

APPENDIX A

Examples

These are the appendices to the course “Notes on Geometry and 3-Manifolds” given by Walter Neumann at the Turán Workshop on Low Dimensional Topology in Budapest, August 1998. The notes for that course can be found on

<http://neumann.maths.mu.oz.au/preprints.html>.

The following examples were presented during tutorials at the Turán Workshop.

1. Trefoil complement

We will put two geometric structures on the trefoil complement. Since the trefoil is a torus knot its complement is Seifert fibred with an orbifold quotient F given by the disk with two orbifold points of orders 2 and 3. Thus $\chi(F) = 1 - (1 - 1/2) - (1 - 1/3) = -1/6$ so F admits a hyperbolic structure. We can realise this explicitly with the orbifold $F \cong PSL(2, \mathbb{Z}) \backslash \mathbb{H}^2$. Now pull back the metric using the map $S^3 - K \rightarrow F$. This construction is not so explicit and in fact we can pull back the metric in different ways to get $\mathbb{H}^2 \times \mathbb{R}$ structures or a PSL structure.

A better way to see things is as follows. We use the fact that the trefoil is a fibred knot. This can be seen from the fundamental group

$$\begin{aligned} \pi_1(S^3 - K) &= \{a_1, a_2, a_3 \mid a_1 a_2 = a_2 a_3 = a_3 a_1\} \\ &\cong \{a, x, y \mid a^{-1} x a = y, a^{-1} y a = x^{-1} y\} \end{aligned}$$

or as the link of the singularity $Y^2 + X^3 = 0$. We can perturb this to $Y^2 = -X^3 + \epsilon$ which describes the Milnor fibre, or the Seifert surface S , of the trefoil as a branched cover of the disk with branch points $X_0, \zeta X_0, \zeta^2 X_0$ where $\zeta^3 = 1$. This makes the holonomy of the fibration $S^3 - K \rightarrow S^1$ clear—it is given by rotation of the disk by $2\pi/3$.

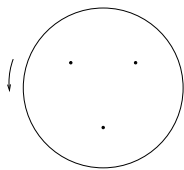


FIGURE 1

Thus if we have a hyperbolic metric on the orbifold given by the disk with degree 2 orbifold points at the branch points that is invariant under the holonomy this gives an explicit $\mathbb{H}^2 \times \mathbb{R}$ structure on $S^3 - K$. Notice that if we quotient S by the $\mathbb{Z}/3$ action we get the orbifold F described above. The hyperbolic metric on F lifts to a hyperbolic metric on S invariant under rotation by $2\pi/3$ so we see the $\mathbb{H}^2 \times \mathbb{R}$ structure explicitly.

We can also get the $\mathbb{P}SL$ structure as follows. If we put a left invariant metric on $PSL(2, \mathbb{R})$ then it pushes down to the quotient $PSL(2, \mathbb{Z}) \backslash PSL(2, \mathbb{R})$. This quotient is the complement of the trefoil. We can see this by studying the action of $SO(2)$ on the right and seeing that the quotient $PSL(2, \mathbb{Z}) \backslash PSL(2, \mathbb{R}) / SO(2)$ is a disk with two orbifold points of degrees 2 and 3. By the classification of Seifert fibred spaces this must be the trefoil complement. Notice that the Seifert surface is totally geodesic in the $\mathbb{H}^2 \times \mathbb{R}$ case, but not so in the $\mathbb{P}SL$ case.

This example is one of the few cases when two types of geometries can be put on the manifold. We cannot put a hyperbolic structure on the trefoil complement due to the following general fact about Seifert fibred spaces. A fibre of the Seifert fibring gives a normal \mathbb{Z} in the fundamental group of the trefoil complement. Thus, the square of any element of the fundamental group commutes with the element represented by a fibre. This gives two parabolic elements and hence a cusp. But there are more commuting parabolic elements and they also define the same cusp which contradicts the fact that a lattice in the boundary \mathbb{C} cannot have more than two generators.

2. JSJ decomposition

We will give the JSJ decomposition of the link of the singularity

$$f = (x^2 + y^3)^2 + xy^5 + z^5.$$

The plumbing graph of this link is given in the following appendix. It is equivalent to the splice diagram in figure 2. For this example it is sufficient to understand

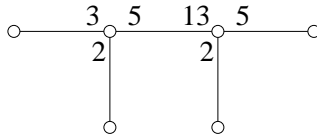


FIGURE 2

the following fact about splice diagrams. The simplest type of splice diagram is of the form in figure 3 which represents a Seifert fibred space with three singular

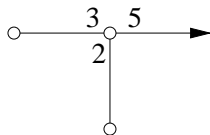


FIGURE 3

fibres of multiplicities 2, 3 and 5, respectively. The arrow indicates that we wish to specify the singular fibre of multiplicity 5 as a link component in the Seifert fibred space. (In this case the link is a knot.) The splice diagram in figure 2 is obtained by “splicing” together the Seifert fibred spaces represented in figures 3 and 4 along the two specified knots. We splice by removing a neighbourhood of each knot and identifying the respective boundary tori where the longitude and meridian of one torus is mapped to the meridian and longitude of the other torus.

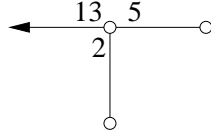


FIGURE 4

This is exactly the canonical decomposition of the three-manifold into two Seifert fibred spaces. They can each be given either the geometry $\mathbb{H} \times \mathbb{R}$ or PSL .

3. Ideal tetrahedra and the invariant trace field

We will show that the invariant trace field of an ideally triangulated hyperbolic manifold with cusps is generated by its tetrahedra parameters.

Given M^3 non-compact and ideally triangulated denote its set of cusps by $C \subset \partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$. Define k_Δ to be the field generated by the simplex parameters of the ideal tetrahedra. Take one of the ideal tetrahedra of M^3 and put three of the vertices of the tetrahedron at $0, 1$ and ∞ .

(i) First we will show that $C \subset k_\Delta \cup \{\infty\}$ and C generates $k_\Delta \cup \{\infty\}$.

This is proved by induction using the fact that if three of z_1, \dots, z_4 are in $k_\Delta \cup \{\infty\}$ then the last one is in $k_\Delta \cup \{\infty\}$ if and only if the cross ratio is in $k_\Delta \cup \{\infty\}$. Thus, we begin with the tetrahedron with three vertices at $0, 1$ and ∞ and use the preceding fact to see that the fourth vertex lies in $k_\Delta \cup \{\infty\}$. Any ideal tetrahedron that shares a face with this tetrahedron now has three of its vertices in $k_\Delta \cup \{\infty\}$ so the induction continues to this tetrahedron. Thus $C \subset k_\Delta \cup \{\infty\}$ and C generates $k_\Delta \cup \{\infty\}$ since each simplex parameter is the cross ratio of four cusp points.

Thus it is sufficient to show that $k_\Delta = k(\Gamma)$.

(ii) $k_\Delta \subset k(\Gamma)$.

Since $\Gamma \subset k(\Gamma) \cdot \bar{\Gamma} = Q(\Gamma)$, the quaternion algebra associated to Γ , then the existence of parabolic elements implies that $Q(\Gamma) \subset M(2, \mathbb{C})$ has zero divisors— if P is parabolic then $(P - I)^2 = 0$ —so $Q(\Gamma) \cong M(2, k(\Gamma))$. Therefore we can conjugate Γ to get a subgroup of $PGL(2, k(\Gamma))$. Notice that if $g \in PGL(2, k(\Gamma))$ then $g^2/\det(g) \in PSL(2, k(\Gamma))$ and thus we can conjugate $\Gamma^{(2)}$ to a subset of $PSL(2, k(\Gamma))$. Now, a cusp is a solution of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

where the matrix lies in $PGL(2, k(\Gamma))$ and in fact by using the square of an element of Γ we may assume that the matrix lies in $PSL(2, k(\Gamma))$. Therefore $x/y \in k(\Gamma)$.

(iii) If Γ is a group fixing a subset of $k_\Delta \cup \{\infty\}$ then $\Gamma \subset PGL(2, k_\Delta)$.

We can see this simply by observing that the following equation:

$$\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{matrix} \gamma(0) = \alpha \\ \gamma(1) = \beta \\ \gamma(\infty) = \gamma \end{matrix} \right) \in k_\Delta \cup \{\infty\}.$$

has a solution in k_Δ :

$$b = \alpha d, \quad a + b = \beta c + \beta d, \quad a = c\gamma.$$

Thus, $\Gamma \subset PGL(2, k_\Delta)$ and $\Gamma^{(2)} \subset PSL(2, k_\Delta)$ so it follows that $k(\Gamma) \subset k_\Delta$.

4. Scissors congruence

Here we will prove that two Euclidean polygons are scissors congruent if and only if they have the same area.

Given two polygons of equal area we can reduce the problem of scissors congruence to two equal area triangles as follows. Assume we can solve the equal area triangle problem. Cut the two polygons into triangles and take a triangle from each polygon. If they have equal area then we use the solution of the equal area triangle problem to cut the two triangles into equal pieces. If they have unequal area then cut the bigger one into two triangles one of which has the same area as the triangle from the other polygon. (We can do this by the intermediate value theorem.) Now cut the two equal area triangles into identical pieces. This has reduced the problem. In the first case the total number of triangles is reduced by two and in the second case the number is reduced by one.

Scissors congruence is an equivalence relation since we can cut a polygon further to prove transitivity. Thus to solve the equal area triangle problem it is enough to show that any triangle is scissors congruent to the square of equal area. From the diagram we see how to go from a triangle to a rectangle.

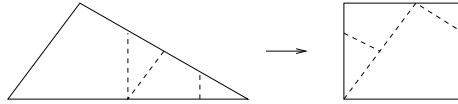


FIGURE 5

To go from the rectangle to the square we cut the rectangle as shown until the ratio of the two sides is not greater than two.

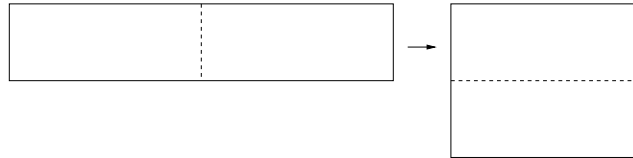


FIGURE 6

Then the following diagram gives the equivalence with the square.

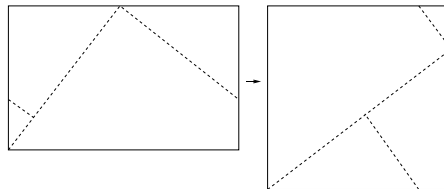


FIGURE 7

APPENDIX B

Problems

The following set of problems were used during the Turán Workshop. They are sometimes answered in the course notes and the worked examples.

1. Geometries

QUICK

(i) Take a family of non-conjugate representations of $\pi_1 \Sigma$ into $SL(2, \mathbb{R})$. Does Mostow-Prasad rigidity imply these representations are conjugate inside $SL(2, \mathbb{C})$?

(ii) Find all integer solutions $\{p_1, \dots, p_s\}$ of

$$2 - \sum_1^s (1 - 1/p_i) = 0 \quad (= \chi(F)).$$

(These give all flat compact genus 0 orbifolds, where the $\{p_i\}$ give the multiplicities of the orbifold points.)

(iii) Put a hyperbolic structure on the surface of genus g .

UNDERSTANDING

(i) Describe the complement of the trefoil in terms of the link of the singularity $y^2 = x^3$ and hence describe the holonomy.

(ii) Put geometries on each of: the trefoil complement; figure-8 complement.

(iii) Why is the type of the geometry that can be put on a closed manifold unique? (Hint: look for topological invariants.)

(iv) Describe the isometry groups of the 8 geometries in dimension 3.

(v) Describe the space of geodesics for each geometry.

THOUGHTFUL

(i) Given any two Euclidean polygons of the same area, show that you can cut one of them into a finite number of Euclidean polygons and reassemble the pieces to get the other polygon.

(ii) We have seen that the trefoil complement admits a geometric structure of type $H^2 \times \mathbb{R}$ and also of type $\mathbb{P}SL$. How can we see that it does *not* admit a hyperbolic structure?

2. Decomposition

QUICK

(i) What does an incompressible torus in Y^3 imply about the fundamental group $\pi_1 Y^3$? If the incompressible torus is boundary parallel what does this say in terms of the fundamental group?

(ii) Give an example to show what is wrong with this definition:

an embedded annulus $(A, \partial A) \hookrightarrow (Y, \partial Y)$ is *boundary parallel* if there exists an annulus $\tilde{A} \subset \partial Y$ with $\partial \tilde{A} = \partial A$ and $A \cup \tilde{A} = \partial H$ for an embedded solid torus $H \hookrightarrow Y$ with $H \cap \partial Y = \tilde{A}$.

(iii) Describe canonical circles in two dimensions.

UNDERSTANDING

(i) Find the maximal set of canonical tori, and hence the JSJ-decomposition, of the following 3-manifolds: trefoil complement; figure-8 complement; trefoil cabled on figure-8 complement; link of the singularity

$$f = (x^2 + y^3)^2 + xy^5 + z^5.$$

(Hint: use the algorithm described in Nemethi's course to verify that the resolution of the singularity has plumbing graph

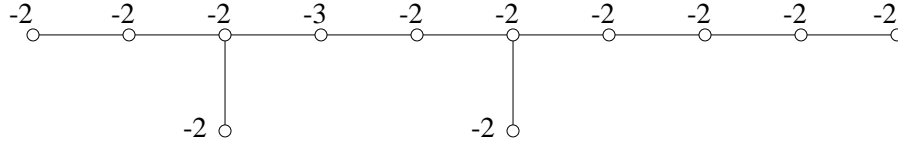


FIGURE 1

where each curve is a \mathbb{P}^1 . Since the link is given by the boundary of a neighbourhood of the divisor and hence a circle bundle over the smooth points of the divisor any embedded circle in the divisor pulls back to a torus.)

Here are examples of decompositions where uniqueness fails:

(ii) Connect sum in two dimensions: prove that $\mathbb{R}\mathbb{P}^2 \# \mathbb{R}\mathbb{P}^2 \# \mathbb{R}\mathbb{P}^2 \cong T^2 \# \mathbb{R}\mathbb{P}^2$.

(iii) Connect sum in three dimensions: $Y \cong M_1 \# M_2 \# \dots \# M_k$ decomposes an orientable 3-manifold Y into prime manifolds M_i , unique up to order. What is not unique here?

(iv) Connect sum of non-orientable 3-manifolds: give an example to show non-uniqueness.

(v) Connect sum in four dimensions: give an example to show non-uniqueness.

(vi) What is the key point in the proof of uniqueness of JSJ-decompositions that fails in the previous three cases. Show this explicitly with the example of the blown up projective plane and torus (in (ii) above).

(vii) Show that every Seifert fibred space has a normal \mathbb{Z} in its fundamental group and thus a Seifert fibred space does not admit a hyperbolic structure.

(viii) Put a geometry on the torus with one orbifold point.

THOUGHTFUL

(i) Canonical decomposition of maps of a surface: given a diffeomorphism $f : \Sigma \rightarrow \Sigma$, we say an embedded circle $S^1 \hookrightarrow \Sigma$ has finite order if some iterate of f takes the circle to an isotopic circle. Define a *canonical* circle to be a finite order circle with the property that all other finite order circles can be isotoped off the canonical circle. Use this to get a canonical decomposition of (Σ, f) and show that it is unique.

(ii) Given any two hyperbolic or spherical polygons of the same area, show that you can cut one of them into a finite number of hyperbolic, respectively spherical, polygons and reassemble the pieces to get the other polygon.

(iii) Prove that if both $\cos \theta$ and θ/π are rational then θ is a multiple of $\pi/2$ or $\pi/3$.

Scissors congruence**QUICK**

- (i) Show that scissors congruence is an equivalence relation.
- (ii) Show $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \cong \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q}$.
- (iii) Calculate the Dehn invariant of the Euclidean polyhedron $\delta(\Sigma \times I)$ for any 2-dimensional polygon Σ .
- (iv) What is the Dehn invariant in dimension 2 (Euclidean, hyperbolic and spherical) and why doesn't it obstruct scissors congruence?

UNDERSTANDING

- (i) Prove that Euclidean polygons of the same area are scissors congruent:
 - (a) show that it is enough to prove this for two triangles;
 - (b) show that a triangle is scissors congruent to a rectangle;
 - (c) show that any rectangle is scissors congruent to a square;
 - (d) conclude the equivalence of any two polygons of equal area.
- (ii) Prove that the Dehn invariant doesn't vanish on a regular tetrahedron and create your own example with $\delta \neq 0$.
- (iii) Prove: $\text{im}(vol|_{\ker \delta}) = \mathbb{R}$ and $\text{im}(\delta|_{\ker vol}) = \text{im}(\delta)$.
- (iv) Recall that scissors congruence for finite area hyperbolic polygons with ideal vertices is allowed. Show that stable scissors congruence is not the same as scissors congruence. Show that two polygons are scissors congruent if and only if they have the same area and they are either both compact or both non-compact and stably scissors congruent if and only if they have the same area.

THOUGHTFUL

- (i) Where does the argument that Euclidean polygons of the same area are scissors congruent (in Understanding(i) above) fail for hyperbolic and spherical geometry? Can it be fixed?
- (ii) Where does the argument that Euclidean polygons of the same area are scissors congruent (in Understanding(i) above) fail in dimension 3? Does it succeed on a limited class of polyhedra?
- (iii) Show that for polygons P and Q , $2P \sim 2Q$ implies $P \sim Q$. Since this is a general fact in any dimension and geometry, one can argue without using the equality of area.
- (iv) Define a Dehn invariant in dimension 4. Is it unique?

Commensurability**QUICK**

- (i) Show that all ideal triangles in \mathbb{H}^2 are isometric.
- (ii) Show that not all ideal tetrahedra in \mathbb{H}^3 are isometric.
- (iii) Construct a rotation of \mathbb{R}^2 that acts on $\mathbb{Z}[i]$ giving a commensurable lattice. Show that a dense set of rotations has this property.

UNDERSTANDING

- (i) Show that the group of symmetries of the tessellation of \mathbb{H}^3 by ideal tetrahedra is arithmetic.
- (ii) Show that $Comm^+(PSL(2, \mathbb{Z}[i])) = PGL(2, \mathbb{Q}[i])$.
- (iii) Prove that there are exactly two quaternion algebras over \mathbb{R} .
- (iv) Show that the figure 8 knot and the Whitehead link are arithmetic.

THOUGHTFUL

- (i) Does arithmeticity exist in dimension 2?

Bloch group.**QUICK**

(i) Show explicitly that the figure 8 knot lies in the kernel of the complex Dehn invariant. (Hint: the parameter for the figure 8 knot is given by $2[\zeta]$ where $\zeta^6 = 1$.)

(ii) Does hyperbolic Dehn surgery on the figure 8 knot change its scissors congruence class in the Bloch group, $\beta_0(Y) \in \mathcal{B}(\mathbb{C})$?

(iii) Are the hyperbolic Dehn surgeries on the figure 8 knot arithmetic?

UNDERSTANDING

(i) Show that the invariant trace field of an ideally triangulated hyperbolic manifold with cusps is generated by its tetrahedra parameters.

(ii) Show that two hyperbolic 3-manifolds with imaginary quadratic trace field are scissors congruent commensurable if and only if they have the same invariant trace fields.

(iii) Prove that the five-term scissors congruence relation implies that $[z] = [1 - 1/z]$.