Volumes of moduli spaces of hyperbolic surfaces with cone points

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Abstract

In this paper we study volumes of moduli spaces of hyperbolic surfaces with geodesic, cusp and cone boundary components. We compute the volumes in some new cases, in particular when there exists a large cone angle. This allows us to give geometric meaning to Mirzakhani's polynomials under substitution of imaginary valued boundary lengths, corresponding to hyperbolic cone angles, and to study the behaviour of the volume under the 2π limit of a cone angle.

Contents

1 Introduction

Define the moduli space of hyperbolic surfaces with labeled boundary components

$$\mathcal{M}^{\text{hyp}}_{g,n}(L_1,...,L_n)=\Big\{(\Sigma,\beta_1,...,\beta_n)\mid\Sigma\text{ oriented hyperbolic surface,}\\ \text{genus }\Sigma=g,\partial\Sigma=\sqcup\beta_j,2\cosh(L_j/2)=|\mathrm{tr}(A_j)|\Big\}/\sim$$

where $A_j \in PSL(2,\mathbb{R})$ represents the conjugacy class defined by the holonomy of the metric around the boundary component β_j. The quotient is by isometries preserving each β_j and its associated conjugacy class. Each $L_j \in \mathbb{R}_{\geq 0} \cup i\,\mathbb{R}_{>0}$ and the boundary component is geometrically realised by a geodesic of length $L_j \in \mathbb{R}_{>0}$, or a cusp when $L_j = 0$, or a cone angle $\theta_j = -iL_j \in \mathbb{R}_{>0}$.

The moduli space $\mathcal{M}_{g,n}^{\text{hyp}}(L_1, ..., L_n)$ comes equipped with a measure which can be constructed in different ways. It is given by torsion of the natural complex defined by the associated flat $PSL(2,\mathbb{R})$ connection [28], or via a symplectic form $\omega^{WP}(\mathbf{L})$ defined naturally on the character variety [12] and via Fenchel-Nielsen coordinates when they exist [32]. See [26, equation 3.44] for a proof that the measure constructed via torsion coincides with the measure induced from the symplectic form defined via Fenchel-Nielsen coordinates. There is a natural homeomorphism from $\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})$, when $\mathbf{L} \in \mathbb{R}^n_{\geq 0}$, to the moduli space of genus g curves with n labeled points $\mathcal{M}_{g,n}$. The family of symplectic forms $\omega^{WP}(\mathbf{L})$ defines a deformation of the Weil-Petersson form $\omega^{WP} = \omega^{WP}(\mathbf{0})$ naturally defined on $\mathcal{M}_{g,n}$.

When $\mathbf{L} = (L_1, ..., L_n) \in \mathbb{R}^n_{\geq 0}$, Mirzakhani [18] proved that the volume

$$\text{Vol}(\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})) = \int_{\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})} \exp \omega^{WP}(\mathbf{L})$$

is finite and given by a symmetric polynomial $V_{g,n}(L_1, ..., L_n) \in \mathbb{R}[L_1, ..., L_n]$. She proved this in two ways:

(I) via a recursion [18] which uniquely determines the volumes from the initial calculations $\text{Vol}(\mathcal{M}_{0,3}^{\text{hyp}}(L_1, L_2, L_3)) = 1$ and $\text{Vol}(\mathcal{M}_{1,1}^{\text{hyp}}(L)) = \frac{1}{48}(L^2 + 4\pi^2)$; and

(II) via a symplectic reduction argument [19] combined with the result of Wolpert [32] that ω^{WP} extends to a smooth closed form on the compactification $\overline{\mathcal{M}}_{g,n}$ and represents the cohomology class $2\pi^2\kappa_1 \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. Wolpert's argument generalises to show that $\omega^{WP}(\mathbf{L})$ extends to a compactification of $\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})$ homeomorphic to the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$. Mirzakhani proved that its cohomology class satisfies

$$[\omega^{WP}(\mathbf{L})] = [\omega^{WP}] + \sum_1^n \tfrac{1}{2} L_j^2 \psi_j \in H^2(\overline{\mathcal{M}}_{g,n}) \tag{1}$$

so that

$$\int_{\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})} \exp \omega^{WP}(\mathbf{L}) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(2\pi^2\kappa_1 + \sum_1^n \tfrac{1}{2} L_j^2 \psi_j\right) \in \mathbb{R}[L_1, ..., L_n] \tag{2}$$

In the equality (1) we identify $\omega^{WP}(\mathbf{L})$ and ω^{WP} with their extensions to $\overline{\mathcal{M}}_{g,n}$—see Section 2.

We *define* the polynomial $V_{g,n}(\mathbf{L})$ using Mirzakhani's algebraic geometric formula (2)

$$V_{g,n}(\mathbf{L}) := \int_{\overline{\mathcal{M}}_{g,n}} \exp\big(2\pi^2\kappa_1 + \sum_1^n \tfrac{1}{2}L_j^2\psi_j\big).$$

It is natural to ask whether the evaluation of $V_{g,n}(\mathbf{L})$ at any $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i\,\mathbb{R}_{>0}]^n$ corresponds to the volume of the moduli space $\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathbf{L})$. Indeed, Mirzakhani's results show that the answer is yes for $\mathbf{L} \in \mathbb{R}^n_{\geq 0}$, and her argument can be generalised to also give an affirmative answer when $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i[0,\pi)\}^n$ via a McShane identity on hyperbolic surfaces with cone angles, in which imaginary arguments appear [27], together with Mirzakhani's recursion argument (I). The argument in [27] relies on the existence of a pants decomposition and Fenchel-Nielsen coordinates and fails when any $\theta_j > \pi$, since a pants decomposition is no longer guaranteed—see Appendix A.

The answer is sometimes no, i.e. evaluation of a polynomial $V_{g,n}(\mathbf{L})$ at $L_j = i\theta$ for $\theta > \pi$ does not necessarily correspond to an actual volume. For example, Mirzakhani's polynomial $V_{0,4}(L_1, L_2, L_3, L_4) = 2\pi^2 + \frac{1}{2}\sum_{j=1}^4 L_j^2$ evaluates to be negative in the following example,

$$V_{0,4}(0, 0, \theta i, (2\pi - \epsilon)i) < 0 \quad \Leftarrow \quad 4\pi\epsilon < \theta^2 < 4\pi^2$$

whereas the volume of the non-empty moduli space $\mathcal{M}_{0,4}^{\mathrm{hyp}}(0, 0, \theta i, (2\pi - \epsilon)i)$ is necessarily positive.

Via relations between intersection numbers on $\overline{\mathcal{M}}_{g,n}$, the polynomials $V_{g,n}(\mathbf{L})$ were proven in [7] to satisfy the following relations:

$$V_{g,n+1}(\mathbf{L}, 2\pi i) = \sum_{k=1}^n \int_0^{L_k} L_k V_{g,n}(\mathbf{L}) dL_k, \qquad \frac{\partial V_{g,n+1}}{\partial L_{n+1}}(\mathbf{L}, 2\pi i) = 2\pi i(2g-2+n)V_{g,n}(\mathbf{L}) \quad (3)$$

These relations were generalised in [8]. The relations (3) evoke the idea of a 2π limit of a cone angle, however until now no geometric meaning has been deduced from these relations. This is because evaluation of the polynomials does not necessarily correspond to volumes close to the limit $\theta \to 2\pi$. One outcome of this paper is to describe occurrences when these relations do and do not have geometric meaning.

When $\mathbf{L} \in \{i[0, 2\pi)\}^n$, $\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathbf{L}) \cong \mathcal{M}_{g,n}$ which is proven in [15] by proving existence of conical metrics on any conformal surface. Further, $\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathbf{L})$ possesses a natural Kähler metric [25] which gives rise to a symplectic structure $\omega^{WP}(\mathbf{L})$ on $\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathbf{L})$. As mentioned above, Mirzakhani's symplectic reduction argument (II) in [19] and its generalisation using results of [27] crucially relies on two properties of the symplectic form $\omega^{WP}(\mathbf{L})$ for $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i[0,\pi)\}^n$: its smooth extension to the natural compactification $\overline{\mathcal{M}}_{g,n}^{\mathrm{hyp}}(\mathbf{L})$ by nodal cusp surfaces; and a homeomorphism of this compactification with the Deligne-Mumford compactification which extends the homeomorphism of moduli spaces of smooth surfaces, or equivalently commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{M}_{g,n} & \xrightarrow{\cong} & \mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L}) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \xrightarrow{\cong} & \overline{\mathcal{M}}_{g,n}^{\text{hyp}}(\mathbf{L}) \end{array} \tag{4}$$

The following two theorems extend to conical hyperbolic surfaces, possibly with a cone angle $\theta > \pi$, the two properties of a smooth extension of the Weil-Petersson form to a natural compactification, and its relation to the Deligne-Mumford compactification. Theorem 1.1 gives a smooth extension of $\omega^{WP}(\mathbf{L})$ to a natural compactification $\overline{\mathcal{M}}_{g,n}^{\text{hyp}}(\mathbf{L})$ by nodal cusp surfaces in the known case of $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i[0,\pi)\}^n$ and the new case of conical hyperbolic surfaces with $\mathbf{L} \in \{i[0,2\pi)\}^n$ proven in Proposition 2.5. The compactification $\overline{\mathcal{M}}_{g,n}^{\text{hyp}}(\mathbf{L})$ is not necessarily homeomorphic to the Deligne-Mumford compactification. Instead, it coincides with a compactification via weighted curves due to Hassett [14], who proves that the commutative diagram (4) exists, although the bottom arrow is not necessarily a homeomorphism.

Theorem 1.1. *When* $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i[0,\pi)\}^n$ *or* $\mathbf{L} \in \{i[0,2\pi)\}^n$ *then* $\omega^{WP}(\mathbf{L})$ *extends smoothly to the natural compactification by nodal hyperbolic surfaces* $\mathcal{M}_{g,n}^{hyp} \to \overline{\mathcal{M}}_{g,n}^{hyp}(\mathbf{L})$.

Hassett's compactification of the moduli space of weighted curves is isomorphic to the Deligne-Mumford compactification when the weights are chosen nicely, which in terms of cone angles is equivalent to $\sum_{j=1}^n \theta_j < 2\pi$. Applying Theorem 1.1 to this case produces new examples of the volume of the moduli space coinciding with evaluation of Mirzakhani's polynomial.

Theorem 1.2. *For* $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i[0,\pi)\}^n$ *or* $\mathbf{L} \in \left\{(i\theta_1, ..., i\theta_n) \in \{i[0,2\pi)\}^n \mid \sum_{j=1}^n \theta_j < 2\pi\right\}$

$$Vol(\mathcal{M}_{g,n}^{hyp}(\mathbf{L})) = V_{g,n}(\mathbf{L}) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(2\pi^2\kappa_1 + \tfrac{1}{2}\mathbf{L}\cdot\mathbf{L}\right)$$

i.e. the volume of the moduli space is obtained by evaluation of Mirzakhani's polynomial.

Remark. Theorems 1.1 and 1.2 do not allow the coexistence of boundary geodesics of positive length together with cone points of cone angle greater than π. In this case neither theorem holds because Mumford's compactness criterion fails, i.e. a compactification by nodal cusp surfaces does not exist. This is described in Section 2. The essential idea affecting the different behaviours of compactifications can be understood via studying two boundary components coming together, although the compactification involves several boundary components coming together. There is a positive lower bound on the distance between any two boundary components of type $L \in \mathbb{R}_{\geq 0} \cup i[0,\pi)$ and as the positive lower bound is approached, a cusp forms elsewhere on the surface. Whereas, no such positive lower bound occurs between a boundary component of type $L = i\theta \in i(\pi, 2\pi)$, i.e. a large cone angle, and a boundary component of type $L' \in \mathbb{R}_{>0}$, i.e. a positive length geodesic, for L' sufficiently large (depending on θ), resulting in the failure of Mumford's compactness criterion. In the case of two boundary components of type $L \in i[0, 2\pi)$, i.e. two cone angles, a positive lower bound on the distance between the cone points occurs when the sum of their cone angles is less than or equal to 2π, and a cusp forms elsewhere on the surface as they approach their lower bound. No positive lower bound on the distance between two cone points occurs when the sum of their cone angles is greater than 2π.

When we set all but one angle to be 0, we can consider the limit as the angle approaches 2π, and thus achieve geometric meaning from the relations (3).

Corollary 1.3. *In the 2π cone angle limit, the volumes satisfy*

$$\lim_{\theta \to 2\pi} Vol(\mathcal{M}^{hyp}_{g,n+1}(0^n, i\theta)) = 0, \quad \lim_{\theta \to 2\pi} \frac{\partial}{\partial \theta} Vol(\mathcal{M}^{hyp}_{g,n+1}(0^n, i\theta)) = 2\pi \chi_{g,n} \, Vol(\mathcal{M}^{hyp}_{g,n}(0^n))$$

where $\chi_{g,n} = 2 - 2g - n$.

The relations in Corollary 1.3 can be proven directly via geometry. The next corollary is a positivity result.

Corollary 1.4.

$$\sum_{j=1}^{n} \theta_j < 2\pi \quad \Rightarrow \quad V_{g,n}(i\theta_1, ..., i\theta_n) > 0. \tag{5}$$

The coefficients of Mirzakhani's polynomials are non-negative, hence positivity of the polynomials on $\mathbf{L} \in \mathbb{R}^n_{\geq 0}$ is obvious, even without the volume interpretation. The inequality (5) is not obvious as a property of the polynomials. The general case of imaginary L_j is subtler as shown in the following example.

$$\begin{aligned} V_{0,5}(\mathbf{L}) &= \frac{1}{8}\left(\sum_{j=1}^{5} L_j^4 + 4\sum_{j<k} L_j^2 L_k^2 + 24\pi^2 \sum_{j=1}^{5} L_j^2 + 80\pi^4 \right)\Bigg|_{L_j = i\theta_j} \\ &= \tfrac{1}{8}(4\pi^2 - \sum_{j=1}^{5}\theta_j^2)(20\pi^2 - \sum_{j=1}^{5}\theta_j^2) + \tfrac{1}{4}\sum_{j<k}\theta_j^2\theta_k^2 > 0 \quad \Leftarrow \quad \sum_{j=1}^{5}\theta_j < 2\pi \\ &= \tfrac{1}{4}\sum_{j<k}\Big(3(\pi^2 - \theta_j^2)(\pi^2 - \theta_k^2) + \pi^4 - \theta_j^2\theta_k^2\Big) + \tfrac{1}{8}\sum_{j=1}^{5}\theta_j^4 > 0 \quad \Leftarrow \quad \theta_j \leq \pi, \ \forall j. \end{aligned}$$

The inequalities are only clear from the expression in the second line and $\sum \theta_j^2 < (\sum \theta_j)^2 < 4\pi^2$ or the expression in the third line and $\theta_j^2\theta_k^2 \leq \pi^4$.

In Section 2 we describe the natural compactifications of the moduli spaces $\mathcal{M}^{\text{hyp}}_{g,n}(\mathbf{L})$ obtained by the inclusion of reducible nodal hyperbolic surfaces with cusps at nodes for nice $\mathbf{L}$. We also show how this fails for some $\mathbf{L}$, in particular when the boundary components include both geodesics and cone angles greater than π. When the compactification is described by reducible nodal hyperbolic surfaces we prove that the natural symplectic form extends to the compactification. This extension represents a cohomology class in the compactification and we calculate it in Section 3.

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2 Compactifications

In this section, we study the behaviour of compactifications of the moduli spaces $\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathbf{L})$ and the extension of the natural symplectic form to the compactification. In particular, we discuss conditions on $\mathbf{L}$ which guarantee that the natural compactification of $\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathbf{L})$ is given solely by the inclusion of reducible nodal hyperbolic surfaces with cusps at nodes. A requisite for this behaviour is Mumford's compactness criterion and the existence of a positive lower bound on the distance between boundary components.

Given $\epsilon > 0$, Mumford's compactness criterion states that for certain choices of $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i\,\mathbb{R}_{>0}\}^n$ the subset of $\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathbf{L})$ consisting of all hyperbolic surfaces containing no closed geodesics of length less than ϵ is compact in $\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathbf{L})$. It is proven in [21] for the case of no boundary, i.e. $n = 0$. When $n > 0$, Mumford's compactness criterion follows from the existence of pants decompositions in any isotopy class—see [9] for a proof—so in particular applies to the case $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i[0,\pi)\}^n$. When $L_j \in i[0, 2\pi)$ with $\theta_j = -iL_j$ satisfying $\sum_j \theta_j < 2\pi$, we give a proof of Mumford's compactness criterion in Proposition 2.5 without relying on the existence of pants decompositions. A consequence of Mumford's compactness criterion is that a compactification of the moduli space is obtained by including reducible nodal hyperbolic surfaces with cusps at nodes. The extension argument, which generalises Wolpert's proof [32], applies to such compactifications. For general $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i\,\mathbb{R}_{>0}\}^n$, there does not always exist a compactification by including nodal surfaces.

The compactification of $\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathbf{0})$ by nodal hyperbolic surfaces with cusps at nodes demonstrates nicely the behaviour near the boundary. By Mumford's compactness criterion, a short closed geodesic can in the interior of the hyperbolic surface and produce a cusp in the limit. There is a positive lower bound on the distance between two boundary components, represented by horocycles of fixed radius, over all hyperbolic surfaces—it is $\log(4)$ for radius $1/2$ horocycles. As the horocycles approach each other, a cusp forms elsewhere on the hyperbolic surface. More generally, several boundary components may approach each other, and $\log(4)$ remains a lower bound for the distance between horocycles of radius $1/2$. For more general $\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathbf{L})$, the possible interaction between boundary components can lead to the failure of Mumford's compactness criterion which we describe in this section.

Wolpert's proof [32] that the Weil-Petersson form ω^{WP} extends smoothly from the moduli space of complete hyperbolic surfaces $\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathbf{0}) \cong \mathcal{M}_{g,n}$ to $\overline{\mathcal{M}}_{g,n}$ uses Fenchel-Nielsen coordinates $\{\ell_j, \theta_j \mid j = 1, ..., 3g - 3 + n\}$ for Teichmüller space associated to pants decompositions, and the formula

$$\omega^{\rm WP} = \sum d\ell_j \wedge d\theta_j. \tag{6}$$

The compactification $\mathcal{M}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is realised by allowing some $\ell_j = 0$. Geometrically, this includes reducible nodal hyperbolic surfaces with cusps at nodes. The formula for $\omega^{\rm WP}$ remains smooth when $\ell_j = 0$ and descends to the compactified moduli space. Smoothness uses the fact that $d\ell_j \wedge d\theta_j = \ell_j d\ell_j \wedge d\tau_j$ is smooth at $\ell_j = 0$, where $\tau_j = \theta_j/\ell_j \in [0,1]$ is well-defined since $\theta_j \in [0,\ell_j]$ in the quotient from Teichmüller space to the moduli space. Wolpert's argument in [32] generalises to $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i[0,\pi)\}^n$. This requires two ingredients: firstly, the existence of a pants decomposition in any isotopy class on any surface in $\mathcal{M}^{\rm hyp}_{g,n}(\mathbf{L})$, hence Fenchel-Nielsen coordinates on the Teichmüller space, discussed in Section 2.1. Secondly it requires a construction of the natural compactification of $\mathcal{M}^{\rm hyp}_{g,n}(\mathbf{L})$ via allowing reducible nodal hyperbolic surfaces with cusps at nodes. The formation of cusp nodes requires a proof of the property that there is a positive lower bound on the distance between boundary components (where cusp boundary components are represented by horocycles) which we review in Lemma 2.1 in preparation for the more general case, discussed in Section 2.2.

2.1 Pants decompositions and Fenchel-Nielsen coordinates

Let Σ be a hyperbolic surface with boundary components of type $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i[0,\pi)\}^n$. Any topological pants decomposition of Σ, i.e. a maximal disjoint collection of essential simple closed curves not isotopic to the boundary or each other, can be represented uniquely by a geometric pants decomposition. Existence uses curve shortening flow which preserves simple closed curves and disjointness from each other and the boundary, and uniqueness uses the Gauss-Bonnet formula. Via lengths and rotations along the simple closed curves, any such pants decomposition can be used to define Fenchel-Nielsen coordinates for Teichmüller space. Wolpert's formula (6) defines a symplectic form on the Teichmüller space, which as usual we call the Weil-Petersson form, that is mapping class group invariant hence descends to a well-defined symplectic form on the moduli space. Note that when $\mathbf{L} \in \{i[0,\pi)\}^n$, there are two definitions of the Weil-Petersson form. One using Wolpert's formula and the other defined via the natural Kähler metric on $\mathcal{M}^{\rm hyp}_{g,n}(\mathbf{L})$ constructed in [25]. The definitions are proven to coincide in [2].

In general, a hyperbolic surface with at least one cone angle greater than π is not guaranteed to have a pants decomposition in a given isotopy class. The curve shortening flow argument fails in this case, since a family of curves may move through a point with cone angle $> \pi$ during shortening. A construction of of hyperbolic surfaces of arbitrary genus without a pants decomposition in a chosen isotopy class is given in Appendix A.

The moduli space of hyperbolic surfaces is homeomorphic to the moduli space of curves $\mathcal{M}^{\rm hyp}_{g,n}(\mathbf{L}) \cong \mathcal{M}_{g,n}$ for $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i[0,\pi)\}^n$ or $\mathbf{L} \in \{i[0,2\pi)\}^n$. In the case that $\mathbf{L}$ is purely imaginary the homeomorphism assigns to any hyperbolic surfaces its conformal structure and for the inverse map it uses the existence of conical metrics proven in [15]. When a hyperbolic surface contains geodesic boundary components, i.e. some $L_j > 0$, a less direct construction is needed. Given a hyperbolic surface X with geodesic boundary components, glue a conformal disk D_j with marked point at its centre along each geodesic boundary component β_j to produce an element $\tilde{X} = X \cup D_{j_1} \cup ... \cup D_{j_k} \in \mathcal{M}_{g,n}$. A conformal disk is glued along β_j canonically as follows. A hyperbolic geodesic of length ℓ naturally lives inside a hyperbolic funnel—a complete, hyperbolic, infinite area annulus—which is conformally equivalent to the annulus $\{z \in \mathbb{C} \mid R^{-1} < |z| < R\}$ for $R = \exp(\pi^2/\ell)$. The circle $|z| = 1$ inside the annulus corresponds to the geodesic of length ℓ. Glue the disk $D = \{|z| \leq 1\}$ onto this annulus, and take $0 \in D$ to be the marked point. For the inverse map, existence can be proven using a double of the surface that produces the correct length geodesic in the corresponding hyperbolic surface. Alternatively, the map is open and one-to-one onto its image due to uniqueness of a hyperbolic metric in its conformal class. This can be used to prove the map is onto.

2.2 Compactification by nodal hyperbolic surfaces

Via the character variety, Bers [4] studied the compactification of $\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})$ by including reducible nodal hyperbolic surfaces with cusps at nodes. The compactification is also a consequence of Mumford's compactness criterion together with a lower bound for distances between boundary components, where nodes are represented by horocycles.

The existence of pants decompositions in arbitrary isotopy classes can be used to prove Mumford's compactness criterion. The following argument can be found in [9]. Given $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i[0, \pi)\}^n$, Bers proves in [3] that there exists a constant $C_{g,n}(\mathbf{L})$ such that any $X \in \mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})$ admits a pants decomposition consisting of simple closed curves of length less than $C_{g,n}(\mathbf{L})$. This is proven by induction on the number $k = 0, ..., 3g - 3 + n$ of disjoint non-isotopic simple closed curves in X. Cut X along k simple closed curves, satisfying the inductive hypothesis and use the known area of X to find a new small length closed essential geodesic in one of the resulting components. This new closed geodesic is proven to exist by existence of geodesics in any isotopy class, which uses $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i[0, \pi)\}^n$. Then Fenchel-Nielsen coordinates taking values in the compact set $[\epsilon, C_{g,n}(\mathbf{L})]$ can be used to prove compactness of the subset of $\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})$ consisting of all hyperbolic surfaces containing no closed geodesics of length less than ϵ. The argument fails for $\mathbf{L} \in \{i[0, 2\pi)\}^n$ and is proven in another way in Proposition 2.5.

Lemmas 2.1 to 2.4 characterise possible interaction between boundary components. They give a lower bound for distances between boundary components, when they exist.

Lemma 2.1. *Given lengths $L_j \in \mathbb{R}_{>0} \cup i[0, \pi)$, for any surface in $\mathcal{M}_{g,n}^{hyp}(L_1, \ldots, L_n)$ there exists a positive lower bound on distances between boundary components.*

Proof. Let $L_1, \ldots, L_n \in \mathbb{R}_{>0} \cup i[0,\pi)$ correspond to boundary components $\beta_1, \ldots, \beta_n$, respectively. Consider the functions on Teichmüller space

$$\begin{aligned} f_{jk} : \mathcal{T}^{\text{hyp}}_{g,n}(L_1, \ldots, L_n) &\longrightarrow \mathbb{R}_{\geq 0} \\ X &\longmapsto \min_{\gamma \subset X} \ell(\gamma), \end{aligned}$$

where min is taken over lengths of all paths $p \subset X$ connecting β_j and β_k. The proof of the Lemma amounts to showing that f_{jk} is bounded below by some positive number.

Let $X \in \mathcal{T}^{\text{hyp}}_{g,n}(L_1, \ldots, L_n)$, let γ be any path from L_1 to L_2, and let K be a closed tubular neighbourhood of $\beta_1 \cap \beta_2 \cap \gamma \subset X$. Its boundary ∂K is a simple closed loop, and since $L_1, L_2 \in \mathbb{R}_{\geq 0} \cup i[0,\pi)$ we can shorten ∂K to produce a simple, closed geodesic $c \subset X$. This produces a hyperbolic pair-of-pants with boundary components β_1, β_2 and c. In the pair-of-pants, let γ' denote the shortest curve in the isotopy class of γ, and suppose that $\ell(\gamma') = \delta$. We show that δ bounded below by some positive number. There are three possibilities for the boundary components L_1 and L_2.

Case 1. $L_1, L_2 \in \mathbb{R}_{\geq 0}$:

Cutting along γ' and the other two seams of the hyperbolic pair-of-pants, we obtain two right angle hyperbolic hexagons. One such hexagon is shown in Figure 1. The hyperbolic

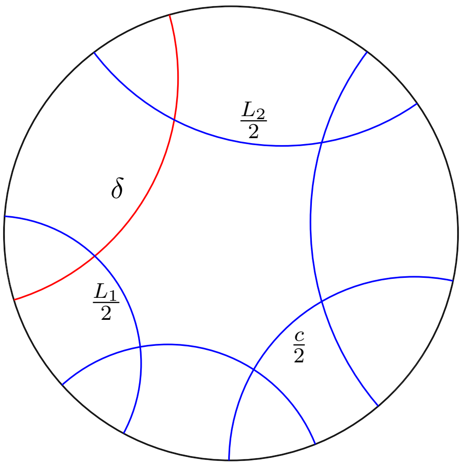

Figure 1. Right angled hyperbolic hexagon.

hexagon formula $\cosh(c) = \sinh(a)\sinh(b)\cosh(\delta) - \cosh(a)\cosh(b)$ [5, Theorem 2.4.1, (i)] implies

$$\begin{aligned} \cosh(\delta) &= \frac{\cosh\left(\frac{c}{2}\right) + \cosh\left(\frac{L_1}{2}\right)\cosh\left(\frac{L_2}{2}\right)}{\sinh\left(\frac{L_1}{2}\right)\sinh(b)} \\ &\geq \frac{1 + \cosh\left(\frac{L_1}{2}\right)\cosh\left(\frac{L_2}{2}\right)}{\sinh\left(\frac{L_1}{2}\right)\sinh\left(\frac{L_2}{2}\right)} \\ &> \frac{1}{\sinh\left(\frac{L_1}{2}\right)\sinh\left(\frac{L_2}{2}\right)} + 1. \end{aligned} \tag{7}$$

This means that $\delta > 1 + \varepsilon(L_1, L_2)$, for some positive function ε, depending on the lengths of the boundary geodesics.

Case 2. $L_1 \in \mathbb{R}_{>0}$ and $L_2 \in i[0, \pi)$:

Let $\theta_2 = -iL_2$ be the cone angle defined by L_2. Cutting along γ' and the other two seams of the hyperbolic pair-of-pants, we obtain two hyperbolic pentagons, each with 4 right angles and one angle $\frac{\theta_2}{2}$ at the cone point. One such pentagon is shown in Figure 2. A hyperbolic pentagon formula from [5, Example 2.2.7, (iv), (v)] gives

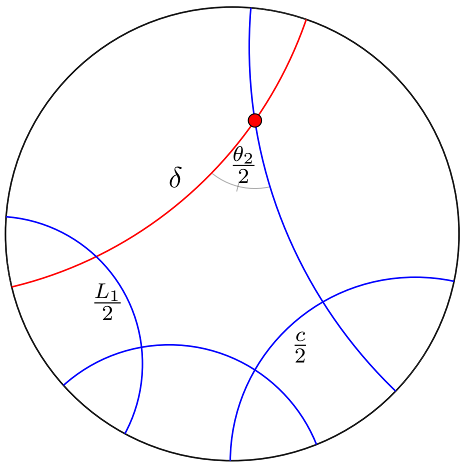

Figure 2. Hyperbolic pentagon with all but one right angle.

$$\begin{aligned}\cosh(\delta) &= \frac{1}{\sin\left(\frac{\theta_2}{2}\right)\sinh\left(\frac{L_1}{2}\right)}\sqrt{\cos^2\left(\frac{\theta_2}{2}\right) + 2\cos\left(\frac{\theta_2}{2}\right)\cosh\left(\frac{L_1}{2}\right) + \cosh^2\left(\frac{L_1}{2}\right) + K} \qquad (8)\\ &\geq \frac{\cos\left(\frac{\theta_2}{2}\right) + \cosh\left(\frac{L_1}{2}\right)}{\sin\left(\frac{\theta_2}{2}\right)\sinh\left(\frac{L_1}{2}\right)}\\ &> \frac{\cosh\left(\frac{L_1}{2}\right)}{\sinh\left(\frac{L_1}{2}\right)}\end{aligned}$$

where $K = 2\cos\left(\frac{\theta_2}{2}\right)\cosh\left(\frac{L_1}{2}\right)\left(\cosh\left(\frac{c}{2}\right) - 1\right) + \cosh^2\left(\frac{c}{2}\right) - 1 > 0$. This means that $\delta > 1 + \varepsilon(L_1)$, for some positive function ε, depending on the length of the boundary geodesic.

Case 3. $L_1, L_2 \in i[0, \pi)$:

Let $\theta_1 = -iL_1$ and $\theta_2 = -iL_2$ be the cone angles defined by L_1 and L_2, respectively. Cutting along γ' and the other two seams of the hyperbolic pair-of-pants, we obtain two hyperbolic quadrilaterals, each with 2 right angles and two angles $\frac{\theta_1}{2}$ and $\frac{\theta_2}{2}$ at the cone points. One such quadrilateral is shown in Figure 3. For any hyperbolic quadrilateral the formula $\cosh\left(\frac{c}{2}\right) = \sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\cosh(\delta) - \cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)$ holds [5, Theorem 2.3.1, (i)],

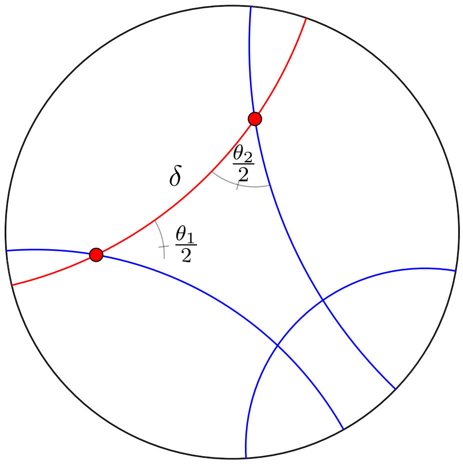

Figure 3. Hyperbolic quadrilateral with two right angles and two non-right angles at cone points.

so

$$\cosh(\delta) = \frac{\cosh\left(\frac{c}{2}\right) + \cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)}{\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)} \qquad (9)$$
$$\geq \frac{1 + \cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)}{\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)}$$

Since θ_1 and θ_2 are fixed, $1 + \cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) > \sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)$. This means that $\delta > 1 + \varepsilon(\theta_1, \theta_2)$, for some positive function ε, depending on the angles of the cone points. This last inequality does not require $\theta_j \in [0, \pi)$, rather it requires only $\frac{1}{2}(\theta_1 + \theta_2) < \pi$ so that the hyperbolic quadrilateral exists. This is needed in the proof of Lemma 2.2. □

Remark. Lemma 2.1 actually holds under slightly stronger conditions, by also allowing at most one cone angle to be equal to π.

Remark. As $\theta \to 0$, notice that the estimates in Lemma 2.1 imply that $\cosh \delta$ is unbounded. This corresponds to a cone point approaching the behaviour of a cusp. Here one would measure distance between boundary components via a horocyle around the cusp. Limiting behaviour of separation estimates imply that a suitably chosen horocyle would give an arbitrarily large separation between cusps and any other boundary components.

A consequence of Lemma 2.1 together with Mumford's compactness criterion is that when $\mathbf{L} \in \{\mathbb{R}_{\geq 0} \cup i[0, \pi)\}^n$, the homeomorphism $\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L}) \to \mathcal{M}_{g,n}$ extends to a homeomorphism between $\overline{\mathcal{M}}_{g,n}^{\text{hyp}}(\mathbf{L})$ and the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$.

In the case $L_j \in i[0, 2\pi)$ such that the angle sum is strictly less than 2π, we must use a different method to extend the result of Lemma 2.1.

Lemma 2.2. *Given lengths $L_j \in i[0, 2\pi)$ with $\theta_j = -iL_j$ and $\sum_j \theta_j < 2\pi$, for any surface in $\mathcal{M}_{g,n}^{hyp}(L_1, \ldots, L_n)$ there exists a positive lower bound on distances between cone points.*

Proof. Let $L_1, \ldots, L_n \in i[0, \pi)$ correspond to cone points $p_1, \ldots, p_n$ with cone angles $\theta_1 = -iL_1, \ldots, \theta_n = -iL_n$, respectively. Let $X \in \mathcal{M}_{g,n}^{\text{hyp}}(L_1, \ldots, L_n)$ and let γ be any path from p_1 to p_2. If $\theta_1, \theta_2 \leq \pi$ then the argument from Lemma 2.1 implies that the length of γ is bounded below and the lemma is proven.

For the remainder of the proof, let us assume without loss of generality that $\theta_1 > \pi$. In the proof of Lemma 2.1 we exploited the fact that any two boundary components together with an isotopy class of paths joining them could be separated from the remaining surface via a disk neighbourhood with geodesic boundary. Note that we still call a disk with cone points a disk. This gave rise to a local model in which the lower bound could be calculated. This no longer holds when $\theta_1 > \pi$, and any local model may contain many cone points close to p_1.

We begin with the case of exactly two points approaching each other, in preparation for many points approaching each other. Suppose that p_1 and p_2 are sufficiently close, so that we may represent a disk neighbourhood $\mathcal{D}$ of p_1 and p_2 locally via a region in the hyperbolic plane. We model $\mathcal{D}$ as a region in the hyperbolic plane, as depicted in Figure 4, where

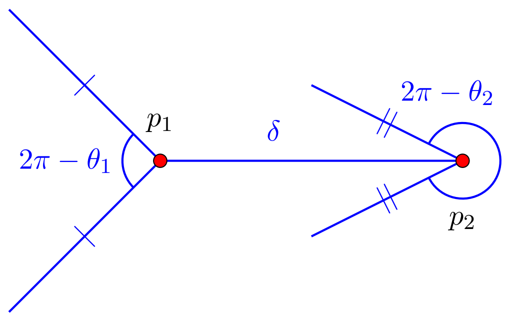

Figure 4. A local depiction in the hyperbolic plane of two nearby cone points p_1 and p_2.

geodesic rays from the cone points are identified, so that the marked angles of $2\pi - \theta_1$ and $2\pi - \theta_2$ are essentially deleted from the plane. Note that geodesics appear as straight lines in the diagram. Let δ be the separation of p_1 and p_2. If δ is small enough, since $\theta_1 + \theta_2 < 2\pi$, there must be a point in which the geodesic rays corresponding to p_1 meet with those of p_2 as depicted in Figure 5, essentially due to the Euclidean picture which approximates the hyperbolic picture at small enough scales. As δ increases, the intersection point moves out to infinity and the endpoints of the geodesic rays on the circle at infinity move together. Then they pass through each so that the geodesic rays do not meet, and a shortest path between them, given by a geodesic meeting the rays perpendicularly, forms and moves towards p_1 and p_2 as the endpoints move apart. In fact, due to identification of the geodesic rays, the perpendicular geodesic represents a simple closed geodesic surrounding p_1 and p_2 in the

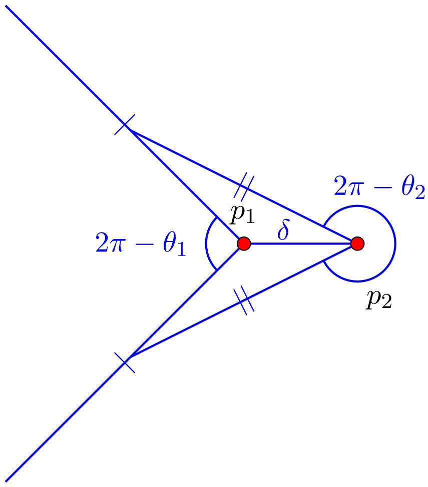

Figure 5. A local depiction in the hyperbolic plane of two cone points p_1 and p_2 whose angle sum is less than 2π arbitrarily close to one another.

surface X. This is shown in red in Figure 6.

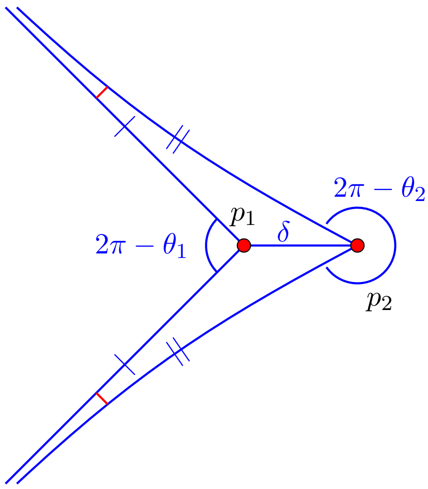

Figure 6. A local depiction in the hyperbolic plane of two nearby cone points p_1 and p_2 whose angle sum is less than 2π and surrounded by the geodesic in red.

We learn two important ideas from the previous construction. Firstly, when δ is sufficiently small, a hyperbolic sphere with three cone points appears around p_1 and p_2. But such a situation cannot occur, as it would imply the existence of a disconnected closed region in the surface. Hence there is a positive lower bound on the distance between the two cone points.

Secondly, when cone points p_1 and p_2 whose angle sum is less than 2π are sufficiently close, to allow the local model, then there exists a disk neighbourhood of p_1 and p_2 with boundary a simple closed geodesic. This holds even when $\theta_1 > \pi$ and contrasts with the behaviour when both θ_1 and θ_2 are less than π, in which case such a geodesically bounded disk neighbourhood exists around arbitrarily long geodesic paths from p_1 to p_2. The simple closed geodesic and two cone points bound a hyperbolic pair of pants formed from gluing two identical quadrilaterals as in in Case 3 of the proof of Lemma 2.1. The inequality (9) applies to angles satisfying $\frac{1}{2}(\theta_1+\theta_2) < \pi$ and gives a positive lower bound on the distance δ from p_1 to p_2. Distances less than this lower bound produce the contradiction of a disconnected closed region in the surface described above.

For the more general case of k cone points p_1,..., p_k say, approaching each other, when they are sufficiently close we produce a similar contradiction of the existence of a disconnected closed region in the surface as in the $k = 2$ case above. We do not achieve this via a local model as in the $k = 2$ case, and instead argue quite differently.

Consider k cone points approaching each other and satisfying $0 < \sum_{j=1}^k \theta_j < 2\pi$. Eventually they all lie in a common disk and remain distinct—if not, then the problem is reduced to fewer than k points. A local model is given by a disk minus distinct points $p_1, ..., p_k$ over which there is a flat $PSL(2,\mathbb{R})$ connection with holonomy A_j around p_j and $|\operatorname{tr} A_j| = |2\cos(\theta_j)|$.

Claim: When the points are close enough, then the holonomy

$$A_1A_2...A_k = A \in PSL(2,\mathbb{R})$$

around the boundary of the disk is not hyperbolic, moreover $|\operatorname{tr} A| < 2$.

Proof of claim: The trace of each A_j is equivalent to the cone angle θ_j and the location of each point p_j in the disk gives more information. It gives the entire matrix $A_j \in PSL(2,\mathbb{R})$. If all points coincide, then $A_j \in SO(2)$ (or a common conjugate of $SO(2)$) so $A \in SO(2)$, and moreover has angle $0 < \sum\theta_j < 2\pi$ hence $|\operatorname{tr}(A)| < 2$ since $|\operatorname{tr}(A)| = 2 \Leftrightarrow A = Id \Leftrightarrow \sum\theta_j = 2n\pi$. As the points p_j move away from each other, and each A_j lives in a different conjugate of $SO(2)$, the function $|\operatorname{tr}(A)|$ is continuous on the configuration space of points $\{(p_1, p_2, ..., p_k)\}$ so since $|\operatorname{tr}(A)| < 2$ on $(p_1, p_1, ..., p_1)$ then $|\operatorname{tr}(A)| < 2$ for points p_j close enough together. Hence the claim is proven.

By the claim, since the boundary holonomy is elliptic, the flat connection above the disk defines a hyperbolic sphere with $k+1$ cone points of area $2\pi - \sum\theta_j$. This contradicts the points coming close together on a more interesting surface, say of type (g, n). As in the $k = 2$ case, just before the sphere with $k+1$ cone points appears, the type (g, n) surface degenerates to a nodal curve. This forces the distinct points $\{(p_1, p_2, ..., p_k)\}$ to have a positive lower bound on their distance. An actual bound can be achieved by studying the genus zero case, generalising a pair of pants. □

A consequence of Lemma 2.2 is that when $\mathbf{L} \in \{i[0, 2\pi)\}^n$ with $\theta_j = -iL_j$ such that $\sum_j \theta_j < 2\pi$, the homeomorphism $\mathcal{M}_{g,n}^{\rm hyp}(\mathbf{L}) \to \mathcal{M}_{g,n}$ extends to a homeomorphism between

$\overline{\mathcal{M}}^{\text{hyp}}_{g,n}(\mathbf{L})$ and the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$. The introduction of boundary geodesics or angles whose sum is greater than 2π breaks down this result. We show this in the following two lemmas.

Lemma 2.3. *Consider $\mathcal{M}^{hyp}_{g,n}(L_1, \dots, L_n)$, with $\theta_j = -iL_j \in (\pi, 2\pi)$ and $L_k \in \mathbb{R}_{>0}$ sufficiently large for some $k \neq j$. There does not exists a lower bound on distances between the corresponding cone point and boundary geodesic.*

Proof. Let p_j and β_k denote a cone point and a geodesic boundary component, respectively, which fit the description given in the lemma. First, suppose that $(g, n) \neq (0, 3)$, so we may find two geodesic curves γ_1 and γ_2, which are not homotopic to each other (other than the $g = 1$ case, where they are equal) or β_k, and together with β_k form a hyperbolic pair-of-pants P with p_j in its interior. Cutting along the seams of P along with unique geodesics from p_j meeting the three boundary geodesics perpendicularly, we decompose P into 3 geodesic pentagons and one geodesic hexagon, as depicted in Figure 7. Here, $\alpha_1 + \alpha_2 + \beta = \theta$ and

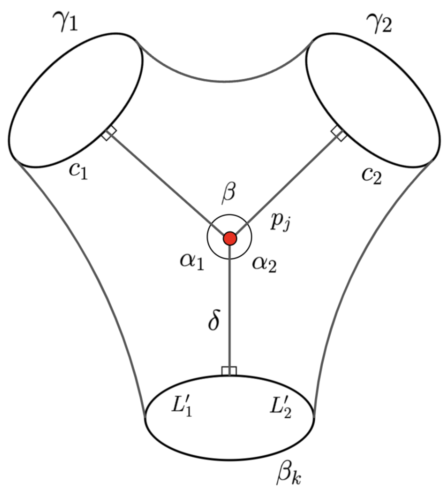

Figure 7. Hyperbolic pair-of-pants with a single cone point of cone angle greater than 2π.

δ is the distance between p_j and β_k. Consider one of the two pentagons with δ as a side length. For $i = 1$ or 2, let $L_i' \in (0, L_k)$ denote the side length of the pentagon coming from β_k and let c_i denote the relevant length component of γ_i which makes up the opposite side length to β_k. This pentagon is depicted in Figure 8. Any hyperbolic pentagon satisfies the following formula from [5, Example 2.2.7, (iv), (v)]

$$\sinh(\delta) = \frac{\cosh(c_i) + \cos(\alpha_i)\cosh(L_i')}{\sin(\alpha_i)\sinh(L_i')}. \tag{10}$$

Since $\theta > \pi$, we may position p_j so that $\cos(\alpha_i) < 0$. Suppose that L_j is sufficiently large, so that $\cos(\alpha_i)\cosh(L_i') < -1$. Then choosing a path in $\mathcal{M}^{\text{hyp}}_{g,n}(L_1, \dots, L_n)$ so that $\cosh(c_i)$ tends towards $|\cos(\alpha_i)\cosh(L_i')| > 1$ will give surfaces with arbitrarily small separation length δ. In this situation, the cone point p_j and boundary geodesic β_k will preferentially

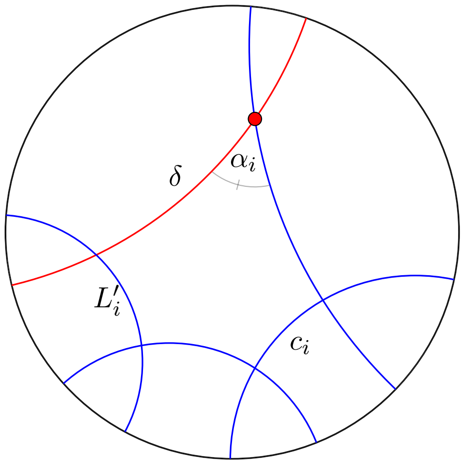

Figure 8. Hyperbolic pentagon with all but one right angle which may be greater than $\frac{\pi}{2}$.

merge, rather than producing a cusp elsewhere in the surface, which would occur if $c_i \to 0$ where possible.

In the case $(g, n) = (0, 3)$, one obtains a single pentagon such as that in Figure 8 by directly cutting along the seam of a given surface and unique geodesics from p_j meeting the two boundary geodesics perpendicularly. From here the proof is as in the general case. □

In the circumstances of Lemma 2.3, the cone point and boundary component come together to form a crowned hyperbolic surface. A diagram of this process is depicted in Figure 9. Here the crown angle is $\phi = \theta_j - \pi$, and boundary holonomy is violated by

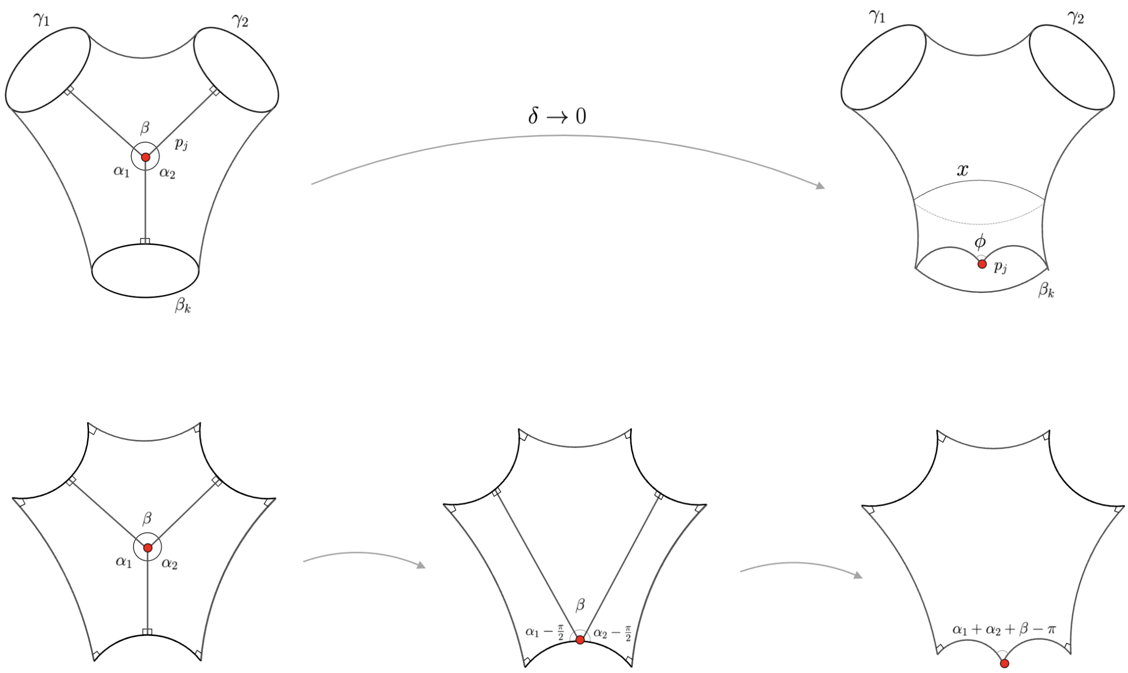

Figure 9. Formation of crowned hyperbolic surface component as $\delta \to 0$.

a shorter simple closed path β' which is homotopic but not equivalent to the boundary geodesic β_k. As a function of θ_j and L_k the length x of β' is uniquely determined by the equation

$$\sin^2(\phi)\cosh(L_k)\tanh^2\left(\frac{x}{2}\right) = \cosh(x)\tanh^2\left(\frac{L_k}{2}\right) - \cos^2(\phi)\,. \tag{11}$$

Notice that as $\theta \to 2\pi$ (so $\phi \to \pi/2$), the limits of each side of equation (11) are given by $\cosh(L_k)\tanh^2\left(\frac{x}{2}\right)$ respectively $\cosh(x)\tanh^2(\frac{L}{2})$, implying $x \sim L$ as we would expect. These crowned surfaces exist in the compactification of the moduli space when there are cone points with cone angles greater than π along with boundary geodesics. This implies that the natural compactification of the corresponding moduli space of hyperbolic surfaces is not homeomorphic to the Deligne-Mumford compactification.

Another situation where we see that the natural compactification of the moduli space of hyperbolic surfaces is not homeomorphic to the Deligne-Mumford compactification is when there are pairs of cone points whose cone angle sum is greater than 2π.

Lemma 2.4. *Consider $\mathcal{M}^{hyp}_{g,n}(L_1,\ldots,L_n)$, with $\theta_j = -iL_j \in [0,2\pi)$ and $\theta_k = -iL_k \in [0,2\pi)$ such that $\theta_j + \theta_k \geq 2\pi$ for some $j \neq k$. There does not exists a lower bound on distances between cone points. In this case, cone points may come together to form a single cone point whose cone angle is $\theta_{j+k} = \theta_j + \theta_k - 2\pi$.*

Proof. Let p_j and p_k denote cone points which fit the description given in the lemma. Let δ be the separation of p_j and p_k, and let $\mathcal{D}$ be a disk neighbourhood of p_j and p_k. We can model $\mathcal{D}$ as a region in the hyperbolic plane, as depicted in the top part of Figure 10. Here, dashed lines are identified at the cone points, so that the marked angles of $2\pi - \theta_j$ and $2\pi - \theta_k$ are essentially deleted from the plane. Since the angle sum is greater than 2π, sending $\delta \to 0$ results in a new cone angle with angle $\theta_j + \theta_k - 2\pi$. In the lower part of Figure 10, all three lines are identified, so that a total angle of $4\pi - \theta_j - \theta_k$ is deleted from the plane. □

In the presence of cone angles greater than π, although a pants decomposition may not exist, Wolpert's extension argument generalises to show that the Weil-Petersson form extends to the compactification.

Proposition 2.5. *Given $\mathbf{L} = (L_1, ..., L_n) \in \{i\,[0,2\pi)\}^n$ with $\mathcal{M}^{hyp}_{g,n}(\mathbf{L})$ non-empty, the Weil-Petersson form $\omega^{WP}(\mathbf{L})$ extends to the natural compactification $\overline{\mathcal{M}}^{hyp}_{g,n}(\mathbf{L})$ by nodal hyperbolic surfaces which fits into the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{M}_{g,n} & \xrightarrow{\cong} & \mathcal{M}^{hyp}_{g,n}(\mathbf{L}) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}^{[\mathbf{a}]}_{g,n} & \xrightarrow{\cong} & \overline{\mathcal{M}}^{hyp}_{g,n}(\mathbf{L}) \end{array} \tag{12}$$

where $\overline{\mathcal{M}}^{[\mathbf{a}]}_{g,n}$ is the Hassett compactification with weights defined by $L_j = 2\pi i(1-a_j)$.

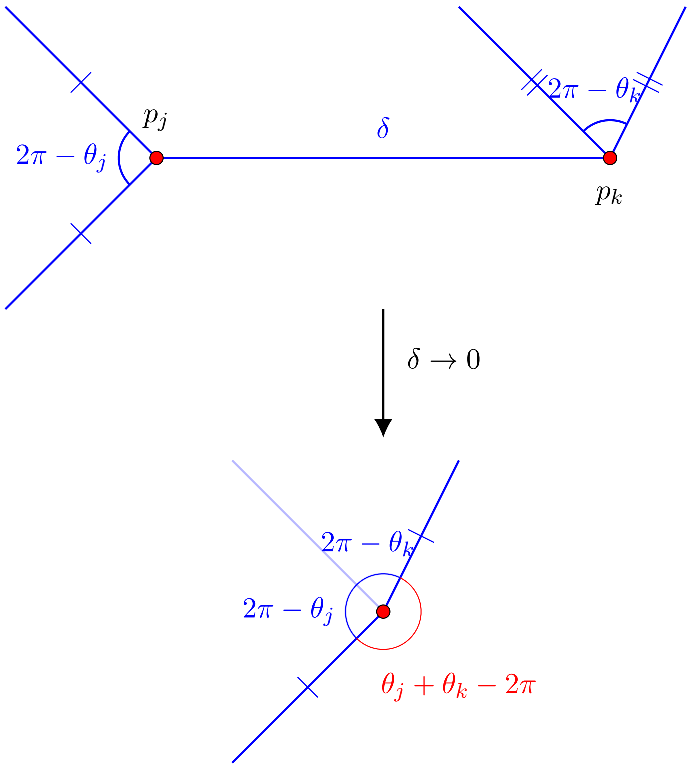

Figure 10. A depiction in the hyperbolic plane of two cone angles p_j and p_k merging together.

Proof. The condition that the moduli space $\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})$ is non-empty is equivalent to the inequality $\sum_{j=1}^n \theta_j < 2\pi(2g-2+n)$ for $L_j = i\theta_j$. In this case, McOwen [15] proved existence of conical hyperbolic metrics in any conformal class, giving the top arrow in (12). The natural compactification of $\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})$ is obtained by including reducible nodal hyperbolic surfaces with cusps at nodes. Hassett's compactification consists of nodal curves with labeled points weighted by $\mathbf{a} = (a_1, ..., a_n)$ and each irreducible component must be stable with respect to these weights. A weight 1 point gives a usual labeled point, and nodes naturally inherit a weight of 1. An irreducible component is **a**-stable if the weighted Euler characteristic $2 - 2g' - \sum_{j \in J} a_j$ is negative. In particular, a set of weighted points $\{p_j \mid j \in J \subset \{1, ..., n\}\}$ for $|J| > 1$ can coincide if $\sum_{j \in J} a_j < 1$, unlike for unweighted stable curves. This behaviour is exactly mirrored by hyperbolic surfaces. The proof of Lemma 2.2 does not require $\sum_{j=1}^n \theta_j < 2\pi$, rather it applies more generally to the local condition $\sum_{j \in J} \theta_j < 2\pi(|J| - 1)$ for $J \subset \{1, ..., n\}$, and $\sum_{j \in J} \theta_j$ is not an integer multiple of 2π, and produces a nodal curve in the limit as the $|J|$ points approach each other. The agreement with Hassett's local condition is shown by:

$$\sum_{j \in J} \theta_j < 2\pi(|J| - 1) \;\Leftrightarrow\; \sum_{j \in J}(1 - a_j) < |J| - 1 \;\Leftrightarrow\; \sum_{j \in J} a_j > 1.$$

Each irreducible component of the limiting nodal curve satisfies McOwen's condition for existence of a conical hyperbolic metric in its conformal class, agreeing with the stratification of Hassett's compactification. On the other hand, if $\sum_{j\in J}\theta_j > 2\pi(|J|-1)$, corresponding to $\sum_{j\in J} a_j < 1$, Figure 10, applied more generally to $|J|>1$ points shows via a local picture that the angles can come together to produce a new angle $\theta=\sum_{j\in J}\theta_j+2\pi(1-|J|)$ corresponding to a new weight $a=\sum_{j\in J} a_j$.

Wolpert proved that $\omega^{\mathrm{WP}}(\mathbf{L})$ extends to the compactification in the case that a pants decomposition and Fenchel-Nielsen coordinates exist. Although a pants decomposition does not necessarily exist for hyperbolic surfaces with cone angles—see Corollary A.2—Wolpert's argument generalises as follows. Close to the boundary divisor of the compactification $\overline{\mathcal{M}}_{g,n}^{\mathrm{hyp}}(\mathbf{L})$, a simple closed geodesic γ of length tending to zero exists. Wolpert proved that the vector field $\partial/\partial\theta$ on the moduli space $\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathbf{L})$ defined by rotation along a simple closed geodesic is dual to the Hamiltonian function ℓ given by the length of the geodesic. He used this to produce the formula (6) for $\omega^{\mathrm{WP}}(\mathbf{L})$. Goldman [12] generalised Wolpert's result to character varieties which includes hyperbolic surfaces with cone angles to get

$$\omega^{\mathrm{WP}}(\mathbf{L}) = d\ell\wedge d\theta+\omega_c$$

where (ℓ,θ) are local coordinates on $\mathcal{M}_{g,n}^{\mathrm{hyp}}(\mathbf{L})$ and ω_c is a closed 2-form independent of (ℓ,θ) and defined with respect to the remaining coordinates. Apply this to the simple closed geodesic γ of length tending to zero. As $\ell_\gamma\to 0$, the differential form $d\ell_\gamma\wedge d\theta+\omega_c$ extends smoothly. Smoothness uses the fact that $\theta\in[0,\ell_\gamma]$ hence $d\ell_\gamma\wedge d\theta=\ell_\gamma d\ell\wedge d\tau$ for $\tau=\theta/\ell_\gamma\in[0,1]$ is smooth at $\ell_\gamma=0$. □

Theorem 1.1 follows from Proposition 2.5 for the cone angle case. The case $\mathbf{L}\in\{\mathbb{R}_{\geq 0}\cup i[0,\pi)\}^n$ is already known and also follows from Mumford's compactness criterion together with Lemma 2.1.

Corollary 2.6. *Consider $\mathcal{M}_{g,n}^{hyp}(\mathbf{L})$ with all $L_j=i\theta_j\in i\mathbb{R}_{\geq 0}$ and $\sum_{j=1}^n\theta_j<2\pi$. Then the natural compactification of $\mathcal{M}_{g,n}^{hyp}(\mathbf{L})$ is homeomorphic to $\overline{\mathcal{M}}_{g,n}$ and $\omega^{WP}(\mathbf{L})$ extends to the compactification.*

Proof. Proposition 2.5 implies that $\omega^{\mathrm{WP}}(\mathbf{L})$ extends to the compactification. The restriction on angles gives the following equivalence of conditions

$$\sum_{j=1}^n\theta_j<2\pi \;\Leftrightarrow\; \sum_{j=1}^n(1-a_j)<1 \;\Leftrightarrow\; \sum_{j\in J}a_j>1,\ \forall J\subset\{1,...,n\}\text{ with }|J|>1.$$

In this case, Hassett proved that his compactification coincides with the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ hence the commutative diagram (12) becomes:

$$\begin{array}{ccc} \mathcal{M}_{g,n} & \stackrel{\cong}{\longrightarrow} & \mathcal{M}^{\text{hyp}}_{g,n}(\mathbf{L}) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \stackrel{\cong}{\longrightarrow} & \overline{\mathcal{M}}^{\text{hyp}}_{g,n}(\mathbf{L}) \end{array}$$

□

Remark. Proposition 2.5 shows that Hassett's compactifications using different stability conditions correspond to compactifications $\overline{\mathcal{M}}^{\text{hyp}}_{g,n}(\mathbf{L})$ using different $\mathbf{L}$. As $\mathbf{L}$ varies, the volumes are polynomials locally in $\mathbf{L}$. The polynomials change as one crosses walls in the space of stability conditions. The calculations of these polynomials is work in progress. Theorem 1.2 shows that the polynomials agree with Mirzakhani's polynomials $V_{g,n}(\mathbf{L})$ in the main chamber of the space of stability conditions. We have found that the polynomials in different chambers have different top degree terms. In [22] polynomials with the same highest degree terms as $V_{g,n}(\mathbf{L})$ are produced via counting lattice points in $\mathcal{M}_{g,n}$. Similarly, polynomials for counting lattice points in the compactification $\overline{\mathcal{M}}_{g,n}$ were produced in [6]. It would be interesting to uncover analogous polynomials with the same highest degree terms for polynomials arising from different chambers in the space of stability conditions using different compactifications of $\mathcal{M}_{g,n}$.

3 Cohomology class of the Weil-Petersson symplectic form

In this section we consider the extension of the Weil-Petersson symplectic form on $\mathcal{M}_{g,n}$ to $\overline{\mathcal{M}}_{g,n}$ and its cohomology class beginning with Wolpert's proof in the classical case.

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of genus g stable curves—curves with only nodal singularities and finite automorphism group—with n labeled points disjoint from nodes, for any $g, n \in \mathbb{N} = \{0, 1, 2, ...\}$ satisfying $2g-2+n > 0$. Define the forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ that forgets the last point, and define $K_{\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}}$ to be the relative canonical sheaf of the map π. There are natural sections $p_j : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ of the map π for each $j = 1, ..., n$. Define $L_j = p_j^* K_{\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}}$ to be the line bundle $L_j \to \overline{\mathcal{M}}_{g,n}$ with fibre above $[(C, p_1, \ldots, p_n)]$ given by $T^*_{p_j}C$ and $\psi_j = c_1(L_j) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ to be its first Chern class. Define the classes $\kappa_m = \pi_* \psi_{n+1}^{m+1} \in H^{2m}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ where π is the forgetful map $\overline{\mathcal{M}}_{g,n+1} \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}$. They were defined by Mumford and in this form by Arbarello-Cornalba [1]. We will refer to ψ_j and κ_m as ψ classes, respectively κ classes. Wolpert proved that the cohomology class of the extension of the Weil-Petersson form is $[\omega^{\text{WP}}] = 2\pi^2 \kappa_1$.

Given any family of hyperbolic surfaces $X \to S$ the relative canonical line bundle $K_{X/S} \to X$ comes equipped with a natural Hermitian metric defined via the unique hyperbolic metric on each fibre. The holomorphic structure and Hermitian metric uniquely determines a connection on $K_{X/S}$ with curvature that represents a Chern form. Wolpert used this idea in [30] to prove an explicit push-forward formula for the Weil-Petersson symplectic form in terms of the curvature, which appears as the $n = 0$ case of (14) below. The line bundle $K_{X/S}$ extends to a compactification $\overline{X}/\overline{S}$ by a family of stable curves (perhaps after stable reduction) and its first Chern class is represented by an extension of the Chern form. This leads to Wolpert's formula $[\omega^{\mathrm{WP}}] = 2\pi^2\kappa_1$ for the cohomology class of the extension of the Weil-Petersson symplectic form to the compactification.

3.1 Wolpert's method

Schumacher and Trapani [25] generalised the work of Wolpert [30] to define a symplectic form $\omega^{\mathrm{W}P}(\mathbf{a})$ on $\mathcal{M}_{g,n}$ for a fixed choice of weights $\mathbf{a} = (a_1, ..., a_n) \in (0, 1]^n$ and relate it to a naturally defined Hermitian metric on the relative log-canonical line bundle over the universal curve. They first used the existence of an incomplete hyperbolic metric with cone angles $\theta_j = 2\pi(1 - a_j)$ on any fibre which varies smoothly in the family S proven by McOwen [15]. The hyperbolic metric is used to define a Kähler metric on $\mathcal{M}_{g,n}$ with Kähler form $\omega^{\mathrm{W}P}(\mathbf{a})$. The relationship of $\omega^{\mathrm{W}P}(\mathbf{a})$ with the curvature of a Hermitian metric on the relative log-canonical line bundle over the universal curve generalises Wolpert's proof as follows. Given any family of hyperbolic surfaces $X \to S$ with n sections having image divisor $D = \sqcup_{j=1}^n D_j \subset X$, fix a choice of weights $\mathbf{a} = (a_1, ..., a_n) \in (0, 1]^n$ and define $\mathbf{a}\cdot D = \sum a_j D_j$. The relative log-canonical line bundle $K_{X/S}(D)$ is equipped with a Hermitian metric, using the the conical hyperbolic metric on fibres determined by $\mathbf{a} \cdot D$, with curvature representing its first Chern form adjusted by the weights $\mathbf{a}$ given by the real $(1, 1)$ form defined over $X \setminus D$ by

$$\Omega_{X/S}(\mathbf{a}) := \tfrac{i}{2}\partial\bar{\partial}\log(g_{\mathbf{a}}) \tag{13}$$

where $g_{\mathbf{a}}$ defines the conical hyperbolic metric $g_{\mathbf{a}}|dz|^2$ on the fibre and the operators $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$ use the same local coordinate z on the fibre.

Theorem 3.1 ([25]).

$$\frac{1}{2}\int_{X/S} \Omega_{X/S}(\mathbf{a})^2 = \omega^{WP}(\mathbf{a}) \tag{14}$$

We write $\omega^{\mathrm{W}P}(\mathbf{a})$ and $\omega^{\mathrm{W}P}(\mathbf{L})$ interchangeably for $L_j = 2\pi i(1 - a_j)$. The theorem generalises the $n = 0$ case proven by Wolpert in [30] and the $n > 0$ case with $\mathbf{a} = (1, ..., 1)$ via an inductive argument in [1]. We have changed the normalisation of the Weil-Petersson form by a factor of $1/2$ to agree with the algebraic geometry relation $\pi_* c_1(K_{X/S})^2 = \kappa_1$ where $[\omega^{\mathrm{W}P}] = 2\pi^2\kappa_1$ and $[\Omega_{X/S}(1, ..., 1)] = 2\pi c_1(K_{X/S})$.

The following corollary applies to weights $\sum_{j=1}^n (1 - a_j) < 1$ which is equivalent to $\sum_{j=1}^n \theta_j < 2\pi$.

Corollary 3.2. *For $\sum_{j=1}^n (1 - a_j) < 1$, the $(1, 1)$-form $\Omega_{X/S}(\mathbf{a})$ extends to the compactifi-*

cation and defines the following cohomology class:

$$[\omega^{WP}(\mathbf{a})] = 2\pi^2\kappa_1 - \tfrac{1}{2}\sum_j \theta_j^2\psi_j \tag{15}$$

Proof 1. The line bundle $K_{X/S}(D)$ extends to the line bundle $K_{\overline{X}/\overline{S}}(D)$ over the compactification $\overline{X}/\overline{S}$ by a family of stable curves. The sections are required to be disjoint in the compactification, hence the divisor D extends naturally. The $(1,1)$ form $\Omega_{X/S}(\mathbf{a})$ defined in (13) extends smoothly to $\overline{X}/\overline{S}$ by [25, Theorem 2.3] where it is proven that $\log(g_{\mathbf{a}})$ depends smoothly on parameters in S except for a term $-\log|z|^2$ which is independent of parameters in S and is annihilated by $\partial\bar{\partial}$. Hence $\partial\bar{\partial}\log(g_{\mathbf{a}})$ depends smoothly on parameters in S at each point in the fibre (parametrised by z), and converges as a fibre tends to a nodal surface since the metric $g_{\mathbf{a}}$ converges smoothly away from nodes. The extension, which we also denote by $\Omega_{X/S}(\mathbf{a})$ represents 2π times the first Chern class twisted by $\mathbf{a}\cdot D$, i.e.

$$[\Omega_{X/S}(\mathbf{a})] = [\Omega_{X/S}] + 2\pi\mathbf{a}\cdot D$$

where $[\Omega_{X/S}] := [\omega_{\overline{X}/\overline{S}}]$ is the first Chern class of the relative canonical line bundle. Then

$$[\Omega_{X/S}(\mathbf{a})]^2 = \big([\Omega_{X/S}] + 2\pi\mathbf{a}\cdot D\big)^2 = [\Omega_{X/S}]^2 + 4\pi\mathbf{a}\cdot D\cdot[\Omega_{X/S}] + (2\pi\mathbf{a}\cdot D)^2$$

hence the left hand side of (14) represents the Gysin homomorphism

$$\begin{aligned}\tfrac{1}{2}\pi_*([\Omega_{X/S}(\mathbf{a})]^2) &= \tfrac{1}{2}\pi_*([\Omega_{X/S}]^2) + 2\pi^2\sum_j(2a_j\psi_j - a_j^2\psi_j)\\ &= [\omega^{\mathrm{WP}}] - 2\pi^2\sum_j(1-a_j)^2\psi_j\end{aligned}$$

which uses $\pi_*(D_j\cdot D_j) = -\psi_j$, $D_j\cdot D_k = 0$, $\pi_*(D_j\cdot\Omega_{X/S}) = 2\pi\psi_j$ and Wolpert's formula $[\omega^{\mathrm{WP}}] = 2\pi^2\kappa_1 = \frac{1}{2}\pi_*([\Omega_{X/S}]^2) + 2\pi^2\sum_j\psi_j$. A consequence is:

$$[\omega^{\mathrm{WP}}(\mathbf{a})] = [\omega^{\mathrm{WP}}] - 2\pi^2\sum_j(1-a_j)^2\psi_j = 2\pi^2\kappa_1 - \tfrac{1}{2}\sum_j\theta_j^2\psi_j \tag{16}$$

for $\theta_j = 2\pi(1-a_j)$.

Proof 2. An alternative proof follows the elegant inductive proof of Wolpert's theorem given by Arbarello and Cornalba in [1]. The main idea is to show that each side of (15) restricts to the boundary strata given by smaller moduli spaces,

$$\overline{\mathcal{M}}_{g-1,n+2}\to\overline{\mathcal{M}}_{g,n},\qquad \overline{\mathcal{M}}_{g_1,n_1+1}\times\overline{\mathcal{M}}_{g_2,n_2+1}\to\overline{\mathcal{M}}_{g,n}$$

via the same formula (15) applied to the smaller moduli spaces. Equivalently the aim is to show that the difference between the two sides of (15) (inductively) vanishes on the boundary.

As usual we denote by $\omega^{WP}(\mathbf{L})$ both the Weil-Petersson 2-form and its extension to the compactification. To understand the restriction of $\omega^{WP}(\mathbf{L})$ to the boundary strata, consider first Wolpert's original proof [32] in the case when Fenchel-Nielsen coordinates exist (such as when all cone angles are small). In such a case, Wolpert's formula (6) holds for $\omega^{WP}(\mathbf{L})$ and the limit of this formula, which defines the extension, shows that $\omega^{WP}(\mathbf{L})$ restricts to $\omega^{WP}(\mathbf{L},0,0)$ defined on $\overline{\mathcal{M}}_{g-1,n+2}$ and $\omega^{WP}(\mathbf{L}_1,0)\times\omega^{WP}(\mathbf{L}_2,0)$ on $\overline{\mathcal{M}}_{g_1,n_1+1}\times\overline{\mathcal{M}}_{g_2,n_2+1}$ where $\mathbf{L}=(\mathbf{L}_1,\mathbf{L}_2)$. When Fenchel-Nielsen coordinates do not necessarily exist, we can adapt Wolpert's proof as follows. As described in the proof of Proposition 2.5, close to the boundary divisor of the compactification $\overline{\mathcal{M}}^{\text{hyp}}_{g,n}(\mathbf{L})$, a simple closed geodesic γ of length ℓ tending to zero exists and leads to the formula $\omega^{\text{WP}}(\mathbf{L})=d\ell\wedge d\theta+\omega_c$ where θ is an angle along γ defining rotation deformations and ω_c is a closed 2-form independent of (ℓ,θ). Moreover, ω_c equals the Weil-Petersson 2-form $\omega^{WP}(\mathbf{L},0,0)$, respectively $\omega^{WP}(\mathbf{L}_1,0)\times\omega^{WP}(\mathbf{L}_2,0)$, due to the argument in proof 1 above, that $\Omega_{X/S}(\mathbf{a})$ converges on fibres to the $(1,1)$-form $\Omega_{X'/S'}(\mathbf{a}')$ on irreducible components of the limit, hence so does $\frac{1}{2}\int_{X/S}\Omega_{X/S}(\mathbf{a})^2$.

The class $2\pi^2\kappa_1-\frac{1}{2}\sum_j\theta_j^2\psi_j$ on the right hand side of (15) also restricts to the same form on the boundary divisor of $\overline{\mathcal{M}}_{g,n}$. Due to pullback properties of κ_1 and the psi classes ψ_j, it pulls back to $2\pi^2\kappa_1-\frac{1}{2}\sum_j\theta_j^2\psi_j$ on the boundary divisor $\overline{\mathcal{M}}_{g-1,n+2}$. It pulls back to $\overline{\mathcal{M}}_{g_1,|I|+1}\times\overline{\mathcal{M}}_{g_2,|J|+1}$ to the sum of $2\pi^2\kappa_1-\frac{1}{2}\sum_{j\in I}\theta_j^2\psi_j$ and $2\pi^2\kappa_1-\frac{1}{2}\sum_{j\in J}\theta_j^2\psi_j$ for $I\sqcup J=\{1,...,n\}$.

Given the inductive assumption that $[\omega^{WP}(\mathbf{L})]$ and $2\pi^2\kappa_1-\frac{1}{2}\sum_j\theta_j^2\psi_j$ agree on $\overline{\mathcal{M}}_{g',n'}$ for $2g'-2+n'<2g-2+n$, by restriction of these classes to the boundary described above the difference between the classes $[\omega^{WP}(\mathbf{L})]$ and $2\pi^2\kappa_1-\frac{1}{2}\sum_j\theta_j^2\psi_j$ vanishes on the boundary of $\overline{\mathcal{M}}_{g,n}$. For moduli spaces of dimension greater than one, this implies that they coincide due to results of Arbarello and Cornalba. So the initial inductive cases requires verifying (15) on $\overline{\mathcal{M}}^{\text{hyp}}_{1,1}(\mathbf{L})$ and $\overline{\mathcal{M}}^{\text{hyp}}_{0,4}(\mathbf{L})$. A degree two cohomology class in this dimension is equivalent to its evaluation, hence equality of the volumes $\text{Vol}(\mathcal{M}^{\text{hyp}}_{1,1}(\mathbf{L}))$ and $\text{Vol}(\mathcal{M}^{\text{hyp}}_{0,4}(\mathbf{L}))$ with Mirzakhani's polynomial $V_{1,1}(L)$, respectively $V_{0,4}(\mathbf{L})$, is enough to prove the equality in cohomology of (15) in these cases. The $(1,1)$ volume is calculated directly in Section 3.2. The $(0,4)$ volume uses proof 1 above, and can likely be calculated directly via a fundamental domain as in the $(1,1)$ case. See [16] for such an approach to the $(0,4)$ case. □

Remark. When some a_j are small and $2g-2+\sum a_j>0$, for example $g>0$ and $\sum a_j<1$, the natural compactification of $\mathcal{M}_{g,n}(\mathbf{L})$ for $L_j=2\pi i(1-a_j)$ is homeomorphic to $\overline{\mathcal{M}}_{g,n}(\mathbf{a})$ defined by Hassett [14]. There is a natural map from the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}\to\overline{\mathcal{M}}_{g,n}(\mathbf{a})$ which collapses some boundary divisors. In such a case, the Weil-Petersson form ω^{WP} does not extend to a smooth form on $\overline{\mathcal{M}}_{g,n}(\mathbf{a})$.

Theorem 1.2 is an immediate consequence of Corollary 3.2 since $\text{Vol}(\mathcal{M}^{\text{hyp}}_{g,n}(\mathbf{L}))$ is given by $\int_{\overline{\mathcal{M}}_{g,n}}\exp\left(2\pi^2\kappa_1+\sum_1^n\frac{1}{2}L_j^2\psi_j\right)$ which agrees with Mirzakhani's polynomial.

Corollary 1.3 now follows from [7] but it can also be proven directly. The vanishing

$$\lim_{\theta\to 2\pi}\mathrm{Vol}(\mathcal{M}^{\mathrm{hyp}}_{g,n+1}(0^n,\theta))=0$$

follows from the degeneration of the Kähler metric in the limit, proven in [25]. One can also see it from Mondello's formula [20] for the Poisson structure. This argument shows that the vanishing in the 2π limit holds more generally, for example when all boundary components are cusps and cone angles, even when the volume is not given by Mirzakhani's polynomial.

The derivative formula can be proven directly as follows. By (15) when $\mathbf{L}=(0^n,\theta)$ then $\omega^{WP}(\mathbf{L})=2\pi^2\kappa_1-\frac{1}{2}\theta^2\psi_{n+1}$ and $\mathrm{Vol}(\mathcal{M}^{\mathrm{hyp}}_{g,n+1}(0^n,i\theta))=\int_{\overline{\mathcal{M}}_{g,n+1}}\frac{\left(2\pi^2\kappa_1-\frac{1}{2}\theta^2\psi_{n+1}\right)^N}{N!}$ where $N=3g-2+n$. Hence

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathrm{Vol}(\mathcal{M}^{\mathrm{hyp}}_{g,n+1}(0^n,i\theta))\Big|_{\theta=2\pi}&=-\int_{\overline{\mathcal{M}}_{g,n+1}}\theta\psi_{n+1}\frac{\left(2\pi^2\kappa_1-\frac{1}{2}\theta^2\psi_{n+1}\right)^{N-1}}{(N-1)!}\Bigg|_{\theta=2\pi}\\&=-2\pi\int_{\overline{\mathcal{M}}_{g,n+1}}\psi_{n+1}\frac{\left(2\pi^2(\kappa_1-\psi_{n+1})\right)^{N-1}}{(N-1)!}\\&=2\pi(2-2g-n)\int_{\overline{\mathcal{M}}_{g,n}}\frac{\left(2\pi^2\kappa_1\right)^{N-1}}{(N-1)!}\\&=2\pi(2-2g-n)\mathrm{Vol}(\mathcal{M}^{\mathrm{hyp}}_{g,n}(0^n))\end{aligned}$$

where the third equality uses the pullback formula $\kappa_1-\psi_{n+1}=\pi^*\kappa$ and the pushforward formula $\pi_*(\psi_{n+1}\pi^*\eta)=(2g-2+n)\eta$ for any $\eta\in H^*(\overline{\mathcal{M}}_{g,n})$.

A general volume $\mathrm{Vol}(\mathcal{M}^{\mathrm{hyp}}_{g,n}(\mathbf{L}))$ is not expected to be given by evaluation of Mirzakhani's polynomial because the Deligne-Mumford compactification is not in general related to a compactification of $\mathcal{M}^{\mathrm{hyp}}_{g,n}(\mathbf{L})$. When there exists boundary geodesics of positive length together with cone points of cone angle greater than π, the results of Section 2 show that the boundary of a compactification necessarily has real codimension one, which is different to the Deligne-Mumford compactification.

To prove that indeed evaluation of Mirzakhani's polynomial does not give the volume, we constructed an example $\mathcal{M}^{\mathrm{hyp}}_{0,4}(0,0,\theta i,(2\pi-\epsilon)i)$ of a non-empty moduli space in the introduction so that $V_{0,4}(0,0,\theta i,(2\pi-\epsilon)i)<0$ for $4\pi\epsilon<\theta^2$. The moduli space is non-empty, hence has positive volume, since $\theta i+2\pi-\epsilon<4\pi$ so existence of conical metrics follows from [15]. A sketch of a proof of further examples goes as follows. If $L_j\to 2\pi i$ then we claim that the volume must tend to zero even in the presence of geodesic boundary components. When $\mathbf{L}$ is chosen so that there exists boundary geodesics of positive length together with cone points of cone angle greater than π, the Weil-Petersson form can be defined via the symplectic reduction argument of Mirzakhani [19]. Glue to each geodesic boundary component a pair of pants with two cusps and a geodesic boundary component.

This defines a new moduli space of hyperbolic surfaces with only cusps and cone angles

$$\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})^+ \to \mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L}).$$

It is a $U(1)^m \times \mathbb{R}_+^n$ bundle ($m =$ the number of geodesic boundary components) constructed by allowing variation of the lengths of the geodesic boundary components and a choice of point on each edge, which keeps track of rotations of the gluing. There is a $U(1)^m$-invariant Kähler symplectic structure on $\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})^+$. In fact $\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})^+$ is an infinite cover of a usual moduli space of conical hyperbolic surfaces, due to a choice of geodesics bounding the glued pairs of pants. Nevertheless, the locally defined symplectic structure lifts to the cover. The symplectic reduction of $\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})^+$ gives the symplectic structure on $\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})$. The symplectic structure uses the moment map together with the Chern forms given by the curvatures of the natural connections m line bundles. Mirzakhani used the Chern *classes* of such line bundles because because she was working over a compactification. The symplectic form on $\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})^+$ degenerates as one of the cone angles tends to 2π. Hence the quotient symplectic form on $\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L})$ tends to zero and the volume tends to zero on compact subsets as a cone angle tends to 2π, suggesting that $\text{Vol}(\mathcal{M}_{g,n}^{\text{hyp}}(\mathbf{L}))$ tends to zero in the limit. But Mirzakhani's polynomials do not tend to zero. For example,

$$V_{0,4}(L_1, L_2, L_3, i\theta) = 2\pi^2 - \frac{1}{2}\theta^2 + \frac{1}{2}\sum_{j=1}^{3} L_j^2 \overset{\theta\to 2\pi}{\longrightarrow} \frac{1}{2}\sum_{j=1}^{3} L_j^2 > 0$$

does not tend to zero for $L_j > 0$. More generally

$$V_{g,n+1}(\mathbf{L}, 2\pi i) = \sum_{k=1}^{n} \int_0^{L_k} L_k V_{g,n}(\mathbf{L}) dL_k > 0$$

for $L_j > 0$. We see that when there exists boundary geodesics of positive length together with cone points of cone angle greater than π, Mirzakhani's polynomials do not have a zero limit as, say $L_n \to 2\pi i$, hence must differ from the actual volume which does limit to zero.

3.2 Volume of $\mathcal{M}_{1,1}^{\text{hyp}}(i\theta)$

In this subsection we prove a volume formula for $\mathcal{M}_{1,1}^{\text{hyp}}(i\theta)$. This is done explicitly by extending the method used by Wolpert [31] in his proof of the cuspidal case.

Let M be a compact, oriented surface of genus 1, with one boundary component. The fundamental group $\pi_1(M)$ is the free group on two generators, $\langle a, b\rangle$. A flat $\text{SL}(2,\mathbb{R})$-connection on M is equivalent to a representative in the equivalence class of $\text{SL}(2,\mathbb{R})$ representations $\rho \in \text{Hom}(\pi_1(M), \text{SL}(2,\mathbb{R}))/\sim$, where the equivalence relation is given by conjugation. The boundary of M is represented by the class $c = [a, b] \in \pi_1(M)$, and this corresponds to a cone point when $\text{tr}(\rho(c)) \in (-2, 2)$. Such a representation gives M the structure of a hyperbolic surface with cone angle θ such that $-2\cos\left(\frac{\theta}{2}\right) = \operatorname{tr}\rho(c)$.

There is an action of $\operatorname{Aut}(\pi_1(M))$ on $\operatorname{Hom}(\pi_1(M), \operatorname{SL}(2,\mathbb{R}))/\sim$, given by $g\cdot[\rho] = [\rho\circ g^{-1}]$. For M a surface of type $(1,1)$, the automorphism group has three generators g_1, g_2 and g_3. The action of generators is given by,

$$\begin{array}{ccc} g_1(a) = a^{-1} & g_2(a) = b & g_3(a) = ab \\ g_1(b) = b & g_2(b) = a & g_3(b) = b. \end{array}$$

One obtains a representation of $\operatorname{Aut}(\pi_1(M))$ into $\operatorname{GL}(2,\mathbb{Z})$ by its action on $\mathbb{Z}^2$, generated by cosets of a and b. Define $\operatorname{Aut}^+(\pi_1(M)) \subset \operatorname{Aut}(\pi_1(M))$ as the preimage of $\operatorname{SL}(2,\mathbb{Z}) \subset \operatorname{GL}(2,\mathbb{Z})$. This fits into a pullback square

$$\begin{array}{ccc} \operatorname{Aut}^+(\pi_1(M)) & \xrightarrow{\subset} & \operatorname{Aut}(\pi_1(M)) \\ \downarrow & & \downarrow \\ \operatorname{SL}(2,\mathbb{Z}) & \xrightarrow{\subset} & \operatorname{GL}(2,\mathbb{Z}). \end{array}$$

Further, let $\operatorname{Inn}(\pi_1(M)) \trianglelefteq \operatorname{Aut}^+(\pi_1(M))$ be the normal subgroup of inner automorphisms, with elements given by conjugation. The action of $\operatorname{Inn}(\pi_1(M))$ on $\operatorname{Hom}(\pi_1(M), \operatorname{SL}(2,\mathbb{R}))/\sim$ is trivial, so there is a well defined action of the quotient group $\operatorname{Out}^+(\pi_1(M)) :=$ $\operatorname{Aut}^+(\pi_1(M))/\operatorname{Inn}(\pi_1(M))$.

Define the character mapping

$$\chi : \operatorname{Hom}(\pi_1(M), \operatorname{SL}(2,\mathbb{R}))/\sim\ \longrightarrow \mathbb{R}^3$$
$$[\rho] \longmapsto \begin{bmatrix} x(\rho) \\ y(\rho) \\ z(\rho) \end{bmatrix} := \begin{bmatrix} \operatorname{tr}(\rho(a)) \\ \operatorname{tr}(\rho(b)) \\ \operatorname{tr}(\rho(ab)) \end{bmatrix}.$$

This is a homeomorphism [10, 11]. Level sets of the trace function

$$\kappa(x,y,z) := \operatorname{tr}(\rho[a,b]) = x^2 + y^2 + z^2 - xyz - 2 \tag{17}$$

describe regions in $\mathbb{R}^3$ corresponding to prescribed boundary holonomy. It is proven in [13] that any fixed $\kappa \in (-2,2)$ defines a surface in $\mathbb{R}^3$ with 5 disconnected components. Further restriction to the region $x, y, z \geq 2$ gives a copy of Teichmüller space $\mathcal{T}^{\text{hyp}}(\theta)$, where $\kappa = -2\cos\left(\frac{\theta}{2}\right)$.

The mapping class group of $\mathcal{T}^{\text{hyp}}(\theta)$ consists of isotopy classes of homeomorphisms of surfaces. In the character variety, these correspond to automorphisms which preserve the polynomial $\kappa(x,y,z)$. A theorem of Dehn-Nielsen (proven in [24]) says that this corresponds to the action of $\operatorname{Out}^+(\pi_1(M)) \cong \operatorname{SL}(2,\mathbb{Z})$ on the image of Teichmüller space in the character variety $\mathcal{T}_{1,1}^{\kappa} = \left\{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 - xyz - 2 = \kappa,\ x \geq 2, y \geq 2, z \geq 2\right\}$.

It will be convenient to work in the homogeneous coordinates $r = \frac{x}{yz}$, $s = \frac{y}{zx}$ and $t = \frac{z}{xy}$. Here for a fixed $\kappa \in (-2, 2)$ the Teichmüller space becomes

$$\mathcal{T}_{1,1}^{\kappa} = \left\{(r, s, t) \in \mathbb{R}^3 \mid r + s + t - 1 = (\kappa + 2)rst,\ r > 0, s > 0, t > 0\right\}.$$

Following [31], define the subgroups

- $\Gamma(2) = \left\{ A \in \mathrm{SL}(2, \mathbb{Z}) \mid A \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mod 2 \right\}$,
- $P\Gamma(2)$ as the image of $\Gamma(2)$ in $\mathrm{PSL}(2, \mathbb{Z})$,
- $\mathcal{M}_2 = \langle \phi_1, \phi_2, \phi_3 \rangle \subset \mathrm{GL}(2, \mathbb{Z})$, where $\phi_1 = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$, $\phi_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\phi_3 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$,
- $\mathcal{M}_2^+ = \mathcal{M}_2 \cap \mathrm{SL}(2, \mathbb{Z})$, corresponding to words of even length in generators of $\mathcal{M}_2$.

From [31] we have the following results; $[\mathrm{PSL}(2, \mathbb{Z}) : \mathcal{M}_2^+] = 6$, $[\mathcal{M}_2 : \mathcal{M}_2^+] = 2$, and the action of $\mathcal{M}_2$ on $\mathcal{T}_{1,1}^{\kappa}$ is given by,

$$\begin{array}{ccc} \phi_1(x) = yz - x & \phi_2(x) = x & \phi_3(x) = x \\ \phi_1(y) = y & \phi_2(y) = y & \phi_3(y) = xz - y \\ \phi_1(z) = z & \phi_2(z) = xy - z & \phi_3(z) = z, \end{array}$$

in (x, y, z)-coordinates and

$$\begin{array}{ccc} \phi_1(r) = 1 - r & \phi_2(r) = \frac{tr}{1-t} & \phi_3(r) = \frac{sr}{1-s} \\ \phi_1(s) = \frac{rs}{1-r} & \phi_2(s) = \frac{ts}{1-t} & \phi_3(s) = 1 - s \\ \phi_1(t) = \frac{rt}{1-r} & \phi_2(t) = 1 - t & \phi_3(t) = \frac{st}{1-s}, \end{array}$$

in (r, s, t)-coordinates.

The main original part of the proof in this section is in computing the fundamental domain Δ for the $\mathcal{M}_2$ action on $\mathcal{T}_{1,1}^{\kappa}$ with $\kappa \in (-2, 2)$. Wolpert does this in the case $\kappa = -2$ corresponding to cusped hyperbolic surfaces, and here the idea of proof will be similar.

Claim 3.3. *The fundamental domain Δ for the $\mathcal{M}_2$ action on $\mathcal{T}_{1,1}^{\kappa}$ with $\kappa \in (-2, 2)$ is given by $\Delta = \{(r, s, t) \in \mathbb{R}^3 \mid r + s + t - 1 = (\kappa + 2)rst,\ r \in \left(0, \frac{1}{2}\right],\ s \in \left(0, \frac{1}{2}\right],\ t \in \left(0, \frac{1}{2}\right]\}$.*

Proof of claim. Consider the function $E(x, y, z) = x + y + z$. We will show that for any $X \in \mathcal{T}_{1,1}^{\kappa}$, E admits a unique minimum in Δ, on the orbit $\mathcal{M}_2(X)$. First note that $x, y, x \geq 2$ so $E(x, y, z) > 0$. Also $\mathcal{M}_2$ has discrete orbits, so a minimum for E on $\mathcal{M}_2(X)$ exists.

We show that minimums of E on the orbit exist in Δ. Consider the formulas,

$$\frac{E\circ\phi_1 - E}{2yz} = \frac{1}{2} - \frac{x}{yz}, \qquad \frac{E\circ\phi_2 - E}{2xy} = \frac{1}{2} - \frac{z}{xy}, \qquad \frac{E\circ\phi_3 - E}{2zx} = \frac{1}{2} - \frac{y}{zx}. \tag{18}$$

If $Y \in \mathcal{M}_2(X)$ is a minimum for E, then $E\circ\phi_i(Y) \geq E(Y)$ for any $i = 1, 2$ or 3. So by (18), coordinates of Y are such that $a, b, c \leq \frac{1}{2}$. So E has a minimum in Δ, on the orbit $\mathcal{M}_2(X)$.

In what remains, we prove uniqueness of such a minimum. First, we prove a convexivity property for the function E. Let $w_{n+1}w_n w_{n-1}\cdots w_1$ be a reduced word of length $n+1$ in $\{\phi_1, \phi_2, \phi_3\}$, and let $W = w_n w_{n-1}\cdots w_1$. Note that $\phi_i^2 = 1$ for $i = 1, 2, 3$, so in particular, $w_n W = w_{n-1}\cdots w_1$. For any $S \in \mathcal{T}^{\kappa}_{1,1}$ such that $E(W(S)) \geq E(w_{n-1}\cdots w_1(S)) = E(w_n W(S))$, it is not possible that $E(W(S)) \geq E(w_{n+1}W(S))$.

If it were possible, then applying equations (18) to $W(S)$ would imply that at least two of r, s, t would be greater than or equal to $\frac{1}{2}$. Suppose with out loss of generality that $r \geq \frac{1}{2}$ and $s \geq \frac{1}{2}$, then $z^2 = \frac{1}{rs} \leq 4$ so $z \in [-2, 2]$. By definition of $W(S) \in \mathcal{T}^{\kappa}_{1,1}$, this would mean that $z = 2$, so $x^2 + y^2 + 2^2 - 2xy = \kappa + 2$, giving that $(x-y)^2 = \kappa - 2 < 0$. This is a contradiction, so the convexity property is established.

Now, suppose $S_1, S_2 \in \Delta$ are points in the orbit $\mathcal{M}_2(X)$, which both reach a minimum value for E. Since $S_1, S_2 \in \Delta$ (18) tells us that, $E(\phi_i(S_1)) \geq E(S_1)$ and $E(\phi_i(S_2)) \geq E(S_1)$ for $i = 1, 2, 3$. Convexivity further implies that $E(\phi_j\phi_i(S_1)) > E(\phi_i S_1)$ and $E(\phi_j\phi_i(S_2)) > E(\phi_i S_1)$ for $i = 1, 2, 3$. Repeated application of convexivity and the strictness of these inequalities leave only two possibilities. Either $S_1 = S_2$ or there exists a $k = 1, 2, 3$, such that $S_1 = \phi_k(S_2)$ (which is equivalent to $S_2 = \phi_k(S_1)$). In the second case, $E(S_1) = E(\phi_k(S_2)) = E(S_2) = E(\phi_k(S_1))$. By the definition of E and ϕ, it follows that $S_1 = S_2$. This completes the proof. $\square$

Rewriting this region as depending on only two variables, we obtain

$$\Delta = \left\{ \left(r, s, \frac{r+s-1}{(\kappa+2)rs-2}\right) \in \mathbb{R}^3 \mid \; r \in \left(\frac{1-2s}{2-(\kappa+2)s}, \frac{1}{2}\right], \; s \in \left(0, \frac{1}{2}\right] \right\}.$$

Remark. When $\theta \to 2\pi$ the fundamental domain contracts to a point. This explains why the volume vanishes in this case.

The Weil-Petersson form can be described explicitly on the character variety in (x, y)-coordinates,

$$\omega = \frac{4dx \wedge dy}{xy - z}. \tag{19}$$

In (r, s)-coordinates this becomes

$$\omega = \frac{dr \wedge ds}{(1-(\kappa+2)rs)rst} = \frac{dr \wedge ds}{(1-r-s)rs}. \tag{20}$$

Finally, the volume is given by

$$\begin{aligned}\int_{\mathcal{M}_{1,1}(i\theta)}\omega &= \frac{1}{[\mathrm{PSL}(2,\mathbb{Z}):\mathcal{M}_2^+]}\int_{\mathcal{T}_{1,1}^\kappa/\mathcal{M}_2^+}\omega\\ &= \frac{[\mathcal{M}_2:\mathcal{M}_2^+]}{[\mathrm{PSL}(2,\mathbb{Z}):\mathcal{M}_2^+]}\int_{\mathcal{T}_{1,1}^\kappa/\mathcal{M}_2^+}\omega\\ &= \frac{2}{6}\int_\Delta \frac{dr\wedge ds}{(1-r-s)rs}\\ &= \frac{1}{3}\int_0^{\frac{1}{2}}\int_{\frac{1-2s}{2-(\kappa+2)s}}^{\frac{1}{2}}\frac{1}{(1-r-s)rs}dr\,ds\\ &= \frac{1}{6}\left(\pi^2-\theta^2/4\right) \qquad\qquad \theta\in[0,2\pi),\end{aligned}$$

where $\kappa=-2\cos\left(\frac{\theta}{2}\right)$. A further factor of 1/2 is needed due to the $\mathbb{Z}_2$ automorphism of the generic hyperbolic surface, yielding

$$\mathrm{Vol}(\mathcal{M}_{1,1}^{\mathrm{hyp}}(i\theta)) = \frac{1}{48}(4\pi^2-\theta^2)$$

which agrees with Mirzakhani's polynomial evaluated at $i\theta$.

A Non-existence of pants decompositions

In this appendix, we give a construction of arbitrary genus hyperbolic surfaces without a pants decomposition in a chosen isotopy class. The two hexagons in Figure 11 naturally glue along the edges of given lengths in the case $w=x$ to produce a hyperbolic pair-of-pants with boundary lengths $X(x,y,z)$, $Y(x,y,z)$ and $Z(x,y,z)$ (respectively opposite the arcs of lengths x, y and z). The pentagon in Figure 11 has area given by $\pi-\phi$ and edge lengths satisfying

$$\cosh(w) = -\cosh^2(x/2)\cos\phi+\sinh^2(x/2).$$

When $\phi\approx\pi$ (and $\phi<\pi$), we have the area ≈ 0 and $w\approx x$, due to $\cosh(w)=\cosh^2(w/2)+\sinh^2(w/2)$, so the pentagon is thin. Glue the pentagon and hexagon along the edges of lengths $x/2$ and x to create a cone angle of $\pi+\phi$. Now glue the remaining matching sides of lengths w, y and z to obtain a genus zero hyperbolic surface Σ with four boundary components—the cone point, and three geodesics of lengths X, $Y'>Y$ and $Z'>Z$.

Lemma A.1. *The genus zero hyperbolic surface Σ constructed from gluing the polygons in Figure 11 does not admit a geometric pants decomposition.*

Proof. Choose $\phi\approx\pi$ and consider a pants decomposition of Σ defined by the isotopy class of a simple closed loop that separates Σ into two pairs-of-pants consisting of the pair-of-pants P containing the cone point and the boundary geodesic of length Z' and its complement that contains the boundaries of lengths X and Y'. Then $\mathrm{area}(P)=\pi-\phi\approx 0$. Cut along

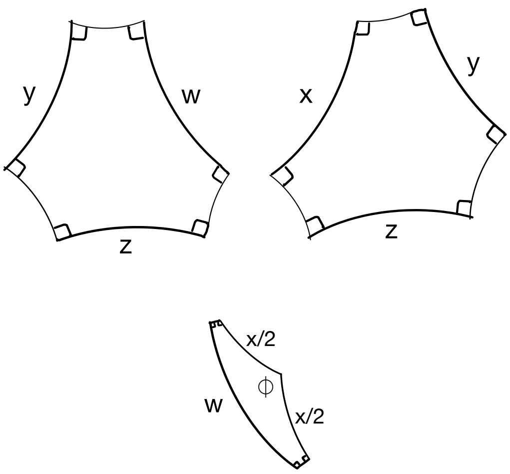

Figure 11. Glue to produce surface with cone angle $\pi + \phi$

the unique non-trivial closed geodesic from the cone point to itself to decompose P into two annuli including $A \subset P$ that contains the boundary geodesic of length Z'. Then A has large area because Z' and $x/2$ are large. (Draw picture: can estimate area.) The large area contradicts area$(P) = \pi - \phi \approx 0$. Thus Σ does not admit a a pants decomposition in the given isotopy class.

More generally, choose any topological pants decomposition of Σ and suppose it admits a geometric pants decomposition in the same isotopy class consisting of the pair-of-pants P containing the cone point and its complement. Then the distance from the cone point to the boundary component of Σ in P is large and the argument above again produces a large area for an annulus inside P hence a contradiction and the proposition is proven. □

Corollary A.2. *There exist hyperbolic surfaces of arbitrary genus and arbitrary geometric boundary types, for example all cusps, except for one cone angle close to 2π, without a pants decomposition in a chosen isotopy class.*

Proof. Simply glue hyperbolic surfaces to the example Σ from Lemma A.1 and choose the pants decomposition isotopy class to contain the three geodesic boundary components of Σ. For example, consider a genus 2 surface with a single cone angle and no other boundary components obtained by arranging $Y = Z$, hence $Y' = Z'$, and gluing along these two boundary components. Then attach a genus 1 surface with geodesic boundary of length X to produce a genus 2 hyperbolic surface with one cone point equipped with a topological pants decomposition not realisable geometrically. □

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