Topological recursion for Gaussian means and cohomological field theories

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We use the explicit relation between genus filtrated s-loop means of the Gaussian matrix model and terms of the genus expansion of the Kontsevich–Penner matrix model (KPMM), which is the generating function for volumes of discretized (open) moduli spaces $M_{g,s}^{\mathrm{disc}}$ (discrete volumes), to express Gaussian means in all genera as polynomials in special times weighted by ancestor invariants of an underlying cohomological field theory. We translate topological recursion of the Gaussian model into recurrent relations for coefficients of this expansion proving their integrality and positivity. As an application, we find the coefficients in the first subleading order for $\mathcal{M}_{g,1}$ for all g in three ways: by using the refined Harer–Zagier recursion, by exploiting the Givental-type decomposition of KPMM, and by an explicit diagram counting.

1 Introduction

Multi-trace means $\langle \prod_{i=1}^{s} \mathrm{tr}\, H^{k_i} \rangle^{\mathrm{conn}}$ of the Gaussian Unitary Ensemble (GUE) were under investigation for many years. First, Harer and Zagier obtained [28] the linear recursion formula on genus filtrated one-trace means, which allows obtaining answers for very high genera (unattainable by other tools). Although exact s-fold integral representation for s-trace means valid for all N were obtained by Brezín and Hikami [8] using the replica method ameliorated in [35], producing an effective genus expansion on the base of these formulas still remains an open problem. The interest to multi-trace means was revived after the appearance of *topological recursion* [11, 21] and quantum curves [25, 37, 16]. It was shown in our first paper [5] that Gaussian means are related via the so-called Kontsevich–Penner matrix model (KPMM) [13, 4] to discrete volumes of open moduli spaces and, simultaneously, to generating functions of ancestor invariants of a *cohomological field theory* [32].

We come to the KPMM using explicit combinatorial formulas. It is known since [14] and [33] that the KPMM is equivalent to the Hermitian matrix model with the potential whose times (coupling constants) are related to the external-matrix eigenvalues via the Miwa-type transformation and whose matrix size is the coefficient of the logarithmic term. The first result of [5] is that the KPMM is a primitive (antiderivative) for the resolvents of the Gaussian matrix model. The resolvents storing the multi-trace Gaussian means are naturally described as meromorphic (multi)differentials with zero residues over a rational Riemann surface, known as the *spectral curve*, hence their primitives are meromorphic functions on the spectral curve. These primitives are conjecturally related (this was proven in the Gaussian case [37], see also [40]) to the so called *quantum curve* which is a linear differential equation that is a non-commutative quantisation of the spectral curve. The spectral and quantum curves are related: the wave function emerging out of the spectral curve is a specialization

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of the free energy for the KPMM which satisfies the second order differential equation that is the quantum curve.

The geometric content of the KPMM is also rich: its free energy was related to structures of *discretized moduli spaces* in [9] and it was identified recently (see [39] and [36]) with the generating function for discrete volumes $N_{g,s}(P_1, \ldots, P_s)$—quasi-polynomials introduced in [38] that count integer points in the interiors $\mathcal{M}_{g,s}$ of moduli spaces of Riemann surfaces of genus g with $s > 0$ holes with the fixed perimeters $P_j \in \mathbb{Z}_+$, $j = 1, \ldots, s$ of holes in the Strebel uniformization. Moreover, it was shown in [10] that in the special times $T_{2n}^{\pm}$ that are discrete Laplace transforms of monomials P_I^{2k}, this model admits a decomposition into two Kontsevich models related by a Bogolyubov canonical transformation, which was the first example of the Givental-type decomposition formulas [24]. We use the approach of [10] for presenting the free-energy expansion terms $\mathcal{F}_{g,s}$ of the KPMM as finite sums over graphs whose nodes are terms of the expansion of the Kontsevich matrix model free energy, internal edges correspond to quadratic terms in the canonical transformation operator, external half edges (dilaton leaves) correspond to the constant shifts of the higher times, and external legs (ordinary leaves) carry the times $T_{2n}^{\pm}$. This graph representation provides another proof of quasi-polynomiality of $N_{g,s}(P_1, \ldots, P_s)$.

From [20] and [18] we know that the terms of topological recursion [19],[11],[12],[1] based on a certain spectral curve satisfying a compatibility condition (relating the $w_{0,1}$ and $w_{0,2}$ invariants) describe ancestor invariants of a cohomological field theory (CohFT), or equivalently a Frobenius manifold.

A fundamental family of Frobenius manifolds described by Dubrovin are Hurwitz spaces. For $\mu = (\mu_1, \ldots, \mu_n)$, the Hurwitz space $H_{g,\mu}$ consists of homotopy classes of genus g branched covers of the sphere with n labeled points over ∞ of ramification profile $(\mu_1, \ldots, \mu_n)$ and simple ramification over $\mathbb{P}^1 - \infty$. It has dimension $|\mu| + n + 2g - 2$ where $|\mu| = \mu_1 + \cdots + \mu_n$.

The 2-dimensional Hurwitz–Frobenius manifold $H_{0,(1,1)}$ consists of double branched covers of the sphere with two branch points and no ramification at infinity. Its free energy is

$$F_0(t_{0,1}, t_{0,0}) = \frac{1}{2} t_{0,0}^2 t_{0,1} + \frac{1}{2} t_{0,1}^2 \log t_{0,1} - \frac{3}{4} t_{0,1}^2 \tag{1.1}$$

with the Euler vector field $E = t_{0,0} \frac{\partial}{\partial t_{0,0}} + 2 t_{0,1} \frac{\partial}{\partial t_{0,1}}$. Note that expression (1.1) appears as a standard term (the perturbative part) in the expansion of any matrix model upon identification of $t_{0,1}$ with the normalized number of eigenvalues and $t_{0,0}$ with the first time; we have that

$$\log \int \prod_{i=1}^{t_{0,1}N} dx_i \prod_{i<j}(x_i - x_j)^2 \, e^{-N \sum_{i=1}^{t_{0,1}N} \left(\frac{1}{2} x_i^2 - t_{0,0} x_i \right)} = N^2 F_0(t_{0,1}, t_{0,0}) + \sum_{g=1}^{\infty} N^{2-2g} F_g(t_{0,1}),$$

where the leading term of the $1/N$-expansion of the free energy of the above Gaussian matrix model is exactly (1.1).

In [5], we related the discrete volumes to the Gaussian means $W_s^{(g)}(x_1, \ldots, x_s)$ and used the CohFT description further relating the discrete volumes to ancestor invariants of a CohFT. These ancestor invariants are evaluated already in terms of the *closed* moduli spaces $\mathcal{M}_{g,s}$ compactified by Deligne and Mumford.

The paper is organized as follows. In Sec. 2 we establish the equivalence between the Gaussian means (the correlation functions) and the terms of expansion of the KPMM free energy.

In Sec. 3, we describe the results of [9], [10], and [38] for open discrete moduli spaces, which we used in [5] to relate the above Gaussian means and the discrete volumes in a purely combinatorial way. The quantum curve can then be obtained as a specialization of the KPMM to the case of unit

size matrices. We describe the Givental-type decomposition formulas for the KPMM obtained in [10] representing them in terms of graph expansions for the free energy terms. This graph representation also implies the quasi-polynomiality of the discrete volumes and provides a link to a CohFT.

In Sec. 4, we identify the Gaussian means expansion terms with the ancestor invariants of a cohomological field theory using the results of [17] and [18]. The decomposition thus obtained has a canonical Givental form. The coefficients of this decomposition, or Laplace transforms of the quasi-polynomials $N_{g,s}(P_1, \ldots, P_s)$, are the special coefficients $\widehat{b}^{(g)}_{\vec{k}, \vec{\beta}}$, which in a sense represent in the "most economic" way the genus filtered s-loop means $W_s^{(g)}(x_1, \ldots, x_s)$ and are linear combinations of the CohFT ancestor invariants of neighbouring levels.

In Sec. 5, we develop the topological recursion for Gaussian means, present the general recursion relations for $\widehat{b}^{(g)}_{\vec{k}, \vec{\beta}}$, and prove that in the range of admissibility all these coefficients are positive integers.

In Sec. 6, we concentrate on the case of a one-loop mean. We find the first subleading coefficient $b^{(g)}_{g-2}$ in three ways: using the modified Harer–Zagier (HZ) recurrence relation, by the graph description of Givental-type decomposition in Sec.3, and by an explicit diagram counting.

2   The effective matrix model for the multi-loop Gaussian means

We consider a sum of connected *chord diagrams* based on s backbones, or loop insertions, carrying the variables u_i, $i = 1, \ldots, s$. We first provide an effective matrix model description for all genus-g contributions in terms of *shapes* —the connected fatgraphs of genus g with s faces and with vertices of arbitrary order greater or equal three; from the Euler characteristic formula, for a fixed g and s, only a finite number of such fat graphs exist, and we let $\Gamma_{g,s}$ denote this finite set. This set $\Gamma_{g,s}$ enumerates cells in the canonical Strebel–Penner ideal cell decomposition of moduli space $\mathcal{M}_{g,s}$. In accordance with [6], $\Gamma_{g,s}$ is in bijection with circular chord diagrams which are also "shapes" in the terminology of [6], that is chord diagrams which are seeds and which has no one-chords.

The correlation functions, or means, are given by the integrals

$$\left\langle \prod_{i=1}^s (\operatorname{tr} H^{k_i}) \right\rangle = \int_{H \in \mathcal{H}_N} \left(\prod_{i=1}^s \operatorname{tr} H^{k_i} \right) e^{-\frac{N}{2} \operatorname{tr} H^2} DH, \tag{2.1}$$

where $\mathcal{H}_N$ is the set of Hermitian $N \times N$ matrices. By Wick's theorem, any correlation function (2.1) can be presented as the sum over all possible (complete) pairings between matrix entries M_{ij}, where the pairings are two-point correlation functions $\langle H_{i,j} H_{k,l} \rangle = \frac{1}{N} \delta_{il} \delta_{jk}$. These pairing are customarily represented by *edges*: double lines of indices. The corresponding index lines run along faces of *fatgraphs* containing ordered set of s vertices of valencies k_i, $i = 1, \ldots, s$, and $\sum_{i=1}^s k_i/2$ edges. For each vertex, we fix a cyclic order of edges incident to this vertex. Furthermore for each vertex we also have a first incident edge given. We denote this set of fatgraphs $\widehat{\Gamma}(k_1, \ldots, k_s)$. Then the sum in (2.1) becomes $\sum_{\gamma \in \widehat{\Gamma}(k_1, \ldots, k_s)} N^{b(\gamma) - \sum_{i=1}^s k_i/2}$, where $b(\gamma)$ is the number of boundary components of γ.

Let $\widehat{\Gamma}(k_1, \ldots, k_s)^c$ be the subset of $\widehat{\Gamma}(k_1, \ldots, k_s)$ which consist of connected fatgraphs and let $\left\langle \prod_{i=1}^s (\operatorname{tr} H^{k_i}) \right\rangle^{\operatorname{conn}}$ be the part of the sum comprising only connected diagrams. The connected correlation functions then admit the $1/N$-expansion,

$$N^{s-2} \left\langle \prod_{i=1}^s (\operatorname{tr} H^{k_i}) \right\rangle^{\operatorname{conn}} = \sum_{g=0}^{\infty} N^{-2g} \left\langle \prod_{i=1}^s (\operatorname{tr} H^{k_i}) \right\rangle_g^{\operatorname{conn}},$$

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to segregate its part where

$$\left\langle \prod_{i=1}^{s} (\operatorname{tr} H^{k_i}) \right\rangle_{g}^{\mathrm{conn}} = |\widehat{\Gamma}_g(k_1, \ldots, k_s)^c|, \tag{2.2}$$

is the part corresponding to the set $\widehat{\Gamma}_g(k_1, \ldots, k_s)^c$ of connected fat graphs of genus g with ciliated vertices.

For nonciliated vertices, we then have the following formula

$$(-1)^s \left\langle \prod_{i=1}^{s} \operatorname{tr} \log(1 - u_i H) \right\rangle_{g}^{\mathrm{conn}} = \sum_{\{k_1, \ldots, k_s\} \in \mathbb{Z}_+^s} \prod_{i=1}^{s} \left(\frac{u_i^{k_i}}{k_i} \right) \left\langle \prod_{i=1}^{s} (\operatorname{tr} H^{k_i}) \right\rangle_{g}^{\mathrm{conn}}$$

$$= \sum_{\{k_1, \ldots, k_s\} \in \mathbb{Z}_+^s} \sum_{\gamma \in \Gamma_g(k_1, \ldots, k_s)^c} \frac{1}{|\operatorname{Aut}(\gamma)|} \prod_{i=1}^{s} u_i^{k_i}, \tag{2.3}$$

where $\Gamma_g(k_1, \ldots, k_s)^c$ is the set of connected fat graphs of genus g with s nonciliated ordered vertices of valencies $k_1, \ldots, k_s$ and $\operatorname{Aut}(\gamma)$ is the automorphism group of the fatgraph γ with ordered vertices. We pass from expressions with nonciliated vertices to those with ciliated vertices, or chord diagrams, by differentiation:

$$\left\langle \prod_{i=1}^{s} \operatorname{tr} \frac{1}{I - u_i H} \right\rangle_{g}^{\mathrm{conn}} \equiv \left\langle \prod_{i=1}^{s} \operatorname{tr} \left[\sum_{k_i=1}^{\infty} u_i^{k_i} H^{k_i} \right] \right\rangle_{g}^{\mathrm{conn}} = (-1)^s \left[\prod_{i=1}^{s} u_i \frac{\partial}{\partial u_i} \right] \left\langle \prod_{i=1}^{s} \operatorname{tr} \log(1 - u_i H) \right\rangle_{g}^{\mathrm{conn}}. \tag{2.4}$$

By combining formula (2.2) with (2.4), we find that

$$\left[\prod_{i=1}^{s} u_i \frac{\partial}{\partial u_i} \right] \left\langle \prod_{i=1}^{s} \operatorname{tr} \log(1 - u_i H) \right\rangle_{g}^{\mathrm{conn}} = \sum_{\gamma \in \widehat{\Gamma}_{g,s}^c} N^{2-2g} \prod_{i=1}^{s} u_i^{k_i}.$$

2.1 Summing up planar subgraphs—formulating the matrix model

We first perform a partial resummation over planar subgraphs in (2.3). A planar chord diagram on an interval is a *rainbow diagram* (see examples in Fig. 1). Rainbow diagrams with a given number of chords are enumerated by the Catalan numbers whose generating function is

$$f(u_i) := \frac{1 - \sqrt{1 - 4u_i^2}}{2u_i^2}, \tag{2.5}$$

so we effectively replace the original edge of a chord diagram by a thickened edge carrying the factor $f(u_i)$ thus stripping out all "pimps," or rainbow subgraphs.
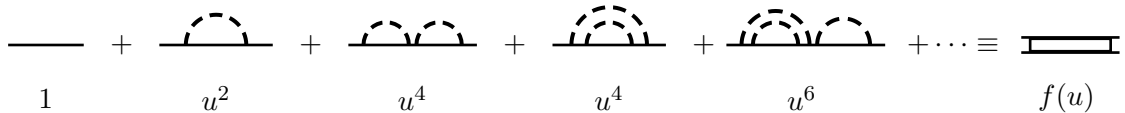

Figure 1: Summing up rainbow diagrams of chords (dashed lines) for a single backbone (a solid line). The result is the new (thickened) edge of the backbone.

We next proceed to summing up ladder-type diagrams, where a "rung" of the ladder joins two cycles that carry (either distinct or coinciding) indices i and j (see an example in Fig. 2). Each

ladder contains at least one rung, which is a chord carrying the factor $u_i u_j$. We obtain an effective fat graph with new *edges* and *vertices* by blowing up cycles of thickened backbone edges until they will be joined pairwise along rungs (each containing at least one rung); disjoint parts of these cycles will then constitute loops of lengths $2r_k \geq 6$ alternatively bounded by r_k rungs (the chords) and r_k thickened edges of circular backbones; these loops then become *vertices* of the respective orders $r_k \geq 3$ of the *new fat graph.*
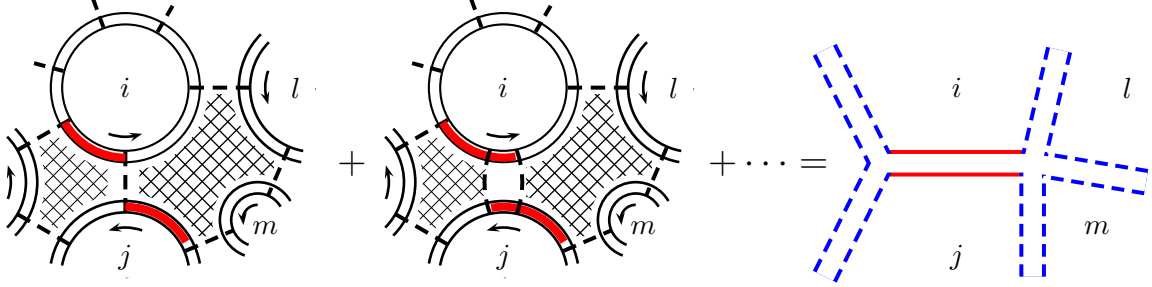
Figure 2: Performing a resummation over ladder diagrams. The thickened edges associated with the selected ladder, which becomes an edge of a new fat graph, are painted dark. The crosshatched domains will become the respective three- and four-valent vertices of the new fat graph representing a *shape*.

Introducing $e^{\lambda_i} = \frac{1+\sqrt{1-4u_i^2}}{2u_i}$, or $u_i = \frac{1}{e^{\lambda_i}+e^{-\lambda_i}}$, for each ladder subgraph, we have a sum

$$\sum_{k=1}^{\infty} (u_i u_j f(u_i) f(u_j))^k = \frac{1}{(u_i f(u_i) u_j f(u_j))^{-1} - 1} := \frac{1}{e^{\lambda_i+\lambda_j} - 1}, \quad (2.6)$$

We therefore attain the effective description.

Theorem 1. [5] *The genus-g term of the (nonciliated) s-backbone connected diagrams is given by the following (finite!) sum over fatgraph shapes $\Gamma_{g,s}$ of genus g with s faces whose vertices have valences at least three:*

$$\left\langle \prod_{i=1}^{s} \mathrm{tr}\, \log(e^{\lambda_i} + e^{-\lambda_i} - H) \right\rangle_g^{\mathrm{conn}} = \sum_{\substack{\text{all fatgraphs} \\ \gamma \,\in\, \Gamma_{g,s}}} \frac{1}{|\mathrm{Aut}(\gamma)|} \prod_{\text{edges}} \frac{1}{e^{\lambda_e^{(+)}+\lambda_e^{(-)}} - 1} := F_s^{(g)}(\lambda_1, \ldots, \lambda_s), \quad (2.7)$$

where $\pm$ denotes the two sides (faces) of the edge e. The quantity $F_s^{(g)}(\lambda_1, \ldots, \lambda_s)$ in the right-hand side is the term in the diagrammatic expansion of the free energy of the Kontsevich–Penner matrix model [13] described by the normalised integral over Hermitian $N \times N$-matrices X:

$$\mathcal{Z}[\Lambda] := e^{\sum_{g,s} N^{2-2g}(\alpha/2)^{2-2g-s} F_s^{(g)}(\lambda)} = \frac{\int DX e^{-\alpha N \mathrm{tr}\left[\frac{1}{4}\Lambda X\Lambda X + \frac{1}{2}\log(1-X)+X/2\right]}}{\int DX e^{-\alpha N \mathrm{tr}\left[\frac{1}{4}\Lambda X\Lambda X - \frac{1}{4}X^2\right]}}. \quad (2.8)$$

Here the sum ranges all stable curves ($2g + s > 2$) and $\Lambda = diag\left(e^{\lambda_1}, \ldots, e^{\lambda_N}\right)$.

Differentiating the relation (2.7) w.r.t. λ_i in the right-hand side we obtain the standard loop means, or (connected) correlation functions $W_s^{(g)}(x_1, \ldots, x_s)$, $x_i = e^{\lambda_i} + e^{-\lambda_i}$, of the Gaussian matrix model enjoying the standard topological recursion relations [19], [11]. We therefore obtain the exact relation between resolvents and terms of the expansion of the KPMM free energy:

$$W_s^{(g)}(e^{\lambda_1} + e^{-\lambda_1}, \ldots, e^{\lambda_s} + e^{-\lambda_s}) = \prod_{i=1}^{s} \left[\frac{1}{e^{\lambda_i} - e^{-\lambda_i}} \frac{\partial}{\partial \lambda_i} \right] F_s^{(g)}(\lambda_1, \ldots, \lambda_s). \quad (2.9)$$

The quantities $W_s^{(g)}(x_1, \ldots, x_s)$ here enjoy the standard topological recursion [11], [1] for the spectral curve $x = e^{\lambda} + e^{-\lambda}$, $y = \frac{1}{2}\left(e^{\lambda} - e^{-\lambda}\right)$.

3 Kontsevich–Penner matrix model and discrete moduli spaces

3.1 The Kontsevich matrix model

We turn now to the cell decomposition of moduli spaces of Riemann surfaces of genus g with $s > 0$ marked points proved independently by Harer [27] using Strebel differentials [43] and by Penner [41], [42] using hyperbolic geometry. This cell decomposition theorem states that strata in the cell decomposition of the direct product $\mathcal{M}_{g,s} \times \mathbb{R}_+^s$ of the open moduli space and the s-dimensional space of strictly positive perimeters of holes are in one-to-one correspondence with fat graphs of genus g with s faces (those are the shapes from Sec. 2) whose edges are decorated with strictly positive numbers $l_i \in \mathbb{R}_+$. The perimeters P_I, $I = 1, \ldots, s$ are the sums of l_i taken (with multiplicities) over edges incident to the corresponding face (boundary component, or hole). So it is natural to call them the *lengths* of the corresponding edges.

The fundamental theorem by Kontsevich [31] establishes the relation between the *intersection indices* $\langle \tau_{d_1} \cdots \tau_{d_s} \rangle_g := \int_{\overline{\mathcal{M}}_{g,s}} \prod_{I=1}^s \psi_I^{d_I}$ and the Kontsevich matrix-model integral. Here ψ_I is a ψ-class, or a Chern class, associated with the Ith marked point, and integrals of these classes (intersection indices) do not depend on actual values of P_I being purely cohomological objects. Multiplying every $\psi_I^{d_I}$ by $P_I^{2d_I}$ and performing the *Laplace transformation* w.r.t. all P_I, we obtain

$$\iint_0^\infty dP_1 \cdots dP_s e^{-\sum_I P_I \lambda_I} \int_{\overline{\mathcal{M}}_{g,s}} \prod_{I=1}^s P_I^{2d_I} \psi_I^{d_I} = \langle \tau_{d_1} \cdots \tau_{d_s} \rangle_g \prod_{I=1}^s \prod_{I=1}^s \frac{(2d_I)!}{\lambda_I^{2d_I+1}}. \tag{3.1}$$

Using the explicit representation of ψ-classes from [31] we can present the left-hand side of (3.1) as the sum over three-valent fat graphs with the weights $1/(\lambda_{I_1} + \lambda_{I_2})$ on edges where I_1 and I_2 are indices of two (possibly coinciding) cycles incident to a given edge. Also a factor $2^{|L|-|V|}$ appears (where $|V|$ and $|L|$ are the cardinalities of the respective sets of vertices and edges). The generating function is then the celebrated *Kontsevich matrix model*

$$e^{\sum_{g=0}^\infty \sum_{s=1}^\infty N^{2-2g} \alpha^{2-2g-s} \mathcal{F}_K^{(g,s)}(\{\xi_k\})} := \frac{\int DX e^{-\alpha N \mathrm{tr} \left[\frac{1}{2} X^2 \Lambda + X^3/6 \right]}}{\int DX e^{-\alpha N \mathrm{tr} \left[\frac{1}{2} X^2 \Lambda \right]}}, \tag{3.2}$$

where

$$\xi_k := \frac{1}{N} \sum_{i=1}^N \frac{(2k)!}{\lambda_i^{2k+1}} = \frac{1}{N} \sum_{i=1}^N \int_0^\infty dP_| P_I^{2k} e^{-\lambda_I P_I} \tag{3.3}$$

are the *times* of the Kontsevich matrix model.

3.2 Open discrete moduli spaces and KPMM

As was proposed in [9], we set all the lengths of edges of the Penner–Strebel graphs to be nonnegative *integers* $l_i \in \mathbb{Z}_+$, $i = 1, \ldots, |L| \le 6g - 6 + 3s$. Instead of integrations over $\mathcal{M}_{g,s}$ we take summations over integer points inside $\mathcal{M}_{g,s}$.

Because the length l_i of every edge appears exactly twice in the sum $\sum_{I=1}^s P_I$, this sum is always a positive even number, and we must take this restriction into account when performing the discrete Laplace transformations with the measure $e^{-\sum_{I=1}^s \lambda_I P_I}$. By analogy with the continuous Laplace transformation in the Kontsevich model, we introduce the new times

$$T_{2k}^\pm(\lambda_I) := \frac{\partial^{2k}}{\partial \lambda_I^{2k}} \frac{1}{\mp e^{\lambda_I} - 1} = \sum_{P_I=1}^\infty (\mp 1)^{P_I} P_I^{2k} e^{-\lambda_I P_I} \tag{3.4}$$

as discrete Laplace transforms; the above $\mathbb{Z}_2$ restrictions ensure the existence of two sets of times.

Following [38] we thus define the *discrete volumes* $N_{g,s}(P_1, \ldots, P_s)$ which is a weighted count of the integer points inside $\mathcal{M}_{g,s}^{\mathrm{disc}} \times \mathbb{Z}_+^s$ for fixed positive integers P_I, $I = 1, \ldots, s$, which are the perimeters of the holes (cycles). These discrete volumes are equal (modulo the standard factors of volumes of automorphism groups) to the numbers of all fat graphs with vertices of valencies three and higher and with positive integer lengths of edges subject to the restriction that the lengths of all cycles (the perimeters) are fixed. Using the identity $\sum_{I=1}^s \lambda_I P_I = \sum_{e \in L} l_e(\lambda_{I_1^{(e)}} + \lambda_{I_2^{(e)}})$, where l_e is the length of the eth edge and $I_1^{(e)}$ and $I_2^{(e)}$ are the indices of two (possibly coinciding) cycles incident to the eth edge, we obtain that

$$\sum_{\{P_I\} \in \mathbb{Z}_+^s} N_{g,s}(P_1, \ldots, P_s) e^{-\sum_{I=1}^s P_I \lambda_I} = \sum_{\Gamma_{g,s}} \frac{1}{|\mathrm{Aut}\,\Gamma_{g,s}|} \prod_{e=1}^{|L|} \frac{1}{e^{\lambda_{I_1^{(e)}} + \lambda_{I_2^{(e)}}} - 1}. \tag{3.5}$$

We recognize in (3.5) the genus expansion of the KPMM (2.8). We thus have the lemma

Lemma 1. [5] *The generating function for the Laplace transformed discrete volumes $N_{g,s}(P_1, \ldots, P_s)$ is the KPMM (2.8). The correspondence (3.5) is given by the formula*

$$e^{\sum_{g,s,P_j \in \mathbb{Z}_+}' N^{2-2g} \alpha^{2-2g-s} N_{g,s}(P_1,\ldots,P_s) e^{-\sum_{I=1}^s P_I \lambda_I}} = \frac{\int DX e^{-\alpha N \mathrm{tr} \left[\frac{1}{2} \Lambda X \Lambda X + \log(1-X) + X \right]}}{\int DX e^{-\alpha N \mathrm{tr} \left[\frac{1}{2} \Lambda X \Lambda X - \frac{1}{2} X^2 \right]}}, \tag{3.6}$$

where the sum ranges all stable curves with $2g - 2 + s > 0$ and strictly positive perimeters P_l.

Remark 1. The formula (3.6) is valid at all values of N and λ_l. Specializing it to the case $N = 1$ (when we have just an ordinary integral instead of the matrix one) and setting $\lambda_l = \lambda$, $\alpha = 1/\hbar$, and $x = e^\lambda + e^{-\lambda}$, we obtain

$$e^{\sum_{g,s,P_j \in \mathbb{Z}_+}' \hbar^{2g+s-2} N_{g,s}(P_1^2,\ldots,P_s^2) e^{-\lambda \sum_{I=1}^s P_I}} = \sqrt{\frac{1 - e^{-2\lambda}}{\pi \hbar}} e^{-(2\hbar)^{-1} e^{2\lambda} + \hbar^{-1}\lambda} F(\hbar, x),$$

where the function

$$F(\hbar, x) := \int_{-\infty}^\infty dt \, e^{-\frac{1}{\hbar}(t^2/2 + xt + \log t)}$$

satisfies the second-order differential equation

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} + x\hbar \frac{\partial}{\partial x} + (1 - \hbar) \right] F(\hbar, x) = 0.$$

We thus reproduce the equation of the quantum curve from [16].

Note that the discrete volumes are quasi-polynomials: their coefficients depend on the mutual parities of the P_I's and we present one more proof of this fact below (see Corollary 1). Because the generating function (2.8) is related by (2.9) to the standard s-loop Gaussian means $W_s^{(g)}$, we have the following lemma.

Lemma 2. [5]. *The correlation functions $W_s^{(g)}(x_1, \ldots, x_s)$ of the Gaussian matrix model subject to the standard topological recursion based on the spectral curve $x = e^\lambda + e^{-\lambda}$, $y = \frac{1}{2}(e^\lambda - e^{-\lambda})$ are related to the discrete volumes by the following explicit relation:*

$$W_s^{(g)}(e^{\lambda_1} + e^{-\lambda_1}, \ldots, e^{\lambda_s} + e^{-\lambda_s}) = \prod_{I=1}^s \left[\frac{1}{e^{\lambda_I} - e^{-\lambda_I}} \sum_{P_I=1}^\infty P_I e^{-P_I \lambda_I} \right] N_{g,s}(P_1, \ldots, P_s). \tag{3.7}$$

The matrix model (2.8) manifests many remarkable properties. Besides being the generating function for the discrete volumes related to Gaussian means, it is also equivalent [14, 30] to the Hermitian matrix model with the potential determined by the Miwa change of the variables $t_k = \frac{1}{k}\mathrm{tr}\,(e^\Lambda + e^{-\Lambda})^{-k} + \frac{1}{2}\delta_{k,2}$, it is the generating function for the number of clean Belyi functions, or for the corresponding Grothendieck *dessins d'enfant* [3] (see also [2]) and, finally, in the special times $T_{2r}^\pm$, $r = 0, 1, \ldots,$ (3.4), it is equal to the product of two Kontsevich matrix models [10], intertwined by a canonical transformation of the variables. We now turn to this last property.

Lemma 3. ([10]) *The partition function $\mathcal{Z}[\Lambda]$ (2.8) expressed in the times $T_k^\pm(\lambda)$ (3.4) depends only on the even times $T_{2k}^\pm(\lambda)$ and satisfies the following exact relation:*

$$\mathcal{Z}[\Lambda] = e^{\mathcal{F}_{\mathrm{KP}}[\{T_{2n}^\pm\}]} = e^{C(\alpha N)} e^{-N^{-2}\mathcal{A}} e^{\mathcal{F}_{\mathrm{K}}[\{T_{2n}^+\}] + \mathcal{F}_{\mathrm{K}}[\{T_{2n}^-\}]}, \tag{3.8}$$

where $\mathcal{F}_{\mathrm{K}}[\{T_{2n}^\pm\}]$ is a free energy of the Kontsevich matrix model (3.2), $T_{2n}^\pm$ given by (3.4) are therefore the times of the KdV hierarchies, and $\mathcal{A}$ is the canonical transformation operator

$$\mathcal{A} = \sum_{m,n=0}^\infty \frac{B_{2(n+m+1)}}{4(n+m+1)} \frac{1}{(2n+1)!(2m+1)!} \left\{ \frac{\partial^2}{\partial T_{2n}^+ \partial T_{2m}^+} + \frac{\partial^2}{\partial T_{2n}^- \partial T_{2m}^-} + 2(2^{2(n+m+1)} - 1)\frac{\partial^2}{\partial T_{2n}^+ \partial T_{2m}^-} \right\}$$

$$+ \sum_{n=2}^\infty \alpha N^2 \frac{2^{2n-1}}{(2n+1)!} \left(\frac{\partial}{\partial T_{2n}^-} + \frac{\partial}{\partial T_{2n}^+} \right). \tag{3.9}$$

Here $C(\alpha N)$ is a function depending only on αN that ensures that $\mathcal{F}_{\mathrm{KP}}[\{T_{2n}^\pm\}] = 0$ for $T_{2n}^\pm \equiv 0$ and B_{2k} are the Bernoulli numbers generated by $t/(e^t - 1) = \sum_{m=0}^\infty B_m t^m/(m!)$.

From this canonical transformation we immediately obtain the (ordinary) graph representation for the term $\mathcal{F}_{g,s}[\{T_{2n}^\pm\}]$ of the expansion of

$$\mathcal{F}_{\mathrm{KP}}[\{T_{2n}^\pm\}] = \sum_{g,s} N^{2-2g} \alpha^{2-2g-s} \mathcal{F}_{g,s}[\{T_{2n}^\pm\}].$$

Lemma 4. ([10, 5]) *We can present a term $\mathcal{F}_{g,s}[\{T_{2n}^\pm\}]$ of the genus expansion of the KPMM (2.8) as a sum of a finite set of graphs $G_{g,s}$ described below; each graph contributes the factor also described below divided by the order of the automorphism group of the graph.*

- *each node (a vertex) v_i, $i = 1, \ldots, q$, of a graph $G_{g,s}$ is decorated by the marking "+" or "−", by the genus $g_i \geq 0$, and has s_i endpoints of edges incident to it ($2g_i - 2 + s_i > 0$, i.e., all nodes are stable); each endpoint of an edge carries a nonnegative integer $k_{r,i}^\pm$, $r = 1, \ldots, s_i$; these integers are subject to restriction that $\sum_{r=1}^{s_i} k_{r,i}^\pm = 3g_i - 3 + s_i$ where the superscript $+$ or $-$ is determined by the marking of the vertex;*

- *edges can be external legs (ordinary leaves) with $k_{r,i}^\pm \geq 0$ (we let $a_i \geq 0$ denote the number of such legs incident to the ith vertex), half-edges (dilaton leaves) with $k_{r,i}^\pm \geq 2$ (we let $b_i \geq 0$ denote the number of such legs incident to the ith node), or internal edges incident either to two different nodes or to the same node (their two endpoints carry in general different numbers $k_{r_1,i_1}^\pm$ and $k_{r_2,i_2}^\pm$) (we let l_i denote the number of internal edge endpoints incident to the ith node);*

- *each node contributes the Kontsevich intersection index $\left\langle \tau_{k_{1,i}^\pm} \cdots \tau_{k_{s_i,i}^\pm} \right\rangle_{g_i}$;*

- *every internal edge with endpoint markings (k_1^+, k_2^+) or (k_1^-, k_2^-) (two endpoints of such an edge can be incident to the same node) contribute the factor*

$$-\frac{B_{2(k_1^\pm + k_2^\pm + 1)}}{2(k_1^\pm + k_2^\pm + 1)} \frac{1}{(2k_1^\pm + 1)!(2k_2^\pm + 1)!}$$

and every internal edge with endpoint markings (k_1^+, k_2^-) (two endpoints of such an edge can be incident only to distinct nodes having different markings $+$ and $-$) contributes the factor

$$-\frac{B_{2(k_1^+ + k_2^- + 1)}}{2(k_1^+ + k_2^- + 1)} \frac{2^{2(k_1^+ + k_2^- + 1)} - 1}{(2k_1^+ + 1)!(2k_2^- + 1)!};$$

- every half-edge with the marking $r^\pm \geq 2$ contributes the factor $-\frac{2^{2r^\pm - 1}}{(2r^\pm + 1)!}$;

- every external leg with the marking $k_{r,i}^\pm$ contributes the corresponding time $T_{2k_{r,i}^\pm}^\pm$;

- $\sum_{i=1}^q (g_i + l_i/2 - 1) + 1 = g$ (the total genus g is equal to the sum of internal genera plus the number of loops in the graph);

- $\sum_{i=1}^q a_i = s$ (the total number of external legs is fixed and equal to s);

From the above formulas, we have that

$$\sum_{j=1}^s k_j^{\text{Ext}} = 3g - 3 + s - \sum_{j=1}^{|L|}(1 + k_{j,1}^{\text{Int}} + + k_{j,2}^{\text{Int}}) - \sum_{j=1}^{|B|}(k_j^{\text{Half}} - 1), \qquad (3.10)$$

where, disregarding the node labels, $k_j^{\text{Ext}} \geq 0$ are indices of the external edges, $k_{j,1}^{\text{Int}} \geq 0$ and $k_{j,2}^{\text{Int}} \geq 0$ are indices of endpoints of the internal edges, $k_j^{\text{Half}} \geq 2$ are indices of half-edges, and $|L|$ and $|B|$ are the cardinalities of the respective sets of internal edges and half-edges of the graph.

The proof is just another application of Wick's theorem, now in the form of exponential of a linear-quadratic differential operator (3.9); for the typical form in the above sum, see Fig. 3.
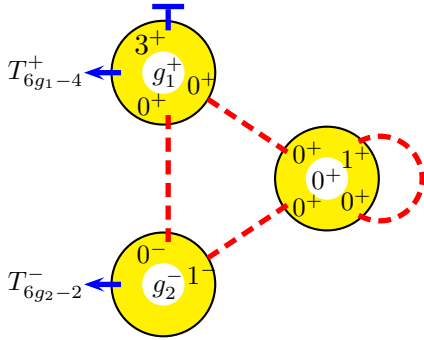
Figure 3: The typical diagram from the graph expansion $G_{g,s}$.

This lemma immediately implies the corollary

Corollary 1. *The quantities $\mathcal{F}_{g,s}[\{T_{2n}^\pm\}]$ are polynomials such that, for every monomial $T_{2n_1}^+ \cdots T_{2n_s}^-$ we have that $\sum_{i=1}^s n_i \leq 3g - 3 + s$, and the highest term with $\sum_{i=1}^s n_i = 3g - 3 + s$ is*

$$\langle \tau_{n_1} \cdots \tau_{n_s} \rangle_g \Big(\prod_{i=1}^s T_{2n_i}^+ + \prod_{i=1}^s T_{2n_i}^- \Big).$$

This also implies that all discrete volumes $N_{g,s}(P_1, \ldots, P_s)$ are $\mathbb{Z}_2$-quasi-polynomials in P_I^2.

Proof. The discrete volumes $N_{g,s}(P_1, \ldots, P_s)$ depend only on even powers of P_I because $\mathcal{F}_{g,s}$ depend only on even times $T_{2n}^\pm$; the quasi-polynomiality follows immediately from the fact that $\mathcal{F}_{g,s}$ are polynomials in T_{2n}^+ and T_{2n}^-.

Remark 2. Note that the quadratic part of the differential operator (3.9) manifests the alternating structure because the Bernoulli numbers B_{2n} are positive for odd n and negative for even n,

$$B_{2n} = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \left[1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots \right].$$

3.3 The times for the multi-resolvents

We first consider $N_{g,1}(P)$, which are polynomials of degree $3g - 2$ in P^2, are nonzero only for even P, and must vanish for all $P = 2, \ldots, 4g - 2$ (because the minimum number of edges of the genus g shape with one face is $2g$, and the minimum nonzero P is therefore $4g$). We thus have that, for even P, $N_{g,1}(P)$ has the form $\prod_{k=1}^{2g-1} (P^2 - (2k)^2) Pol_{g-1}(P^2)$, where $Pol_{g-1}(x)$ is a polynomial of degree $g - 1$ and $N_{g,1}(P)$ vanishes for odd P, so its Laplace transform in formula (3.7) is

$$W_1^{(g)}(e^\lambda + e^{-\lambda}) = \frac{-1}{e^\lambda - e^{-\lambda}} \sum_{i=0}^{g-1} \frac{b_i^{(g)}}{2^{4g+2i-1}(4g+2i-1)!} \prod_{k=1}^{2g+i-1} \left(\frac{\partial^2}{\partial \lambda^2} - (2k)^2 \right) \frac{\partial}{\partial \lambda} \frac{1}{e^{2\lambda} - 1} \quad (3.11)$$

for some coefficients $b_i^{(g)}$. Using that $-\frac{\partial}{\partial \lambda} \frac{1}{e^{2\lambda}-1} = \frac{2}{(e^\lambda - e^{-\lambda})^2}$ and the relation

$$\left(\frac{\partial^2}{\partial \lambda^2} - (m)^2 \right) \frac{1}{(e^\lambda - e^{-\lambda})^m} = \frac{4(m)(m+1)}{(e^\lambda - e^{-\lambda})^{m+2}}, \quad m \geq 1, \quad (3.12)$$

we obtain the general representation for the one-loop mean,

$$W_1^{(g)}(e^\lambda + e^{-\lambda}) = \frac{1}{e^\lambda - e^{-\lambda}} \sum_{i=0}^{g-1} b_i^{(g)} \frac{1}{(e^\lambda - e^{-\lambda})^{4g+2i}} = \frac{1}{(e^\lambda - e^{-\lambda})^{4g+1}} \sum_{i=0}^{g-1} \frac{b_i^{(g)}}{(e^\lambda - e^{-\lambda})^{2i}}. \quad (3.13)$$

In [5], we have found the transition formulas between $b_i^{(g)}$ and the coefficients $P_{g,i}$ from [7]: the integrality of $b_i^{(g)}$ implies that of $P_{g,i}$ and vice versa, but the positivity conjecture for $P_{g,s}$ put forward in [7] requires an additional work.

We now consider the general s-resolvent case. From (2.9) we have that the (stable) loop means (with $2g + s - 2 \geq 1$) are polynomials $W_s^{(g)}(e^{\lambda_1} + e^{-\lambda_1}, \ldots, e^{\lambda_s} + e^{-\lambda_s}) = F_{g,s}(\{t_{2n_j+1}^\pm(\lambda_j)\})$ in times obtained by the substitution

$$T_{2d}^\pm \to t_{2d+1}^\pm(\lambda_j) := \frac{1}{e^{\lambda_j} - e^{-\lambda_j}} \left(\frac{\partial}{\partial \lambda_j} \right)^{2d+1} \frac{1}{e^{\lambda_i} \pm 1}, \quad (3.14)$$

All the times $t_{2d+1}^\pm(\lambda)$ are strictly skew-symmetric with respect to the change of variables $\lambda \to -\lambda$.

Using (3.11) and the fact that

$$t_{2d+1}^-(\lambda) + t_{2d+1}^+(\lambda) = \frac{1}{e^\lambda - e^{-\lambda}} \left(\frac{\partial}{\partial \lambda} \right)^{2d+1} \frac{2}{e^{2\lambda} - 1} = \sum_{j=1}^{d+1} q_{j,d} \frac{1}{(e^\lambda - e^{-\lambda})^{2j+1}} \quad (3.15)$$

and

$$t_{2d+1}^-(\lambda) - t_{2d+1}^+(\lambda) = \frac{1}{e^\lambda - e^{-\lambda}} \left(\frac{\partial}{\partial \lambda} \right)^{2d+1} \frac{2}{e^\lambda - e^{-\lambda}} = \sum_{j=1}^{d+1} \tilde{q}_{j,d} \frac{e^\lambda + e^{-\lambda}}{(e^\lambda - e^{-\lambda})^{2j+1}} \quad (3.16)$$

with some integer coefficients $q_{j,d}$ and $\tilde{q}_{j,d}$, where relation (3.16) follows from that $\frac{1}{e^\lambda - 1} + \frac{1}{e^\lambda + 1} = \frac{2}{e^\lambda - e^{-\lambda}}$ and from another useful representation

$$\frac{1}{e^\lambda - e^{-\lambda}} \frac{\partial}{\partial\lambda} \prod_{k=1}^{d}\left(\frac{\partial^2}{\partial\lambda^2} - (2k-1)^2\right) \frac{2}{e^\lambda - e^{-\lambda}} = \frac{1}{e^\lambda - e^{-\lambda}} \frac{\partial}{\partial\lambda} \frac{2^{2d+1}(2d)!}{(e^\lambda - e^{-\lambda})^{2d+1}}$$

$$= -2^{2d+1}(2d+1)!\frac{e^\lambda + e^{-\lambda}}{(e^\lambda - e^{-\lambda})^{2d+3}}, \quad (3.17)$$

we can equivalently expand $F_{g,s}\big(\{t^\pm_{2n_j+1}(\lambda_j)\}\big)$ in the variables

$$s_{k,\beta}(\lambda) := \frac{(e^\lambda + e^{-\lambda})^\beta}{(e^\lambda - e^{-\lambda})^{2k+3}}, \quad k = 0, \ldots, 3g+s-3, \ \beta = 0, 1. \quad (3.18)$$

In the next section we demonstrate that the coefficients of these expansions are related to the ancestor invariants of a CohFT.

We now present the general structure of the multiloop means.

Lemma 5. *The general expression for a stable ($2g+s-3 \geq 0$) loop mean $W_s^{(g)}(e^{\lambda_1}+e^{-\lambda_1}, \ldots, e^{\lambda_s}+e^{-\lambda_s})$ in terms of the variables $s_{k,\beta}(\lambda)$ given by (3.18) reads:*

$$W_s^{(g)}(e^{\lambda_1} + e^{-\lambda_1}, \ldots, e^{\lambda_s} + e^{-\lambda_s}) = \sum_{\vec{k},\vec{\beta}} \widehat{b}^{(g)}_{\vec{k},\vec{\beta}} \prod_{j=1}^{s} s_{k_j,\beta_j}(\lambda_j), \quad (3.19)$$

where k_j and β_j are subject to the restrictions:

$$2g - 1 + \frac{1}{2}\sum_{j=1}^{s}\beta_j \leq \sum_{j=1}^{s}k_j \leq 3g+s-3, \qquad \sum_{j=1}^{s}\beta_j = 0 \ mod \ 2. \quad (3.20)$$

The two nonstable loop means are

$$W_1^{(0)}(e^\lambda + e^{-\lambda}) = e^{-\lambda}, \quad (3.21)$$

$$W_2^{(0)}(e^{\lambda_1} + e^{-\lambda_1}, e^{\lambda_2} + e^{-\lambda_2}) = \prod_{i=1,2}\prod_{j=1,2}\frac{1}{e^{\lambda_i} - e^{-\lambda_j}} \quad (3.22)$$

We prove restrictions (3.20) using two considerations: first, if we scale $\lambda_j \to \infty$ uniformly for all j, $\lambda_j \to \lambda_j + R$, every edge contributes a factor e^{-2R} plus s factors e^{-R} due to the derivatives. The minimum number of edges (for a shape with one vertex) is $2g+s-1$, so the minimum factor appearing is $e^{(-4g-3s+2)R}$ whereas $s_{k,\beta}(\lambda)$ scale as $e^{(-3-2k+\beta)R}$, which results in the lower estimate. The upper estimate emerges out of the pole behaviour at $\lambda_j = 0$. On the one hand, $s_{k,\beta}(\lambda) \sim \lambda^{-2k-3}$ as $\lambda \to 0$ irrespectively on β; on the other hand, from the relation to the Kontsevich model we can conclude that the pole structure of the derivatives of the Kontsevich KdV times is $t_{d_j}(\lambda_j) \sim \lambda_j^{-2d_j-3}$ with $\sum_j d_j \leq 3g+s-3$ and therefore $\sum_j d_j = \sum_j k_j$, which leads to the upper estimate. That the sum of the β_j factors is even follows from the symmetricity of the total expression with respect to the total change of the times $T^\pm \to T^\mp$; under this change, the variables $s_{k,\beta}(\lambda)$ behave as $s_{k,\beta}(\lambda) \to (-1)^\beta s_{k,\beta}(\lambda)$, so the sum of the beta factors must be even.

In Sec. 5, we use the topological recursion to prove that all admissible by (3.20) coefficients $\widehat{b}^{(g)}_{\vec{k},\vec{\beta}}$ are positive integers (see Theorem 6).

4 Cohomological field theory from discrete volumes

We now describe a cohomological field theory (CohFT) associated to the discrete volumes. A dimension d Frobenius manifold structure is equivalent to a CohFT for a dimension d vector space H with a basis $\{e_\alpha\}$ and a metric η. We show that the quasi-polynomial discrete volumes are equivalent to the correlation functions of the CohFT associated to the Hurwitz Frobenius manifold $H_{0,(1,1)}$ described in the introduction. We give two accountings of the genus 0 case: the first approach is constructive and the other generalises to all genera. The constructive approach also implies that we deal with a homogenous CohFT. The primary correlation functions of our CohFT turn out to be virtual Euler characteristics $\chi(\mathcal{M}_{g,n})$ of moduli spaces.

4.1 Cohomological field theories

Given a complex vector space H equipped with a complex metric η, a CohFT is a sequence of S_s-equivariant linear maps

$$I_{g,s} : H^{\otimes s} \to H^*(\overline{\mathcal{M}}_{g,s}),$$

which satisfy the following compatibility conditions with respect to inclusion of strata. Any partition into two disjoint subsets $I \sqcup J = \{1, \ldots, s\}$ defines a map $\phi_I : \overline{\mathcal{M}}_{g_1,|I|+1} \times \overline{\mathcal{M}}_{g_2,|J|+1} \to \overline{\mathcal{M}}_{g,s}$ such that

$$\phi_I^* I_{g,s}(v_1 \otimes \cdots \otimes v_s) = I_{g_1,|I|+1} \otimes I_{g_2,|J|+1}\Big(\bigotimes_{i \in I} v_i \otimes \Delta \otimes \bigotimes_{j \in J} v_j\Big)$$

where $\Delta = \sum_{\alpha,\beta} \eta^{\alpha\beta} e_\alpha \otimes e_\beta$ with respect to a basis $\{e_\alpha\}$ of H. The map $\psi : \overline{\mathcal{M}}_{g-1,s+2} \to \overline{\mathcal{M}}_{g,s}$ induces

$$\psi^* I_{g,s}(v_1 \otimes \cdots \otimes v_s) = I_{g-1,s+2}(v_1 \otimes \cdots \otimes v_s \otimes \Delta).$$

The three-point function $I_{0,3}$ together with the metric η induces a product $\bullet$ on H, $u \bullet v = \sum_{\alpha,\beta} I_{0,3}(u \otimes v \otimes e_\alpha)\eta^{\alpha\beta} e_\beta$, where $I_{0,3}$ takes its values in $\mathbb{C}$. A vector e_0 satisfying

$$I_{0,3}(v_1 \otimes v_2 \otimes e_0) = \eta(v_1 \otimes v_2), \quad \forall v_1, v_2 \in H$$

is the identity element for the product on H.

An extra condition satisfied both by the CohFT under consideration and by Gromov–Witten invariants pertains to the forgetful map for $s \geq 3$, $\pi : \overline{\mathcal{M}}_{g,s+1} \to \overline{\mathcal{M}}_{g,s}$, which induces

$$I_{g,s+1}(v_1 \otimes \cdots \otimes v_s \otimes e_0) = \pi^* I_{g,s}(v_1 \otimes \cdots \otimes v_s). \tag{4.1}$$

4.2 Quasipolynomials and ancestor invariants

The discrete volumes $N_{g,s}(P_1, \ldots, P_s)$ are mod 2 even quasi-polynomials, i.e. it is an even polynomial on each coset of $2\mathbb{Z}^s \subset \mathbb{Z}^s$. Define a basis of mod 2 even quasi-polynomials induced (via tensor product) from the following single-variable basis $p_{k,\alpha}(b)$ for $k = 0, 1, 2, \ldots$ and $\alpha = 0, 1$.

$$p_{0,0}(b) = \begin{cases} 1, & b \text{ even} \\ 0, & b \text{ odd} \end{cases}, \quad p_{0,1}(b) = \begin{cases} 0, & b \text{ even} \\ 1, & b \text{ odd} \end{cases}, \quad p_{k+1,\alpha}(b) = \sum_{m=0}^{b} m p_{k,\alpha}(m), \quad k \geq 0.$$

Then

$$p_{k,\alpha}(b) = \frac{p_{0,k+\alpha}(b)}{4^k k!} \prod_{\substack{0 < m \leq k \\ m = k+\alpha \ (\text{mod } 2)}} (b^2 - m^2) \tag{4.2}$$

where in the second subscript we mean $k + \alpha \pmod 2$.

Put $\vec{k} = (k_1, \ldots, k_s)$ and $\vec{\alpha} = (\alpha_1, \ldots, \alpha_s)$.

Theorem 2. *We have that*

$$N_{g,s}(P_1, \ldots, P_s) = \sum_{\vec{k}, \vec{\alpha}} c^g_{\vec{k}, \vec{\alpha}} \prod_{i=1}^{s} p_{k_i, \alpha_i}(P_i)$$

where the coefficients are ancestor invariants:

$$c^g_{\vec{k}, \vec{\alpha}} = \int_{\overline{\mathcal{M}}_{g,s}} I_{g,s}(e_{\alpha'_1} \otimes \cdots \otimes e_{\alpha'_s}) \prod_{i=1}^{s} \psi_i^{k_i}. \tag{4.3}$$

The *proof* is an application of [18] where theories with spectral curves satisfying special conditions were identified with semisimple CohFTs. The outcome of applying [18] is non-constructive so we prove the genus zero case in a different way that provides an explicit realisation of the CohFT.

4.3 A homogeneous CohFT in genus zero

The *primary* correlators of a CohFT are $Y_{g,s} := \int_{\overline{\mathcal{M}}_{g,s}} I_{g,s} : H^{\otimes s} \to \mathbb{C}$, and we assemble them into the generating function

$$F(t_0, \ldots, t_{D-1}) = \sum N^{2-2g} \frac{1}{s!} Y_{g,s} = \sum N^{2-2g} F_g$$

where $(t_0, \ldots, t_{D-1})$ in H^* is the dual basis of $\{e_0, \ldots, e_{D-1}\}$. The genus 0 part F_0 is the *prepotential* of the CohFT.

Theorem 3 Manin [32] Theorem III.4.3. *One can uniquely reconstruct a genus 0 CohFT from abstract correlation functions.*

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,s}$ possesses a natural stratification indexed by *dual graphs*. The dual graph of $\Sigma \in \overline{\mathcal{M}}_{g,s}$ has vertices corresponding to the irreducible components of Σ with specified genera, edges corresponding to the nodes (cusps) of Σ, and a *tail*—an edge with an open end (no vertex)—corresponding to each labeled point of Σ. If Γ is a dual graph of type (g, s), then the collection of curves D_Γ whose associated dual graph is Γ forms a stratum of $\overline{\mathcal{M}}_{g,s}$. The closure $\overline{D}_\Gamma = \cup_{\Gamma' < \Gamma} D_{\Gamma'}$, where the partial ordering is given by edge contraction, represents an element of $H^*(\overline{\mathcal{M}}_{g,s})$. Keel [29] proved that $H^*(\overline{\mathcal{M}}_{0,s})$ is generated by $\overline{D}_\Gamma$ and derived all relations.

The proof of Theorem 3 uses that

$$\int_{\overline{D}_\Gamma} I_{0,s}(v_1 \otimes \cdots \otimes v_s) = \bigotimes_{v \in V_\Gamma} Y_{0,|v|} \left(\bigotimes_{i=1}^{s} v_i \otimes \Delta^{\otimes |E_\Gamma|} \right).$$

which defines evaluation of a cohomology class on boundary strata tautologically from the definition of a CohFT. Because $H^*(\overline{\mathcal{M}}_{0,s})$ is generated by its boundary strata, and relations in $H^*(\overline{\mathcal{M}}_{0,s})$ agree with the relations satisfied by abstract correlation functions, this suffices for proving the theorem.

In particular, we have the primary invariants

$$Y_{0,3}(e_0 \otimes e_0 \otimes e_1) = 1 = Y_{0,3}(e_1 \otimes e_1 \otimes e_1), \quad Y_{0,s}(e_0 \otimes \text{anything}) = 0, \quad s > 3$$
$$Y_{0,s}(e_1^{\otimes s}) = N_{0,s}(0, \ldots, 0) = \chi(\mathcal{M}_{0,s}) \quad s > 3 \tag{4.4}$$

that define a genus 0 CohFT.

A CohFT is conformal if its prepotential is quasihomogeneous with respect to the *Euler vector field*:

$$E \cdot F_0 = (3 - d)F_0 + Q(t) \tag{4.5}$$

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where Q is a quadratic polynomial in $t = (t_0, \ldots, t_{D-1})$. Using the genus 0 reconstruction in Theorem 3, Manin proved that a conformal CohFT induces the following push-forward condition on the genus 0 CohFT.

Let ξ be any vector field on H treated as a manifold with coordinates $t_0, \ldots, t_{D-1} \in H^*$. The Lie derivative with respect to ξ of the CohFT correlation functions $I_{g,s}$ induces a natural action

$$(\xi \cdot I)_{g,s}(v_1 \otimes \cdots \otimes v_s) = \deg I_{g,s}(v_1 \otimes \cdots \otimes v_s) - \sum_{j=1}^{s} I_{g,s}(v_1 \otimes \cdots \otimes [\xi, v_j] \otimes \cdots \otimes v_s) + \pi_* I_{g,s+1}(v_1 \otimes \cdots \otimes v_s \otimes \xi)$$

where $\pi : \overline{\mathcal{M}}_{g,s+1} \to \overline{\mathcal{M}}_{g,s}$ is the forgetful map, $I_{g,s}$ are ($H^*(\overline{\mathcal{M}}_{0,s})$-valued) tensors on H, and the vector field ξ acts infinitesimally on $I_{g,s}$.

A CohFT is *homogeneous* of weight d if

$$(E \cdot I)_{g,s} = ((g-1)d + s)I_{g,s} \tag{4.6}$$

If a preprotential satisfies the homogeneity condition (4.5), the proof of Theorem 3 implies that the corresponding genus 0 CohFT is homogeneous. The Lie derivative of the bivector Δ dual to the metric η on H can be calculated in flat coordinates

$$\mathcal{L}_E \cdot \Delta = \mathcal{L}_E \cdot \eta^{ij} e_i \otimes e_j = \eta^{ij}([E, e_i] \otimes e_j + e_i \otimes [E, e_j]) = (d-2)\eta^{ij} e_i \otimes e_j = (d-2)\Delta$$

where we have used a choice of flat coordinates [15] with respect to which $\eta = \delta_{i,D-1-i}$ and $E = \sum_i (\alpha_i t_i + \beta_i) \frac{\partial}{\partial t_i}$, where $\alpha_i + \alpha_{D-1-i} = 2 - d$.

4.4 Proof of Theorem 2 in genus 0.

We can now prove the genus 0 case of Theorem 2. For this we produce a prepotential from the primary (constant) terms of $N_{0,s}(P_1, \ldots, P_s)$, which uniquely (and constructively) determines a genus 0 CohFT. Moreover, the quasihomogeneity of the prepotential implies a homogeneous CohFT. The higher coefficients of $N_{0,s}(P_1, \ldots, P_s)$ satisfy a homogeneity condition that makes them the correlation functions of the homogeneous CohFT.

The prepotential

$$F_0 = \sum \frac{1}{s!} Y_{0,s} = \frac{1}{2} t_0^2 t_1 + \sum_{s \geq 3} \frac{1}{s!} N_{0,s}(\vec{0}) t_1^s = \frac{1}{2} t_0^2 t_1 + \frac{1}{2}(1 + t_1)^2 \log(1 + t_1) - \frac{1}{2} t_1 - \frac{3}{4} t_1^2 \tag{4.7}$$

assembled from $N_{0,s}(\vec{0}) = (-1)^{s-3}(s-3)!$ is quasihomogeneous with respect to the Euler vector field $E = t_0 \frac{\partial}{\partial t_0} + 2(1 + t_1) \frac{\partial}{\partial t_1}$:

$$E \cdot F_0 = 4F_0 + t_1^2 + t_0^2.$$

This ensures that the genus 0 CohFT $I_{0,s}$ produced from Theorem 3 satisfies

$$\pi_* I_{g,s+1}(e_S \otimes e_1) = \frac{1}{2}\left(1 - g + s - \deg - \sum \alpha_{i_k}\right) I_{g,s}(e_S) \tag{4.8}$$

where $e_S = e_{i_1} \otimes \ldots \otimes e_{i_s}$, and $\alpha_0 = 1$, $\alpha_1 = 2$ are the coefficients of E. The CohFT also satisfies the pull-back condition (4.1).

Theorem 4. [Teleman [44]] *A semi-simple homogenous CohFT with flat identity is uniquely and explicitly reconstructible from genus zero data.*

Thus, given the genus 0 primary invariants $N_{0,s}(\vec{0})$ there is a unique homogenous CohFT with flat identity. Below we demonstrate that its correlation functions agree with the coefficients of $N_{g,s}(P_1, \ldots, P_s)$.

The pushforward relation (4.8) expressed in terms of correlators is [5]

$$\int_{\overline{\mathcal{M}}_{g,s+1}} I_{g,s+1}(e_S \otimes e_1) \prod_{i=1}^s \psi_i^{k_i} = \left(\sum_{i=1}^s \frac{k_i}{2} + \chi_{g,s}\right) \int_{\overline{\mathcal{M}}_{g,s}} I_{g,s}(e_S) \prod_{i=1}^s \psi_i^{k_i} + \sum_{j=1}^s \int_{\overline{\mathcal{M}}_{g,s}} I_{g,s}(e_{S\setminus\{j\}} \otimes e_j^*) \prod_{i=1}^s \psi_i^{k_i - \delta_{ij}}.$$

The condition $E \cdot F_0 = 4F_0 + t_1^2 + t_0^2$ on $N_{0,s}(\vec{0})$ is a specialisation to $g = 0$ and $P_i = 0$ of the divisor equation [39]

$$N_{g,s+1}(0, P_1, \ldots, P_s) = \sum_{j=1}^s \sum_{k=1}^{P_j-1} k N_{g,s}(P_1, ..., P_s)|_{P_j=k} + \left(\frac{1}{2} \sum_{j=1}^s P_j + \chi_{g,s}\right) N_{g,s}(P_1, \ldots, P_s). \quad (4.9)$$

The flat identity pull-back condition is known as the *string equation* on correlators for $2g - 2 + s > 0$:

$$\int_{\overline{\mathcal{M}}_{g,s+1}} I_{g,s+1}(v_1 \otimes \cdots \otimes v_s \otimes e_0) \prod_{i=1}^s \psi_i^{k_i} = \sum_{j=1}^s \int_{\overline{\mathcal{M}}_{g,s}} I_{g,s}(v_1 \otimes \cdots \otimes v_s) \prod_{i=1}^s \psi_i^{k_i - \delta_{i,j}}$$

and agrees with the recursion [39]

$$N_{g,s+1}(1, P_1, \ldots, P_s) = \sum_{j=1}^s \sum_{k=1}^{P_j} k N_{g,s}(P_1, \ldots, P_s)|_{P_j=k} \quad (4.10)$$

In particular, this proves the genus 0 case of Theorem 2 since the recursions (4.9) and (4.10) uniquely determine the correlation functions of $I_{0,s}$ and $N_{0,s}(P_1, \ldots, P_s)$.

This constructive proof describes explicitly the genus 0 classes $I_{0,s}(e_S) \in H^*(\overline{\mathcal{M}}_{0,s})$:

$$\int_{\overline{\mathcal{M}}_{0,s}} I_{0,s}(e_S) = \begin{cases} \chi(\mathcal{M}_{0,s}) & e_S = e_1^{\otimes s} \\ 0 & \text{otherwise} \end{cases}.$$

4.5 General proof of Theorem 2 using DOSS method [18].

We establish the correspondence between correlation functions of the CohFT and discrete volumes in higher genera applying the results of [18], where it was shown that for spectral curves satisfying a compatibility condition, the Givental reconstruction of higher genus correlation functions can be formulated in terms of graphs, and the same graphs can be used to calculate topological recursion.

Dunin-Barkowsky, Orantin, Shadrin, and Spitz [18] using Eynard's technique of [20] associated to any semi-simple CohFT a local spectral curve (Σ, B, x, y). The Givental R-matrix gives rise to the bidifferential B on the spectral curve

$$\sum_{p,q} \check{B}_{p,q}^{i,j} z^p w^q = \frac{\delta^{ij} - \sum_{k=1}^N R_k^i(-z) R_k^j(-w)}{z + w} \quad (4.11)$$

where $\check{B}_{p,q}^{i,j}$ are coefficients of an asymptotic expansion of the Laplace transform of the regular part of the Bergmann bidifferential B expressed in terms of the local coordinates $s_i = \sqrt{x - x(a_i)}$ where $dx(a_i) = 0$. The R-matrix together with the transition matrix Ψ from a flat to a normalised canonical bases expresses the meromorphic differential ydx in terms of s_i. In particular, this implies a compatibility condition (4.13) between the differential ydx and the bifferential B.

One can apply [18] in either direction, beginning with a semi-simple CohFT or a spectral curve. The prepotential F_0 (4.7) gives rise to a semi-simple CohFT thus generating the R-matrix and the transition matrix Ψ and hence the spectral curve. But having in hands a candidate for the spectral curve, we can start with the spectral curve and apply [18] to obtain the coefficients of $N_{g,s}(P_1, \ldots, P_s)$ as ancestor invariants of a CohFT. Because it agrees with the above CohFT in genus 0, by uniqueness it is the same CohFT produced by Teleman's theorem.

The spectral curves for the discrete volumes and Gromov–Witten invariants of $\mathbb{P}^1$ are similar:

$$\begin{array}{llll} \text{discrete volumes} & x = z + 1/z, & y = z, & B = \frac{dz dz'}{(z-z')^2} \\ \text{GW invariants} & x = z + 1/z, & y = \log z, & B = \frac{dz dz'}{(z-z')^2} \end{array} \qquad (4.12)$$

and because x and B determine the R-matrix uniquely, it is the same for the both curves. The R-matrix for the Gromov–Witten invariants of $\mathbb{P}^1$ reads [18]:

$$R(u) = \sum_{k=0}^{\infty} R_k u^k, \quad R_k = \frac{(2k-1)!!(2k-3)!!}{2^{4k} k!} \begin{pmatrix} -1 & (-1)^{k+1} 2ki \\ 2ki & (-1)^{k+1} \end{pmatrix}.$$

The results of [18] can be applied to those spectral curves for which a Laplace transform of ydx is related to this R-matrix (which is essentially the Laplace transform of the regular part of the bidifferential).

For local coordinates s_i, $i = 1, 2$ near $x = \pm 2$ given by $x = s_i^2 \pm 2$

$$y = 1 + s_1 + \frac{1}{2} s_1^2 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(2k-3)!!}{2^{3k} k!} s_1^{2k+1}, \qquad y = -1 + i s_2 + \frac{1}{2} s_2^2 - i \sum_{k=1}^{\infty} \frac{(2k-3)!!}{2^{3k} k!} s_2^{2k+1},$$

so we obtain

$$(y\check{d}x)_1 = \frac{\sqrt{u}}{2\sqrt{\pi}} \int_{\gamma_1} e^{-u(x-2)} y dx \sim \sum_{k=0}^{\infty} (-1)^{k-1} \frac{(2k+1)!!(2k-3)!!}{2^{4k+1} k!} u^{-(k+1)}$$

$$(y\check{d}x)_2 = \frac{\sqrt{u}}{2\sqrt{\pi}} \int_{\gamma_2} e^{-u(x+2)} y dx \sim -i \sum_{k=0}^{\infty} \frac{(2k+1)!!(2k-3)!!}{2^{4k+1} k!} u^{-(k+1)},$$

where $(-1)!! = 1$, $(-3)!! = -1$, and we let $\sim$ denote the Poincare asymptotic in the parameter u.

The compatibility condition between the differential ydx and the bifferential B reads

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} \cdot \frac{1}{\sqrt{2}} R(u) = \begin{pmatrix} (y\check{d}x)_1 & (y\check{d}x)_2 \end{pmatrix} \qquad (4.13)$$

which uses the first row of the transition matrix $\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$. A direct verification indicates that it is satisfied for $x = z + 1/z$, $y = z$, $B = dz dz'/(z-z')^2$.

From this, [18] supplies the times

$$\xi_0^0 = \frac{1}{2} \left(\frac{1}{1-z} - \frac{1}{1+z} \right), \qquad \xi_0^1 = \frac{1}{2} \left(\frac{1}{1-z} + \frac{1}{1+z} \right), \qquad \xi_k^i = \left(\frac{d}{dx} \right)^k \xi_0^i = \sum_k p_{k,i} z^k \qquad (4.14)$$

and the main result

$$W_s^{(g)}(x_1, \ldots, x_s) = \sum_{\vec{k}, \vec{\alpha}} c_{\vec{k}, \vec{\alpha}}^g \prod_{i=1}^{n} \xi_{k_i, \alpha_i}$$

where the coefficients are the ancestor invariants (4.3). As remarked above, the CohFT produced this way necessarily coincides with the homogeneous CohFT produced by Teleman's theorem since they both use Givental reconstruction and the same initial data.

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4.6 Ancestor invariants and Gaussian means

Lemma 2 and formulas (3.11) and (3.17) straightforwardly express the loop means in terms of the ancestor invariants.

Theorem 5. *We have the following explicit relation between the ancestor invariants (4.3) of a CohFT and the Gaussian means:*

$$W_s^{(g)}\big(e^{\lambda_1} + e^{-\lambda_1}, \ldots, e^{\lambda_s} + e^{-\lambda_s}\big) = \sum_{\vec{k},\vec{\alpha}} c_{\vec{k},\vec{\alpha}}^g \prod_{j=1}^{s} \widehat{p}_{k_j,\alpha_j}(\lambda_j), \tag{4.15}$$

where

$$\widehat{p}_{k,\alpha}(\lambda) = \begin{cases} 2^{1-2r}(2r+1)s_{r,0}(\lambda), & k = 2r, & \alpha = 0; \\ 2^{-2r}(2r+1)s_{r,1}(\lambda), & k = 2r, & \alpha = 1; \\ 2^{-2r+2}2r(2r+1)s_{r,1}(\lambda), & k = 2r-1, & \alpha = 0; \\ 2^{-2r-1}s_{r,0}(\lambda), & k = 2r+1, & \alpha = 1, \end{cases} \tag{4.16}$$

and $s_{r,\beta}(\lambda)$, $\beta = 0, 1$, are defined in (3.18).

Example 1. The topological (degree zero) part of the CohFT is

$$I_{g,s}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_s}) = \epsilon(\vec{\alpha})2^g + \text{ higher degree terms}$$

where $\epsilon(\vec{\alpha}) \equiv \sum_{i=1}^{s} \alpha_i \pmod 2$ is 0 or 1. This explains the asymptotic behaviour of the topological invariants $W_s^{(g)}$ at their poles.

Example 2. If $\{e_0, e_1\}$ is a basis of H corresponding to flat coordinates then

$$\fint_{\overline{\mathcal{M}}_{g,s}} I_{g,s}(e_1^{\otimes s}) = \chi(\mathcal{M}_{g,s}).$$

This uses the fact that $N_{g,s}(0,0,\ldots,0) = \chi(\mathcal{M}_{g,s})$ and

$$p_{k,\alpha}(0) = \begin{cases} 1, & (k,\alpha) = (0,0) \\ 0, & \text{otherwise.} \end{cases}$$

We thus identify the coefficients $\widehat{b}_{\vec{k},\vec{\beta}}^g$ of the expansions (3.19) with (linear combinations) of the ancestor invariants $c_{\vec{k},\vec{\alpha}}^g$ using the identification (4.16): for $s = 1$, we have

$$\widehat{b}_{r,0}^g = 2^{1-2r}(2r+1)c_{2r,0}^g + 2^{-1-2r}c_{2r+1,1}^g,$$
$$\widehat{b}_{r,1}^g = 2^{-2r}(2r+1)c_{2r,1}^g + 2^{2-2r}2r(2r+1)c_{2r-1,0}^g,$$

and for general s we have up to 2^s terms $c_{\vec{k},\vec{\alpha}}^{(g)}$ with all admissible substitutions $(k_i, 1) \leftrightarrow (k_i - 1, 0)$.

5 The topological recursion

In this section, we present main ingredients of the topological recursion method developed in [19, 11, 12, 21]. In parallel, we adapt the general construction to the Gaussian means $W_s^{(g)}(x_1, \ldots, x_s)$:

(i) We have a spectral curve $\Sigma_{x,y} = 0$ with two meromorphic differentials, dx and dy, on this curve. The zeros of dx are the *branching points*. For the Gaussian means, this curve is the sphere $yx - y^2 = 1$, and we use the convenient local coordinates:

$$x = e^\lambda + e^{-\lambda}, \qquad y = e^\lambda, \qquad dx = (e^\lambda - e^{-\lambda})d\lambda. \tag{5.1}$$

We consider the covering of this sphere by two maps: $y = e^\lambda$ and $\overline{y} = e^{-\lambda}$; the sphere is represented as a cylinder obtained from the strip $\text{Im}\lambda \in [0, 2\pi]$ by identifying points $(x, 0)$ of the real line $\text{Im}\lambda = 0$ with the points $(x, 2i\pi)$ of the line $\text{Im}\lambda = 2\pi$. We have two branching points $\lambda = 0, i\pi$.

(ii) We have the Bergmann 2-differential $B(p, q)$ that is a symmetric differential with zero A-cycles (which are absent in a genus zero case here) and with the double pole at coinciding p and q. We also need its antiderivative $E(p, q)$ which is a 1-differential in p and a function of q defined as $\int_{\overline{q}}^q B(p, \bullet)$. For the Gaussian means,

$$B(p, q) = \frac{de^\lambda de^\mu}{(e^\lambda - e^\mu)^2}, \qquad E(p, q) = \frac{de^\lambda}{e^\lambda - e^\mu}, \quad p = e^\lambda, \ q = e^\mu. \tag{5.2}$$

(iii) We have the *recursion kernel* $K(p, q)$ defined to be a $(1, -1)$-differential $K(p, q) = E(p, q)\frac{1}{(y(q) - \overline{y}(q))dx}$; for the Gaussian means,

$$K(p, q) = \frac{de^\lambda}{e^\lambda - e^\mu} \frac{1}{(e^\mu - e^{-\mu})^2 d\mu}, \quad p = e^\lambda, \ q = e^\mu, \tag{5.3}$$

where one power of the difference $e^\mu - e^{-\mu}$ in the denominator comes from the difference $y(q) - \overline{y}(q)$ and another one comes from dx.

(iv) We introduce the *correlation functions* $W_s^{(g)}(p_1, \ldots p_s)$ to be symmetric s-differentials determined recurrently as follows: we choose one of the variables, p_1, as a *root*. Then,

$$W_3^{(0)}(p_1, p_2, p_3) = \sum_{\text{res } dx = 0} K(p_1, q)[B(p_2, q) + B(\overline{p}_2, q][B(p_3, q) + B(\overline{p}_3, q], \tag{5.4}$$

$$W_1^{(1)}(p_1) = \sum_{\text{res } dx = 0} K(p_1, q)B(q, \overline{q}), \tag{5.5}$$

$$W^{(g)}(p_1, p_2, \ldots, p_s) = \sum_{\text{res } dx = 0} K(p_1, q)\left[\sum_{k=2}^s [B(p_k, q) + B(\overline{p}_k, q)]W_{s-1}^{(g)}(q, p_2, \ldots, \widehat{p}_k, \ldots, p_s)\right.$$

$$\left. + W_{s+1}^{(g-1)}(q, q, p_2, \ldots, p_s) + \sideset{}{'}\sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{p_2, \ldots, p_s\}}} W_{|I|+1}^{(g_1)}(q, \{p_i\}_{i \in I})W_{|J|+1}^{(g_2)}(q, \{p_j\}_{j \in J})\right], \tag{5.6}$$

where the right-hand side is explicitly symmetric w.r.t. all $p_2, \ldots, p_s$ but not w.r.t. p_1, $\sum'$ means that we take only stable terms (those with $2g - 2 + s > 0$) explicitly segregating the only nonstable contribution (the term with $[B(p_k, q) + B(\overline{p}_k, q)]$). The hat over a symbol indicates its omission from the list of arguments and in the last term we take the sum over all partitions of the set of arguments $\{p_2, \ldots, p_s\}$ into two nonintersecting subsets I and J. We depict the recursion relation schematically in Fig. 4.
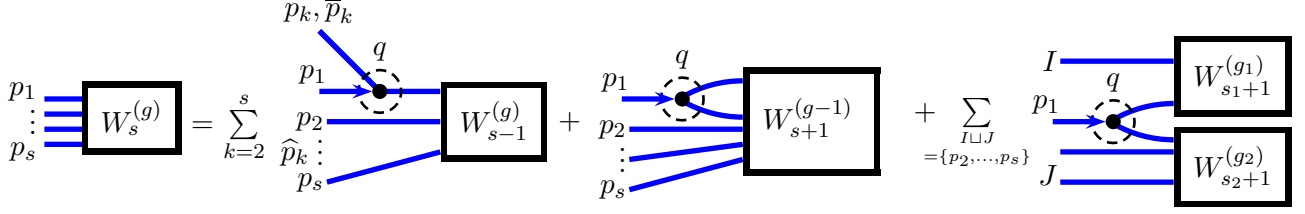
Figure 4: The graphical representation of recursion in (5.6). It clearly indicates the *breeding* (the first term in the right-hand side) and *convolution* (the second and third terms in the right-hand side) processes; arrowed line is the recursion kernel $K(p_1, q)$ and nonarrowed lines are Bergmann kernels $B(p_k, q)$ and $B(\overline{p}_k, q)$.

Using (5.6) we construct all higher $W_s^{(g)}$ out of $W_3^{(0)}(p_1, p_2, p_3)$ and $W_1^{(1)}(p_1)$.

The lemma from [11] states that, although recursion relations (5.6) are not explicitly symmetric w.r.t. permutations of all $p_1, \ldots, p_s$, the whole sum in the right-hand side of (5.6) is actually symmetric.

5.1 The topological recursion for the Gaussian means

In any local theory satisfying the topological recursion, all stable $W_s^{(g)}(x_1, \ldots, x_s)$ have singularities only at the branch points. In the Gaussian case, we therefore conclude that the only singularities in the right-hand side of (5.6) besides poles of high orders at the branching points (for $W_s^{(g)}$, the highest possible order of a pole is $6g + 2s - 3$) are simple poles at $q = p_1$ arising from $K(p_1, q)$ and double poles at $q = p_k$, $q = \overline{p}_k$ arising from $[B(p_k, q) + B(\overline{p}_k, q)]$. We can thus perform the integration w.r.t. q in the right-hand side by evaluating residues at these points, not at the branch points, which drastically simplifies actual calculations.

The actual Gaussian means $W_s^{(g)}(x_1, \ldots, x_s)$ are the coefficients of one-differentials

$$W_s^{(g)}(p_1, p_2, \ldots, p_s) = W_s^{(g)}(x_1, \ldots, x_s) dx_1 \cdots dx_s.$$

Here

$$W_s^{(g)}(x_1, \ldots, x_s) = W_s^{(g)}(e^{\lambda_1} + e^{-\lambda_1}, \ldots, e^{\lambda_s} + e^{-\lambda_s})$$

$$= \sum_{\bullet = \text{perm.}\{1,\ldots,s\}} \sum_{\substack{|Y_0| + |Y_1| = s \\ |Y_1| \in 2\mathbb{Z}}} \widehat{b}_{Y_0, Y_1}^{(g)} \prod_{i, \sum d_i = |Y_0|} \frac{s_{t_i, 0}(\lambda_\bullet)^{d_i}}{d_i!} \prod_{j, \sum k_j = |Y_1|} \frac{s_{r_j, 1}(\lambda_\bullet)^{k_j}}{k_j!}, \quad (5.7)$$

where we enumerate the expansion coefficients $\widehat{b}_{Y_0, Y_1}^{(g)}$ by two Young tableauxes, Y_0 and Y_1, whose total length (total number of columns) is s, Y_1 necessarily has an even length, the tableaux Y_0 contains d_i columns of height $t_i + 1$, the tableaux Y_1 contains k_j columns of height $r_j + 1$, and we perform the total symmetrisation w.r.t. all permutations of s arguments λ_i (indicated as $\lambda_\bullet$) distributed over the both tableauxes (all terms of the products have different arguments).

Recurrent relations described by (5.6) (or graphically in Fig. 4) are governed by two operations on the basic functions $s_{k,0}(\lambda)$ and $s_{k,1}(\lambda)$: "convolution" and "breeding." The convolution process occurs in the second and third terms in (5.6): it produces a linear combination of basis functions $s_{k,\beta}(\lambda_1)$ out of two basis functions, $s_{k_1, \beta_1}(\lambda_q)$ and $s_{k_2, \beta_2}(\lambda_q)$ given by the integral

$$\text{"convolution"}: \quad \sum_{\text{res } dx_q = 0} K(p_1, q) s_{k_1, \beta_1}(\lambda_q) s_{k_2, \beta_2}(\lambda_q) dx_q^2. \quad (5.8)$$

Recalling that $dx_q = (e^{\lambda_q} - e^{-\lambda_q})d\lambda_q$ and that, instead of evaluating this integral by residues at the branch points we can evaluate it at its only simple pole $p_1 = q$ outside the branch points, we obtain with accounting for explicit form (3.18) of the basic vectors

$$\sum_{\text{res } dx_q = 0} K(p_1, q)s_{k_1,\beta_1}(\lambda_q)s_{k_2,\beta_2}(\lambda_q)dx_q^2 = -\text{res}_{p_1=q}K(p_1, q)s_{k_1,\beta_1}(\lambda_q)s_{k_2,\beta_2}(\lambda_q)dx_q^2$$

$$= \frac{(e^{\lambda_1} + e^{-\lambda_1})^{\beta_1+\beta_2}}{(e^{\lambda_1} - e^{-\lambda_1})^{6+2k_1+2k_2}}d\lambda_1 = \frac{(e^{\lambda_1} + e^{-\lambda_1})^{\beta_1+\beta_2}}{(e^{\lambda_1} - e^{-\lambda_1})^{7+2k_1+2k_2}}dx_1,$$

so, recalling that $(e^\mu + e^{-\mu})^2 = (e^\mu - e^{-\mu})^2 + 4$, we obtain the following rule for the convolution operation:

$$\text{"convolution"}: \quad \underline{s_{k_1,\beta_1}(\lambda_q)s_{k_2,\beta_2}(\lambda_q)} = \begin{cases} s_{k_1+k_2+2,\beta_1+\beta_2}(\lambda_1) & \beta_1 + \beta_2 < 2, \\ s_{k_1+k_2+1,0}(\lambda_1) + 4s_{k_1+k_2+2,0}(\lambda_1) & \beta_1 = \beta_2 = 1. \end{cases} \quad (5.9)$$

The second operation we need is the "breeding" operation, which we encounter in the first term in the right-hand side of (5.6). This operation produces a term bilinear in $s_{k_1,\beta_1}(\lambda_1)$ and $s_{k_2,\beta_2}(\lambda_p)$ out of $s_{k,\beta}(\lambda_q)$:

$$\text{"breeding"}: \quad \sum_{\text{res } dx_q = 0} K(p_1, q)\big[B(p_k, q) + B(\overline{p}_k, q)\big]s_{k,\beta}(\lambda_q)dx_q, \quad (5.10)$$

where we can again do the integration by residues at $q = p_1$ and $q = p_k$ (for the term with $B(p_k, q)$) and at $q = p_1$ and $q = \overline{p}_k$ (for the term with $B(\overline{p}_k, q)$). The calculations involve combinatorics of geometric progression type but are otherwise straightforward. Two cases, $\beta = 0$ and $\beta = 1$, are rather different, so two integrations give

$$\sum_{\text{res } dx_q = 0} K(p_1, q)\big[B(p_k, q) + B(\overline{p}_k, q)\big]s_{k,0}(\lambda_q)dx_q = \sum_{m=0}^{k}(2 + 2k - 2m)s_{m,0}(\lambda_1)s_{k-m,0}(\lambda_p)dx_1dx_p$$

$$+ \sum_{m=0}^{k+1}(3 + 2k - 2m)4s_{m,0}(\lambda_1)s_{k+1-m,0}(\lambda_p)dx_1dx_p + \sum_{m=0}^{k+1}(3 + 2k - 2m)s_{m,1}(\lambda_1)s_{k+1-m,1}(\lambda_p)dx_1dx_p$$

and

$$\sum_{\text{res } dx_q = 0} K(p_1, q)\big[B(p_k, q) + B(\overline{p}_k, q)\big]s_{k,1}(\lambda_q)dx_q = \sum_{m=0}^{k}(2 + 2k - 2m)s_{m,1}(\lambda_1)s_{k-m,0}(\lambda_p)dx_1dx_p$$

$$+ \sum_{m=0}^{k}(1 + 2k - 2m)s_{m,0}(\lambda_1)s_{k-m,1}(\lambda_p)dx_1dx_p$$

$$+ \sum_{m=0}^{k+1}4(3 + 2k - 2m)\Big[s_{m,0}(\lambda_1)s_{k+1-m,1}(\lambda_p) + s_{m,1}(\lambda_1)s_{k+1-m,0}(\lambda_p)\Big]dx_1dx_p.$$

The thus obtained "breeding" operation reads:

$$\text{"breeding"}: \quad \big(s_{k,0}(\lambda_q)\big)^\uparrow = \sum_{m=0}^{k}(2 + 2k - 2m)s_{m,0}(\lambda_1)s_{k-m,0}(\lambda_p)$$

$$+ \sum_{m=0}^{k+1}(3 + 2k - 2m)4s_{m,0}(\lambda_1)s_{k+1-m,0}(\lambda_p) + \sum_{m=0}^{k+1}(3 + 2k - 2m)s_{m,1}(\lambda_1)s_{k+1-m,1}(\lambda_p) \quad (5.11)$$

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and

$$\left(s_{k,1}(\lambda_q)\right)^{\uparrow} = \sum_{m=0}^{k}(2+2k-2m)s_{m,1}(\lambda_1)s_{k-m,0}(\lambda_p) + \sum_{m=0}^{k}(1+2k-2m)s_{m,0}(\lambda_1)s_{k-m,1}(\lambda_p)$$

$$+ \sum_{m=0}^{k+1} 4(3+2k-2m)\Big[s_{m,0}(\lambda_1)s_{k+1-m,1}(\lambda_p) + s_{m,1}(\lambda_1)s_{k+1-m,0}(\lambda_p)\Big] \qquad (5.12)$$

The two starting correlation functions are

$$W_3^{(0)}(x_1, x_2, x_3) = 4s_{0,0}(\lambda_1)s_{0,0}(\lambda_2)s_{0,0}(\lambda_3)dx_1 dx_2 dx_3$$
$$+ \big[s_{0,1}(\lambda_1)s_{0,1}(\lambda_2)s_{0,0}(\lambda_3) + s_{0,1}(\lambda_1)s_{0,0}(\lambda_2)s_{0,1}(\lambda_3) + s_{0,0}(\lambda_1)s_{0,1}(\lambda_2)s_{0,1}(\lambda_3)\big]dx_1 dx_2 dx_3 \quad (5.13)$$

and

$$W_1^{(1)}(x) = s_{1,0}(\lambda)dx. \qquad (5.14)$$

5.2 Recursion relations determining $\widehat{b}_{\vec{k},\vec{\beta}}^{(g)}$

The convenient graphical form of representing the breeding and convolution relations from the previous subsection is by using Young tableauxes (as usual, to save the space, if one of the tableauxes is empty, we just omit it); we let the first (white) tableaux denote Y_0 and the second (grey) tableaux denote Y_1, for example, $W_3^{(0)} = 4\,\square\square\square + 1\,\square\blacksquare\blacksquare$ and $W_1^{(1)} = 1\,\blacksquare$, and the coefficients of products of "white" and "grey" tableauxes are precisely $\widehat{b}_{Y_0,Y_1}^{(g)}$.

The *breeding* operation then produces two columns (labeled "1" and "p") out of one column of every sort in accordance with the following rules:

- We take exactly one column of every sort for the breeding (the sorts differ by heights and colours of columns);

- We absorb the obtained columns labeled "1" and "p" into the obtained product of Young tableauxes (other columns remain unaltered); if, among the remaining columns, we have k columns of the same sort as the column labeled "p", we multiply the resulting tableaux by $k+1$, after which we erase the label p but retain the label "1".

So, as the result of breeding, we obtain a linear combination of products of two Young tableauxes with positive integer coefficients. Exactly one column in each product is labeled "1".

Example 3. We first calculate $W_4^{(0)}$. Because in this case no convolution operations are possible, the whole answer is obtained by the breeding of $W_3^{(0)}$. For elements of this Young tableauxes we have:

$$(\square)^{\uparrow} = 2\,\underset{1}{\square}\cdot\underset{p}{\square} + 4\,\underset{1}{\boxminus}\cdot\underset{p}{\square} + 12\,\underset{1}{\square}\cdot\underset{p}{\boxminus} + 3\,\underset{1}{\square}\cdot\underset{p}{\boxminus} + 1\,\underset{1}{\boxminus}\cdot\underset{p}{\square} \qquad (5.15)$$

$$(\blacksquare)^{\uparrow} = 2\,\underset{1}{\blacksquare}\cdot\underset{p}{\square} + 4\,\underset{1}{\boxminus}\cdot\underset{p}{\square} + 1\,\underset{1}{\square}\cdot\underset{p}{\blacksquare} + 4\,\underset{1}{\boxminus}\cdot\underset{p}{\square} + 12\,\underset{1}{\square}\cdot\underset{p}{\boxminus} + 12\,\underset{1}{\square}\cdot\underset{p}{\boxminus}. \qquad (5.16)$$

So, for elements of $W_3^{(0)}$, we obtain (we explicitly segregate the multipliers appearing due to the

symmetrisation w.r.t. p):

$$(W_3^{(0)})^\uparrow = 4(\square)^\uparrow\,\square\square + (\square)^\uparrow\,\square\square + \square\,(\square)^\uparrow\,\square$$

$$= 8\cdot 3\underset{1\ p}{\square\square\square\square} + 16\cdot 3\underset{1\ p}{\boxed{\square}\square\square} + 48\underset{p\ 1}{\boxed{\square}\square\square} + 12\underset{p\ 1}{\square\square\boxed{\square}} + 4\underset{1\ p}{\square\square\boxed{\square}}$$

$$+2\underset{1\ p}{\square\square\,\square\square} + 4\underset{1\ p}{\boxed{\square}\,\square\square} + 12\underset{p\ 1}{\boxed{\square}\,\square\square} + 3\underset{p\ 1}{\boxed{\square}\square\square} + 1\cdot 3\underset{1\ p}{\boxed{\square}\square\square}$$

$$+2\cdot 2\underset{p\ 1}{\square\square\,\square\square} + 4\cdot 2\underset{p\ 1}{\square\square\boxed{\square}} + 1\cdot 2\underset{1\ p}{\square\square\,\square\square} + 4\cdot 2\underset{1\ p}{\boxed{\square}\,\square\square} + 12\underset{1\ p}{\square\square\boxed{\square}} + 12\underset{p\ 1}{\boxed{\square}\,\square\square}$$

$$= 24\underset{1}{\square\square\square\square} + 48\left(\underset{1}{\boxed{\square}\square\square} + \underset{1}{\boxed{\square}\square\square}\right) + 12\left(\underset{1}{\square\square\boxed{\square}} + \underset{1}{\square\square\boxed{\square}} + \underset{1}{\square\square\boxed{\square}}\right)$$

$$+12\left(\underset{1}{\boxed{\square}\,\square\square} + \underset{1}{\boxed{\square}\,\square\square} + \underset{1}{\boxed{\square}\,\square\square}\right) + 3\left(\underset{1}{\boxed{\square}\square\square} + \underset{1}{\boxed{\square}\square\square}\right) + 4\left(\underset{1}{\square\square\,\square\square} + \underset{1}{\square\square\square\square}\right). \qquad (5.17)$$

We see that we have automatically obtained *symmetrized* expressions w.r.t. p_1: every term in brackets contains exactly one appearance of label "1" for every sort of columns. The total answer is therefore totally symmetric in all its arguments and reads

$$W_4^{(0)} = 24\underset{1}{\square\square\square\square} + 48\underset{1}{\boxed{\square}\square\square} + 12\underset{1}{\square\square\boxed{\square}} + 12\underset{1}{\boxed{\square}\,\square\square} + 3\underset{1}{\boxed{\square}\square\square} + 4\underset{1}{\square\square\,\square\square}. \qquad (5.18)$$

Here the coefficients of summands in this expression are precisely $\widehat{b}_{\vec{k},\vec{\beta}}$ for the planar four-backbone case.

The *convolution* operation produces one column labeled "1" (or a linear combination of such columns) out of two columns by the following rules:

$$\underset{n_1}{\updownarrow}\boxed{\square}\,\boxed{\square}\underset{n_2}{\updownarrow} = \underset{1}{\boxed{\square}}^{\,n_1+n_2+1}, \qquad \underset{n_1}{\updownarrow}\boxed{\square}\,\boxed{\square}\underset{n_2}{\updownarrow} = \underset{1}{\boxed{\square}}^{\,n_1+n_2+1}, \qquad \underset{n_1}{\updownarrow}\boxed{\square}\,\boxed{\square}\underset{n_2}{\updownarrow} = 4\,\underset{1}{\boxed{\square}}^{\,n_1+n_2+1} + \boxed{\square}^{\,n_1+n_2} \qquad (5.19)$$

1. When we do convolution *inside* the same product $Y_0\,Y_1$, we must make all possible convolutions between different types of columns (one convolution per every pair of types) and convolutions inside the same type (if we have more than one column of this type in Y_0 or in Y_1). The additional factors are:

- we have a factor of two if we make the convolution between different types of columns;

- we have an additional factor of two if we make a convolution in a term of $W_s^{(g)}$ with $s > 2$, i.e., if the result of convolution inside a Young tableaux product $Y_0\,Y_1$ contains more than one column.

2. When we do convolution between *two separate* products $Y_0\,Y_1$ and $Y_0'\,Y_1'$, we must make all possible convolutions between all column types in the first product and in the second product (one convolution per every pair of types from different products) unless $Y_0\,Y_1 = Y_0'\,Y_1'$; in the latter case we take into account every type of pairings between entries of the tableaux $Y_0\,Y_1$ only once. We then consider the union of two products; as the result, we obtain a linear combination of Young tableauxes of the form

$$\underbrace{}_{s_k}\ \underbrace{}_{s_k'}\cdots\underset{1}{\underbrace{}}\,\underbrace{}_{s_l}\ \underbrace{}_{s_l'}\cdots\underbrace{}_{s_2}\ \underbrace{}_{s_2'}\ \underbrace{}_{s_1}\ \underbrace{}_{s_1'}\cdots\underbrace{}_{t_r}\ \underbrace{}_{t_r'}\cdots\underbrace{}_{t_1}\ \underbrace{}_{t_1'}, \qquad (5.20)$$

where we have exactly one column labeled "1" and in every term s_j columns come from Y_0, s_j' columns come from Y_0' and, correspondingly, t_i columns come from Y_1 and t_i' columns come from Y_1'.

The combinatorial factors are:

- we multiply the obtained Young tableaux by the product of binomial factors:

$$\prod_{j=1}^{k}\binom{s_j + s_j'}{s_j}\prod_{i=1}^{r}\binom{t_i + t_i'}{t_i};$$

- we multiply by a factor of two if we make the convolution between different types of columns and/or if we make a convolution between two different tableauxes (i.e., if $Y_0 \neq Y_0'$ and/or $Y_1 \neq Y_1'$ (in other words, the only situation when we do not have this factor is when we make a convolution between two equal Young tableauxes, $Y_0 = Y_0'$ and $Y_1 = Y_1'$ and we convolve terms of the same type in these two tableauxes);

- we multiply by an additional factor of two if the result of convolution between Young tableauxes $Y_0 Y_1$ and $Y_0' Y_1'$ contains more than one column, that is, if $|Y_0 Y_1| > 1$ and/or $|Y_0' Y_1'| > 1$ (in other words, the only situation when we do not have this factor is when $s_1 = s_2 = 1$).

Example 4. We next calculate $W_2^{(1)}$ (in the third line, we explicitly indicate the combinatorial factors due to the convolution process):

$$
\begin{aligned}
W_2^{(1)} &= (W_1^{(1)})^{\uparrow} + \underline{W_3^{(0)}} = (\boxbox)^{\uparrow} + 4\,\square\square\square + \square\square\square + \square\,\square\square \\
&= 4\,\square + 12\,\boxbox + 2\,\square + 4\,\square + 20\,\square + 5\,\square + 3\,\boxbox + \square \\
&\quad + 4\cdot 2\,\square + 1\cdot 4\,\square + 4\cdot 2\,\square + 1\cdot 2\,\square \\
&= 4\big(\square + \square\big) + 12\,\boxbox + 20\big(\square + \square\big) + 5\big(\square + \square\big) + 3\,\boxbox \\
&= 4\,\square + 12\,\boxbox + 20\,\square + 5\,\square + 3\,\boxbox.
\end{aligned}
\tag{5.21}
$$

Having this expression and $W_1^{(1)}$, we can now calculate $W_1^{(2)}$:

$$
\begin{aligned}
W_1^{(2)} &= \underline{W_2^{(1)}} + \underline{W_1^{(1)} \times W_1^{(1)}} = 4\,\square + 12\,\boxbox + 20\,\square + 5\,\square + 3\,\boxbox + \square \times \square \\
&= 4\cdot 2\,\square + 12\,\square + 20\cdot 2\,\square + 5\cdot 2\big(4\,\square + \square\big) + 3\big(4\,\square + \square\big) + 1\cdot 1\,\square \\
&= 3\cdot 5\cdot 7\,\square + 3\cdot 7\,\square.
\end{aligned}
\tag{5.22}
$$

The same answer follows from the Harer–Zagier recursion relation: $b_1^{(2)} = 3\cdot 5\cdot 7$, $b_0^{(2)} = 3\cdot 7$.

Example 5. The first example in which we have all three above operations is calculating $W_3^{(1)}$:

$$W_3^{(1)} = \big(W_2^{(1)}\big)^{\uparrow} + \underline{W_4^{(0)}} + \underline{W_3^{(0)} \times W_1^{(1)}}.
\tag{5.23}$$

Here, the first term (with breeding) contains 67 summands, the second contains 21 summands, and the third contains only three summands, and performing the summation we again obtain the result that is totally symmetric in all p_i including p_1 and reads

$$
\begin{aligned}
W_3^{(1)} &= 24\,\square + 192\,\square + 240\,\square + 288\,\boxbox + 480\,\square + 560\,\square + 30\,\square + 18\,\square + 24\,\square \\
&\quad + 120\,\square + 72\,\boxbox + 140\,\square + 120\,\square + 120\,\square + 30\,\square + 140\,\square.
\end{aligned}
\tag{5.24}
$$

One more example is $W_5^{(0)}$:

$$W_5^{(0)} = \left(W_4^{(0)}\right)^\uparrow + \underline{W_3^{(0)} \times W_3^{(0)}}.\tag{5.25}$$

The first term comprises 85 summands and the second term comprises seven summands presenting below for clarifying the symmetry coefficients count (we omit unit binomial coefficients)

$$\underline{W_3^{(0)} \times W_3^{(0)}} \; = \; 16\,\square\square\square \times \square\square\square + 4\,\square\square\square \times \square\;\square\square + 4\,\square\square\square \times \square\square\square + \square\square\square \times \square\;\square\square + \square\square\square \times \square\square\square + \square\;\square\square \times \square\square\square$$

$$= \; 16\cdot 2\cdot\binom{4}{2}\square\square\square\square\square + 4\cdot 2\cdot 2\,\square\square\square\square + 4\cdot 2\cdot 2\cdot\binom{3}{1}\square\square\square\square$$

$$+\,2\cdot\binom{4}{2}\square\square\square\square\square + 2\cdot 2\cdot\binom{3}{1}\square\square\square\square\square + 2\cdot\binom{2}{1}\cdot\binom{2}{1}\square\square\left(4\square + \square\right)\square\square.$$

Here, because we are convolving identical objects, we take into account every type of convolution only once, but if we convolve different entries or different types of columns inside the same entry, we have to multiply by two. One factor of two is always present because the result contains more than one column.

The sum in (5.25) is totally symmetric in all p_i including p_1 and reads

$$W_5^{(0)} \; = \; 192\,\square\square\square\square\square + 768\,\square\square\square\square + 1152\,\square\square\square\square + 960\,\square\square\square\square + 144\,\square\square\square\square + 240\,\square\square\square\square$$

$$+288\,\square\square\square\square + 288\,\square\square\square\square + 144\,\square\square\square\square + 288\,\square\square\square\square + 240\,\square\square\square\square + 24\,\square\square\square\square$$

$$+72\,\square\square\square\square + 60\,\square\square\square\square + 18\,\square\square\square\square + 72\,\square\square\square\square + 12\,\square\square\square\square + 60\,\square\square\square\square\tag{5.26}$$

We therefore arrive at the main statement of this section.

Theorem 6. *All the coefficients $\widehat{b}^{(g)}_{\vec{k},\vec{\beta}}$ of expansions of the loop means from Lemma 5 in the range determined by relations (3.20) are positive integers.*

The **proof** follows from that all the coefficients in the breeding and convolution relations are positive integers, the result is automatically symmetric w.r.t. permutations of all arguments including p_1, and in the breeding and convolution relations we have all terms provided these terms satisfy restrictions in (3.20).

6 The one-backbone case

6.1 The Harer–Zagier recursion and the graph decomposition from Sec. 3

In the one-backbone case, we have the representation (3.13) and the alternative representation

$$W_1^{(g)}(e^\lambda + e^{-\lambda}) = \sum_{r=0}^{3g-2}(-1)^r\frac{\varkappa_{g,1,r}}{2^{d-r}(d-r)!}\frac{1}{e^\lambda - e^{-\lambda}}\left(\frac{\partial}{\partial\lambda}\right)^{2d-2r+1}\frac{2}{e^{2\lambda}-1},\quad d = 3g-2,\tag{6.1}$$

where on the base of reasonings related to stratification of closed moduli spaces, $\varkappa_{g,1,r}$ are (conjecturally positive) rational numbers, $\varkappa_{g,1,0} = \langle\tau_{3g-2}\rangle_g$.

For the coefficients $b_i^{(g)}$ of (3.13) based on the Harer and Zagier recurrent formula [28], we have obtained the recurrence relation (also found in [26])

Proposition 1. [5] *The coefficients $b_k^{(g)}$ from (3.13) satisfy the three-term recurrence relation:*

$$(4g + 2k + 6)b_k^{(g+1)} = (4g + 2k + 1)(4g + 2k + 3)\left[(4g + 2k + 2)b_k^{(g)} + 4(4g + 2k - 1)b_{k-1}^{(g)}\right]. \quad (6.2)$$

All these coefficients are positive integers.

(Of course, the positive integrality of $b_k^{(g)}$ is a particular case of the general Theorem 6.)

In [5], we used recursion (6.2) to develop a method allowing determining $b_{g-1-k}^{(g)}$ for any fixed $k \geq 0$ and for all g. For example, we have just two-term relations for the boundary coefficients

$$(4g + 6)b_0^{(g+1)} = (4g - 1)(4g + 3)(4g + 2)b_0^{(g)},$$
$$(6g + 6)b_g^{(g+1)} = 4(6g + 1)(6g + 3)(6g - 1)b_{g-1}^{(g)},$$

which immediately give

$$b_{g-1}^{(g)} = \frac{2^{g-1}(6g - 3)!!}{3^g\, g!}, \qquad b_0^{(g)} = \frac{(4g)!}{8^g\, g!\,(2g + 1)!!}. \quad (6.3)$$

Substituting $b_{g-1}^{(g)}$ into (6.1) and evaluating the leading term ($r = 0$) we obtain the highest Kontsevich coefficient $\varkappa_{g,1,0} = \langle \tau_{3g-2} \rangle_g = \frac{1}{2^{3g}\, 3^g\, g!}$.

Solving recursion (6.2) for the first subleading term, we have obtained

$$b_{g-2}^{(g)} = \frac{1}{5}\frac{2^{g-2}(6g - 5)!!}{3^{g-2}(g - 2)!}, \quad \text{or} \quad \varkappa_{g,1,1} = \frac{1}{5}[12g^2 - 7g + 5]\varkappa_{g,1,0}, \quad g \geq 2. \quad (6.4)$$

For the next term, we have

$$b_{g-3}^{(g)} = \frac{(2g - 1)\, 2^{g-3}(6g - 7)!!}{5^2\, 3^{g-3}(g - 3)!} - \frac{7\, 2^{g-3}(6g - 7)!!}{10\,(3g - 2)!!!}, \quad \text{where } (3g - 2)!!! \equiv \prod_{k=3}^{g}(3k - 2), \quad (6.5)$$

etc. The complete multi-step procedure was described in [5].

We can alternatively derive $b_{g-2}^{(g)}$ from the graph representation of Lemma 4. For this, it suffices to take only the part with the times T_{2k}^+. The highest term for genus g is $\langle \tau_{3g-2} \rangle_g T_{6g-4}^+$

Following Lemma 4, the first-order correction, or the coefficient of T_{6g-6}^+, comes only from two terms: from the graph with one vertex and one internal edge with endpoint markings $(0, 0)$ and from the graph with one vertex and one half-edge with marking 2 (see Fig. 5): the corresponding coefficient is then

$$\frac{B_2}{4}\langle \tau_{3g-3}\tau_0\tau_0 \rangle_{g-1} + \frac{2^3}{5!}\langle \tau_{3g-3}\tau_2 \rangle_g, \quad (6.6)$$

and we need only to know the corresponding intersection indices. Whereas $\langle \tau_{3g-3}\tau_0\tau_0 \rangle_{g-1} = \langle \tau_{3g-5} \rangle_{g-1}$, in [5] we have calculated the intersection index $\langle \tau_{3g-3}\tau_2 \rangle_g$ using the Virasoro conditions for the Kontsevich matrix model; the result is

$$\langle \tau_2 \tau_{3g-3} \rangle_g = \frac{1}{5}[12g(g - 1) + 5]\langle \tau_{3g-2} \rangle_g, \quad g \geq 2. \quad (6.7)$$

Using formula (6.7) and that $B_2 = 1/24$, we obtain that the coefficient of T_{6g-6}^+ is

$$\frac{1}{5}[12g^2 - 7g + 5] \quad (6.8)$$

in full agreement with (6.4).

Below we present the third calculation of the same quantity using the explicit diagram counting.
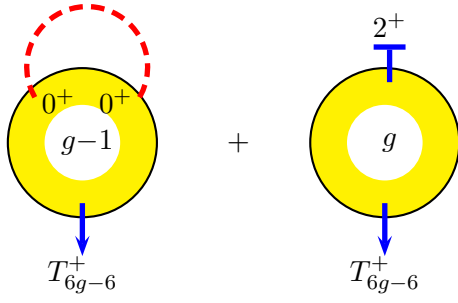
Figure 5: The two diagrams contributing to $b_{g-2}^{(g)}$.

6.2 2-cycles and the recursion for $b_{g-1}^{(g)}$ and $b_{g-2}^{(g)}$ terms

6.2.1 Contracting edges in genus-g graphs

We now find $b_{g-2}^{(g)}$ using the explicit fat graph counting. For this, we consider the set of shapes with one boundary component and one marked edge. We let $\Gamma^{(g)}$ denote the sets of combinatorial types of the corresponding shapes of genus g and let $V^{(g)}$ denote cardinalities of these sets.

We first consider the procedure of edge contraction in the genus g graphs. We let $\Gamma_{q;3-3}^{(g)}$ denote the set of genus-g shapes with the marked edge with all vertices having valence three *and with q 2-cycles* (all these 2-cycles are of the form as in the rightmost diagram in Fig. 6). We let $V_{q;3-3}^{(g)}$ denote the number of such diagrams. We let $\Gamma_{4,3-3}^{(g)}$, $\Gamma_{4,4,3-3}^{(g)}$, and $\Gamma_{5,3-3}^{(g)}$ denote the respective sets of of genus-g shapes with the marked edge and with one four-valent vertex, two four-valent vertices, and one five-valent vertex and with all other vertices having valence three. The numbers of the corresponding shapes are $V_{4,3-3}^{(g)}$, $V_{4,4,3-3}^{(g)}$, and $V_{5,3-3}^{(g)}$.

We now consider the contraction process. We never contract the marked edge corresponding to the ends of the backbone and can contract any other edge in any graph from $\Gamma_{q;3-3}^{(g)}$ (there are $6g-4$ contractible edges in total) every time obtaining a graph from $\Gamma_{4,3-3}^{(g)}$. Vice versa, every graph from $\Gamma_{4,3-3}^{(g)}$ can be obtained from two graphs in $\Gamma_{q;3-3}^{(g)}$; we therefore have the equality

$$(6g-4)\sum_{q=0}^{\max} V_{q;3-3}^{(g)} = 2V_{4,3-3}^{(g)} = (6g-4)b_{g-1}^{(g)}.$$

A more interesting situation occurs when we want to contract two edges. We have three possible outcomes:

1 when we contract two disjoint edges we obtain a graph from $V_{4,4,3-3}^{(g)}$;

2 when we contract two edges with incidence one we obtain a graph from $V_{5,3-3}^{(g)}$;

3 we do not allow contracting two edges with incidence two (which therefore constitute a 2-loop).

We consider the first case first. The total number of disjoint pair of edges is

$$\frac{1}{2}(6g-4)(6g-5) - \# \text{ of incident pairs of edges}. \tag{6.9}$$

The number of edges of incidence one and two can be easily counted: this is three times the number of vertices minus 4 because of the marked edge minus twice the number of 2-loops in a graph from

$V_{q,3-3}^{(g)}$, i.e.,

$$3(4g-2) - 4 - 2q;$$

the number of pairs of incidence two is obviously q. Then the total number of nonincident pairs can be easily counted to be

$$(3g-4)(6g-5) + q.$$

Note that from each such pair we produce a graph in $V_{4,4,3-3}^{(g)}$, and each graph from $V_{4,4,3-3}^{(g)}$ can be produced exactly in four ways from the graphs from $V_{q;3-3}^{(g)}$ with some q (it might be the same graph from $V_{q;3-3}^{(g)}$ that produces a graph from $V_{4,4,3-3}^{(g)}$, we then count this case with the corresponding multiplicity. The resulting relation reads

$$\sum_{q=0}^{\max} \big[(3g-4)(6g-5) + q\big] V_{q;3-3}^{(g)} = 4 V_{4,4,3-3}^{(g)}. \tag{6.10}$$

Analogously, each graph from $V_{5,3-3}^{(g)}$ can be obtained by contracting two edges with incidence one by exactly five ways from graphs in $V_{q;3-3}^{(g)}$, that is, we obtain that

$$\sum_{q=0}^{\max} \big[12g - 10 - 2q\big] V_{q;3-3}^{(g)} = 5 V_{5,3-3}^{(g)}. \tag{6.11}$$

The total number of diagrams with $6g-6$ nonmarked edges is precisely the sum of $V_{4,4,3-3}^{(g)}$ and $V_{5,3-3}^{(g)}$, and it is given by a combination of b factors, so we obtain

$$V_{4,4,3-3}^{(g)} + V_{5,3-3}^{(g)} = b_{g-2}^{(g)} + \frac{(3g-2)(3g-3)}{2} b_{g-1}^{(g)}, \tag{6.12}$$

and we have three above equations on three unknowns $V_{4,4,3-3}^{(g)}$, $V_{5,3-3}^{(g)}$, and $\sum_{q=1}^{\max} q V_{q;3-3}^{(g)}$. The solution reads

$$V_{4,4,3-3}^{(g)} = \frac{1}{4}\Big[(3g-4)(6g-5) b_{g-1}^{(g)} + g b_{g-1}^{(g)} - \frac{20}{3} b_{g-2}^{(g)}\Big]; \tag{6.13}$$

$$V_{5,3-3}^{(g)} = 2(g-1) b_{g-1}^{(g)} + \frac{8}{3} b_{g-2}^{(g)}; \tag{6.14}$$

$$\sum_q q V_{q;3-3}^{(g)} = g b_{g-1}^{(g)} - \frac{20}{3} b_{g-2}^{(g)}. \tag{6.15}$$

Observe that there is another particular combination of V's that produce an interesting relation

$$(2!) 2^2 V_{4,4,3-3}^{(g)} + 5 V_{5,3-3}^{(g)} = (6g-5)(6g-6) V_{3-3}^{(g)}, \quad \text{where} \quad V_{3-3}^{(g)} = \sum_{q=0}^{\max} V_{q;3-3}^{(g)}. \tag{6.16}$$

(We have verified the validity of this relation for $g = 3$ using the data from [34].)

6.2.2 Blowing up process, $g \to g+1$

We now consider the "inverse" process depicted in Fig. 6, which enables us to blow up a handle from a pair of marked sides of edges in a graph from $\Gamma_{q;3-3}^{(g)}$; the number of 2-cycles is irrelevant here. At the first stage we allow "bubbling" as in the middle diagram in Fig. 6 of two sides of edges; we must now allow this bubbling on the marked edge as well. We also must present this marked edge as a

subdiagram comprising three edges joined in a single vertex: two edges are incident to the rest of the diagram (their ends are the ends of the marked edge and are therefore always different), the third edge is the tail. (We can consider bubbling process in order, then, on the first stage, we have $2(6g-1)$ possibilities of setting a bubble on an edge side whereas on the second stage we have already $2(6g+1)$ such possibilities because we increased the total number of edges by two in the first process. So, the total number of possibilities is

$$\frac{1}{2}2^2(6g-1)(6g+1) = 2(6g-1)(6g+1).$$

Every time we bubble a graph from $\Gamma_{3-3}^{(g)}$ we obtain a graph from $\Gamma_{q;3-3}^{(g+1)}$ with $q \neq 0$. Vice versa, every graph from $\Gamma_{q;3-3}^{(g+1)}$ with nonzero q can be obtained in exactly q ways from graphs from $\Gamma_{3-3}^{(g)}$. (Note that the number of 2-cycles does not necessarily increase in this process: if we bubble a side of an edge entering a 2-cycle in the initial graph, we destroy this 2-cycle, so, in principle, we can even reduce the number of 2-cycles in this process, but every time we obtain a graph of genus $g+1$ with at least one 2-cycle.
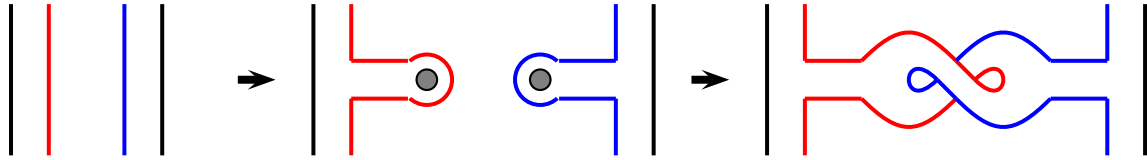

Figure 6: The procedure of gluing the handle into two sides of two arbitrary edges of a three-valent graph $\Gamma_{3...3}$, which increase the genus by one. We can think about it as of blowing up a handle from a pair of punctures.

We therefore have the relation

$$2(6g-1)(6g+1)V_{3-3}^{(g)} = \sum_{q=0}^{\max} q V_{q;3-3}^{(g+1)}, \tag{6.17}$$

from which, substituting the result in (6.15) and recalling that $V_{3-3}^{(g)}$ is merely $b_{g-1}^{(g)}$, we obtain the new relation on b's:

$$2(6g-1)(6g+1)b_{g-1}^{(g)} = (g+1)b_g^{(g+1)} - \frac{20}{3}b_{g-1}^{(g+1)}, \tag{6.18}$$

from which we immediately find that

$$b_{g-2}^{(g)} = \frac{3}{10}\frac{g(g-1)}{2g-1}b_{g-1}^{(g)}, \tag{6.19}$$

which coincides with (6.4).

Conclusion

Application of topological recursion (TR) to constructing generating functions for cohomological field theories is becoming an important issue in contemporary mathematical physics (see, e.g., the recent paper [23] where all genus all descendants equivariant Gromov-Witten invariants of $\mathbb{P}^1$ were constructed using TR). In this respect, it seems interesting to understand the status of Givental-type decompositions in the quantum spectral curve approach.

Acknowledgments

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