A NEW COHOMOLOGY CLASS ON THE MODULI SPACE OF CURVES

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ABSTRACT. We define a collection of cohomology classes $\Theta_{g,n}\in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$ for $2g-2+n>0$ that restrict naturally to boundary divisors. We prove that a generating function for the intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}$ is a tau function of the KdV hierarchy. This is analogous to the theorem conjectured by Witten and proven by Kontsevich that a generating function for the intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^n\psi_i^{m_i}$ is a tau function of the KdV hierarchy.

CONTENTS

1. INTRODUCTION

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of genus g stable curves—curves with only nodal singularities and finite automorphism group—with n labeled points disjoint from nodes. Define $\psi_i=c_1(L_i)\in H^2(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ the first Chern class of the line bundle $L_i\to\overline{\mathcal{M}}_{g,n}$ with fibre above $[(C,p_1,\dots,p_n)]$ given by $T^*_{p_i}C$. Consider the natural maps given by the forgetful map $\overline{\mathcal{M}}_{g,n+1}\xrightarrow{\pi}\overline{\mathcal{M}}_{g,n}$ and the gluing maps $\overline{\mathcal{M}}_{g-1,n+2}\xrightarrow{\phi_{\rm irr}}\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{h,|I|+1}\times\overline{\mathcal{M}}_{g-h,|J|+1}\xrightarrow{\phi_{h,I}}\overline{\mathcal{M}}_{g,n}$ for $I\sqcup J=\{1,...,n\}$.

In this paper we construct cohomology classes $\Theta_{g,n}\in H^*(\overline{\mathcal{M}}_{g,n})$ for $g\geq 0$, $n\geq 0$ and $2g-2+n>0$ satisfying the following four properties:

(i) $\Theta_{g,n}\in H^*(\overline{\mathcal{M}}_{g,n})$ is of pure degree,
(ii) $\phi_{\rm irr}^*\Theta_{g,n}=\Theta_{g-1,n+2}$, $\quad\phi_{h,I}^*\Theta_{g,n}=\pi_1^*\Theta_{h,|I|+1}\cdot\pi_2^*\Theta_{g-h,|J|+1}$,
(iii) $\Theta_{g,n+1}=\psi_{n+1}\cdot\pi^*\Theta_{g,n}$,
(iv) $\Theta_{1,1}=3\psi_1$.

The properties (i)-(iv) uniquely define intersection numbers of the classes $\Theta_{g,n}$ with the classes ψ_i. It is not clear if they uniquely define the classes $\Theta_{g,n}$ themselves.

Remark 1.1. One can replace (ii) by the equivalent property

$$\phi_\Gamma^*\Theta_{g,n}=\Theta_\Gamma.$$

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for any stable graph Γ of genus g and with n external edges. Here

$$\phi_\Gamma:\overline{\mathcal{M}}_\Gamma=\prod_{v\in V(\Gamma)}\overline{\mathcal{M}}_{g(v),n(v)}\to\overline{\mathcal{M}}_{g,n},\quad \Theta_\Gamma=\prod_{v\in V(\Gamma)}\pi_v^*\Theta_{g(v),n(v)}\in H^*(\overline{\mathcal{M}}_\Gamma)$$

where π_v is projection onto the factor $\overline{\mathcal{M}}_{g(v),n(v)}$. This generalises (ii) from 1-edge stable graphs where $\phi_{\Gamma_{\rm irr}}=\phi_{\rm irr}$ and $\phi_{\Gamma_{h,I}}=\phi_{h,I}$. See Section 4.1.1 for more on stable graphs.

Remark 1.2. The sequence of classes $\Theta_{g,n}$ satisfies many properties of a cohomological field theory (CohFT). It is essentially a 1-dimensional CohFT with vanishing genus zero classes, not to be confused with the Hodge class which is trivial in genus zero but does not vanish there. The trivial cohomology class $1\in H^0(\overline{\mathcal{M}}_{g,n})$, which is a trivial example of a CohFT, satisfies conditions (i) and (ii). In this case, the forgetful map property (iii) is replaced by $\Theta_{g,n+1}=\pi^*\Theta_{g,n}$ and the initial value property (iv) is replaced by $\Theta_{1,1}=1$.

Remark 1.3. The existence proof of $\Theta_{g,n}$—see Section 2—requires the initial value property (iv) given by $\Theta_{1,1}=3\psi_1$. The existence of $\Theta_{g,n}$ with (iv) replaced by $\Theta_{1,1}=\lambda\psi_1$ for general $\lambda\in\mathbb{C}$ is unknown. One can of course replace $\Theta_{g,n}$ by $\lambda^{2g-2+n}\Theta_{g,n}$ but this would change property (iii).

Theorem 1. *There exists a class $\Theta_{g,n}$ satisfying (i) - (iv) and furthermore any such class satisfies the following properties.*

(I) $\Theta_{g,n}\in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$.

(II) $\Theta_{0,n}=0$ *for all n and* $\phi_\Gamma^*\Theta_{g,n}=0$ *for any Γ with a genus 0 vertex.*

(III) $\Theta_{g,n}\in H^*(\overline{\mathcal{M}}_{g,n})^{S_n}$, *i.e. it is symmetric under the S_n action.*

(IV) The intersection numbers $\displaystyle\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}\prod_{j=1}^N\kappa_{\ell_j}$ *are uniquely determined.*

(V) $\displaystyle Z^\Theta(\hbar,t_0,t_1,...)=\exp\sum_{g,n,\vec{k}}\frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\cdot\prod_{j=1}^n\psi_j^{k_j}\prod t_{k_j}$ *is a tau function of the KdV hierarchy.*

The main content of Theorem 1 is the existence of $\Theta_{g,n}$ which is constructed via the push-forward of a class over the moduli space of spin curves in Section 2, and the KdV property (V) proven in Section 4. The non-constructive uniqueness result (IV)—which relies on the existence of non-explicit tautological relations—follows from the more general property that the intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}$ are uniquely determined by *any* initial value $\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}\in\mathbb{C}$. For the initial value $\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}=\frac{1}{8}$, (IV) is strengthened by (V) which allows one to recursively calculate all intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}$ via recursive relations coming out of the KdV hierarchy. The proofs of properties (I) - (IV) are straightforward and presented in Section 3.

The proof of (V) does not directly use the KdV hierarchy. Instead it identifies the proposed KdV tau function Z^Θ with a known KdV tau function—the Brezin-Gross-Witten KdV tau function $Z^{\rm BGW}$ defined in [2, 20]. This identification of $Z^\Theta(\hbar,t_0,t_1,...)$ with $Z^{\rm BGW}(\hbar,t_0,t_1,...)$ is stated as Theorem 3 in Section 4.4.3. The proof of Theorem 3 uses a set of tautological relations, known as Pixton's relations, obtained from the moduli space of 3-spin curves and proven in [31]. Just as tautological relations give topological recursion relations for Gromov-Witten invariants, the intersections of $\Theta_{g,n}$ with Pixton's relations produce topological recursion relations satisfied by $\Theta_{g,n}$ that are enough to uniquely determine all intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}\prod_{j=1}^N\kappa_{\ell_j}$. The strategy of the proof of (V) is to show that the coefficients of $Z^{\rm BGW}(\hbar,t_0,t_1,...)$ satisfy the same relations as the corresponding coefficients of $Z^\Theta(\hbar,t_0,t_1,...)$, given by $\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}$.

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2. Existence

The existence of a cohomology class $\Theta_{g,n}\in H^*(\overline{\mathcal{M}}_{g,n})$ satisfying (i) - (iv) can be proven using the moduli space of stable twisted spin curves $\overline{\mathcal{M}}_{g,n}^{\rm spin}$ which consist of pairs (Σ,θ) given by a twisted stable curve Σ

equipped with an orbifold line bundle θ together with an isomorphism $\theta^{\otimes 2} \cong \omega_\Sigma^{\log}$. See precise definitions below. We first construct a cohomology class on $\overline{\mathcal{M}}_{g,n}^{\rm spin}$ and then push it forward to a cohomology class on $\overline{\mathcal{M}}_{g,n}$.

A stable twisted curve, with group $\mathbb{Z}_2$, is a 1-dimensional orbifold, or stack, $\mathcal{C}$ such that generic points of $\mathcal{C}$ have trivial isotropy group and non-trivial orbifold points have isotropy group $\mathbb{Z}_2$. A stable twisted curve is equipped with a map which forgets the orbifold structure $\rho : \mathcal{C} \to C$ where C is a stable curve known as the the coarse curve of $\mathcal{C}$. We say that $\mathcal{C}$ is smooth if its coarse curve C is smooth. Each nodal point of $\mathcal{C}$ (corresponding to a nodal point of C) has non-trivial isotropy group and all other points of $\mathcal{C}$ with non-trivial isotropy group are labeled points of $\mathcal{C}$.

A line bundle L over $\mathcal{C}$ is a locally equivariant bundle over the local charts, such that at each nodal point there is an equivariant isomorphism of fibres. Hence each orbifold point p associates a representation of $\mathbb{Z}_2$ on $L|_p$ acting by multiplication by $\exp(2\pi i\lambda_p)$ for $\lambda_p = 0$ or $\frac{1}{2}$. One says L is *banded* by λ_p. The equivariant isomorphism at nodes guarantees that the representations agree on each local irreducible component at the node.

The sheaf of local sections $\mathcal{O}_\mathcal{C}(L)$ pushes forward to a sheaf $|L| := \rho_*\mathcal{O}_\mathcal{C}(L)$ on C which can be identified with the local sections of L invariant under the $\mathbb{Z}_2$ action. Away from nodal points $|L|$ is locally free, hence a line bundle. The pull-back bundle $\rho^*(|L|) = L \otimes \bigotimes_{i\in I} \mathcal{O}(-p_i)$ where L is banded by the non-trivial representation precisely at p_i for $i \in I$. Hence $\deg |L| = \deg L - \frac{1}{2}|I|$. At nodal points, the push-forward $|L|$ is locally free when L is banded by the trivial representation, and $|L|$ is a torsion-free sheaf that is not locally free when L is banded by the non-trivial representation. See [16] for a nice description of these ideas.

The canonical bundle $\omega_\mathcal{C}$ of $\mathcal{C}$ is generated by dz for any local coordinate z. At an orbifold point $x = z^2$ the canonical bundle $\omega_\mathcal{C}$ is generated by dz hence it is banded by $\frac{1}{2}$ i.e. $dz \mapsto -dz$ under $z \mapsto -z$. Over the coarse curve ω_C is generated by $dx = 2zdz$. In other words $\rho^*\omega_C \not\cong \omega_\mathcal{C}$ however $\omega_C \cong \rho_*\omega_\mathcal{C}$. Moreover, $\deg \omega_C = -\chi = 2g-2$ and

$$\deg \omega_\mathcal{C} = -\chi^{\rm orb} = 2g - 2 + \frac{1}{2}n.$$

For $\omega_\mathcal{C}^{\log} = \omega_\mathcal{C}(p_1, ..., p_n)$, locally $\frac{dx}{x} = 2\frac{dz}{z}$ so $\rho^*\omega_C^{\log} \cong \omega_\mathcal{C}^{\log}$ and $\deg \omega_C^{\log} = 2g-2+n = \deg \omega_\mathcal{C}^{\log}$.

Following [1], define

$$\overline{\mathcal{M}}_{g,n}^{\rm spin} = \{(\mathcal{C}, \theta, p_1, ..., p_n, \phi) \mid \phi : \theta^2 \xrightarrow{\cong} \omega_\mathcal{C}^{\log}\}.$$

Here $\omega_\mathcal{C}^{\log}$ and θ are line bundles over the stable twisted curve $\mathcal{C}$ with labeled orbifold points p_j and $\deg\theta = g-1+\frac{1}{2}n$. The relation $\theta^2 \xrightarrow{\cong} \omega_\mathcal{C}^{\log}$ is possible because the representation associated to $\omega_\mathcal{C}^{\log}$ at p_i is trivial—$dz/z \xrightarrow{z\mapsto -z} dz/z$. We require the representations associated to θ at each p_i to be non-trivial, i.e $\lambda_{p_i} = \frac{1}{2}$. At nodal points p, both types $\lambda_p = 0$ or $\frac{1}{2}$ can occur. The equivariant isomorphism of fibres over nodal points forces the balanced condition $\lambda_{p_+} = \lambda_{p_-}$ for $p_\pm$ corresponding to p on each irreducible component. Among the 2^{2g} different spin structures on a twisted curve $\mathcal{C}$, some will have $\lambda_p = 0$ and some will have $\lambda_p = \frac{1}{2}$.

The forgetful map

$$f : \overline{\mathcal{M}}_{g,n+1}^{\rm spin} \to \overline{\mathcal{M}}_{g,n}^{\rm spin}$$

is defined via $f(\mathcal{C}, \theta, p_1, ..., p_{n+1}, \phi) = (\rho(\mathcal{C}), \rho_*\theta, p_1, ..., p_n, \rho_*\phi)$ where ρ forgets the label and orbifold structure at p_{n+1}. As described above, the push-forward $\rho_*\theta$ consists of local sections invariant under the $\mathbb{Z}_2$ action. Since the representation at p_{n+1} is given by multiplication by -1, any invariant local section must vanish at p_{n+1}. In other words $\rho_*\theta = \rho_*\{\theta(-p_{n+1})\}$, $\rho^*\rho_*\theta = \{\theta(-p_{n+1})\}$ and $\deg \rho_*\theta = \deg\theta - \frac{1}{2}$.

Tautological line bundles $L_{p_i} \to \overline{\mathcal{M}}_{g,n}^{\rm spin}$, $i = 1, ..., n$ are defined analogously to those defined over $\overline{\mathcal{M}}_{g,n}$. For a family $\pi : \mathcal{C} \to S$ with sections $p_i : S \to \mathcal{C}$, $i = 1, ..., n$, they are defined by $L_{p_i} := p_i^*(\omega_{\mathcal{C}/S})$. The spin structure produces another collection of line bundles $\gamma_{p_i} \to \overline{\mathcal{M}}_{g,n}^{\rm spin}$ defined by $\gamma_{p_i} := p_i^*(\theta)$ where θ is a spin structure, i.e. $\theta^2 \cong \omega_{\mathcal{C}/S}^{\log}$. In particular, $\gamma_{p_i}^2 = L_{p_i}$.

We can now define a vector bundle over $\overline{\mathcal{M}}_{g,n}^{\rm spin}$ using the dual bundle $\theta^\vee$ on each stable twisted curve. Denote by $\mathcal{E}$ the universal spin structure over $\overline{\mathcal{M}}_{g,n}^{\rm spin}$. Given a map $S \to \overline{\mathcal{M}}_{g,n}^{\rm spin}$, $\mathcal{E}$ pulls back to θ giving a family $(\mathcal{C},\theta,p_1,...,p_n,\phi)$ where $\pi:\mathcal{C}\to S$ has stable twisted curve fibres, $p_i:S\to\mathcal{C}$ are sections with orbifold isotropy $\mathbb{Z}_2$ and $\phi:\theta^2\xrightarrow{\cong}\omega^{\log}_{\mathcal{C}/S}=\omega_{\mathcal{C}/S}(p_1,..,p_n)$. Consider the push-forward sheaf $\pi_*\mathcal{E}^\vee$ over $\overline{\mathcal{M}}^{\rm spin}_{g,n}$. Since

$$\deg\theta^\vee=1-g-\frac{1}{2}n<0$$

then $R^0\pi_*\mathcal{E}^\vee=0$ and the following definition makes sense.

Definition 2.1. Define a bundle $E_{g,n}=R^1\pi_*\mathcal{E}^\vee$ over $\overline{\mathcal{M}}^{\rm spin}_{g,n}$ with fibre $H^1(\theta^\vee)^\vee$.

It is a bundle of rank $2g-2+n$ by the following Riemann-Roch calculation. Orbifold Riemann-Roch takes into account the representation information in terms of its band λ_{p_i}.

$$h^0(\theta^\vee)-h^1(\theta^\vee)=1-g+\deg\theta^\vee-\sum_{i=1}^n\lambda_{p_i}=1-g+1-g-\frac{1}{2}n-\frac{1}{2}n=2-2g-n.$$

Here we used the requirement that $\lambda_{p_i}=\frac12$ for $i=1,...,n$. Since $\deg\theta^\vee=1-g-\frac12 n<0$ we have $h^0(\theta^\vee)=0$ and hence $h^1(\theta^\vee)=2g-2+n$. Thus $H^1(\theta^\vee)^\vee$ gives fibres of a rank $2g-2+n$ vector bundle.

The bundle $E_{g,n}$ restricts naturally to the boundary divisors:

$$\phi_{\rm irr}:\overline{\mathcal{M}}^{\rm spin}_{g-1,n+2}\to\overline{\mathcal{M}}^{\rm spin}_{g,n},\quad \phi_{h,I}:\overline{\mathcal{M}}^{\rm spin}_{h,|I|+1}\times\overline{\mathcal{M}}^{\rm spin}_{g-h,|J|+1}\to\overline{\mathcal{M}}^{\rm spin}_{g,n}$$

where $I\sqcup J=\{1,...,n\}$. These maps are obtained by considering a node $p\in\mathcal{C}$ for $(\mathcal{C},\theta,p_1,...,p_n,\phi)\in\overline{\mathcal{M}}^{\rm spin}_{g,n}$. Denote the normalisation by $\nu:\tilde{\mathcal{C}}\to\mathcal{C}$ with points $p_\pm\in\tilde{\mathcal{C}}$ that map to the node $p=\nu(p_\pm)$. When $\tilde{\mathcal{C}}$ is not connected, then since θ must be banded by $\lambda_p=0$ at an even number of orbifold points, $\lambda_{p_i}=\frac12$ forces $\lambda_{p_\pm}=\frac12$. Hence it decomposes into two spin structures θ_1 and θ_2, and conversely any two spin structures θ_1 and θ_2 glue to give θ thus inducing $\phi_{h,I}$. When $\tilde{\mathcal{C}}$ is connected, there are two cases. In the first case, when θ is banded by $\lambda_{p_\pm}=\frac12$, it decomposes into a spin structure $\tilde\theta=\nu^*\theta$, and conversely any spin structure $\tilde\theta$ glues to give θ thus inducing $\phi_{\rm irr}$. In the second case when θ is banded by $\lambda_{p_\pm}=0$, it gives rise to one further boundary divisor

$$\phi_{{\rm irr},2}:\overline{\mathcal{M}}^{\rm spin}_{g-1,n,2}\to\overline{\mathcal{M}}^{\rm spin}_{g,n}$$

where $\overline{\mathcal{M}}^{\rm spin}_{g-1,n,2}$ consists of spin structures banded, as usual, by $\lambda_{p_i}=\frac12$ for $i=1,...,n$ and banded by $\lambda_{p_i}=0$ for $i=n+1,n+2$. The bundle $E_{g-1,n,2}\to\overline{\mathcal{M}}^{\rm spin}_{g-1,n,2}$ is still defined, with fibre $H^1(\tilde\theta^\vee)$, because $H^0(\tilde\theta^\vee)=0$ since the band $\lambda_{p_{n+1}}=0=\lambda_{p_{n+2}}$ does not affect $\deg\tilde\theta^\vee=1-(g-1)-\frac12(n+2)<0$. It does affect the rank of the bundle. By Riemann-Roch $h^0(\tilde\theta^\vee)-h^1(\tilde\theta^\vee)=1-(g-1)+\deg\tilde\theta^\vee-\frac12 n=2-2g-n+1$ hence $\dim H^1(\tilde\theta^\vee)=\dim H^1(\theta^\vee)-1$ when $\tilde\theta=\nu^*\theta$.

Lemma 2.2.

$$\phi_{irr}^*E_{g,n}\equiv E_{g-1,n+2},\quad \phi_{h,I}^*E_{g,n}\cong\pi_1^*E_{h,|I|+1}\oplus\pi_2^*E_{g-h,|J|+1}$$

where π_i is projection from $\overline{\mathcal{M}}^{\rm spin}_{h,|I|+1}\times\overline{\mathcal{M}}^{\rm spin}_{g-h,|J|+1}$ onto the ith factor, $i=1,2$.

Proof. A spin structure $\tilde\theta$ on a connected normalisation $\tilde{\mathcal{C}}$ has $\deg\tilde\theta^\vee=1-(g-1)-\frac12(n+2)<0$ hence $H^0(\tilde\theta^\vee)=0$. By Riemann-Roch $h^0(\tilde\theta^\vee)-h^1(\tilde\theta^\vee)=1-(g-1)+\deg\tilde\theta^\vee-\frac12(n+2)=2-2g-n$. Hence $\dim H^1(\tilde\theta^\vee)=\dim H^1(\theta^\vee)$ and the gluing map

$$0\to H^1(\tilde\theta^\vee)\to H^1(\theta^\vee)$$

is an isomorphism. In other words $\phi_{\rm irr}^*E_{g,n}\equiv E_{g-1,n+2}$.

The argument is analogous when $\tilde{\mathcal{C}}$ is not connected and $\lambda_{p_\pm}=\frac12$. Again $\deg\theta_i^\vee<0$ hence $H^0(\theta_i^\vee)=0$ for $i=1,2$. By Riemann-Roch $\dim H^1(\theta_1^\vee)+\dim H^1(\theta_2^\vee)=\dim H^1(\theta^\vee)$ so the gluing map

$$0\to H^1(\theta_1^\vee)\oplus H^1(\theta_2^\vee)\to H^1(\theta^\vee)$$

is an isomorphism. In other words $\phi_{h,I}^*E_{g,n}\cong\pi_1^*E_{h,|I|+1}\oplus\pi_2^*E_{g-h,|J|+1}$. $\square$

The following lemma describes the restriction of $E_{g,n}$ to $\overline{\mathcal{M}}^{\rm spin}_{g-1,n,2}$ in terms of the bundle $E_{g-1,n,2} \to \overline{\mathcal{M}}^{\rm spin}_{g-1,n,2}$ defined above.

Lemma 2.3.

$$0 \to \mathcal{O}_{\overline{\mathcal{M}}^{\rm spin}_{g-1,n,2}} \to \phi^*_{irr,2}E_{g,n} \to E_{g-1,n,2} \to 0.$$

Proof. When the bundle θ is banded by $\lambda_{p_\pm} = 0$, the map between sheaves of local holomorphic sections

$$\mathcal{O}_C(\theta, U) \to \mathcal{O}_{\tilde{C}}(\nu^*\theta, \nu^{-1}U)$$

is not surjective whenever $U \ni p$. The image consists of local sections that agree, under an identification of fibres, at p_+ and p_-. The dual bundle $\theta^\vee$ arises as a quotient sheaf

$$0 \to I \to \nu^*\theta^\vee \to \theta^\vee \to 0 \tag{1}$$

where $\mathcal{O}_{\tilde{C}}(I, U)$ is generated by the element of the dual that sends a local section $s \in \mathcal{O}_{\tilde{C}}(\nu^*\theta, \nu^{-1}U)$ to $s(p_+) - s(p_-)$. Note that evaluation $s(p_\pm)$ only makes sense after a choice of trivialisation of $\nu^*\theta$ at p_+ and p_-, but the ideal I is independent of this choice. The complex (1) splits as follows. We can choose a representative ϕ upstairs of any element from the quotient space so that $\phi(p_+) = 0$, i.e. $\mathcal{O}_C(\theta^\vee, U)$ corresponds to elements of $\mathcal{O}_{\tilde{C}}(\nu^*\theta^\vee, \nu^{-1}U)$ that vanish at p_+. This is achieved by adding the appropriate multiple of $s(p_+) - s(p_-)$ to a given $\phi \in \mathcal{O}_{\tilde{C}}(\nu^*\theta^\vee, \nu^{-1}U)$. (Note that $\phi(p_-)$ is arbitrary. One could instead arrange $\phi(p_-) = 0$ with $\phi(p_+)$ arbitrary.) In other words we can identify $\theta^\vee$ with $\nu^*\theta^\vee(-p_+)$ in the complex:

$$0 \to \nu^*\theta^\vee(-p_+) \to \nu^*\theta^\vee \to \nu^*\theta^\vee|_{p_+} \to 0.$$

In a family $\pi : C \to S$, $R^0\pi_*(\nu^*\theta^\vee) = 0 = R^0\pi_*(\nu^*\theta^\vee(-p_+))$ since $\deg \nu^*\theta^\vee < 0$. Also $R^1\pi_*(\nu^*\theta^\vee|_{p_+}) = 0$ since p_+ has relative dimension 0. Thus

$$0 \to R^0\pi_*(\nu^*\theta^\vee|_{p_+}) \to R^1\pi_*(\nu^*\theta^\vee(-p_+)) \to R^1\pi_*(\nu^*\theta^\vee) \to 0.$$

Furthermore, we have $\nu^*L|_{p_+} \cong \mathbb{C}$ canonically via evaluation, hence $R^0\pi_*(\nu^*L|_{p_+}) \cong \mathcal{O}_S$. Since $E_{g-1,n,2} = R^1\pi_*(\nu^*\theta^\vee)$ and $\phi^*_{\rm irr,2}E_{g,n} = R^1\pi_*(\nu^*\theta^\vee(-p_+))$ the result follows. □

Remark 2.4. When $\lambda_{p_\pm} = \frac{1}{2}$ we have $\lambda_{p_+} + \lambda_{p_-} = 1$. We see from above that $\lambda_{p_\pm} = 0$ really wants one of $\lambda_{p_\pm}$ to be 1 to preserve $\lambda_{p_+} + \lambda_{p_-} = 1$.

Definition 2.5. For $2g - 2 + n > 0$ define the Euler class

$$\Omega_{g,n} := (-1)^n c_{2g-2+n}(E_{g,n}) \in H^{4g-4+2n}(\overline{\mathcal{M}}^{\rm spin}_{g,n}, \mathbb{Q}).$$

Note that $\Omega_{0,n} = 0$ for $n = 3, 4, ...$ because $\text{rank}(E_{0,n}) = n - 2$ is greater than $\dim \overline{\mathcal{M}}^{\rm spin}_{0,n} = n - 3$ so its top Chern class vanishes. Nevertheless, it would be interesting to know if the bundles $E_{0,n}$ carry non-trivial information.

The cohomology classes $\Omega_{g,n}$ behave well with respect to inclusion of strata.

Lemma 2.6.

$$\phi^*_{irr}\Omega_{g,n} = \Omega_{g-1,n+2}, \quad \phi^*_{h,I}\Omega_{g,n} = \pi_1^*\Omega_{h,|I|+1} \cdot \pi_2^*\Omega_{g-h,|J|+1}, \quad \phi^*_{irr,2}\Omega_{g,n} = 0.$$

Proof. This is an immediate application of Lemma 2.2

$$\phi^*_{\rm irr}E_{g,n} \equiv E_{g-1,n+2}, \quad \phi^*_{h,I}E_{g,n} \cong \pi_1^*E_{h,|I|+1} \oplus \pi_2^*E_{g-h,|J|+1}$$

and the naturality of $c_{2g-2+n} = c_{\rm top}$. We have

$$\phi^*_{\rm irr}c_{\rm top}(E_{g,n}) = c_{\rm top}(E_{g-1,n+2}), \quad \phi^*_{h,I}c_{\rm top}(E_{g,n}) = \pi_1^*c_{\rm top}(E_{h,|I|+1}) \cdot \pi_2^*c_{\rm top}(E_{g-h,|J|+1}).$$

The power of $(-1)^n$ in Definition 2.5 is the same after pull-back by $\phi^*_{\rm irr}$ and $\phi^*_{h,I}$ because the rank of the bundles is the same on both sides and $(-1)^n = (-1)^{2g-2+n}$. For example, $\phi^*_{\rm irr}(-1)^n c_{\rm top}(E_{g,n}) = (-1)^{n+2}c_{\rm top}(E_{g-1,n+2})$.

Also, $\phi^*_{\rm irr,2}\Omega_{g,n} = 0$ follows immediately from the exact sequence

$$0 \to \mathcal{O}_{\overline{\mathcal{M}}^{\rm spin}_{g-1,n,2}} \to \phi^*_{\rm irr,2}E_{g,n} \to E_{g-1,n,2} \to 0$$

which implies $\phi_{\text{irr},2}^* c_{2g-2+n}(E_{g,n}) = c_{2g-3+n}(E_{g-1,n,2}) \cdot c_1(\mathcal{O}_{\overline{\mathcal{M}}^{\text{spin}}_{g-1,n,2}}) = 0$. □

A consequence of Lemma 2.6 is that $\phi_\Gamma^*\Omega_{g,n} = 0$ for any stable graph Γ (defined as for $\overline{\mathcal{M}}_{g,n}$ with extra data on edges) that contains a genus 0 vertex.

Remark 2.7. We can allow all types of band in the moduli space, but as in the proof of Lemma 2.6 the class $\Omega_{g,n}$ vanishes when $\lambda_{p_i} = 0$ for some i.

Consider the map $f : \overline{\mathcal{M}}^{\text{spin}}_{g,n+1} \to \overline{\mathcal{M}}^{\text{spin}}_{g,n}$ that forgets the point p_{n+1}.

Lemma 2.8.
$$\Omega_{g,n+1} = \frac{1}{2}\psi_{p_{n+1}} \cdot f^*\Omega_{g,n}.$$

Proof. Recall that the forgetful map $(\mathcal{C}, \theta, p_1, ..., p_{n+1}, \phi) \mapsto (\rho(\mathcal{C}), \rho_*\theta, p_1, ..., p_n, \rho_*\phi)$ pushes forward θ via ρ which forgets the orbifold structure at p_{n+1}. As described above, $\rho_*\theta = \rho_*\theta(-p_{n+1})$ so $\rho_*\theta^\vee = \rho_*\theta^\vee(p_{n+1})$. Consider the exact sequence of sheaves over a family $\pi : \mathcal{C} \to S$ where $S \to \overline{\mathcal{M}}^{\text{spin}}_{g,n+1}$:
$$0 \to \theta^\vee \to \theta^\vee(p_{n+1}) \to \theta^\vee(p_{n+1})|_{p_{n+1}} \to 0.$$
As in the proof of Lemma 2.3, $\deg\theta^\vee(p_{n+1}) < 0$ so $R^0\pi_*(\theta^\vee) = 0 = R^0\pi_*(\theta^\vee(p_{n+1}))$ and p_{n+1} has relative dimension 0 so $R^1\pi_*(\theta^\vee(p_{n+1})|_{p_{n+1}}) = 0$. Hence
$$0 \to R^0\pi_*(\theta^\vee(p_{n+1})|_{p_{n+1}}) \to R^1\pi_*(\theta^\vee) \to R^1\pi_*(\theta^\vee(p_{n+1})) \to 0.$$
Now $\pi_*(\theta^\vee(p_{n+1})|_{p_{n+1}}) = \pi_*(\theta^\vee|_{p_{n+1}}) = \gamma^\vee_{p_{n+1}}$, $R^1\pi_*(\theta^\vee) = E_{g,n+1}$ and $R^1\pi_*(\theta^\vee(p_{n+1})) = f^*E_{g,n}$ hence
$$0 \to \gamma^\vee_{p_{n+1}} \to E_{g,n+1} \to f^*E_{g,n} \to 0.$$
Thus $c_{2g-2+n+1}(E_{g,n+1}) = -\frac{1}{2}\psi_{p_{n+1}} \cdot f^*c_{2g-2+n}(E_{g,n})$ since $c_1(\gamma_{p_{n+1}}) = \frac{1}{2}\psi_{p_{n+1}}$. Include the power of $(-1)^n$ to get $(-1)^{n+1}c_{2g-2+n+1}(E_{g,n+1}) = \frac{1}{2}\psi_{p_{n+1}} \cdot f^*(-1)^n c_{2g-2+n}(E_{g,n})$ which implies the result. □

Definition 2.9. For $p : \overline{\mathcal{M}}^{\text{spin}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ define
$$\Theta_{g,n} = 2^n p_*\Omega_{g,n} \in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n}).$$

Lemma 2.8 together with the factor of 2^n in the definition of $\Omega_{g,n}$ immediately gives property (iii) of $\Theta_{g,n}$
$$\Theta_{g,n+1} = \psi_{n+1} \cdot \pi^*\Theta_{g,n}.$$
Property (iv) of $\Theta_{g,n}$ is given by the following calculation.

Proposition 2.10. $\Theta_{1,1} = 3\psi_1 \in H^2(\overline{\mathcal{M}}_{1,1})$.

Proof. A one-pointed twisted elliptic curve $(\mathcal{E}, p)$ is a one-pointed elliptic curve (E, p) such that p has isotropy $\mathbb{Z}_2$. The degree of the divisor p in $\mathcal{E}$ is $\frac{1}{2}$ and the degree of every other point in $\mathcal{E}$ is 1. If dz is a holomorphic differential on E (where $E = \mathbb{C}/\Lambda$ and z is the identity function on the universal cover $\mathbb{C}$) then locally near p we have $z = t^2$ so $dz = 2tdt$ vanishes at p. In particular, the canonical divisor $(\omega_\mathcal{E}) = p$ has degree $\frac{1}{2}$ and $(\omega^{\log}_\mathcal{E}) = (\omega_\mathcal{E}(p)) = 2p$ has degree 1.

A spin structure on $\mathcal{E}$ is a degree $\frac{1}{2}$ line bundle $\mathcal{L}$ satisfying $\mathcal{L}^2 = \omega^{\log}_\mathcal{E}$. Line bundles on $\mathcal{E}$ correspond to divisors on $\mathcal{E}$ up to linear equivalence. Note that meromorphic functions on $\mathcal{E}$ are exactly the meromorphic functions on E. The four spin structures on $\mathcal{E}$ are given by the divisors $\theta_0 = p$ and $\theta_i = q_i - p$, $i = 1, 2, 3$, where q_i is a non-trivial order 2 element in the group E with identity p. Clearly $\theta_0^2 = 2p = \omega^{\log}_\mathcal{E}$. For $i = 1, 2, 3$, $\theta_i^2 = 2q_i - 2p \sim 2p$ since there is a meromorphic function $\wp(z) - \wp(q_i)$ on E with a double pole at p and a double zero at q_i. Its divisor on $\mathcal{E}$ is $2q_i - 4p$, since p has isotropy $\mathbb{Z}_2$, hence $2q_i - 2p \sim 2p$.

Since $H^2(\overline{\mathcal{M}}_{1,1})$ is generated by ψ_1 it is enough to calculate $\int_{\overline{\mathcal{M}}_{1,1}} \Theta_{1,1}$. The Chern character of the push-forward bundle $E_{1,1}$ is calculated via the Grothendieck-Riemann-Roch theorem
$$\text{Ch}(R\pi_*\mathcal{E}^\vee) = \pi_*(\text{Ch}(\mathcal{E}^\vee)\text{Td}(\omega^\vee_\pi)).$$
In fact we need to use the orbifold Grothendieck-Riemann-Roch theorem [36]. The calculation we need is a variant of the calculation in [17, Theorem 6.3.3] which applies to $\mathcal{E}$ such that $\mathcal{E}^2 = \omega^{\log}_\mathcal{C}$ instead of $\mathcal{E}^\vee$.

Importantly, this means that the Todd class has been worked out, and it remains to adjust the $\mathrm{Ch}(\mathcal{E}^\vee)$ term. We get

$$\int_{\overline{\mathcal{M}}_{1,1}} p_*c_1(E_{1,1}) = 2\int_{\overline{\mathcal{M}}_{1,1}}\left[\frac{11}{24}\kappa_1+\frac{1}{24}\psi_1+\frac{1}{2}\left(-\frac{1}{24}+\frac{1}{12}\right)(i_\Gamma)_*(1)\right] = 2\left(\frac{11}{24^2}+\frac{1}{24^2}+\frac{1}{2}\cdot\frac{1}{24}\cdot\frac{1}{2}\right)=\frac{1}{16}$$

which agrees with

$$\int_{\overline{\mathcal{M}}_{1,1}}\frac{3}{2}\psi_1=\frac{3}{2}\cdot\frac{1}{24}=\frac{1}{16}.$$

Hence $p_*c_1(E)=\frac{3}{2}\psi_1$ and $\Theta_{1,1}=2p_*c_1(E)=3\psi_1$. $\square$

Proposition 2.11. *The pushforward $p_*2^n\Omega_{g,n}=\Theta_{g,n}\in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$ satisfies property (ii).*

Proof. The two properties (ii) of $\Theta_{g,n}$, follow from the analogous properties for $\Omega_{g,n}$. This uses the relationship between compositions of pull-backs and push-forwards in the following diagrams:

$$\begin{tikzcd} \overline{\mathcal{M}}^{\mathrm{spin}}_{g-1,n+2} \arrow[r, "\phi_{\mathrm{irr}}"] \arrow[d, "p"] & \overline{\mathcal{M}}^{\mathrm{spin}}_{g,n} \arrow[d, "p"] \\ \overline{\mathcal{M}}_{g-1,n+2} \arrow[r, "\phi_{\mathrm{irr}}"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathcal{M}}^{\mathrm{spin}}_{h,|I|+1}\times\overline{\mathcal{M}}^{\mathrm{spin}}_{g-h,|J|+1} \arrow[r, "\phi_{h,I}"] \arrow[d, "p"] & \overline{\mathcal{M}}^{\mathrm{spin}}_{g,n} \arrow[d, "p"] \\ \overline{\mathcal{M}}_{h,|I|+1}\times\overline{\mathcal{M}}_{g-h,|J|+1} \arrow[r, "\phi_{h,I}"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

We have $\phi_{\mathrm{irr}}^*p_*=4p_*\phi_{\mathrm{irr}}^*$ and $\phi_{h,I}^*p_*=4p_*\phi_{h,I}^*$ where the factor of 4 is due to ramification of p—see (39) in [21]. Hence

$$\phi_{\mathrm{irr}}^*\Theta_{g,n}=\phi_{\mathrm{irr}}^*p_*2^n\Omega_{g,n}=4p_*\phi_{\mathrm{irr}}^*2^n\Omega_{g,n}=p_*2^{n+2}\Omega_{g-1,n+2}=\Theta_{g-1,n+2}$$

and similarly $\phi_{h,I}^*\Theta_{g,n}=\pi_1^*\Theta_{h,|I|+1}\cdot\pi_2^*\Theta_{g-h,|J|+1}$ uses $4\cdot 2^n=2^{n+2}=2^{|I|+1+|J|+1}$. $\square$

Remark 2.12. The construction of $\Omega_{g,n}$ should also follow from the cosection construction in [3] using the moduli space of spin curves with fields

$$\overline{\mathcal{M}}_{g,n}(\mathbb{Z}_2)^p=\{(C,\theta,\rho)\mid (C,\theta)\in\overline{\mathcal{M}}^{\mathrm{spin}}_{g,n},\ \rho\in H^0(\theta)\}.$$

A cosection of the pull-back of $E_{g,n}$ to $\overline{\mathcal{M}}_{g,n}(\mathbb{Z}_2)^p$ is given by ρ^{-3} since it pairs well with $H^1(\theta)$—we have $\rho^{-3}\in H^0((\theta^\vee)^3)$ while $H^1(\theta)\cong H^0(\omega\otimes\theta^\vee)^\vee=H^0((\theta^\vee)^3)^\vee$. Using the cosection ρ^{-3} a virtual fundamental class is constructed in [3] that likely gives rise to $\Omega_{g,n}\in H^{4g-4+2n}(\overline{\mathcal{M}}^{\mathrm{spin}}_{g,n})$. The virtual fundamental class is constructed away from the zero set of ρ.

3. Uniqueness

The degree property (I) of Theorem 1, $\Theta_{g,n}\in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$, proven below, enables a reduction argument which leads to uniqueness of intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}\prod_{j=1}^N\kappa_{\ell_j}$ which is property (IV) of Theorem 1. In this section we prove these two results together with properties (II) and (III) which are immediate consequences. We leave the proof of the final property (V) of Theorem 1 until Section 4.4.3.

Proof of (I). Write $d(g,n)=\mathrm{degree}(\Theta_{g,n})$ which exists by (i). The degree here is half the cohomological degree so $\Theta_{g,n}\in H^{2d(g,n)}(\overline{\mathcal{M}}_{g,n})$. Using (ii), $\phi_{\mathrm{irr}}^*\Theta_{g,n}=\Theta_{g-1,n+2}$ implies that $d(g,n)=d(g-1,n+2)$ hence $d(g,n)=f(2g-2+n)$ is a function of $2g-2+n$. Then again using (ii), $\phi_{h,I}^*\Theta_{g,n}=\Theta_{h,|I|+1}\otimes\Theta_{g-h,|J|+1}$ implies that $f(a+b)=f(a)+f(b)=(a+b)f(1)$. Hence $d(g,n)=(2g-2+n)k$ for an integer k. But $d(g,n)\le 3g-3+n$ implies $k\le 1$. When $k=0$, this gives $\Theta_{g,n}\in H^0(\overline{\mathcal{M}}_{g,n})$, and $\Theta_{g,n}$ is a topological field theory. But $\deg\Theta_{g,n}=0$ contradicts (iv) hence $k=1$ and $\deg\Theta_{g,n}=2g-2+n$.

The proof above used only properties (i), (ii) and (iv). Alternatively, one can use (iii) in place of the second part of (ii) given by $\phi_{h,I}^*\Theta_{g,n}=\Theta_{h,|I|+1}\otimes\Theta_{g-h,|J|+1}$. $\square$

Proof of (II). This is an immediate consequence of (I) since $\deg \Theta_{0,n} = n-2 > n-3 = \dim \overline{\mathcal{M}}_{0,n}$ hence $\Theta_{0,n} = 0$. For any stable graph Γ with a genus 0 vertex, Remark 1.1 gives $\phi_\Gamma^*\Theta_{g,n} = \Theta_\Gamma$. Furthermore $0 = \Theta_\Gamma = \prod_{v\in V(\Gamma)} \pi_v^*\Theta_{g(v),n(v)}$ since the genus 0 vertex contributes a factor of 0 to the product. $\square$

Proof of (III). Property (iii) implies that $\Theta_{g,n} = \pi^*\Theta_g \cdot \prod_{i=1}^n \psi_i$ where $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$ is the forgetful map. Since $\pi^*\omega \in H^*(\overline{\mathcal{M}}_{g,n})^{S_n}$ for any $\omega \in H^*(\overline{\mathcal{M}}_g)$ and clearly $\prod_{i=1}^n \psi_i \in H^*(\overline{\mathcal{M}}_{g,n})^{S_n}$ hence we have $\Theta_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n})^{S_n}$ as required. $\square$

Proof of (IV). The uniqueness result follows from the more general result given in the following proposition. $\square$

Proposition 3.1. *For any $\Theta_{g,n}$ satisfying* (i) - (iii) *above, the intersection numbers*

$$\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^n \psi_i^{m_i} \prod_{j=1}^N \kappa_{\ell_j} \tag{2}$$

are uniquely determined from the initial condition $\Theta_{1,1} = \lambda\psi_1$ for $\lambda \in \mathbb{C}$.

Proof. For $n > 0$, we will push forward the integral (2) via the forgetful map $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ as follows. Consider first the case when there are no κ classes. The presence of ψ_n in $\Theta_{g,n} = \psi_n \cdot \pi^*\Theta_{g,n-1}$ gives

$$\Theta_{g,n}\psi_k = \Theta_{g,n}\pi^*\psi_k, \quad k < n$$

since $\psi_n\psi_k = \psi_n\pi^*\psi_k$ for $k < n$. Hence

$$\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^n \psi_i^{m_i} = \int_{\overline{\mathcal{M}}_{g,n}} \pi^*\Big(\Theta_{g,n-1} \prod_{i=1}^{n-1} \psi_i^{m_i}\Big)\psi_n^{m_n+1} = \int_{\overline{\mathcal{M}}_{g,n-1}} \pi_*\left\{\pi^*\Big(\Theta_{g,n-1} \prod_{i=1}^{n-1} \psi_i^{m_i}\Big)\psi_n^{m_n+1}\right\}$$
$$= \int_{\overline{\mathcal{M}}_{g,n-1}} \Theta_{g,n-1} \prod_{i=1}^{n-1} \psi_i^{m_i}\kappa_{m_n}$$

so we have reduced an intersection number over $\overline{\mathcal{M}}_{g,n}$ to an intersection number over $\overline{\mathcal{M}}_{g,n-1}$. In the presence of κ classes, replace κ_{ℓ_j} by $\kappa_{\ell_j} = \pi^*\kappa_{\ell_j} + \psi_n^{\ell_j}$ and repeat the push-forward as above on all summands. By induction, we see that

$$\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^n \psi_i^{m_i} \prod_{j=1}^N \kappa_{\ell_j} = \int_{\overline{\mathcal{M}}_g} \Theta_g \cdot p(\kappa_1, \kappa_2, ..., \kappa_{3g-3})$$

i.e. the intersection number (2) reduces to an intersection number over $\overline{\mathcal{M}}_g$ of Θ_g times a polynomial in the κ classes.

By (I), $\deg \Theta_g = 2g-2$, so we may assume the polynomial p consists only of terms of homogeneous degree $g-1$ (where $\deg \kappa_r = r$). But by Looijenga's theorem [27], and the stronger result given by Pixton's relations [31], a homogeneous degree $g-1$ monomial in the κ classes is equal in cohomology to the sum of boundary terms. Now property (ii) of Θ_g, shows that the pull-back of Θ_g to these boundary terms is $\Theta_{g',n'}$ for $g' < g$ so we have expressed (2) as a sum of integrals of $\theta_{g',n'}$ against ψ and κ classes. By induction, one can reduce to integrals $\int_{\overline{\mathcal{M}}_{1,1}} \Theta_{1,1} = \frac{\lambda}{24}$ and the proposition is proven. $\square$

Remark 3.2. The intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^n \psi_i^{m_i} \prod_{j=1}^N \kappa_{\ell_j}$ are directly related to the intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^n \psi_i^{m_i}$ with no κ classes. This essentially reverses the reduction shown in the proof of Proposition 3.1. Explicitly, for $\pi : \overline{\mathcal{M}}_{g,n+N} \to \overline{\mathcal{M}}_{g,n}$ and $\mathbf{m} = (m_1, ..., m_N)$ define a polynomial in κ classes by

$$R_{\mathbf{m}}(\kappa_1, \kappa_2, ...) = \pi_*\Big(\psi_{n+1}^{m_1+1} ... \psi_{n+N}^{m_N+1}\Big)$$

so, for example, $R_{(m_1,m_2)} = \kappa_{m_1}\kappa_{m_1} + \kappa_{m_1+m_2}$. Then

$$\Theta_{g,n} \cdot R_{\mathbf{m}} = \Theta_{g,n} \cdot \pi_*\Big(\psi_{n+1}^{m_1+1} ... \psi_{n+N}^{m_N+1}\Big) = \pi_*\Big(\pi^*\Theta_{g,n} \cdot \psi_{n+1}^{m_1+1} ... \psi_{n+N}^{m_N+1}\Big) = \pi_*\Big(\Theta_{g,n+N} \cdot \psi_{n+1}^{m_1} ... \psi_{n+N}^{m_N}\Big). \tag{3}$$

The polynomials $R_{\mathbf{m}}(\kappa_1, \kappa_2, ...)$ generate all polynomials in the κ_i so (3) can be used to remove any κ class.

Remark 3.3. Existence of classes $\Theta_{g,n}$ satisfying (i) - (iii) and the initial condition $\Theta_{1,1} = \lambda\psi_1$ for a general $\lambda \in \mathbb{C}$ is an interesting problem. One approach is to look inside the strata algebra—see Section 4.1.1—which consists of push-forwards of classes involving κ and ψ classes from lower strata.

The intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \cdot \prod_{j=1}^n \psi_j^{k_j}$ satisfy a dilaton equation which is realised as a homogeneity condition on the partition function

$$Z^{\Theta}(\hbar, t_0, t_1, ...) = \exp \sum_{g,n,\vec{k}} \frac{\hbar^{g-1}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \cdot \prod_{j=1}^n \psi_j^{k_j} \prod t_{k_j}. \tag{4}$$

Proposition 3.4. *The function $Z^{\Theta}(\hbar, t_0, t_1, ...)$ is homogeneous of degree 0 with respect to $\{q = 1 - t_0, t_1, t_2, ...\}$ with $\deg q = 1$ and $\deg t_i = 2i + 1$ for $i > 0$. Equivalently it satisfies the dilaton equation:*

$$\frac{\partial}{\partial t_0} Z(\hbar, t_0, t_1, ...) = \sum_{i=0}^{\infty} (2i+1) t_i \frac{\partial}{\partial t_i} Z(\hbar, t_0, t_1, ...) \tag{5}$$

Proof. We have

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{g,n+1}} \Theta_{g,n+1} \cdot \prod_{j=1}^n \psi_j^{k_j} &= \int_{\overline{\mathcal{M}}_{g,n+1}} \pi^*\Theta_{g,n} \cdot \psi_{n+1} \cdot \prod_{j=1}^n \psi_j^{k_j} = \int_{\overline{\mathcal{M}}_{g,n+1}} \pi^*\Theta_{g,n} \cdot \psi_{n+1} \cdot \prod_{j=1}^n \pi^*\psi_j^{k_j} \\ &= \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \cdot \prod_{j=1}^n \psi_j^{k_j} \cdot \pi_*\psi_{n+1} = (2g-2+n) \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \cdot \prod_{j=1}^n \psi_j^{k_j}. \end{aligned}$$

where we have used $\psi_{n+1} \cdot \psi_j = \psi_{n+1} \cdot \pi^*\psi_j$ for $j = 1, ..., n$ and $\pi_*(\pi^*\omega \cdot \psi_{n+1}) = \omega \cdot \pi_*\psi_{n+1}$. But this exactly agrees with the dilaton equation (5) via the correspondence (4). □

Proposition 3.4 together with the initial condition $\Theta_{1,1} = \lambda\psi_1$, or $\int_{\overline{\mathcal{M}}_{1,1}} \Theta_{1,1} = \frac{\lambda}{24}$, gives

$$\log Z^{\Theta} = -\frac{\lambda}{24} \log(1 - t_0) + O(\hbar). \tag{6}$$

The following example demonstrates Proposition 3.1 with an explicit genus 2 relation.

Example 3.5. *A genus two Pixton relation first proven by Mumford* [30, equation (8.5)], *relating κ_1 and the divisors $\mathcal{M}_{\Gamma_1} \cong \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1}$ and $\mathcal{M}_{\Gamma_2} \cong \overline{\mathcal{M}}_{1,2}$ in $\overline{\mathcal{M}}_2$, labeled by stable graphs Γ_i is given by*

$$\kappa_1 - \frac{7}{5}[\mathcal{M}_{\Gamma_1}] - \frac{1}{5}[\mathcal{M}_{\Gamma_2}] = 0$$

which induces the relation

$$\Theta_2 \cdot \kappa_1 - \frac{7}{5}\Theta_2 \cdot [\mathcal{M}_{\Gamma_1}] - \frac{1}{5}\Theta_2 \cdot [\mathcal{M}_{\Gamma_2}] = 0.$$

Property (ii) *of $\Theta_{g,n}$ yields*

$$\int_{\overline{\mathcal{M}}_2} \Theta_2 \cdot [\mathcal{M}_{\Gamma_1}] = \int_{\mathcal{M}_{\Gamma_1}} \Theta_2 = \int_{\overline{\mathcal{M}}_{1,1}} \Theta_{1,1} \cdot \int_{\overline{\mathcal{M}}_{1,1}} \Theta_{1,1} \cdot \frac{1}{|Aut(\Gamma_1)|}, \quad \int_{\mathcal{M}_{\Gamma_2}} \Theta_2 = \int_{\overline{\mathcal{M}}_{1,2}} \Theta_{1,2} \cdot \frac{1}{|Aut(\Gamma_2)|}$$

hence the relation on the level of intersection numbers is given by

$$\int_{\overline{\mathcal{M}}_2} \Theta_2 \cdot \kappa_1 - \frac{7}{5} \cdot \int_{\overline{\mathcal{M}}_{1,1}} \Theta_{1,1} \cdot \int_{\overline{\mathcal{M}}_{1,1}} \Theta_{1,1} \cdot \frac{1}{|Aut(\Gamma_1)|} - \frac{1}{5} \cdot \int_{\overline{\mathcal{M}}_{1,2}} \Theta_{1,2} \cdot \frac{1}{|Aut(\Gamma_2)|} = 0.$$

We have $\int_{\overline{\mathcal{M}}_{1,1}} \Theta_{1,1} = \frac{\lambda}{24} = \int_{\overline{\mathcal{M}}_{1,2}} \Theta_{1,1}$ from (6)*, and $|Aut(\Gamma_1)| = 2 = |Aut(\Gamma_2)|$. Hence*

$$\begin{aligned} \int_{\overline{\mathcal{M}}_2} \Theta_2 \cdot \kappa_1 &= \frac{7}{5} \cdot \int_{\overline{\mathcal{M}}_{1,1}} \Theta_{1,1} \cdot \int_{\overline{\mathcal{M}}_{1,1}} \Theta_{1,1} \cdot \frac{1}{|Aut(\Gamma_1)|} + \frac{1}{5} \cdot \int_{\overline{\mathcal{M}}_{1,2}} \Theta_{1,2} \cdot \frac{1}{|Aut(\Gamma_2)|} \\ &= \frac{7}{5} \cdot \left(\frac{\lambda}{24}\right)^2 \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{\lambda}{24} \cdot \frac{1}{2} = \frac{7\lambda^2 + 24\lambda}{5760}. \end{aligned}$$

From now on we will specialise to the case $\lambda = 3$ for which we have a proof of existence of $\Theta_{g,n}$. From example 3.5, we have $\int_{\overline{\mathcal{M}}_{2,1}} \Theta_{2,1} \cdot \psi_1 = \int_{\overline{\mathcal{M}}_{2,1}} \pi^*\Theta_2 \cdot \psi_1^2 = \int_{\overline{\mathcal{M}}_2} \Theta_2 \cdot \kappa_1 = \frac{3}{128}$. All genus 2 terms can be obtained from $\int_{\overline{\mathcal{M}}_{2,1}} \Theta_{2,1} \cdot \psi_1$ and (5). Combining this with (6) we have

$$\log Z^{\Theta} = -\frac{1}{8}\log(1-t_0) + \hbar\frac{3}{128}\frac{t_1}{(1-t_0)^3} + O(\hbar^2). \tag{7}$$

4. KdV tau functions

In this section we prove property V of Theorem 1 which states that a generating function for the intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^n \psi_i^{m_i}$ is a tau function of the KdV hierarchy, analogous to the theorem conjectured by Witten and proven by Kontsevich for the intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{m_i}$.

A tau function $Z(t_0, t_1, ...)$ of the KdV hierarchy (equivalently the KP hierarchy in odd times $p_{2k+1} = t_k/(2k+1)!!$) gives rise to a solution $U = \hbar\frac{\partial^2}{\partial t_0^2}\log Z$ of the KdV hierarchy

$$U_{t_1} = UU_{t_0} + \frac{\hbar}{12}U_{t_0t_0t_0}, \quad U(t_0, 0, 0, ...) = f(t_0). \tag{8}$$

The first equation in the hierarchy is the KdV equation (8), and later equations $U_{t_k} = P_k(U, U_{t_0}, U_{t_0t_0}, ...)$ for $k > 1$ determine U uniquely from $U(t_0, 0, 0, ...)$. See [29] for the full definition.

The Brezin-Gross-Witten solution $U^{\mathrm{BGW}} = \hbar\frac{\partial^2}{\partial t_0^2}\log Z^{\mathrm{BGW}}$ of the KdV hierarchy arises out of a unitary matrix model studied in [2, 20]. It is defined by the initial condition

$$U^{\mathrm{BGW}}(t_0, 0, 0, ...) = \frac{\hbar}{8(1-t_0)^2}.$$

The low genus g terms (= coefficient of $\hbar^{g-1}$) of $\log Z^{\mathrm{BGW}}$ are

$$\begin{aligned}\log Z^{\mathrm{BGW}} &= -\frac{1}{8}\log(1-t_0) + \hbar\frac{3}{128}\frac{t_1}{(1-t_0)^3} + \hbar^2\frac{15}{1024}\frac{t_2}{(1-t_0)^5} + \hbar^2\frac{63}{1024}\frac{t_1^2}{(1-t_0)^6} + O(\hbar^3) \\ &= (\frac{1}{8}t_0 + \frac{1}{16}t_0^2 + \frac{1}{24}t_0^3 + ...) + \hbar(\frac{3}{128}t_1 + \frac{9}{128}t_0t_1 + ...) + \hbar^2(\frac{15}{1024}t_2 + \frac{63}{1024}t_1^2 + ...) + ...\end{aligned} \tag{9}$$

The tau function $Z^{\mathrm{BGW}}(\hbar, t_0, t_1, ...)$ shares many properties of the famous Kontsevich-Witten tau function $Z^{\mathrm{KW}}(\hbar, t_0, t_1, ...)$ introduced in [37]. The Kontsevich-Witten tau function Z^{KW} is defined by the initial condition $U^{\mathrm{KW}}(t_0, 0, 0, ...) = t_0$ for $U^{\mathrm{KW}} = \hbar\frac{\partial^2}{\partial t_0^2}\log Z^{\mathrm{KW}}$. The low genus terms of $\log Z^{\mathrm{KW}}$ are

$$\log Z^{\mathrm{KW}}(\hbar, t_0, t_1, ...) = \hbar^{-1}(\frac{t_0^3}{3!} + \frac{t_0^3t_1}{3!} + \frac{t_0^4t_2}{4!} + ...) + \frac{t_1}{24} + ...$$

Theorem 2 (Witten-Kontsevich 1992 [26, 37])**.**

$$Z^{KW}(\hbar, t_0, t_1, ...) = \exp\sum_{g,n}\hbar^{g-1}\frac{1}{n!}\sum_{\vec{k}\in\mathbb{N}^n}\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^n \psi_i^{k_i}t_{k_i}$$

is a tau function of the KdV hierarchy.

The main aim of this section is to prove that $Z^{\Theta}(\hbar, t_0, t_1, ...) = Z^{\mathrm{BGW}}(\hbar, t_0, t_1, ...)$ which implies property V of Theorem 1. Agreement of $Z^{\Theta}(\hbar, t_0, t_1, ...)$ and $Z^{\mathrm{BGW}}(\hbar, t_0, t_1, ...)$ up to genus 2 is clear from (7) and (9) and can be extended to genus 3 using Appendix A. Furthermore, $Z^{\mathrm{BGW}}(\hbar, t_0, t_1, ...)$ also satisfies the homogeneity property (5)—see [6]—which is apparent in the low genus terms given in (9). The homogeneity property satisfied by both Z^{Θ} and Z^{BGW} reduces the proof that they are equal to checking only finitely many coefficients for each genus.

Theorem 3.

$$Z^{\Theta}(\hbar, t_0, t_1, ...) = Z^{BGW}(\hbar, t_0, t_1, ...).$$

Outline of the proof of Theorem 3. We summarise here Sections 4.1-4.4 which contain the proof of Theorem 3. The equality of Z^Θ and $Z^{\rm BGW}$ up to genus two used the relation between coefficients of $Z^\Theta(\hbar,t_0,t_1,...)$

$$\int_{\overline{\mathcal{M}}_{2,1}}\Theta_{2,1}\cdot\psi_1-\frac{7}{10}\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}-\frac{1}{10}\cdot\int_{\overline{\mathcal{M}}_{1,2}}\Theta_{1,2}=0 \tag{10}$$

arising from the genus two Pixton relation. The main idea of the proof is to show that the coefficients of $Z^{\rm BGW}(\hbar,t_0,t_1,...)$ also satisfy (10), and more generally a set of relations arising from Pixton relations.

Associated to the A_2 Frobenius manifold is a partition function $Z_{A_2}(\hbar,\{t^\alpha_k\})$, defined in Section 4.2. Pixton's relations on the level of intersection numbers arise due to unexpected vanishing of some of the coefficients in $Z_{A_2}(\hbar,\{t^\alpha_k\})$. Briefly, the partition function $Z_{A_2}(\hbar,\{t^\alpha_k\})$ is constructed in two ways—out of intersections of cohomology classes in $H^*(\overline{\mathcal{M}}_{g,n})$ which form a CohFT and out of a graphical construction that associates coefficients of $Z^{\rm KW}(\hbar,t_0,t_1,...)$ to vertices of the graphs, described in Sections 4.1 and 4.3. The vanishing of coefficients in $Z_{A_2}(\hbar,\{t^\alpha_k\})$ is clear only from one of these constructions hence leads to relations among intersection numbers. There is a third construction of $Z_{A_2}(\hbar,\{t^\alpha_k\})$ via topological recursion, defined in Section 4.4, applied to a spectral curve naturally associated to the A_2 Frobenius manifold. This construction gives a second method to prove vanishing of certain coefficients of $Z_{A_2}(\hbar,\{t^\alpha_k\})$.

The partition function $Z^\Theta_{A_2}(\hbar,\{t^\alpha_k\})$ is built out of the A_2 CohFT cohomology classes in $H^*(\overline{\mathcal{M}}_{g,n})$ times $\Theta_{g,n}$. It stores relations between intersection numbers involving $\Theta_{g,n}$ such as (10). So for example, the coefficient of $\hbar t^1_1$ in $\log Z^\Theta_{A_2}(\hbar,\{t^\alpha_k\})$ vanishes and is also the relation (10). We can construct $Z^\Theta_{A_2}(\hbar,\{t^\alpha_k\})$ via the graphical construction that produces $Z_{A_2}(\hbar,\{t^\alpha_k\})$ although with vertex contributions coming from coefficients of $Z^\Theta(\hbar,t_0,t_1,...)$ in place of coefficients of $Z^{\rm KW}(\hbar,t_0,t_1,...)$.

The idea is to produce a partition function $Z^{\rm BGW}_{A_2}(\hbar,\{t^\alpha_k\})$ using the graphical construction of $Z_{A_2}(\hbar,\{t^\alpha_k\})$ with $Z^{\rm KW}$ replaced by $Z^{\rm BGW}$. Vanishing of coefficients of $Z^{\rm BGW}_{A_2}(\hbar,\{t^\alpha_k\})$ are relations between coefficients of $Z^{\rm BGW}$ analogous to each of the relations satisfied by the coefficients of $Z^\Theta_{A_2}(\hbar,\{t^\alpha_k\})$ such as (10). To prove vanishing of primary coefficients of $Z^{\rm BGW}_{A_2}(\hbar,\{t^\alpha_k\})$ for $n\le g-1$, which is enough to prove that the coefficients of $Z^{\rm BGW}$ satisfy the same relations as the coefficients of Z^Θ, we use the third construction of $Z_{A_2}(\hbar,\{t^\alpha_k\})$ via topological recursion. We construct a spectral curve to get $Z^{\rm BGW}_{A_2}(\hbar,\{t^\alpha_k\})$ and this construction allows us to prove vanishing of certain coefficients. To conclude, we have vanishing of certain coefficients of $Z^\Theta_{A_2}(\hbar,\{t^\alpha_k\})$ due to a cohomological viewpoint, and we have vanishing of corresponding coefficients of $Z^{\rm BGW}_{A_2}(\hbar,\{t^\alpha_k\})$ due to the topological recursion viewpoint, and this shows that coefficients of $Z^{\rm BGW}$ and Z^Θ satisfy the same relations.

4.1. **Twisted loop group action.** Givental [22] showed how to build partition functions of cohomological field theories which are sequences of cohomology classes in $H^*(\overline{\mathcal{M}}_{g,n})$—see Section 5—out of the basic building block $Z^{\rm KW}(\hbar,t_0,t_1,...)$. This construction can be immediately adapted to allow one to use in place of $Z^{\rm KW}(\hbar,t_0,t_1,...)$ the building blocks $Z^{\rm BGW}(\hbar,t_0,t_1,...)$ and $Z^\Theta(\hbar,t_0,t_1,...)$ (where the latter two will eventually be shown to coincide). Givental defined an action by elements of the twisted loop group and translations on the basic building block partition functions above. This action was interpreted as an action on sequences of cohomology classes in $H^*(\overline{\mathcal{M}}_{g,n})$ independently, by Katzarkov-Kontsevich-Pantev, Kazarian and Teleman—see [31, 33].

Consider an element of the loop group $LGL(N,\mathbb{C})$ given by a formal series

$$R(z)=\sum_{k=0}^\infty R_kz^k$$

where R_k are $N\times N$ matrices and $R_0=I$. We further require $R(z)$ to lie in the twisted loop group $L^{(2)}GL(N,\mathbb{C})\subset LGL(N,\mathbb{C})$, the subgroup which is defined to consist of elements satisfying

$$R(z)R(-z)^T=I.$$

Define

$$\mathcal{E}(w,z)=\frac{I-R^{-1}(z)R^{-1}(w)^T}{w+z}=\sum_{i,j\ge0}\mathcal{E}_{ij}w^iz^j$$

which has the power series expansion on the right since the numerator $I - R^{-1}(z)R^{-1}(w)^T$ vanishes at $w = -z$ since $R^{-1}(z)$ is also an element of the twisted loop group.

Givental's action is defined via weighted sums over graphs. Consider the following set of decorated graphs with vertices labeled by the set $\{1, ..., N\}$.

Definition 4.1. For a graph γ denote by

$$V(\gamma), \quad E(\gamma), \quad H(\gamma), \quad L(\gamma) = L^*(\gamma) \sqcup L^\bullet(\gamma)$$

its set of vertices, edges, half-edges and leaves. The disjoint splitting of $L(\gamma)$ into ordinary leaves L^* and dilaton leaves $L^\bullet$ is part of the structure on γ. The set of half-edges consists of leaves and oriented edges so there is an injective map $L(\gamma) \to H(\gamma)$ and a multiply-defined map $E(\gamma) \to H(\gamma)$ denoted by $E(\gamma) \ni e \mapsto \{e^+, e^-\} \subset H(\gamma)$. The map sending a half-edge to its vertex is given by $v : H(\gamma) \to V(\gamma)$. Decorate γ by functions:

$$\begin{aligned} g &: V(\gamma) \to \mathbb{N} \\ \alpha &: V(\gamma) \to \{1, ..., N\} \\ p &: L^*(\gamma) \xrightarrow{\cong} \{1, 2, ..., n\} \\ k &: H(\gamma) \to \mathbb{N} \end{aligned}$$

such that $k|_{L^\bullet(\gamma)} > 1$ and $n = |L^*(\gamma)|$. We write $g_v = g(v)$, $\alpha_v = \alpha(v)$, $\alpha_\ell = \alpha(v(\ell))$, $p_\ell = p(\ell)$, $k_\ell = k(\ell)$. The *genus* of γ is $g(\gamma) = b_1(\gamma) + \sum_{v \in V(\gamma)} g(v)$. We say γ is *stable* if any vertex labeled by $g = 0$ is of valency ≥ 3 and there are no isolated vertices labeled by $g = 1$. We write n_v for the valency of the vertex v. Define $\Gamma_{g,n}$ to be the set of all stable, connected, genus g, decorated graphs with n ordinary leaves.

Given a sequence of classes $\Omega = \{\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}) \otimes (V^*)^{\otimes n} \mid g, n \in \mathbb{N}, 2g - 2 + n > 0\}$, following [31, 33] define a new sequence of classes $R\Omega = \{(R\Omega)_{g,n}\}$ by a weighted sum over stable graphs, with weights defined as follows.

(i) *Vertex weight:* $w(v) = \Omega_{g(v),n_v}$ at each vertex v

(ii) *Leaf weight:* $w(\ell) = R^{-1}(\psi_{p(\ell)})$ at each leaf ℓ

(iii) *Edge weight:* $w(e) = \mathcal{E}(\psi'_e, \psi''_e)$ at each edge e

Then

$$(R\Omega)_{g,n} = \sum_{\Gamma \in G_{g,n}} \frac{1}{|\mathrm{Aut}(\Gamma)|} (p_\Gamma)_* \prod_{\substack{v \in V(\Gamma) \\ \ell \in L^*(\Gamma) \\ e \in E(\Gamma)}} w(v) w(\ell) w(e)$$

This defines an action of the twisted loop group on sequences of cohomology classes. It is applied in [31, 33] to cohomological field theories $\Omega_{g,n}$ whereas a generalised notion of cohomological field theory is used here.

Define a translation action of $T(z) \in zV[[z]]$ on the sequence $\Omega_{g,n}$ as follows.

(iv) *Dilaton leaf weight:* $w(\ell) = T(\psi_{p(\ell)})$ at each dilaton leaf $\ell \in L^\bullet$.

So the translation action is given by

$$(T\Omega)_{g,n}(v_1 \otimes ... \otimes v_n) = \sum_{m \geq 0} \frac{1}{m!} p_* \Omega_{g,n+m}(v_1 \otimes ... \otimes v_n \otimes T(\psi_{n+1}) \otimes ... \otimes T(\psi_{n+m})) \tag{11}$$

where $p : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map. To ensure the sum (11) is finite, one usually requires $T(z) \in z^2 V[[z]]$, so that $\dim \overline{\mathcal{M}}_{g,n+m} = 3g - 3 + n + m$ grows more slowly in m than the degree $2m$ coming from T. Instead, we control growth of the degree of $\Omega_{g,n}$ in n to ensure $T(z) \in zV[[z]]$ produces a finite sum.

The partition function of a sequence $\Omega = \{\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}) \otimes (V^*)^{\otimes n} \mid g, n \in \mathbb{N}, 2g - 2 + n > 0\}$ is defined by:

$$Z_\Omega(\hbar, \{t_k^\alpha\}) = \exp \sum_{g,n,\vec{k}} \frac{\hbar^{g-1}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(e_{\alpha_1} \otimes ... \otimes e_{\alpha_n}) \cdot \prod_{j=1}^n \psi_j^{k_j} \prod t_{k_j}^{\alpha_j}$$

where $\{e_1,...,e_N\}$ is a basis of V, $\alpha\in\{1,...,N\}$ and $k\in\mathbb{N}$. For $\dim V=1$ and $\Omega_{g,n}=1\in H^*(\overline{\mathcal{M}}_{g,n})$, $Z_\Omega(\hbar,\{t_k\})=Z^{\rm KW}(\hbar,\{t_k\})$. Similarly, for $\Omega_{g,n}=\Theta_{g,n}\in H^*(\overline{\mathcal{M}}_{g,n})$, $Z_\Omega(\hbar,\{t_k\})=Z^\Theta(\hbar,\{t_k\})$.

The action on sequences of cohomology classes above immediately gives rise to an action on partition functions, which store correlators of the sequence of cohomology classes, known as the Givental action [11, 19, 33]. It gives a graphical construction of the partition functions $Z_{R\Omega}$ and $Z_{T\Omega}$ obtained from Z_Ω.

The graphical expansions can be conveniently expressed via the action of differential operators $\hat{R}$ and $\hat{T}$: $Z_{R\Omega}=\hat{R}Z_\Omega$, $Z_{T\Omega}=\hat{T}Z_\Omega$ as follows. Givental used $R(z)$ to produce a differential operator $\hat{R}$, a so-called quantisation of $R(z)$, which acts on a product of tau-functions to produce a generating series for the correlators of the cohomological field theory. Put $R(z)=\exp(\sum\limits_{\ell>0}r_\ell z^\ell)$.

$$\hat{R}=\exp\left\{\sum_{\ell=1}^\infty\sum_{\alpha,\beta}\left(\sum_{k=0}^\infty u^{k,\beta}(r_k)^\alpha_\beta\frac{\partial}{\partial u^{k+\ell,\alpha}}+\frac{\hbar}{2}\sum_{m=0}^{\ell-1}(-1)^{m+1}(r_\ell)^\alpha_\beta\frac{\partial^2}{\partial u^{m,\alpha}\partial u^{\ell-m-1,\beta}}\right)\right\}.$$

The action of the differential operator $\hat{R}$ on $Z_\Omega(\hbar,\{t_k^\alpha\})$ is equivalent to the weighted sum over graphs. The first term $u^{k,\beta}(r_k)^\alpha_\beta\frac{\partial}{\partial u^{k+\ell,\alpha}}$ gives ordinary leaf contributions, and the second term $\frac{\hbar}{2}(-1)^{m+1}(r_\ell)^\alpha_\beta\frac{\partial^2}{\partial u^{m,\alpha}\partial u^{\ell-m-1,\beta}}$ gives edge contributions.

Associated to $T(z)=\sum_{\ell>0}v_\ell z^\ell=\exp(t_\ell z^\ell)-I\in zV[[z]]$ is the differential operator

$$\hat{T}=\exp\left\{\sum_{\ell=1}^\infty\sum_{\alpha,\beta}\sum_{k=0}^\infty(t_k)^\alpha_\beta\frac{\partial}{\partial u^{k+\ell,\alpha}}\right\}$$

which is a translation operator. It gives dilaton leaf contributions to the weighted sum of graphs. The differential operators $\hat{R}$ and $\hat{T}$ act on a product of functions $Z^{\rm BGW}(\hbar,t_0,t_1,...)$, $Z^{\rm KW}(\hbar,t_0,t_1,...)$ and $Z^\Theta(\hbar,t_0,t_1,...)$ where the TFT is encoded via rescaling of the variables.

4.1.1. *Pixton's relations.* Dual to any point $(C,p_1,...,p_n)\in\overline{\mathcal{M}}_{g,n}$ is its stable graph Γ with vertices $V(\Gamma)$ representing irreducible components of C, internal edges representing nodal singularities and a (labeled) external edge for each p_i. Each vertex is labeled by a genus $g(v)$ and has valency $n(v)$. The genus of a stable graph is $g(\Gamma)=h_1(\Gamma)+\sum\limits_{v\in V(\Gamma)}g(v)$. For a given stable graph Γ of genus g and with n external edges we have

$$\phi_\Gamma:\overline{\mathcal{M}}_\Gamma=\prod_{v\in V(\Gamma)}\overline{\mathcal{M}}_{g(v),n(v)}\to\overline{\mathcal{M}}_{g,n}.$$

The *strata algebra* $S_{g,n}$ is a finite-dimensional vector space over $\mathbb{Q}$ with basis given by isomorphism classes of pairs (Γ,ω), for Γ a stable graph of genus g with n external edges and $\omega\in H^*(\overline{\mathcal{M}}_\Gamma)$ a product of κ and ψ classes in each $\overline{\mathcal{M}}_{g(v),n(v)}$ for each vertex $v\in V(\Gamma)$. There is a natural map $q:S_{g,n}\to H^*(\overline{\mathcal{M}}_{g,n})$ defined by the push-forward $q(\Gamma,\omega)=\phi_{\Gamma*}(\omega)\in H^*(\overline{\mathcal{M}}_{g,n})$. The map q allows one to define a multiplication on $S_{g,n}$, essentially coming from intersection theory in $\overline{\mathcal{M}}_{g,n}$, which can be described purely graphically. The image $q(S_{g,n})\subset H^*(\overline{\mathcal{M}}_{g,n})$ is the tautological ring $RH^*(\overline{\mathcal{M}}_{g,n})$ and an element of the kernel of q is a tautological relation. See [31], Section 0.3 for a good description of $S_{g,n}$.

The main result of [31] is the construction of elements $R^d_{g,A}\in S_{g,n}$ for $A=(a_1,...,a_n)$, $a_i\in\{0,1\}$ satisfying $q(R^d_{g,A})=0$ which push forward to tautological relations in $H^{2d}(\overline{\mathcal{M}}_{g,n})$. The elements $R^d_{g,A}$ are constructed out of the elements $R\Omega)_{g,n}$ defined above. The element $R^1_2\in H^2(\overline{\mathcal{M}}_2)$ is given in Example 3.5.

4.2. **Construction of elements of the twisted loop group.** Consider the linear system

$$\left(\frac{d}{dz}-U-\frac{V}{z}\right)\Psi=0. \tag{12}$$

where $\Psi(z)\in\mathbb{C}^N$, $U=\mathrm{diag}(u_1,...,u_N)$ for u_i distinct and V is skew symmetric. An asymptotic solution of (12) as $z\to\infty$

$$\Psi=R(z^{-1})e^{zU},\quad R(z)=I+R_1z+R_2z^2+...$$

defines a power series $R(z)$ with coefficients given by $N\times N$ matrices which is easily shown to satisfy $R(z)R^T(-z)=I$ hence $R(z)\in L^{(2)}GL(N,\mathbb{C})$. Substitute $\Psi=R(z^{-1})e^{zU}$ into (12) and send $z\mapsto z^{-1}$ to get

$$0=\left(\frac{d}{dz}+\frac{U}{z^2}+\frac{V}{z}\right)R(z)e^{U/z}=\left(\frac{d}{dz}R(z)+\frac{1}{z^2}[U,R(z)]+\frac{1}{z}VR(z)\right)e^{U/z}$$

or equivalently

$$[R_{k+1},U]=(k+V)R_k,\quad k=0,1,... \tag{13}$$

We will describe two natural solutions of (12) using the data of a Frobenius manifold, and using the data of a Riemann surface equipped with a meromorphic function. These will give two constructions of elements $R(z)$ of the twisted loop group. The construction of an element $R(z)$ in two ways using both of these viewpoints will be crucial to proving that Z^{Θ} and Z^{BGW} coincide.

Dubrovin [7] proved that there is a Frobenius manifold structure on the space of pairs (U,V) where V varies in the u_i so that the monodromy of Ψ around 0 remains fixed. Conversely, he associated a family of linear systems (12) depending on the canonical coordinates $(u_1,...,u_N)$ of any semisimple Frobenius manifold M as follows.

Recall that a Frobenius manifold is a complex manifold M equipped with an associative product on its tangent bundle compatible with a flat metric—a nondegenerate symmetric bilinear form—on the manifold [7]. It is encoded by a single function $F(t_1,...,t_N)$, known as the *prepotential*, that satisfies a nonlinear partial differential equation known as the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation:

$$F_{ijm}\eta^{mn}F_{k\ell n}=F_{i\ell m}\eta^{mn}F_{jkn},\quad \eta_{ij}=F_{0ij}$$

where $\eta^{ik}\eta_{kj}=\delta_{ij}$, $F_i=\frac{\partial}{\partial t_i}F$ and $\{t_1,...,t_N\}$ are (flat) local coordinates of M which correspond to the $k=0$ coordinates above via $t_\alpha=t_0^\alpha$, $\alpha=1,...,N$. It comes equipped with the two vector fields $1\!\!1$, the unit for the product, and the Euler vector field E which describes symmetries of the Frobenius manifold, neatly encoded by $E\cdot F(t_1,...,t_N)=c\cdot F(t_1,...,t_N)$ for some $c\in\mathbb{C}$. Multiplication by the Euler vector field E produces an endomorphism U with eigenvalues $\{u_1,...,u_N\}$ known as *canonical coordinates* on M. They give rise to vector fields $\partial/\partial u_i$ with respect to which the metric η, product $\bullet$ and Euler vector field E are diagonal:

$$\frac{\partial}{\partial u_i}\bullet\frac{\partial}{\partial u_j}=\delta_{ij}\frac{\partial}{\partial u_i},\quad \eta\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right)=\delta_{ij}\Delta_i,\quad E=\sum u_i\frac{\partial}{\partial u_i}.$$

The differential equation (12) in z is defined at any point of the Frobenius manifold using U, the endomorphism defined by multiplication by the Euler vector field E, and the endomorphism $V=[\Gamma,U]$ where $\Gamma_{ij}=\frac{\partial_{u_i}\Delta_j}{2\sqrt{\Delta_i\Delta_j}}$ for $i\neq j$ are the so-called rotation coefficients of the metric η in the normalised canonical basis. Hence associated to each point of the Frobenius manifold is an element $R(z)=\sum R_kz^k$ of the twisted loop group. From $(H=T_pM,\bullet,E,V)$ the endomorphisms R_k of H are defined recursively from R_1 by $R_0=I$ and (13).

Example 4.2. *The 2-dimensional versal deformation space of the A_2 singularity has a Frobenius manifold structure [7, 32] defined by the prepotential*

$$F(t_1,t_2)=\frac{1}{2}t_1^2t_2+\frac{1}{72}t_2^4.$$

The prepotential determines the flat metric with respect to the basis $\{\frac{\partial}{\partial t_1},\frac{\partial}{\partial t_2}\}$ by $\eta_{ij}=F_{1ij}$ hence

$$\eta=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

and the (commutative) product by $\eta(\partial/\partial t_i\bullet\partial/\partial t_j,\partial/\partial t_k)=F_{ijk}$ hence

$$\frac{\partial}{\partial t_1}\bullet\frac{\partial}{\partial t_1}=\frac{\partial}{\partial t_1},\quad \frac{\partial}{\partial t_1}\bullet\frac{\partial}{\partial t_2}=\frac{\partial}{\partial t_2},\quad \frac{\partial}{\partial t_2}\bullet\frac{\partial}{\partial t_2}=\frac{1}{3}t_2\frac{\partial}{\partial t_1}.$$

The unit and Euler vector fields are

$$1\!\!1=\frac{\partial}{\partial t_1},\quad E=t_1\frac{\partial}{\partial t_1}+\frac{2}{3}t_2\frac{\partial}{\partial t_2}.$$

and $E\cdot F(t_1,t_2)=\frac{8}{3}F(t_1,t_2)$*. The canonical coordinates are*

$$u_1=t_1+\frac{2}{3\sqrt{3}}t_2^{3/2},\quad u_2=t_1-\frac{2}{3\sqrt{3}}t_2^{3/2}.$$

With respect to the normalised canonical basis, the rotation coefficients $\Gamma_{12}=\dfrac{-i\sqrt{3}}{8}t_2^{-3/2}=\Gamma_{21}$ *give rise to* $V=[\Gamma,U]=\frac{i\sqrt{3}}{2}t_2^{-3/2}\begin{pmatrix}0&-1\\1&0\end{pmatrix}$*. In canonical coordinates we have*

$$U=\begin{pmatrix}u_1&0\\0&u_2\end{pmatrix},\quad V=\frac{2i}{3(u_1-u_2)}\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \tag{14}$$

The metric η *applied to the vector fields* $\partial/\partial u_i=\frac{1}{2}\left(\partial/\partial t_1-(-1)^i(3/t_2)^{1/2}\partial/\partial t_2\right)$ *is* $\eta\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right)=\delta_{ij}\Delta_i$ *where* $\Delta_1=\frac{\sqrt{3}}{2}t_2^{-1/2}=-\Delta_2$.

We consider three natural bases of the tangent space $H=T_pM$ at any point p of a Frobenius manifold. The flat basis $\{\partial/\partial t_i\}$ which gives a constant metric η, the canonical basis $\{\partial/\partial u_i\}$ which gives a trivial product $\bullet$, and the normalised canonical basis $\{v_i\}$, for $v_i=\Delta_i^{-1/2}\partial/\partial u_i$, which gives a trivial metric η. (A different choice of square root of Δ_i would simply give a different choice of normalised canonical basis.) The transition matrix Ψ from flat coordinates to normalised canonical coordinates sends the metric η to the dot product, i.e. $\Psi^T\Psi=\eta$. The TFT structure on H induced from η and $\bullet$ is diagonal in the normalised canonical basis. It is given by

$$\Omega_{g,n}(v_i^{\otimes n})=\Delta_i^{1-g-\frac{1}{2}n}$$

and vanishes on mixed products of v_i and v_j, $i\neq j$. We find the normalised canonical basis most useful for comparisons with topological recursion—see Section 4.4.

Example 4.3. *On the* A_2 *Frobenius manifold restrict to the point* $(u_1,u_2)=(2,-2)$*, or equivalently* $(t_1,t_2)=(0,3)$*. Then* $\Delta_1=1/2=-\Delta_2$ *determines the TFT. (In the flat basis the TFT structure at* $(t_1,t_2)=(0,3)$ *on* H *is given by* $\Omega_{g,n}((\partial/\partial t_1)^{\otimes n_0}\otimes(\partial/\partial t_2)^{\otimes n_1})=2^g\cdot\delta^{\text{odd}}_{g+n_1}$*.) At the point* $(u_1,u_2)=(2,-2)$ *the element* $R(z)\in L^{(2)}GL(2,\mathbb{C})$ *satisfying* (13) *with* U *and* V *given by*

$$U=\begin{pmatrix}2&0\\0&-2\end{pmatrix},\quad V=\frac{1}{6}\begin{pmatrix}0&i\\-i&0\end{pmatrix}$$

is

$$R(z)=\sum_m\frac{(6m)!}{(6m-1)(3m)!(2m)!}\begin{pmatrix}-1&(-1)^m6mi\\-6mi&(-1)^{m-1}\end{pmatrix}\left(\frac{z}{1728}\right)^m.$$

We also have

$$T_0(z)=\mathbb{1}-R^{-1}(z)(\mathbb{1}),\quad \mathbb{1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$
$$T(z)=zT_0(z).$$

Remark 4.4. Equation (13) defines $R(z)$ as an endomorphism valued power series over an N-dimensional vector space H equipped with a symmetric, bilinear, nondegenerate form η. The matrix $R(z)$ here—using a basis for H so that η is dot product—is related to the matrix $R(z)$ in [31] by conjugation by the transition matrix Ψ from flat coordinates to normalised canonical coordinates

$$R(z)=\Psi\sum_m\frac{(6m)!}{(3m)!(2m)!}\begin{pmatrix}\frac{1+6m}{1-6m}&0\\0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}^m\left(\frac{z}{1728}\right)^m\Psi^{-1},\quad \Psi=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\i&-i\end{pmatrix}.$$

4.2.1. *Spectral curves and the twisted loop group.* An element of the twisted loop group $R(z)\in L^{(2)}GL(N,\mathbb{C})$ can be naturally defined from a Riemann surface Σ equipped with a bidifferential $B(p_1,p_2)$ on $\Sigma\times\Sigma$ and a meromorphic function $x:\Sigma\to\mathbb{C}$, for $N=$ the number of zeros of dx. A basic example is the function $x=z^2$ on $\Sigma=\mathbb{C}$ which gives rise to the constant element $R(z)=1\in GL(1,\mathbb{C})$. More generally, any function x that looks like this example locally—$x=s^2+c$ for s a local coordinate around a zero of dx and $c\in\mathbb{C}$—gives $R(z)=I+R_1z+...\in L^{(2)}GL(N,\mathbb{C})$ which is in some sense a deformation of $I\in GL(N,\mathbb{C})$, or N copies of the basic example.

Definition 4.5. On any compact Riemann surface with a choice of $\mathcal{A}$-cycles $(\Sigma,\{\mathcal{A}_i\}_{i=1,\dots,g})$, define a *fundamental normalised bidifferential of the second kind* $B(p,p')$ to be a symmetric tensor product of differentials on $\Sigma\times\Sigma$, uniquely defined by the properties that it has a double pole on the diagonal of zero residue, double residue equal to 1, no further singularities and normalised by $\int_{p\in\mathcal{A}_i}B(p,p')=0$, $i=1,\dots,g$, [18]. On a rational curve, which is sufficient for this paper, B is the Cauchy kernel

$$B(z_1,z_2)=\frac{dz_1dz_2}{(z_1-z_2)^2}.$$

The bidifferential $B(p,p')$ acts as a kernel for producing meromorphic differentials on the Riemann surface Σ via $\omega(p)=\int_\Lambda\lambda(p')B(p,p')$ where λ is a function defined along the contour $\Lambda\subset\Sigma$. Depending on the choice of (Λ,λ), ω can be a differential of the 1st kind (holomorphic), 2nd kind (zero residues) or 3rd kind (simple poles). A fundamental example is $B(\mathcal{P},p)$ which is a normalised (trivial $\mathcal{A}$-periods) differential of the second kind holomorphic on $\Sigma\backslash\mathcal{P}$ with a double pole at a simple zero $\mathcal{P}$ of dx. The expression $B(\mathcal{P},p)$ is an integral over a closed contour around $\mathcal{P}$ defined as follows.

Definition 4.6. For a Riemann surface equipped with a meromorphic function (Σ,x) we define evaluation of any meromorphic differential ω at a simple zero $\mathcal{P}$ of dx by

$$\omega(\mathcal{P}):=\operatorname*{Res}_{p=\mathcal{P}}\frac{\omega(p)}{\sqrt{2(x(p)-x(\mathcal{P}))}}$$

where we choose a branch of $\sqrt{x(p)-x(\mathcal{P})}$ once and for all at $\mathcal{P}$ to remove the ±1 ambiguity.

Shramchenko [34] constructed a solution of (12) with $V=[\mathcal{B},U]$ for $\mathcal{B}_{ij}=B(\mathcal{P}_i,\mathcal{P}_j)$ (defined for $i\neq j$) given by

$$\Psi(z)^i_j=-\frac{\sqrt{z}}{\sqrt{2\pi}}\int_{\Gamma_j}B(\mathcal{P}_i,p)\cdot e^{\frac{-x(p)}{z}}.$$

The proof in [34] is indirect, showing that $\Psi(z)^i_j$ satisfies an associated set of PDEs in u_i, and using the Rauch variational formula to calculate $\partial_{u_k}B(\mathcal{P}_i,p)$. Rather than giving this proof, we will work directly with the associated element $R(z)$ of the twisted loop group.

Definition 4.7. Define the asymptotic series for z near 0 by

$$R^{-1}(z)^i_j=-\frac{\sqrt{z}}{\sqrt{2\pi}}\int_{\Gamma_j}B(\mathcal{P}_i,p)\cdot e^{\frac{u_j-x(p)}{z}}\tag{15}$$

where Γ_j is a path of steepest descent for $-x(p)/z$ containing $u_j=x(\mathcal{P}_j)$.

Note that the asymptotic expansion of the contour integral (15) for z near 0 depends only the intersection of Γ_j with a neighbourhood of $p=\mathcal{P}_j$. When $i=j$, the integrand has zero residue at $p=\mathcal{P}_j$ so we deform Γ_j to go around $\mathcal{P}_j$ to get a well-defined integral. Locally, this is the same as defining $\int_{\mathbb{R}}s^{-2}\exp(-s^2)ds=-2\sqrt{\pi}$ by integrating the analytic function $z^{-2}\exp(-z^2)$ along the real line in $\mathbb{C}$ deformed to avoid 0.

Lemma 4.8 ([34])**.** *The asymptotic series $R(z)$ defined in (15) satisfies the twisted loop group condition*

$$R(z)R^T(-z)=Id.\tag{16}$$

Proof. The proof here is taken from [9]. We have

$$\begin{aligned}\sum_{i=1}^N\operatorname*{Res}_{q=\mathcal{P}_i}\frac{B(p,q)B(p',q)}{dx(q)}&=-\operatorname*{Res}_{q=p}\frac{B(p,q)B(p',q)}{dx(q)}-\operatorname*{Res}_{q=p'}\frac{B(p,q)B(p',q)}{dx(q)}\\&=-d_p\left(\frac{B(p,p')}{dx(p)}\right)-d_{p'}\left(\frac{B(p,p')}{dx(p')}\right)\end{aligned}\tag{17}$$

where the first equality uses the fact that the only poles of the integrand are $\{p,p',\mathcal{P}_i,i=1,\dots,N\}$, and the second equality uses the Cauchy formula satisfied by the Bergman kernel. Define the Laplace transform of the Bergman kernel by

$$\check{B}^{i,j}(z_1,z_2)=\frac{e^{\frac{u_i}{z_1}+\frac{u_j}{z_2}}}{2\pi\sqrt{z_1z_2}}\int_{\Gamma_i}\int_{\Gamma_j}B(p,p')e^{-\frac{x(p)}{z_1}-\frac{x(p')}{z_2}}.$$

The Laplace transform of the LHS of (17) is

$$\frac{e^{\frac{u_i}{z_1}+\frac{u_j}{z_2}}}{2\pi\sqrt{z_1z_2}}\int_{\Gamma_i}\int_{\Gamma_j}e^{-\frac{x(p)}{z_1}-\frac{x(p')}{z_2}}\sum_{k=1}^{N}\operatorname*{Res}_{q=\mathcal{P}_k}\frac{B(p,q)B(p',q)}{dx(q)}=\sum_{k=1}^{N}\frac{e^{\frac{u_i}{z_1}+\frac{u_j}{z_2}}}{2\pi\sqrt{z_1z_2}}\int_{\Gamma_i}e^{-\frac{x(p)}{z_1}}B(p,\mathcal{P}_k)\int_{\Gamma_j}e^{-\frac{x(p')}{z_2}}B(p',\mathcal{P}_k)$$
$$=\sum_{k=1}^{N}\frac{\left[R^{-1}(z_1)\right]_i^k\left[R^{-1}(z_2)\right]_j^k}{z_1z_2}.$$

Since the Laplace transform satisfies $\int_{\Gamma_i}d\left(\frac{\omega(p)}{dx(p)}\right)e^{-\frac{x(p)}{z}}=\frac{1}{z}\int_{\Gamma_i}\omega(p)e^{-\frac{x(p)}{z}}$ for any differential $\omega(p)$, by integration by parts, then the Laplace transform of the RHS of (17) is

$$-\frac{e^{\frac{u_i}{z_1}+\frac{u_j}{z_2}}}{2\pi\sqrt{z_1z_2}}\int_{\Gamma_i}\int_{\Gamma_j}e^{-\frac{x(p)}{z_1}-\frac{x(p')}{z_2}}\left\{d_p\left(\frac{B(p,p')}{dx(p)}\right)+d_{p'}\left(\frac{B(p,p')}{dx(p')}\right)\right\}=-\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\check{B}^{i,j}(z_1,z_2).$$

Putting the two sides together yields the following result due to Eynard [13]

$$\check{B}^{i,j}(z_1,z_2)=-\frac{\sum_{k=1}^{N}\left[R^{-1}(z_1)\right]_i^k\left[R^{-1}(z_2)\right]_j^k}{z_1+z_2}. \tag{18}$$

Equation (16) is an immediate consequence of (18) and the finiteness of $\check{B}^{i,j}(z_1,z_2)$ at $z_2=-z_1$. □

Example 4.9. *Let $\Sigma\cong\mathbb{C}$ be a rational curve equipped with the meromorphic function $x=z^3-3z$ and bidifferential $B(z_1,z_2)=dz_1dz_2/(z_1-z_2)^2$.*

Choose a local coordinate t around $z=-1=\mathcal{P}_1$ so that $x(t)=\frac{1}{2}t^2+2$. Then

$$B(\mathcal{P}_1,t)=\frac{-i}{\sqrt{6}}\frac{dz}{(z+1)^2}=dt\left(t^{-2}-\frac{1}{144}+\frac{35}{41472}t^2+...+\text{odd terms}\right)$$
$$B(\mathcal{P}_2,t)=\frac{1}{\sqrt{6}}\frac{dz}{(z-1)^2}=dt\left(-\frac{i}{24}+\frac{35i}{3456}t^2+...+\text{odd terms}\right)$$

Choose a local coordinate s around $z=1=\mathcal{P}_2$ so that $x(s)=\frac{1}{2}s^2-2$. Then

$$B(\mathcal{P}_1,s)=\frac{-i}{\sqrt{6}}\frac{dz}{(z+1)^2}=ds\left(-\frac{i}{24}-\frac{35i}{3456}s^2+...+\text{odd terms}\right)$$
$$B(\mathcal{P}_2,s)=\frac{1}{\sqrt{6}}\frac{dz}{(z-1)^2}=ds\left(s^{-2}+\frac{1}{144}+\frac{35}{41472}s^2+...+\text{odd terms}\right)$$

The odd terms are annihilated by the Laplace transform, and we get

$$R^{-1}(z)_1^1=-\frac{\sqrt{z}}{\sqrt{2\pi}}\int_{\Gamma_1}B(\mathcal{P}_1,t)\cdot e^{\frac{-\frac{1}{2}t^2}{z}}=1+\frac{1}{144}z-\frac{35}{41472}z^2+...$$
$$R^{-1}(z)_2^1=-\frac{\sqrt{z}}{\sqrt{2\pi}}\int_{\Gamma_1}B(\mathcal{P}_2,t)\cdot e^{\frac{-\frac{1}{2}t^2}{z}}=\frac{i}{24}z-\frac{35i}{3456}z^2+...$$
$$R^{-1}(z)_1^2=-\frac{\sqrt{z}}{\sqrt{2\pi}}\int_{\Gamma_2}B(\mathcal{P}_1,s)\cdot e^{\frac{-\frac{1}{2}s^2}{z}}=\frac{i}{24}z+\frac{35i}{3456}z^2+...$$
$$R^{-1}(z)_2^2=-\frac{\sqrt{z}}{\sqrt{2\pi}}\int_{\Gamma_2}B(\mathcal{P}_2,s)\cdot e^{\frac{-\frac{1}{2}s^2}{z}}=1-\frac{1}{144}z-\frac{35}{41472}z^2+...$$

Hence $R^{-1}(z)=I-R_1z+(R_1^2-R_2)z^2+...=I-R_1^Tz+R_2^Tz^2+...$ gives

$$R_1=\frac{1}{144}\begin{pmatrix}-1&-6i\\-6i&1\end{pmatrix},\quad R_2=\frac{35}{41472}\begin{pmatrix}-1&12i\\-12i&-1\end{pmatrix}$$

which agrees with Example 4.3 for the A_2 Frobenius manifold.

4.3. **Partition Functions from the A_2 singularity.** Using the twisted loop group action and the translation action defined in the previous section, we construct in (19), (20) and (21) below, three partition functions out of $Z^{\rm BGW}(\hbar,t_0,t_1,...)$, $Z^{\rm KW}(\hbar,t_0,t_1,...)$ and $Z^{\Theta}(\hbar,t_0,t_1,...)$, denoted by

$$Z_{A_2}(\hbar,\{t_k^\alpha\}),\quad Z^{\Theta}_{A_2}(\hbar,\{t_k^\alpha\}),\quad Z^{\rm BGW}_{A_2}(\hbar,\{t_k^\alpha\})$$

where $\alpha\in\{1,2\}$, $k\in\mathbb{N}$, i.e. $\{t_k^\alpha\}=\{t_0^1,t_0^2,t_1^1,t_1^2,t_2^1,t_2^2,...\}$.

The three partition functions above use the element $R(z)$ of the twisted loop group arising out of the A_2 Frobenius manifold. The partition function $Z_{A_2}(\hbar,\{t_k^\alpha\})$ is defined via the graphical action defined in Section 4.1 (and represented here via the equivalent action by differential operators) using $R(z)$ and $T(z)$ defined from the A_2 Frobenius manifold in Example 4.3

$$\begin{aligned}Z_{A_2}(\hbar,\{t_k^\alpha\}) &:= \hat{\Psi}\cdot\hat{R}\cdot\hat{T}\cdot Z^{\rm KW}(\tfrac12\hbar,\{\frac{1}{\sqrt2}v_k^1\})Z^{\rm KW}(-\tfrac12\hbar,\{\frac{i}{\sqrt2}v_k^2\})\\ &=\exp\sum_{g,n,\vec{k}}\frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\Omega^{A_2}_{g,n}(e_{\alpha_1}\otimes...\otimes e_{\alpha_n})\cdot\prod_{j=1}^n\psi_j^{k_j}\prod t^{\alpha_j}_{k_j}\end{aligned}\tag{19}$$

where $\hat{R}$ and $\hat{T}$ are differential operators acting on copies of $Z^{\rm KW}$, which can be expressed as a sum over stable graphs, and $\hat{\Psi}$ is the linear change of coordinates $v_k^i=\Psi^i_\alpha t_k^\alpha$ (also expressible as a differential operator). Similarly, define

$$\begin{aligned}Z^{\Theta}_{A_2}(\hbar,\{t_k^\alpha\}) &:= \hat{\Psi}\cdot\hat{R}\cdot\hat{T}_0\cdot Z^{\Theta}(\tfrac12\hbar,\{\frac{1}{\sqrt2}v_k^1\})Z^{\Theta}(-\tfrac12\hbar,\{\frac{i}{\sqrt2}v_k^2\})\\ &=\exp\sum_{g,n,\vec{k},\vec{\alpha}}\frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\cdot\Omega^{A_2}_{g,n}(e_{\alpha_1},...,e_{\alpha_n})\cdot\prod_{j=1}^n\psi_j^{k_j}\prod t^{\alpha_j}_{k_j}.\end{aligned}\tag{20}$$

The operators $\hat{\Psi}$ and $\hat{R}$ in (20) coincide with the operators in (19) associated to the A_2 singularity. The translation operator $\hat{T}_0=z^{-1}\hat{T}$ is related to the translation operator in (19) essentially by the shift $m_i+1\to m_i$ we saw in (3). The graphical expression (22) for $\Omega^{A_2}_{g,n}(e_{\alpha_1},...,e_{\alpha_n})$ immediately gives rise to a graphical expression for $\Theta_{g,n}\cdot\Omega^{A_2}_{g,n}(e_{\alpha_1},...,e_{\alpha_n})$. This produces a graphical expression for $Z^{\Theta}_{A_2}$ which is expressed in (20) as a differential operator acting on two copies of Z^{Θ} in place of $Z^{\rm KW}$. This is a sum over graphs $G'_{g,n}$ via the structure shown in (24) (which also applies when the expression is non-zero).

Finally, define

$$Z^{\rm BGW}_{A_2}:=\hat{\Psi}\cdot\hat{R}\cdot\hat{T}_0\cdot Z^{\rm BGW}(\tfrac12\hbar,\{\frac{1}{\sqrt2}v_k^1\})Z^{\rm BGW}(-\tfrac12\hbar,\{\frac{i}{\sqrt2}v_k^2\}).\tag{21}$$

Equivalently, we have replaced each $\int_{\overline{\mathcal{M}}_{g,n+N}}\Theta_{g,n+N}\prod_{i=1}^{n+N}\psi_i^{k_i}$ in $Z^{\Theta}_{A_2}$ with the corresponding coefficients from $F_g^{\rm BGW}$. To prove Theorem 3, we will show that $Z^{\rm BGW}_{A_2}$ has many vanishing terms from which it will follow that $Z^{\rm BGW}_{A_2}=Z^{\Theta}_{A_2}$ and $Z^{\rm BGW}=Z^{\Theta}$.

The A_2 Frobenius manifold and its associated cohomological field theory $\Omega^{A_2}_{g,n}\in H^*(\overline{\mathcal{M}}_{g,n})\otimes(V^*)^{\otimes n}$ for $H=\mathbb{C}^2$ is used to produce Pixton's relations among tautological cohomology classes over $\overline{\mathcal{M}}_{g,n}$. The class $\Omega^{A_2}_{g,n}\in H^*(\overline{\mathcal{M}}_{g,n})\otimes(V^*)^{\otimes n}$ is defined by the Givental-Teleman theorem—see [11, 19, 35]—via a sum over stable graphs:

$$\Omega^{A_2}_{g,n}=\sum_{\Gamma\in G_{g,n}}(\phi_\Gamma)_*\omega^R_\Gamma\in H^*(\overline{\mathcal{M}}_{g,n})\tag{22}$$

where ω^R_Γ is built out of contributions to edges and vertices of Γ using R, ψ and κ classes and a topological field theory at vertices that takes in vectors of H.

The key idea behind Pixton's relations among tautological classes [31] is a degree bound on the cohomological classes $\deg\Omega^{A_2}_{g,n}\leq\frac13(g-1+n)<3g-3+n$. The construction of $\Omega^{A_2}_{g,n}$ using R in (22) does not know about this degree bound and produces classes in the degrees where $\Omega^{A_2}_{g,n}$ vanishes. This leads to sums of tautological classes representing the zero class, i.e. relations, expressed as sums over stable graphs $G_{g,n}$:

$$0=\sum_{\Gamma\in G_{g,n}}(\phi_\Gamma)_*\omega_\Gamma^R\in H^*(\overline{\mathcal{M}}_{g,n})=\sum_{\Gamma\in G'_{g,n}}(\phi_\Gamma)_*\tilde{\omega}_\Gamma^R \tag{23}$$

for classes $\omega_\Gamma^R,\tilde{\omega}_\Gamma^R\in H^*(\overline{\mathcal{M}}_\Gamma)$. Here we denote by $G'_{g,n}$ the set of all stable graphs of genus g with n labeled points and any number of extra leaves, known as *dilaton leaves* at each vertex. So the set $G_{g,n}$ is finite whereas $G'_{g,n}$ is infinite. Nevertheless, all sums in (23) are finite. The classes ω_Γ appearing in (23) consist of products of ψ and κ classes associated to each vertex of Γ. The classes $\tilde{\omega}_\Gamma^R$ consist of products of only ψ classes, again associated to each vertex of Γ.

Pixton's relations (23) induce relations between intersection numbers of ψ classes with $\Theta_{g,n}$:

$$0=\Theta_{g,n}\cdot\sum_{\Gamma\in G_{g,n}}(\phi_\Gamma)_*\omega_\Gamma^R=\sum_{\Gamma\in G_{g,n}}(\phi_\Gamma)_*\left(\Theta_\Gamma\cdot\omega_\Gamma^R\right)=\sum_{\Gamma\in G'_{g,n}}(\phi_\Gamma)_*\left(\Theta_\Gamma\cdot\tilde{\omega}_\Gamma^R\right). \tag{24}$$

The second equality uses $\Theta_{g,n}\cdot(\phi_\Gamma)_*=(\phi_\Gamma)_*\Theta_\Gamma$. The final equality uses Remark 3.2 to replace κ classes by ψ classes, such as in Example 3.5 below where the κ_1 term is replaced by $\int_{\overline{\mathcal{M}}_{2,1}}\Theta_{2,1}\cdot\psi_1=\int_{\overline{\mathcal{M}}_2}\Theta_2\cdot\kappa_1$. The classes ω_Γ in (24) are linear combinations of monomials $\prod\psi_i^{k_i}\cdot R_{\mathbf{m}}$, which form a basis for products of ψ and κ classes. The final equality in (24) is obtained by substituting (linear combinations of) the following expression into the sum over $G_{g,n}$

$$\Theta_{g,n}(\phi_\Gamma)_*\Big(\prod\psi_i^{k_i}\cdot R_{\mathbf{m}}\Big)=(\phi_\Gamma)_*\Big(\Theta_\Gamma\prod\psi_i^{k_i}\cdot R_{\mathbf{m}}\Big)=(\phi_\Gamma)_*\Big(\prod\psi_i^{k_i}\cdot\pi_*(\Theta_{\Gamma_{(N)}}\psi_{n+1}^{m_1}\cdots\psi_{n+N}^{m_N})\Big)$$

where $\Gamma_{(N)}$ is obtained from Γ by adding N (dilaton) leaves to the vertex of Γ on which $\prod\psi_i^{k_i}\cdot R_{\mathbf{m}}$ is defined. The final term in (24) is a sum over graphs of intersection numbers of $\Theta_{g,n}$ with ψ classes only.

Primary invariants of a partition function are those coefficients of $\prod_{i=1}^n t_{k_i}^{\alpha_i}$ with all $k_i=0$. They correspond to intersections in $\overline{\mathcal{M}}_{g,n}$ with no ψ classes. The primary invariants of $Z_{A_2}^{\Theta}(\hbar,\{t_k^\alpha\})$ vanish for $n<2g-2$. This uses $\deg\Omega_{g,n}^{A_2}\leq\frac{1}{3}(g-1+n)$ so $\deg\Omega_{g,n}^{A_2}\cdot\Theta_{g,n}\leq\frac{1}{3}(g-1+n)+2g-2+n<3g-3+n$ when $n<2g-2$. These vanishing coefficients correspond to top intersections of ψ classes with the relations (24). For $n\leq g-1$ these relations are sufficient to uniquely determine the intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}$ via the recursive method in Section 3. For $n\geq g$ the dilaton equation (5) determines the intersection numbers from the lower ones.

4.4. **Topological recursion.** Topological recursion is a procedure which takes as input a spectral curve, defined below, and produces a collection of symmetric tensor products of meromorphic 1-forms $\omega_{g,n}$ on C^n. The correlators store enumerative information in different ways. Periods of the correlators store top intersection numbers of tautological classes in the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ and local expansions of the correlators can serve as generating functions for enumerative problems.

A spectral curve $S=(C,x,y,B)$ is a Riemann surface C equipped with two meromorphic functions $x,y:C\to\mathbb{C}$ and a bidifferential $B(p_1,p_2)$ defined in (4.5), which is the Cauchy kernel in this paper. Topological recursion, as developed by Chekhov, Eynard, Orantin [4, 14], is a procedure that produces from a spectral curve $S=(C,x,y,B)$ a symmetric tensor product of meromorphic 1-forms $\omega_{g,n}$ on C^n for integers $g\geq0$ and $n\geq1$, which we refer to as *correlation differentials* or *correlators*. The correlation differentials $\omega_{g,n}$ are defined by

$$\omega_{0,1}(p_1)=-y(p_1)\,dx(p_1)\qquad\text{and}\qquad\omega_{0,2}(p_1,p_2)=B(p_1,p_2)$$

and for $2g-2+n>0$ they are defined recursively via the following equation.

$$\omega_{g,n}(p_1,p_L)=\sum_{dx(\alpha)=0}\operatorname*{Res}_{p=\alpha}K(p_1,p)\Bigg[\omega_{g-1,n+1}(p,\hat{p},p_L)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=L}}^{\circ}\omega_{g_1,|I|+1}(p,p_I)\,\omega_{g_2,|J|+1}(\hat{p},p_J)\Bigg]$$

Here, we use the notation $L=\{2,3,\dots,n\}$ and $p_I=\{p_{i_1},p_{i_2},\dots,p_{i_k}\}$ for $I=\{i_1,i_2,\dots,i_k\}$. The outer summation is over the zeroes α of dx and $p\mapsto\hat{p}$ is the involution defined locally near α satisfying $x(\hat{p})=x(p)$

and $\hat{p} \neq p$. The symbol $\circ$ over the inner summation means that we exclude any term that involves $\omega_{0,1}$. Finally, the recursion kernel is given by

$$K(p_1, p) = \frac{1}{2} \frac{\int_{\hat{p}}^{p} \omega_{0,2}(p_1, \cdot\,)}{[y(p) - y(\hat{p})]\, dx(p)}.$$

which is well-defined in the vicinity of each zero of dx. It acts on differentials in p and produces differentials in p_1 since the quotient of a differential in p by the differential $dx(p)$ is a meromorphic function. For $2g - 2 + n > 0$, each $\omega_{g,n}$ is a symmetric tensor product of meromorphic 1-forms on C^n with residueless poles at the zeros of dx and holomorphic elsewhere. A zero α of dx is *regular*, respectively irregular, if y is regular, respectively has a simple pole, at α. The order of the pole in each variable of $\omega_{g,n}$ at a regular, respectively irregular, zero of dx is $6g - 4 + 2n$, respectively $2g$. Define $\Phi(p)$ up to an additive constant by $d\Phi(p) = y(p)dx(p)$. For $2g - 2 + n > 0$, the invariants satisfy the dilaton equation [14]

$$\sum_{\alpha} \operatorname*{Res}_{p=\alpha} \Phi(p)\, \omega_{g,n+1}(p, p_1, \ldots, p_n) = (2g - 2 + n)\, \omega_{g,n}(p_1, \ldots, p_n),$$

where the sum is over the zeros α of dx. This enables the definition of the so-called *symplectic invariants*

$$F_g = \sum_{\alpha} \operatorname*{Res}_{p=\alpha} \Phi(p)\omega_{g,1}(p).$$

The correlators $\omega_{g,n}$ are normalised differentials of the second kind in each variable—they have zero $\mathcal{A}$-periods, and poles only at the zeros $\mathcal{P}_i$ of dx of zero residue. Their principle parts are skew-invariant under the local involution $p \mapsto \hat{p}$. A basis of such normalised differentials of the second kind is constructed from x and B in the following definition.

Definition 4.10. For a Riemann surface C equipped with a meromorphic function $x : C \to \mathbb{C}$ and bidifferential $B(p_1, p_2)$ define the auxiliary differentials on C as follows. For each zero $\mathcal{P}_i$ of dx, define

$$V_0^i(p) = B(\mathcal{P}_i, p), \quad V_{k+1}^i(p) = d\left(\frac{V_k^i(p)}{dx(p)}\right), \; i = 1, ..., N, \quad k = 0, 1, 2, ... \tag{25}$$

where evaluation $B(\mathcal{P}_i, p)$ at $\mathcal{P}_i$ is given in Definition 4.6.

The correlators $\omega_{g,n}$ are polynomials in the auxiliary differentials $V_k^i(p)$. To any spectral curve S, one can define a partition function Z^S by assembling the polynomials built out of the correlators $\omega_{g,n}$ [10, 13].

Definition 4.11.

$$Z^S(\hbar, \{u_k^\alpha\}) := \exp \sum_{g,n} \frac{\hbar^{g-1}}{n!} \omega_{g,n}^S \bigg|_{V_k^\alpha(p_i) = u_k^\alpha}.$$

As usual define F_g to be the contribution from $\omega_{g,n}$:

$$\log Z^S(\hbar, \{u_k^\alpha\}) = \sum_{g \geq 0} \hbar^{g-1} F_g^S(\{u_k^\alpha\}).$$

4.4.1. *From topological recursion to Givental reconstruction.* The relation between Givental's reconstruction of CohFTs defined in Section 4.1 and topological recursion was proven in [10]. The A_2 case is treated in [8]. Recall that the input data for Givental's reconstruction is an element $R(z) \in L^{(2)}GL(N, \mathbb{C})$ and $T(z) = z\left(\mathbb{1} - R^{-1}(z)\mathbb{1}\right) \in z\,\mathbb{C}^N[[z]]$ defined by a vector $\mathbb{1} \in \mathbb{C}^N$. Its output is a CohFT or its partition function $Z(\hbar, \{t_k^\alpha\})$. The input data for topological recursion is a spectral curve $S = (C, x, y, B)$. Its output is the correlators $\omega_{g,n}$ which can be assembled into a partition function $Z^S(\hbar, \{t_k^\alpha\})$. This is summarised in the following diagram.

$$\begin{array}{ccc} R(z)\in L^{(2)}GL(N,\mathbb{C}) & \xrightarrow{\text{Givental reconstruction}} & Z(\hbar,\{t_k^\alpha\}) \\ T(z)\in z\cdot\mathbb{C}^N[[z]] & & \\ \updownarrow & & \updownarrow \\ S=(C,x,y,B) & \xrightarrow{\text{topological recursion}} & Z^S(\hbar,\{t_k^\alpha\}) \end{array}$$

The left arrow in the diagram, i.e. a correspondence between the input data, which produces the same output $Z(\hbar,\{t_k^\alpha\})=Z^S(\hbar,\{t_k^\alpha\})$ is the main result of [10]. Given $R(z)$ and $T(z)$ arising from a CohFT, it was proven in [10] that there exists a local spectral curve S, which is a collection of disks neighbourhoods of zeros of dx on which B and y are define locally, giving the partition function Z of the CohFT. We will use the converse of this result proven in [9], beginning instead from S, which builds on the construction of [10].

Recall from Section 4.2.1 the construction

$$(C,x,B)\mapsto R(z)\in L^{(2)}GL(N,\mathbb{C})$$

of an element $R(z)$ from part of the data of a spectral curve S. We will now associate a (locally defined) function y on C to the unit vector $\mathbb{1}=\{\Delta_i^{1/2}\}\in\mathbb{C}^N$ in normalised canonical coordinates to get the full data of a spectral curve $S=(C,x,y,B)$. Define dy by

$$\sum_{k=1}^N R^{-1}(z)_k^i\cdot\Delta_k^{1/2}=\frac{1}{\sqrt{2\pi z}}\int_{\Gamma_i}dy(p)\cdot e^{\frac{(u_i-x(p))}{z}}. \tag{26}$$

This defines y locally around each $\mathcal{P}_i$. In fact, it only defines the part of y skew-invariant under the local involution $p\mapsto\hat{p}$ defined by x since the Laplace transform annihilates invariant parts, but topological recursion depends only on this skew-invariant part of y. The $z\to 0$ limit of (26) gives

$$dy(\mathcal{P}_i)=\Delta_i^{1/2} \tag{27}$$

hence conversely y determines the unit $\mathbb{1}=\{\Delta_i^{1/2}\}\in\mathbb{C}^N$ and $R^{-1}(z)\mathbb{1}$ which produces the translation. Equation (27) is a strong condition on y since it shows that $y(p)$ is determined by $dy(\mathcal{P}_i)$, $i=1,...,N$ and $B(p,p')$. If a globally defined differential dy on a compact curve C satisfies the condition

$$d\left(\frac{dy}{dx}(p)\right)=-\sum_{i=1}^n\operatorname*{Res}_{p'=\mathcal{P}_i}\frac{dy}{dx}(p')B(p',p) \tag{28}$$

then dy satisfies (26) for $\Delta_i^{1/2}$ defined by (27) and $R(z)$ defined by (15). This is immediate by taking the Laplace transform of (28) and using the fact that $\operatorname*{Res}_{p'=\mathcal{P}_i}\frac{dy}{dx}(p')B(p',p)=dy(\mathcal{P}_i)B(\mathcal{P}_i,p)$. Note that if the poles of dy are dominated by the poles of dx, equivalently dy/dx has poles only at the zeros of dx, then (28) is satisfied as a consequence of the Cauchy formula. This allows us to produce spectral curves $S=(C,x,y,B)$ that give rise to $R(z)$ and $\mathbb{1}$—see Example 4.15 below.

The result of [10] was generalised in [5] to show that the differential operators $\hat{\Psi}$, $\hat{R}$ and $\hat{T}_0$ acting on copies of Z^{BGW} arises by applying topological recursion to an irregular spectral curve. Equivalently, periods of the correlators of an irregular spectral curve store linear combinations of coefficients of $\log Z^{\mathrm{BGW}}$. The appearance of Z^{BGW} is due to its relationship with topological recursion applied to the curve $x=\frac{1}{2}z^2$, $y=\frac{1}{z}$ [6].

4.4.2. *Examples.* We demonstrate topological recursion with four key examples of rational spectral curves equipped with the bidifferential $B(p_1,p_2)$ given by the Cauchy kernel. The spectral curves in the Examples 4.12 and 4.13, denoted S_{Airy} and S_{Bes}, have partition functions Z^{KW} and Z^{BGW} respectively. Any spectral curve at regular, respectively irregular, zeros of dx is locally isomorphic to S_{Airy}, respectively S_{Bes}. A consequence is that the tau functions Z^{KW} and Z^{BGW} are fundamental to the correlators produced from topological recursion. Moreover, they are built out of $Z^{\rm KW}$ and $Z^{\rm BGW}$ via Givental reconstruction described in Section 4.1 where R and T are obtained from the spectral curve as described in Section 4.4.1.

Example 4.14 with spectral curve S_{A_2} has partition function Z_{A_2} corresponding to the A_2 Frobenius manifold—defined in (19). Example 4.15 with spectral curve $S^{\rm BGW}_{A_2}$ has partition function $Z^{\rm BGW}_{A_2}$ which will be shown to have vanishing primary terms for $n\leq g-1$ which correspond to relations among coefficients of $Z^{\rm BGW}$. In each example we use a global rational parameter z for the curve $C\cong\mathbb{C}$.

Example 4.12. *Topological recursion applied to the Bessel curve*

$$S_{Bes}=\left(\mathbb{C},x=\frac{1}{2}z^2,\ y=\frac{1}{z},\ B=\frac{dzdz'}{(z-z')^2}\right)$$

produces correlators

$$\omega^{Bes}_{g,n}=\sum_{\vec{k}\in\mathbb{Z}^n_+}C_g(k_1,...,k_n)\prod_{i=1}^n(2k_i+1)!!\frac{dz_i}{z_i^{2k_i+2}}$$

where $C_g(k_1,...,k_n)\neq 0$ only for $\sum_{i=1}^n k_i=g-1$. Define $\xi_k(z)=(2k+1)!!z^{-(2k+2)}dz$. It is proven in [6] *that*

$$Z^{BGW}(\hbar,t_0,t_1,...)=\exp\sum_{g,n}\frac{\hbar^{g-1}}{n!}\omega^{Bes}_{g,n}\Big|_{\xi_k(z_i)=t_k}=\exp\sum_{g,n,\vec{k}}\frac{\hbar^{g-1}}{n!}C_g(k_1,...,k_n)\prod t_{k_j}.$$

Example 4.13. *Topological recursion applied to the Airy curve*

$$S_{Airy}=\left(\mathbb{C},x=\frac{1}{2}z^2,\ y=z,\ B=\frac{dzdz'}{(z-z')^2}\right)$$

produces correlators which are proven in [15] *to store intersection numbers*

$$\omega^{Airy}_{g,n}=\sum_{\vec{k}\in\mathbb{Z}^n_+}\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^n\psi_i^{k_i}(2k_i+1)!!\frac{dz_i}{z_i^{2k_i+2}}$$

and the coefficient is non-zero only for $\sum_{i=1}^n k_i=3g-3+n$. Hence

$$Z^{KW}(\hbar,t_0,t_1,...)=\exp\sum_{g,n}\frac{\hbar^{g-1}}{n!}\omega^{Airy}_{g,n}\Big|_{\xi_k(z_i)=t_k}=\exp\sum_{g,n,\vec{k}}\frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^n\psi_i^{k_i}\prod t_{k_j}.$$

For the next two examples, define a collection of differentials $\xi^\alpha_k(z)$ on $\mathbb{C}$ for $\alpha\in\{1,2\}$, $k\in\{0,1,2,...\}$ by

$$\xi^\alpha_0=\frac{dz}{(1-z)^2}-(-1)^\alpha\frac{dz}{(1+z)^2},\quad \xi^\alpha_{k+1}(p)=d\left(\frac{\xi^\alpha_k(p)}{dx(p)}\right),\ \alpha=1,2,\quad k=0,1,2,... \tag{29}$$

These are linear combinations of the $V^i_k(p)$ defined in (25) with $x=z^3-3z$. The $V^i_k(p)$ correspond to normalised canonical coordinates while the $\xi^\alpha_k(p)$ correspond to flat coordinates.

Example 4.14. *Consider the spectral curve*

$$S_{A_2}=\left(\mathbb{C},x=z^3-3z,\ y=z\sqrt{-3},\ B=\frac{dzdz'}{(z-z')^2}\right).$$

Example 4.9 shows that $(\mathbb{C},x=z^3-3z,B)$ produces the $R(z)$ associated to the A_2 Frobenius manifold at the point $(u_1,u_2)=(2,-2)$ calculated in Example 4.3.

$$R^{-1}(z)=I-\frac{1}{144}\begin{pmatrix}-1&-6i\\-6i&1\end{pmatrix}z+\frac{35}{41472}\begin{pmatrix}-1&-12i\\12i&-1\end{pmatrix}z^2+...$$

It remains to show that y gives rise to $1\!\!1$ and $R^{-1}(z)1\!\!1$. The local expansions of $dy=\sqrt{-3}dz$ around $z=-1=\mathcal{P}_1$ and $z=1=\mathcal{P}_2$ in the respective local coordinates t and s, such that $x(t)=\frac{1}{2}t^2+2$ and $x(s)=\frac{1}{2}s^2-2$ are:

$$dy=\sqrt{-3}dz=\left(\frac{1}{\sqrt{2}}-\frac{5}{144\sqrt{2}}t^2+\frac{385}{124416\sqrt{2}}t^4+...+\text{odd terms}\right)dt$$

$$dy=\left(\frac{i}{\sqrt{2}}+\frac{5i}{144\sqrt{2}}s^2+\frac{385i}{124416\sqrt{2}}s^4+...+\text{odd terms}\right)ds$$

hence the Laplace transforms are:

$$\frac{1}{\sqrt{2\pi z}}\int_{\Gamma_1} dy(p)\cdot e^{\frac{(u_1-x(p))}{z}} = \frac{1}{\sqrt{2}} - \frac{5}{144\sqrt{2}}z + \frac{385}{41472\sqrt{2}}z^2 + \ldots$$

$$\frac{1}{\sqrt{2\pi z}}\int_{\Gamma_2} dy(p)\cdot e^{\frac{(u_2-x(p))}{z}} = \frac{i}{\sqrt{2}} + \frac{5i}{144\sqrt{2}}z + \frac{385i}{41472\sqrt{2}}z^2 + \ldots$$

which indeed gives

$$\left\{\frac{1}{\sqrt{2\pi z}}\int_{\Gamma_k} dy(p)\cdot e^{\frac{(u_k-x(p))}{z}}\right\} = R^{-1}(z)\mathbb{1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} + \frac{5}{144\sqrt{2}}\begin{pmatrix}-1\\ i\end{pmatrix}z + \frac{385}{41472\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}z^2 + \ldots$$

Note that $dy(\mathcal{P}_1) = \frac{1}{\sqrt{2}} = \Delta_1^{1/2}$ and $dy(\mathcal{P}_2) = \frac{i}{\sqrt{2}} = \Delta_2^{1/2}$—the first terms in the expansions in t and s above—gives the unit $\mathbb{1}$ (and the TFT). These first terms are enough to produce the entire local expansions of dy, and hence prove that the series for dy gives $R^{-1}(z)\mathbb{1}$. The poles of dy are dominated by the poles of dx, i.e. dy/dx has poles only at the zeros $\mathcal{P}_1$ and $\mathcal{P}_2$ of dx, hence dy satisfies (28) so its Laplace transform satisfies (26) as required.

Apply topological recursion to S_{A_2} to produce correlators $\omega_{g,n}^{A_2}$. Then the partition function associated to S_{A_2} coincides with the partition function of the A_2 Frobenius manifold:

$$Z^{A_2}(\hbar,\{t_k^\alpha\}) = \exp\sum_{g,n}\frac{\hbar^{g-1}}{n!}\omega_{g,n}^{A_2}\bigg|_{\xi_k^\alpha(z_i)=t_k^\alpha}.$$

This was first proven in [8] via the superpotential construction of Dubrovin [7].

The coefficients of $\log Z^{A_2}(\hbar,\{t_k^\alpha\})$ are obtained from $\omega_{g,n}$ by

$$\frac{\partial^n}{\partial t_{k_1}^{\alpha_1}\ldots\partial t_{k_n}^{\alpha_n}}F_g^{A_2}(\{t_k^\alpha\})\bigg|_{t_k^\alpha=0} = \operatorname*{Res}_{z_1=\infty}\ldots\operatorname*{Res}_{z_n=\infty}\prod_{i=1}^n p_{\alpha_i,k_i}(z_i)\omega_{g,n}(z_1,\ldots,z_n) \tag{30}$$

for the polynomials in z given by $p_{\alpha,k}(z) = \sqrt{-3}\frac{(-1)^\alpha}{\alpha}z^{3k+\alpha}$ + lower order terms. The lower order terms (and the top coefficient) will not be important here because we will only consider vanishing of (30) arising from high enough order vanishing of $\omega_{g,n}(z_1,\ldots,z_n)$ at $z_i=\infty$ so that the integrand in (30) is holomorphic at $z_i=\infty$. Equation (30) is a special case of the more general phenomena, proven in [9], that periods of $\omega_{g,n}$ are dual to insertions of vectors in a CohFT. In this case it follows from the easily verified fact that the residues are dual to the differentials ξ_k^α defined in (29).

Recall that vanishing of coefficients of $\log Z^{A_2}(\hbar,\{t_k^\alpha\})$ correspond to relations among top intersection classes of tautological relations. The correlators $\omega_{g,n}^{A_2}$ are holomorphic at $z=\infty$ (their only poles occur at $z=\pm1$) and in fact have high order vanishing there. The high order vanishing at $z=\infty$ together with (30) proves vanishing of some coefficients of $\log Z^{A_2}(\hbar,\{t_k^\alpha\})$. For example:

$$\omega_{2,1}^{A_2}(z) = \frac{35}{243}\frac{z(11z^4+14z^2+2)}{(z^2-1)^{10}}dz \quad\Rightarrow\quad \operatorname*{Res}_{z=\infty} z^m\omega_{2,1}^{A_2}(z) = 0,\quad m\in\{0,1,2,\ldots,13\}$$

Hence (30) vanishes for $k_1 = 0,1,2,3$ which gives relations between intersection numbers

$$\int_{\overline{\mathcal{M}}_{2,1}} R^{(m)}\psi_1^{4-m} = 0,\quad m = 1,2,3,4$$

where $R^{(m)}$ is a relation between cohomology classes in $H^{2m}(\overline{\mathcal{M}}_{2,1})$ proven in [31], such as $R^{(2)} = \psi_1^2+$ boundary terms $=0$.

Note that if one rescales $y\mapsto\lambda y$ then the correlators rescale by $\omega_{g,n}^{A_2}\mapsto\lambda^{2-2g-n}\omega_{g,n}^{A_2}$ so the change does not affect the vanishing terms. The coefficient $\sqrt{-3}$ is chosen for convenience to get precise agreement with the associated topological field theory and Frobenius manifold.

The next example produces $Z_{A_2}^{\mathrm{BGW}}$, defined in (21), which we recall replaces factors of Z^{KW} with Z^{BGW} in the partition function Z_{A_2} of the A_2 Frobenius manifold.

Example 4.15. *The following spectral curve $S^{BGW}_{A_2}$ shares (C, x, B) with the spectral curve S_{A_2}, since this produces the correct operator $R(z)$ used in the construction of both Z_{A_2} and $Z^{BGW}_{A_2}$. We replace the function y in S_{A_2} by dy/dx which produces the required shift on the Laplace transform.*

$$S^{BGW}_{A_2} = \left(\mathbb{C}, x = z^3 - 3z,\ y = \frac{\sqrt{-3}}{3z^2 - 3},\ B = \frac{dzdz'}{(z - z')^2}\right).$$

As in the previous example, we have:

$$R^{-1}(z) = I - \frac{1}{144}\begin{pmatrix} -1 & -6i \\ -6i & 1 \end{pmatrix} z + \frac{35}{41472}\begin{pmatrix} -1 & -12i \\ 12i & -1 \end{pmatrix} z^2 + \dots$$

The local expansions of $dy = \frac{2}{\sqrt{-3}}\frac{zdz}{(z^2-1)^2}$ around $z = -1 = \mathcal{P}_1$ and $z = 1 = \mathcal{P}_2$ in the respective local coordinates t and s, such that $x(t) = \frac{1}{2}t^2 + 2$ and $x(s) = \frac{1}{2}s^2 - 2$ are:

$$dy = \frac{2}{\sqrt{-3}}\frac{zdz}{(z^2-1)^2} = \left(-\frac{1}{\sqrt{2}}t^{-2} - \frac{5}{144\sqrt{2}} + \frac{385}{41472\sqrt{2}}t^2 + \dots + \text{odd terms}\right) dt$$

$$dy = \left(-\frac{i}{\sqrt{2}}s^{-2} + \frac{5i}{144\sqrt{2}} + \frac{385i}{41472\sqrt{2}}s^2 + \dots + \text{odd terms}\right) ds$$

and the Laplace transforms are a shift of the Laplace transforms in the previous example:

$$\frac{1}{\sqrt{2\pi z}}\int_{\Gamma_1} dy(p)\cdot e^{\frac{(u_1 - x(p))}{z}} = \frac{1}{\sqrt{2}}z^{-1} - \frac{5}{144\sqrt{2}} + \frac{385}{41472\sqrt{2}}z + \dots$$

$$\frac{1}{\sqrt{2\pi z}}\int_{\Gamma_2} dy(p)\cdot e^{\frac{(u_2 - x(p))}{z}} = \frac{i}{\sqrt{2}}z^{-1} + \frac{5i}{144\sqrt{2}} + \frac{385i}{41472\sqrt{2}}z + \dots$$

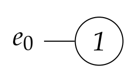
Apply topological recursion to $S^{BGW}_{A_2}$ to produce correlators $\omega_{g,n}$ and partition function $Z^{S^{BGW}_{A_2}}$. It is proven in [5] that topological recursion applied to an irregular spectral curve, i.e. y has simple poles at the zeros of dx, produces a partition function Z^S with translation term encoded in the Laplace transform of dy and in this case $T_0(z) = \mathbb{1} - R^{-1}(z)\mathbb{1}$. This is exactly the construction of $Z^{BGW}_{A_2}$ defined in Section 4.1 hence we have:

$$Z^{BGW}_{A_2}(\hbar, \{t^\alpha_k\}) = Z^{S^{BGW}_{A_2}}(\hbar, \{t^\alpha_k\}).$$

Lemma 4.16 below proves that the sums of the orders of vanishing of $\omega^{BGW,A_2}_{g,n}(z_1, \dots, z_n)$ at $z_i = \infty$ is bounded below by $2g - 2$. Explicitly in low genus,

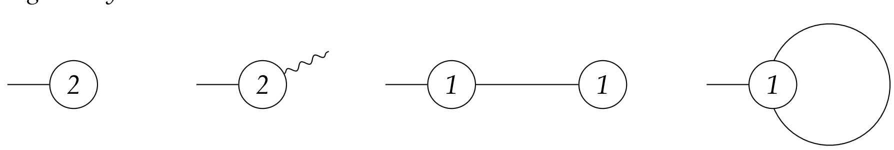
$$\omega^{BGW,A_2}_{1,1}(z) = \frac{z^2 + 1}{4\sqrt{-3}(z^2 - 1)^2}dz, \quad \omega^{BGW,A_2}_{2,1}(z) = \frac{-5z^2 - 1}{16\sqrt{-3}(z-1)^4(z+1)^4}dz.$$

We find that $-\operatorname*{Res}_{z=\infty}\sqrt{-3}\cdot z\cdot\omega_{1,1}(z) = \frac{1}{4}$ agrees with the graphical expansion



which contributes $2^g \cdot \int_{\overline{\mathcal{M}}_{1,1}} \Theta_{1,1} = 2\cdot\frac{1}{8} = \frac{1}{4}$.

From the vanishing of $\omega_{2,1}(z)$ at $z = \infty$ it immediately follows that $\operatorname*{Res}_{z=\infty}\frac{\sqrt{-3}}{2}z\cdot\omega_{2,1}(z) = 0$ which signifies a relation between coefficients of $Z^{BGW}(\hbar, t_0, t_1, \dots)$. We will write the relations using $\Theta_{g,n}$ however the relations are between coefficients of $Z^{BGW}(\hbar, t_0, t_1, \dots)$ and what we are showing here is that these coefficients satisfy the same relations as intersection numbers involving $\Theta_{g,n}$, or equivalently coefficients of $Z^{\Theta}(\hbar, t_0, t_1, \dots)$. The graphical expansion of this is given by:



(plus graphs containing genus 0 vertices on which $\Theta_{2,1}$ vanishes) which contributes

$$2^2\cdot\frac{60}{1728}\cdot\int_{\overline{\mathcal{M}}_{2,1}}\Theta_{2,1}\cdot\psi_1+2^2\cdot\frac{-60}{1728}\cdot\int_{\overline{\mathcal{M}}_{2,1}}\Theta_{2,1}\cdot\kappa_1+2^2\cdot\frac{84}{1728}\cdot\int_{\overline{\mathcal{M}}_{1,2}}\Theta_{1,2}\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}+\frac{2}{2}\cdot\frac{84-60}{1728}\cdot\int_{\overline{\mathcal{M}}_{1,3}}\Theta_{1,3}$$

which agrees with the expansion in weighted graphs of $\operatorname*{Res}_{z=\infty}\dfrac{\sqrt{-3}}{2}z\cdot\omega_{2,1}(z)=0$ *given by*

$$\frac{5}{1536}-\frac{15}{1536}+\frac{7}{2304}+\frac{1}{288}=0.$$

4.4.3. *Intersection numbers and the Brezin-Gross-Witten tau function.* We are finally in a position to prove that $Z^{\rm BGW}(\hbar,t_0,t_1,...)$ is a generating function for the intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{k_i}$, or

$$Z^{\rm BGW}(\hbar,t_0,t_1,...)=Z^{\Theta}(\hbar,t_0,t_1,...)$$

stated as Theorem 3 below. *A priori* the coefficients of $Z^{\rm BGW}_{A_2}$ have nothing to do with integration of cohomology classes over $\overline{\mathcal{M}}_{g,n}$, nevertheless we will see that $Z^{\rm BGW}_{A_2}$ and $Z^{\Theta}_{A_2}$ share the same relations which will be used to prove their coincidence. This relies on the spectral curve $x=z^3-3z$, $y=\frac{1}{\sqrt{-3}(z^2-1)}$ analysed in Example 4.15.

Proof of Theorem 3. The proof is a rather beautiful application of Pixton's relations proven by Pandharipande, Pixton and Zvonkine, [31]. We use Pixton's relations to induce relations among the intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}$. The key idea of the proof of the theorem is to show that the coefficients of $Z^{\rm BGW}$ satisfy the same relations as those, such as (10), induced by Pixton's relations on the intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}$. The proof here brings together all of the results of Section 4.

Recall from Section 4.3 that the partition functions $Z^{\Theta}_{A_2}(\hbar,\{t^\alpha_k\})$ and $Z^{\rm BGW}_{A_2}(\hbar,\{t^\alpha_k\})$ have the same structure since they are built out of the same element $R(z)$, arising from the A_2 Frobenius manifold, and the same translation $T_0(z)=1\!\!1-R^{-1}(z)1\!\!1$, but with *a priori* different vertex contributions. Vanishing coefficients in both partition functions produce the same relations between coefficients of $Z^{\Theta}(\hbar,t_0,t_1,...)$, respectively $Z^{\rm BGW}(\hbar,t_0,t_1,...)$. Enough of these relations will prove that $Z^{\Theta}(\hbar,t_0,t_1,...)=Z^{\rm BGW}(\hbar,t_0,t_1,...)$. Vanishing of certain coefficients of $Z^{\Theta}_{A_2}(\hbar,\{t^\alpha_k\})$ is a consequence of the cohomological viewpoint—it comes from $\Theta_{g,n}R^{(g-1)}=0$ for Pixton relations $0=R^{(g-1)}\in H^{2g-2}(\overline{\mathcal{M}}_{g,n})$. The vanishing of corresponding coefficients of $Z^{\rm BGW}_{A_2}(\hbar,\{t^\alpha_k\})$ uses topological recursion applied to the spectral curve $S^{\rm BGW}_{A_2}$, defined in Example 4.15, which produces the partition function $Z^{\rm BGW}_{A_2}(\hbar,\{t^\alpha_k\})$ out of correlators $\omega^{{\rm BGW},A_2}_{g,n}(z_1,...,z_n)$. We know that $\omega^{{\rm BGW},A_2}_{g,n}(z_1,...,z_n)$ is holomorphic at $z_i=\infty$ for each $i=1,...,n$. The next lemma shows its order of vanishing there.

Lemma 4.16.

$$\sum_{i=1}^n ord_{z_i=\infty}\,\omega^{BGW,A_2}_{g,n}(z_1,...,z_n)\geq 2g-2$$

where $ord_{z=\infty}\,\eta(z)$ is the order of vanishing the differential at $z=\infty$.

Proof. We can make the rational differential

$$\omega^{A_2}_{g,n}(z_1,...,z_n)=\frac{p_{g,n}(z_1,....,z_n)}{\prod_{i=1}^n(z_i^2-1)^{2g}}dz_1...dz_n$$

homogeneous by applying topological recursion to $x(z)=z^3-3Q^2z$ and $y=\sqrt{-3}/x'(z)$ which are homogeneous in z and Q. Then $\omega^{A_2}_{g,n}(Q,z_1,...,z_n)$ is homogeneous in z and Q of degree $2-2g-n$:

$$\omega^{A_2}_{g,n}(Q,z_1,...,z_n)=\lambda^{2-2g-n}\omega^{A_2}_{g,n}(\lambda Q,\lambda z_1,...,\lambda z_n).$$

The degree of homogeneity uses the fact that $(z,Q)\mapsto(\lambda z,\lambda Q)\Rightarrow ydx\mapsto\lambda ydx\Rightarrow\omega_{g,n}\mapsto\lambda^{2-2g-n}\omega_{g,n}$ because ydx appears in the kernel $K(p_1,p)$ with homogeneous degree -1 which easily leads to degree

$2-2g-n$ for $\omega_{g,n}$. The degree $2-2g-n$ homogeneity of

$$\omega_{g,n}^{A_2}(Q,z_1,...,z_n)=\frac{p_{g,n}(Q,z_1,....,z_n)}{\prod_{i=1}^n(z_i^2-Q^2)^{2g}}dz_1...dz_n$$

implies that $\deg p_{g,n}(Q,z_1,....,z_n)=4gn-n+2-2g-n$ so $\deg p_{g,n}(z_1,....,z_n)\leq 4gn-n+2-2g-n$. Note that dz_i is homogeneous of degree 1 but has a pole of order 2 at $z_i=\infty$, hence

$$\sum_{i=1}^n \mathrm{ord}_{z_i=\infty}\,\omega_{g,n}^{\mathrm{BGW},A_2}(z_1,...,z_n)=4gn-\deg p_{g,n}(z_1,....,z_n)-2n\geq 2g-2.$$

□

The primary coefficients of $Z^{\mathrm{BGW}}_{A_2}(\hbar,\{t^\alpha_k\})$, those where $k=0$, correspond to

$$\operatorname*{Res}_{z_1=\infty}...\operatorname*{Res}_{z_n=\infty}\prod_{i=1}^n z_i^{\epsilon_i}\omega_{g,n}^{A_2}(z_1,...,z_n)$$

for $\epsilon_i=1$ or 2. Different choices of ϵ_i give different relations (except half which vanish for parity reasons). We have:

Corollary 4.17.

$$n<2g-2\Rightarrow \operatorname*{Res}_{z_1=\infty}...\operatorname*{Res}_{z_n=\infty}\prod_{i=1}^n z_i^{\epsilon_i}\omega_{g,n}^{A_2}(z_1,...,z_n)=0$$

Proof. Since $n<2g-2$ and $\sum_{i=1}^n \mathrm{ord}_{z_i=\infty}\,\omega_{g,n}^{\mathrm{BGW},A_2}(z_1,...,z_n)\geq 2g-2$ by Lemma 4.16, there exists an i such that $\mathrm{ord}_{z_i=\infty}\,\omega_{g,n}^{\mathrm{BGW},A_2}(z_1,...,z_n)\geq 2$. Hence $z_i^{\epsilon_i}\omega_{g,n}^{A_2}(z_1,...,z_n)$ is holomorphic at $z_i=\infty$, so

$$\operatorname*{Res}_{z_i=\infty} z_i^{\epsilon_i}\omega_{g,n}^{A_2}(z_1,...,z_n)=0$$

and the multiple residue vanishes as required. □

Hence the primary coefficients of $Z^{\mathrm{BGW}}_{A_2}(\hbar,\{t^\alpha_k\})$ vanish for $n<2g-2$ proving that the coefficients of $Z^\Theta(\hbar,t_0,t_1,...)$ and $Z^{\mathrm{BGW}}(\hbar,t_0,t_1,...)$ for $n<2g-2$ satisfy the same relations. Pixton's relations give enough relations among coefficients of $Z^\Theta(\hbar,t_0,t_1,...)$ to calculate them, and hence the same relations also uniquely determine the coefficients of $Z^{\mathrm{BGW}}(\hbar,t_0,t_1,...)$ and we have $Z^\Theta(\hbar,t_0,t_1,...)=Z^{\mathrm{BGW}}(\hbar,t_0,t_1,...)$ up to coefficients with $n<2g-2$. The dilaton equation (5) then proves equality of coefficients for all n giving $Z^\Theta(\hbar,t_0,t_1,...)=Z^{\mathrm{BGW}}(\hbar,t_0,t_1,...)$.

□

5. Cohomological field theories

The class $\Theta_{g,n}$ combines with known enumerative invariants, such as Gromov-Witten invariants, to give rise to new invariants. More generally, $\Theta_{g,n}$ pairs with any cohomological field theory, which is fundamentally related to the moduli space of curves $\overline{\mathcal{M}}_{g,n}$, retaining many of the properties of the cohomological field theory, and is in particular often calculable.

A *cohomological field theory* is a pair (V,η) composed of a finite-dimensional complex vector space V equipped with a metric η and a sequence of S_n-equivariant maps.

$$\Omega_{g,n}:V^{\otimes n}\to H^*(\overline{\mathcal{M}}_{g,n})$$

that satisfy compatibility conditions from inclusion of strata:

$$\phi_{\mathrm{irr}}:\overline{\mathcal{M}}_{g-1,n+2}\to\overline{\mathcal{M}}_{g,n},\quad \phi_{h,I}:\overline{\mathcal{M}}_{h,|I|+1}\times\overline{\mathcal{M}}_{g-h,|J|+1}\to\overline{\mathcal{M}}_{g,n},\quad I\sqcup J=\{1,...,n\}$$

given by

$$\phi_{\mathrm{irr}}^*\Omega_{g,n}(v_1\otimes...\otimes v_n)=\Omega_{g-1,n+2}(v_1\otimes...\otimes v_n\otimes\Delta) \tag{31}$$

$$\phi_{h,I}^*\Omega_{g,n}(v_1\otimes...\otimes v_n)=\Omega_{h,|I|+1}\otimes\Omega_{g-h,|J|+1}\Big(\bigotimes_{i\in I}v_i\otimes\Delta\otimes\bigotimes_{j\in J}v_j\Big) \tag{32}$$

where $\Delta \in V \otimes V$ is dual to the metric $\eta \in V^* \otimes V^*$. When $n = 0$, $\Omega_g := \Omega_{g,0} \in H^*(\overline{\mathcal{M}}_g)$. There exists a vector $1\!\!1 \in V$ compatible with the forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ by

$$\Omega_{g,n+1}(1\!\!1 \otimes v_1 \otimes ... \otimes v_n) = \pi^*\Omega_{g,n}(v_1 \otimes ... \otimes v_n) \tag{33}$$

for $2g - 2 + n > 0$, and

$$\Omega_{0,3}(1\!\!1 \otimes v_1 \otimes v_2) = \eta(v_1, v_2).$$

For a one-dimensional CohFT, i.e. $\dim V = 1$, identify $\Omega_{g,n}$ with the image $\Omega_{g,n}(1\!\!1^{\otimes n})$, so we write $\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n})$. A trivial example of a CohFT is $\Omega_{g,n} = 1 \in H^0(\overline{\mathcal{M}}_{g,n})$ which is a topological field theory as we now describe.

A two-dimensional topological field theory (TFT) is a vector space V and a sequence of symmetric linear maps

$$\Omega^0_{g,n} : V^{\otimes n} \to \mathbb{C}$$

for integers $g \geq 0$ and $n > 0$ satisfying the following conditions. The map $\Omega^0_{0,2} = \eta$ defines a metric η, and together with $\Omega^0_{0,3}$ it defines a product $\bullet$ on V via

$$\eta(v_1 \bullet v_2, v_3) = \Omega^0_{0,3}(v_1, v_2, v_3) \tag{34}$$

with identity $1\!\!1$ given by the dual of $\Omega^0_{0,1} = 1\!\!1^* = \eta(1\!\!1, \cdot)$. It satisfies

$$\Omega^0_{g,n+1}(1\!\!1 \otimes v_1 \otimes ... \otimes v_n) = \Omega^0_{g,n}(v_1 \otimes ... \otimes v_n)$$

and the gluing conditions

$$\Omega^0_{g,n}(v_1 \otimes ... \otimes v_n) = \Omega^0_{g-1,n+2}(v_1 \otimes ... \otimes v_n \otimes \Delta) = \Omega^0_{g_1,|I|+1} \otimes \Omega^0_{g_2,|J|+1}\Big(\bigotimes_{i \in I} v_i \otimes \Delta \otimes \bigotimes_{j \in J} v_j\Big)$$

for $g = g_1 + g_2$ and $I \sqcup J = \{1, ..., n\}$.

Consider the natural isomorphism $H^0(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{C}$. The degree zero part of a CohFT $\Omega_{g,n}$ is a TFT:

$$\Omega^0_{g,n} : V^{\otimes n} \xrightarrow{\Omega_{g,n}} H^*(\overline{\mathcal{M}}_{g,n}) \to H^0(\overline{\mathcal{M}}_{g,n}).$$

We often write $\Omega_{0,3} = \Omega^0_{0,3}$ interchangeably. Associated to $\Omega_{g,n}$ is the product (34) built from η and $\Omega_{0,3}$.

Given a CohFT $\Omega = \{\Omega_{g,n}\}$ and a basis $\{e_1, ..., e_N\}$ of V, the partition function of Ω is defined as in (19).

$$Z_\Omega(\hbar, \{t^\alpha_k\}) = \exp \sum_{g,n,\vec{k}} \frac{\hbar^{g-1}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(e_{\alpha_1} \otimes ... \otimes e_{\alpha_n}) \cdot \prod_{j=1}^n \psi_j^{k_j} \prod t^{\alpha_j}_{k_j} \tag{35}$$

for $\alpha_i \in \{1, ..., N\}$ and $k_j \in \mathbb{N}$.

Remark 5.1. The class $\Theta_{g,n}$ defined in the introduction satisfies properties (31) and (32) of a one-dimensional CohFT. In place of property (33), it satisfies $\Theta_{g,n+1}(1\!\!1 \otimes v_1 \otimes ... \otimes v_n) = \psi_{n+1} \cdot \pi^*\Theta_{g,n}(v_1 \otimes ... \otimes v_n)$ and $\Theta_{0,3} = 0$.

Definition 5.2. For any CohFT Ω defined on (V, η) define $\Omega^\Theta = \{\Omega^\Theta_{g,n}\}$ to be the sequence of S_n-equivariant maps $\Omega^\Theta_{g,n} : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$ given by $\Omega^\Theta_{g,n}(v_1 \otimes ... \otimes v_n) = \Theta_{g,n} \cdot \Omega_{g,n}(v_1 \otimes ... \otimes v_n)$.

This is essentially to the tensor products of CohFTs, albeit involving $\Theta_{g,n}$. The tensor products of CohFTs is obtained as above by cup product on $H^*(\overline{\mathcal{M}}_{g,n})$, generalising Gromov-Witten invariants of target products and the Künneth formula $H^*(X_1 \times X_2) \cong H^*X_1 \otimes H^*X_2$.

Generalising Remark 5.1, $\Omega^\Theta_{g,n}$ satisfies properties (31) and (32) of a CohFT on (V, η). In place of property (33), it satisfies

$$\Omega^\Theta_{g,n+1}(1\!\!1 \otimes v_1 \otimes ... \otimes v_n) = \psi_{n+1} \cdot \pi^*\Omega^\Theta_{g,n}(v_1 \otimes ... \otimes v_n)$$

and $\Omega^\Theta_{0,3} = 0$.

The product defined in (34) is *semisimple* if it is diagonal $V \cong \mathbb{C} \oplus \mathbb{C} \oplus ... \oplus \mathbb{C}$, i.e. there is a canonical basis $\{u_1, ..., u_N\} \subset V$ such that $u_i \cdot u_j = \delta_{ij} u_i$. The metric is then necessarily diagonal with respect to the same basis, $\eta(u_i, u_j) = \delta_{ij}\eta_i$ for some $\eta_i \in \mathbb{C} \setminus \{0\}$, $i = 1, ..., N$. The Givental-Teleman theorem [22, 35] states that the twisted loop group action defined in Section 4.1 acts transitively on semisimple CohFTs. In particular, a semisimple homogeneous CohFT is uniquely determined by its underlying TFT. The tau function Z^{BGW} appears in a generalisation of Givental's decomposition of CohFTs [5] which, combined with the result $Z^{\mathrm{BGW}}=Z^{\Theta}$, generalises Givental's action on CohFTs to allow one to replace the TFT term by the classes $\Theta_{g,n}$.

5.1. **Gromov-Witten invariants.** Let X be a projective algebraic variety and consider $(C,x_1,\dots,x_n)$ a connected smooth curve of genus g with n distinct marked points. For $\beta\in H_2(X,\mathbb{Z})$ the moduli space of stable maps $\mathcal{M}^g_n(X,\beta)$ is defined by:

$$\mathcal{M}_{g,n}(X,\beta)=\{(C,x_1,\dots,x_n)\xrightarrow{\pi}X\mid \pi_*[C]=\beta\}/\sim$$

where π is a morphism from a connected nodal curve C containing distinct points $\{x_1,\dots,x_n\}$ that avoid the nodes. Any genus zero irreducible component of C with fewer than three distinguished points (nodal or marked), or genus one irreducible component of C with no distinguished point, must not be collapsed to a point. We quotient by isomorphisms of the domain C that fix each x_i. The moduli space of stable maps has irreducible components of different dimensions but it has a virtual class of dimension

$$\dim[\mathcal{M}_{g,n}(X,\beta)]^{\mathrm{virt}}=(\dim X-3)(1-g)+\langle c_1(X),\beta\rangle+n.$$

For $i=1,\dots,n$ there exist evaluation maps:

$$ev_i:\mathcal{M}_{g,n}(X,\beta)\longrightarrow X,\quad ev_i(\pi)=\pi(x_i)\tag{36}$$

and classes $\gamma\in H^*(X,\mathbb{Z})$ pull back to classes in $H^*(\mathcal{M}_{g,n}(X,\beta),\mathbb{Q})$

$$ev_i^*:H^*(X,\mathbb{Z})\longrightarrow H^*(\mathcal{M}_{g,n}(X,\beta),\mathbb{Q}).\tag{37}$$

The forgetful map $p:\mathcal{M}_{g,n}(X,\beta)\to\overline{\mathcal{M}}_{g,n}$ maps a stable map to its domain curve followed by contraction of unstable components. The push-forward map p_* on cohomology defines a CohFT Ω_X on the even part of the cohomology $V=H^{\mathrm{even}}(X;\mathbb{C})$ (and a generalisation of a CohFT on $H^*(X;\mathbb{C})$) equipped with the metric

$$\eta(\alpha,\beta)=\int_X\alpha\wedge\beta.$$

We have $(\Omega_X)_{g,n}:H^{\mathrm{even}}(X)^{\otimes n}\to H^*(\overline{\mathcal{M}}_{g,n})$ defined by

$$(\Omega_X)_{g,n}(\alpha_1,\dots\alpha_n)=\sum_\beta p_*\left(\prod_{i=1}^n ev_i^*(\alpha_i)\cap[\mathcal{M}_{g,n}(X,\beta)]^{\mathrm{virt}}\right)\in H^*(\overline{\mathcal{M}}_{g,n}).$$

Note that it is the dependence of $p=p(g,n,\beta)$ on β (which is suppressed) that allows $(\Omega_X)_{g,n}(\alpha_1,\dots\alpha_n)$ to be composed of different degree terms. The partition function of the CohFT Ω_X with respect to a chosen basis e_α of $H^{\mathrm{even}}(X;\mathbb{C})$ is

$$Z_{\Omega_X}(\hbar,\{t^\alpha_k\})=\exp\sum_{\substack{g,n,\vec{k}\\ \vec{\alpha},\beta}}\frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}p_*\left(\prod_{i=1}^n ev_i^*(e_{\alpha_i})\cap[\mathcal{M}_{g,n}(X,\beta)]^{\mathrm{virt}}\right)\cdot\prod_{j=1}^n\psi_j^{k_j}\prod t^{\alpha_j}_{k_j}.$$

It stores *ancestor* invariants. These are different to *descendant* invariants which use in place of $\psi_j=c_1(L_j)$, $\Psi_j=c_1(\mathcal{L}_j)$ for line bundles $\mathcal{L}_j\to\mathcal{M}_{g,n}(X,\beta)$ defined as the cotangent bundle over the ith marked point.

Following Definition 5.2, we define Ω_X^Θ by

$$(\Omega_X^\Theta)_{g,n}(\alpha_1,\dots\alpha_n)=\Theta_{g,n}\cdot\sum_\beta p_*\left(\prod_{i=1}^n ev_i^*(\alpha_i)\right)\in H^*(\overline{\mathcal{M}}_{g,n}).$$

and

$$Z^\Theta_{\Omega_X}(\hbar,\{t^\alpha_k\})=\exp\sum_{\substack{g,n,\vec{k}\\ \vec{\alpha},\beta}}\frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\cdot p_*\left(\prod_{i=1}^n ev_i^*(e_{\alpha_i})\right)\cdot\prod_{j=1}^n\psi_j^{k_j}\prod t^{\alpha_j}_{k_j}.$$

The invariants Ω_X^Θ have not been defined directly from a moduli space $\mathcal{M}^\Theta_{g,n}(X,\beta)$, say obtained by restricting the domain of a stable map to a $g-1$-dimensional subvariety $X_{g,n}\subset\overline{\mathcal{M}}_{g,n}$ arising as the push-forward of the zero set of a section of the bundle $E_{g,n}$ defined in Definition 2.1. Nevertheless, it is instructive for comparison purposes to write the virtual dimension of this yet to be defined $\mathcal{M}^{\Theta}_{g,n}(X,\beta)$. We have

$$\dim[\mathcal{M}^{\Theta}_{g,n}(X,\beta)]^{\mathrm{virt}} = (\dim X - 1)(1-g) + \langle c_1(X), \beta\rangle.$$

Again we emphasise that the invariants of X stored in $Z^{\Theta}_{\Omega_X}(\hbar, \{t^{\alpha}_k\})$ are rigorously defined, and the purpose of the dimension formula is for a comparison with usual Gromov-Witten invariants. We see that the virtual dimension is independent of n. Elliptic curves now take the place of Calabi-Yau 3-folds to give virtual dimension zero moduli spaces, independent of genus and degree. The invariants of a target curve X are trivial when the genus of X is greater than 1 and computable when $X = \mathbb{P}^1$. For $c_1(X) = 0$ and $\dim X > 1$, the invariants vanish for $g > 1$, while for $g = 1$ it seems to predict an invariant associated to maps of elliptic curves to X.

5.1.1. *Weil-Petersson volumes.* A fundamental example of a 1-dimensional CohFT is given by

$$\Omega_{g,n} = \exp(2\pi^2\kappa_1) \in H^*(\overline{\mathcal{M}}_{g,n}).$$

Its partition function stores Weil-Petersson volumes

$$V_{g,n} = \frac{(2\pi^2)^{3g-3+n}}{(3g-3+n)!}\int_{\overline{\mathcal{M}}_{g,n}} \kappa_1^{3g-3+n}$$

and deformed Weil-Petersson volumes studied by Mirzakhani [28]. Weil-Petersson volumes of the subvariety of $\overline{\mathcal{M}}_{g,n}$ dual to $\Theta_{g,n}$ make sense even before we find such a subvariety. They are given by

$$V^{\Theta}_{g,n} = \frac{(2\pi^2)^{g-1}}{(g-1)!}\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n}\cdot\kappa_1^{g-1}$$

which are calculable since they are given by a translation of Z^{BGW}. If we include ψ classes, we get polynomials $V^{\Theta}_{g,n}(L_1,...,L_n)$ which give the deformed volumes analogous to Mirzakhani's volumes.

5.1.2. *ELSV formula.* Another example of a 1-dimensional CohFT is given by

$$\Omega_{g,n} = c(E^{\vee}) = 1 - \lambda_1 + ... + (-1)^g\lambda_g \in H^*(\overline{\mathcal{M}}_{g,n})$$

where $\lambda_i = c_i(E)$ is the ith Chern class of the Hodge bundle $E \to \overline{\mathcal{M}}_{g,n}$ defined to have fibres $H^0(\omega_C)$ over a nodal curve C.

Hurwitz [23] studied the problem of connected curves Σ of genus g covering $\mathbb{P}^1$, branched over $r+1$ fixed points $\{p_1, p_2, ..., p_r, p_{r+1}\}$ with arbitrary profile $\mu = (\mu_1, ..., \mu_n)$ over p_{r+1}. Over the other r branch points one specifies simple ramification, i.e. the partition $(2,1,1,....)$. The Riemann-Hurwitz formula determines the number r of simple branch points via $2-2g-n = |\mu| - r$.

Definition 5.3. Define the simple Hurwitz number $H_{g,\mu}$ to be the weighted count of genus g connected covers of $\mathbb{P}^1$ with ramification $\mu = (\mu_1, ..., \mu_n)$ over ∞ and simple ramification elsewhere. Each cover π is counted with weight $1/|\mathrm{Aut}(\pi)|$.

Coefficients of the partition function of the CohFT $\Omega_{g,n} = c(E^{\vee})$ appear naturally in the ELSV formula [12] which relates the Hurwitz numbers $H_{g,\mu}$ to the Hodge classes. The ELSV formula is:

$$H_{g,\mu} = \frac{r(g,\mu)!}{|\mathrm{Aut}\,\mu|}\prod_{i=1}^{n}\frac{\mu_i^{\mu_i}}{\mu_i!}\int_{\overline{\mathcal{M}}_{g,n}} \frac{1-\lambda_1+...+(-1)^g\lambda_g}{(1-\mu_1\psi_1)...(1-\mu_n\psi_n)}$$

where $\mu = (\mu_1, ..., \mu_n)$ and $r(g,\mu) = 2g-2+n+|\mu|$.

Using $\Omega^{\Theta}_{g,n} = \Theta\cdot c(E^{\vee})$ we can define an analogue of the ELSV formula:

$$H^{\Theta}_{g,\mu} = \frac{(2g-2+n+|\mu|)!}{|\mathrm{Aut}\,\mu|}\prod_{i=1}^{n}\frac{\mu_i^{\mu_i}}{\mu_i!}\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n}\cdot\frac{1-\lambda_1+...+(-1)^{g-1}\lambda_{g-1}}{(1-\mu_1\psi_1)...(1-\mu_n\psi_n)}.$$

It may be that $H^{\Theta}_{g,\mu}$ has an interpretation of enumerating simple Hurwitz covers. Note that it makes sense to set all $\mu_i = 0$, and in particular there are non-trivial primary invariants over $\overline{\mathcal{M}}_g$, unlike for simple Hurwitz numbers. An example calculation:

$$\int_{\overline{\mathcal{M}}_2} \Theta_2\lambda_1 = \frac{1}{5}\cdot\frac{1}{8}\cdot\frac{1}{8}\cdot\frac{1}{2}+\frac{1}{10}\cdot\frac{1}{8}\cdot\frac{1}{2}=\frac{1}{128} \qquad \Leftarrow \quad \lambda_1=\frac{1}{10}(2\delta_{1,1}+\delta_{\rm irr}).$$

Appendix A. Calculations

Here we show explicitly the equality $Z^{\rm BGW}=Z^{\Theta}$ up to genus 3. The coefficients of the Brezin-Gross-Witten tau function are calculated recursively since it is a tau function of the KdV hierarchy. It has low genus g (= coefficient of $\hbar^{g-1}$) terms given by:

$$\begin{aligned}\log Z^{\rm BGW} &= -\frac{1}{8}\log(1-t_0)+\hbar\frac{3}{128}\frac{t_1}{(1-t_0)^3}+\hbar^2\frac{15}{1024}\frac{t_2}{(1-t_0)^5}+\hbar^2\frac{63}{1024}\frac{t_1^2}{(1-t_0)^6}+O(\hbar^8)\\ &=\frac{1}{8}t_0+\frac{1}{16}t_0^2+\frac{1}{24}t_0^3+...+\hbar(\frac{3}{128}t_1+\frac{9}{128}t_0t_1+...)+\hbar^2(\frac{15}{1024}t_2+\frac{63}{1024}t_1^2+...)+...\end{aligned}$$

The intersection numbers of $\Theta_{g,n}$ stored in

$$\log Z^{\Theta}(\hbar,t_0,t_1,...)=\sum_{g,n,\vec{k}}\frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\cdot\prod_{j=1}^n\psi_j^{k_j}\prod t_{k_j}$$

are calculated recursively using relations among tautological classes in $H^*(\overline{\mathcal{M}}_{g,n})$. The calculation of these intersection numbers up to genus 2 can be found throughout the text. We assemble them here for convenience, then present the genus 3 calculations.

$\boxed{g = 0}$ Theorem 1 property (II) gives $\Theta_{0,n}=0$ which agrees with the vanishing of all genus 0 terms in $Z^{\rm BGW}$.

$\boxed{g = 1}$ Proposition 2.10 gives $\Theta_{1,1} = 3\psi_1$ hence $\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}=\frac{1}{8}$. We use this together with the dilaton equation to get $\int_{\overline{\mathcal{M}}_{1,n}}\Theta_{1,n}=\frac{(n-1)!}{8}$. This agrees with $-\frac{1}{8}\log(1-t_0)$ in $\log Z^{\rm BGW}$.

$\boxed{g = 2}$ Using Mumford's relation [30] κ_1 = sum of boundary terms in $\overline{\mathcal{M}}_2$ which coincides with a genus 2 Pixton relation, Example 3.5 produced the genus 2 intersection numbers from the genus 1 intersection numbers.

$$\begin{aligned}\int_{\overline{\mathcal{M}}_2}\Theta_2\cdot\kappa_1 &= \frac{7}{5}\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}\cdot\frac{1}{|{\rm Aut}(\Gamma_1)|}+\frac{1}{5}\cdot\int_{\overline{\mathcal{M}}_{1,2}}\Theta_{1,2}\cdot\frac{1}{|{\rm Aut}(\Gamma_2)|}\\ &=\frac{7}{5}\cdot\frac{1}{8}\cdot\frac{1}{8}\cdot\frac{1}{2}+\frac{1}{5}\cdot\frac{1}{8}\cdot\frac{1}{2}=\frac{3}{128}.\end{aligned}$$

Note that $\int_{\overline{\mathcal{M}}_{2,1}}\Theta_{2,1}\cdot\psi_1=\int_{\overline{\mathcal{M}}_{2,1}}\pi^*\Theta_2\cdot\psi_1^2=\int_{\overline{\mathcal{M}}_2}\Theta_2\cdot\kappa_1$. Using the dilaton equation we then get $\int_{\overline{\mathcal{M}}_{2,n}}\Theta_{2,n}\cdot\psi_1=\frac{3(n+1)!}{256}$ which agrees with the $\hbar\frac{3}{128}\frac{t_1}{(1-t_0)^3}$ term in $\log Z^{\rm BGW}$.

$\boxed{g = 3}$ There are two independent genus 3 Pixton relations expressing κ_2 and κ_1^2 as sums of boundary terms in $\overline{\mathcal{M}}_3$. The relations correspond to sums over stable graphics in $\overline{\mathcal{M}}_3$ hence they contain many terms. In place of these, we use the equivalent relations discovered earlier in [24, 25] which push forward to relations in $\overline{\mathcal{M}}_3$. In $\overline{\mathcal{M}}_{3,1}$ there is a relation ψ_1^3 = sum of boundary terms, which yields

$$\begin{aligned}&\int_{\overline{\mathcal{M}}_{3,1}}\Theta_{3,1}\cdot\psi_1^2=\int_{\overline{\mathcal{M}}_{3,1}}\pi^*\Theta_3\cdot\psi_1^3\\ &\quad=\frac{41}{21}\cdot\int_{\overline{\mathcal{M}}_{2,1}}\Theta_{2,1}\cdot\psi_1\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}+\frac{5}{42}\cdot\int_{\overline{\mathcal{M}}_{2,2}}\Theta_{2,2}\cdot\psi_1-\frac{1}{105}\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}\cdot\int_{\overline{\mathcal{M}}_{1,3}}\Theta_{1,3}\cdot\frac{1}{|{\rm Aut}|}\\ &\qquad+\frac{11}{70}\cdot\int_{\overline{\mathcal{M}}_{1,2}}\Theta_{1,2}\cdot\int_{\overline{\mathcal{M}}_{1,2}}\Theta_{1,2}\cdot\frac{1}{|{\rm Aut}|}-\frac{4}{35}\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}\cdot\int_{\overline{\mathcal{M}}_{1,2}}\Theta_{1,2}\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}\\ &\qquad-\frac{1}{105}\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}\cdot\int_{\overline{\mathcal{M}}_{1,3}}\Theta_{1,3}\cdot\frac{1}{|{\rm Aut}|}-\frac{1}{1260}\cdot\int_{\overline{\mathcal{M}}_{1,4}}\Theta_{1,4}\cdot\frac{1}{|{\rm Aut}|}\end{aligned}$$

$$
\begin{aligned}
=&\frac{41}{21}\cdot\frac{3}{128}\cdot\frac{1}{8}+\frac{5}{42}\cdot\frac{9}{128}-\frac{1}{105}\cdot\frac{1}{8}\cdot\frac{2}{8}\cdot\frac{1}{2}+\frac{11}{70}\cdot\frac{1}{8}\cdot\frac{1}{8}\cdot\frac{1}{2}-\frac{4}{35}\cdot\frac{1}{8}\cdot\frac{1}{8}\cdot\frac{1}{8}-\frac{1}{105}\cdot\frac{1}{8}\cdot\frac{2}{8}\cdot\frac{1}{2}\\
&-\frac{1}{1260}\cdot\frac{6}{8}\cdot\frac{1}{4}\\
=&\frac{15}{1024}
\end{aligned}
$$

In $\overline{\mathcal{M}}_{3,2}$, there is a relation $\psi_1^2\psi_2-\psi_1\psi_2^2=$ sum of boundary terms, which yields

$$
\begin{aligned}
7\int_{\overline{\mathcal{M}}_{3,2}}\Theta_{3,2}\cdot(\psi_1^2-\psi_1\psi_2)=&\,7\int_{\overline{\mathcal{M}}_{3,2}}\pi^*\Theta_{3,1}\cdot(\psi_1^2\psi_2-\psi_1\psi_2^2)\\
=&-\frac{16}{3}\cdot\int_{\overline{\mathcal{M}}_{2,2}}\Theta_{2,2}\cdot\psi_2\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}-5\int_{\overline{\mathcal{M}}_{2,2}}\Theta_{2,2}\cdot\psi_1\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}\\
&-\frac{40}{3}\cdot\int_{\overline{\mathcal{M}}_{2,1}}\Theta_{2,1}\cdot\psi_1\cdot\int_{\overline{\mathcal{M}}_{1,2}}\Theta_{1,2}-\frac{1}{6}\cdot\int_{\overline{\mathcal{M}}_{2,3}}\Theta_{2,3}\cdot\psi_1-\int_{\overline{\mathcal{M}}_{2,3}}\Theta_{2,3}\cdot\psi_1\cdot\frac{1}{|\mathrm{Aut}|}\\
&-\frac{1}{15}\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}\cdot\int_{\overline{\mathcal{M}}_{1,4}}\Theta_{1,4}\cdot\frac{1}{|\mathrm{Aut}|}-\frac{9}{10}\cdot\int_{\overline{\mathcal{M}}_{1,3}}\Theta_{1,3}\cdot\int_{\overline{\mathcal{M}}_{1,2}}\Theta_{1,2}\\
&-\frac{1}{15}\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}\cdot\int_{\overline{\mathcal{M}}_{1,4}}\Theta_{1,4}\cdot\frac{1}{|\mathrm{Aut}|}+\frac{4}{15}\cdot\int_{\overline{\mathcal{M}}_{1,2}}\Theta_{1,2}\cdot\int_{\overline{\mathcal{M}}_{1,3}}\Theta_{1,3}\cdot\frac{1}{|\mathrm{Aut}|}\\
&-\frac{4}{5}\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}\cdot\int_{\overline{\mathcal{M}}_{1,3}}\Theta_{1,3}\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}+\frac{16}{5}\cdot\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}\cdot\int_{\overline{\mathcal{M}}_{1,2}}\Theta_{1,2}\cdot\int_{\overline{\mathcal{M}}_{1,2}}\Theta_{1,2}\\
&-\frac{1}{180}\cdot\int_{\overline{\mathcal{M}}_{1,5}}\Theta_{1,5}\cdot\frac{1}{|\mathrm{Aut}|}\\
=&-\frac{16}{3}\cdot\frac{9}{128}\cdot\frac{1}{8}-5\frac{9}{128}\cdot\frac{1}{8}-\frac{40}{3}\cdot\frac{3}{128}\cdot\frac{1}{8}-\frac{1}{6}\cdot\frac{36}{128}-\frac{36}{128}\cdot\frac{1}{2}-\frac{1}{15}\cdot\frac{1}{8}\cdot\frac{6}{8}\cdot\frac{1}{2}\\
&-\frac{9}{10}\cdot\frac{2}{8}\cdot\frac{1}{8}-\frac{1}{15}\cdot\frac{1}{8}\cdot\frac{6}{8}\cdot\frac{1}{2}+\frac{4}{15}\cdot\frac{1}{8}\cdot\frac{2}{8}\cdot\frac{1}{2}-\frac{4}{5}\cdot\frac{1}{8}\cdot\frac{2}{8}\cdot\frac{1}{8}+\frac{16}{5}\cdot\frac{1}{8}\cdot\frac{1}{8}\cdot\frac{1}{8}-\frac{1}{180}\cdot\frac{24}{8}\cdot\frac{1}{4}\\
=&-\frac{357}{1024}
\end{aligned}
$$

Hence

$$
\int_{\overline{\mathcal{M}}_{3,2}}\Theta_{3,2}\cdot\psi_1\psi_2=\int_{\overline{\mathcal{M}}_{3,2}}\Theta_{3,2}\cdot\psi_1^2+\frac{1}{7}\frac{357}{1024}=\frac{75}{1024}+\frac{51}{1024}=\frac{63}{512}
$$

where $\int_{\overline{\mathcal{M}}_{3,2}}\Theta_{3,2}\cdot\psi_1^2=\frac{75}{1024}$ is obtained from $\int_{\overline{\mathcal{M}}_{3,1}}\Theta_{3,1}\cdot\psi_1^2=\frac{15}{1024}$ via the dilaton equation. The dilaton equation then yields $\int_{\overline{\mathcal{M}}_{3,n}}\Theta_{3,n}\cdot\psi_1^2=\frac{75}{1024}\frac{(n+3)!}{5!}$ and $\int_{\overline{\mathcal{M}}_{3,n}}\Theta_{3,n}\cdot\psi_1\psi_2=\frac{63}{512}\frac{(n+3)!}{5!}$ which agree with the $\hbar^2\frac{15}{1024}\frac{t_2}{(1-t_0)^5}+\hbar^2\frac{63}{1024}\frac{t_1^2}{(1-t_0)^6}$ terms in $\log Z^{\mathrm{BGW}}$.

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