparent from the construction of $\mathcal{F}\left(G\right)$ in Theorem 9.1 and the description in Corollary 7.7 of cofibrant objects in $\mathbf{Str2Grp}$.

By Corollary 7.9, since $\mathcal{F}\left(G\right)$ is cofibrant, for any 2-group $\mathcal{G}$ we have the following equivalence of categories:

$$Hom_{\mathbf{2Grp}}\left(\mathcal{F}\left(G\right), \mathcal{G}\right) \simeq Hom_{\mathbf{Str2Grp}}\left(\mathcal{F}\left(G\right), \bar{\mathcal{G}}\right)$$

Here $\bar{\mathcal{G}}$ is the strict 2-category of Theorem 5.18. Theorem 9.3 gives the following isomorphism of categories:

$$Hom_{\mathbf{Str2Grp}}\left(\mathcal{F}\left(G\right), \bar{\mathcal{G}}\right) \cong Hom_{\mathbf{Grpd}}\left(G, U\left(\bar{\mathcal{G}}\right)\right)$$

Now, since U preserves equivalences, $U\left(\mathcal{G}\right) \simeq U\left(\bar{\mathcal{G}}\right)$. Thus we have the equivalence of categories below:

$$Hom_{\mathbf{Grpd}}\left(G, U\left(\bar{\mathcal{G}}\right)\right) \simeq Hom_{\mathbf{Grpd}}\left(G, U\left(\mathcal{G}\right)\right)$$

Composing these equivalences gives the desired equivalence:

$$Hom_{\mathbf{2Grp}}\left(\mathcal{F}\left(G\right), \mathcal{G}\right) \simeq Hom_{\mathbf{Grpd}}\left(G, U\left(\mathcal{G}\right)\right)$$

We will not prove the pseudonaturality. For an idea of the proof, see the related theorems in [34, 35]. □

Remark 9.5. In [5], the authors define left adjoints to the forgetful functors $U_1 : \mathbf{SQuad} \longrightarrow \mathbf{RQuad}$ and $U_2 : \mathbf{RQuad} \longrightarrow \mathbf{Cross}$ of Remark 3.22. Using the model structures of 7.15, which recover the 2-categorical structures of $\mathbf{RQuad}$ and $\mathbf{SQuad}$, these functors may be extended to 2-functors.

Recall from Remark 5.24 that we have equivalences $\mathbf{RQuad} \simeq \mathbf{BraidStr2Grp}$ and $\mathbf{SQuad} \simeq \mathbf{SymStr2Grp}$. We may thus define the free braided strict 2-group on a strict 2-group and the free symmetric strict 2-group on a braided strict 2-group. Composing these 2-functors with the 2-functor $\mathcal{F} : \mathbf{Grpd} \longrightarrow \mathbf{Str2Grp}$ of Theorem 9.1 allows us to define the free braided or symmetric 2-group on a groupoid.

Further Directions

We have now developed constructions for free 2-groups and kernels of 2-group morphisms. These constructions are of particular interest in the 2-category

Sym2Grp, since here, as noted in Section 8.1, we may construct cokernels and quotients. Moreover, in [38] Vitale defines a notion of 2-exactness for a sequence of symmetric 2-groups. Using these constructions, we may define presentations for symmetric 2-groups. As in the 1-dimensional case, descriptions of symmetric 2-groups in terms of generator and relator groupoids should offer a convenient way of constructing symmetric 2-groups and a concrete way of working with them.

The primary difficulty in producing these constructions in **2Grp** rather than **Sym2Grp** lies in the conditions which must be placed on morphisms. As noted in Section 8.1, the quotient defined in [8] is characterised by a universal property, which holds only with respect to morphisms satisfying certain coherence conditions. This makes quotients in **2Grp** harder to work with than quotients in **Sym2Grp**, which are given simply by cokernels. This, however, is an unavoidable aspect of 2-group theory.

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