# Periodic instantons and the loop group

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**Abstract**

We construct a large class of periodic instantons. Conjecturally we produce all periodic instantons. This confirms a conjecture of Garland and Murray that relates periodic instantons to orbits of the loop group acting on an extension of its Lie algebra.

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## 1 Introduction

Periodic instantons are solutions of the anti-self-dual equations

$$F_B = -*F_B$$

for a connection $B$ on a trivial vector bundle with structure group $G$ over $S^1 \times \mathbf{R}^3$. In this paper, $G$ is a compact Lie group with complexification $G^c$ equipped with a representation acting on $\mathbf{C}^n$ that is unitary on $G$.

Put $B = A + \Phi d\theta$ so

$$*F_A = d_A\Phi - \mu\partial_\theta A \tag{1}$$

where we use the three-dimensional Hodge star operator and $\mu$ is the reciprocal of the radius of the circle. One can think of the connection and Higgs field as defined over $\mathbf{R}^3$ and dependent on the circle-valued $\theta$.

Nahm studied periodic instantons, calling them calorons [17]. Later, Garland and Murray studied periodic instantons from the twistor viewpoint [7]. To remedy the fact that there was so far no existence theorem for periodic instantons nor an understanding of the topology of the moduli space of instantons (if they were to exist), they conjectured that periodic instantons can be constructed using holomorphic spheres in a flag manifold associated to the loop group. This conjecture is confirmed by the main result of this paper, Theorem 1.

Recently the study of super-symmetric Yang-Mills theory over $S^1_{1/\mu} \times \mathbf{R}^3$ has been used as further evidence for the existence of dualities in physical theories. In [21] Seiberg and Witten obtained a result for periodic instantons analogous to their 1994 work on instantons, [20], by studying the limiting behaviour when $\mu \to 0$ and $\mu \to \infty$. This led to the Rozansky-Witten invariants, [19]. We will not discuss these developments here.

# 2 Loop groups.

Define $LG$ to be the group of smooth gauge transformations of the trivial $G$-bundle over the circle. Equivalently, $LG$ is the space of smooth maps from $S^1$ to the compact Lie group $G$. Following [7], intertwine the gauge transformations with the isometries of the circle to get the twisted product $\widehat{LG} = LG \tilde{\times} S^1$ where the action of $S^1$ is given by rotation. It has Lie algebra $\widehat{L\mathbf{g}} \cong L\mathbf{g} \oplus \mathbf{R}d$ with Lie bracket

$$[X + x\mathrm{d}, Y + y\mathrm{d}] = [X, Y] - y\partial X/\partial\theta + x\partial Y/\partial\theta.$$

Put $\hat{A} = A + a\mathrm{d}$, $\hat{\Phi} = \Phi + \phi\mathrm{d}$. Then the Bogomolny equations over $\mathbf{R}^3$ for this pair are given by

$$*F_{\hat{A}} = d_{\hat{A}}\hat{\Phi}. \tag{2}$$

The d-component is given by $*da = d\phi$ so a finite energy condition will force $a = 0$ and $\phi$ =constant= $\mu$, say. The remaining part of (2) is then (1). Thus, one can think of periodic instantons as monopoles over $\mathbf{R}^3$ with structure group $\widehat{LG}$.

Monopoles for finite-dimensional groups are well-studied [10, 16, 18]. In particular, the topology of the moduli space of monopoles is understood. The moduli space of monopoles with structure group $G$ is diffeomorphic to the space of holomorphic maps from the two-sphere to a homogeneous space of $G$, or equivalently to an adjoint orbit of $G$, [4, 12]. In analogy with the finite-dimensional case this led Garland and Murray to conjecture that periodic instantons are in one-to-one correspondence with based holomorphic maps from $S^2$ to orbits of $\widehat{LG}$ in $\widehat{L\mathbf{g}}$. The following theorem addresses half of this conjecture. The action of $\widehat{LG}$ is really an action of $LG$. For $(\xi, \mu) \in \widehat{L\mathbf{g}}$ denote its orbit by $LG \cdot (\xi, \mu)$.

**Theorem 1** *There is an injective map from*
*(i) the space of based holomorphic maps from $S^2$ to $LG \cdot (\xi, \mu)$, to*
*(ii) the moduli space of instantons over $S^1_{1/\mu} \times \mathbf{R}^3$.*

The basing condition on the space of holomorphic maps distinguishes an element of the orbit of $LG$ that is conjecturally the asymptotic value of the Higgs field. See Section 6. The moduli space consists of gauge equivalence classes of connections where the gauge group consists of gauge transformations independent of $\theta$ in the limit at infinity.

The full conjecture, that the map is also surjective, is equivalent to a conjecture for decay properties of finite energy periodic instantons analogous to known decay properties for monopoles. We discuss this in Section 6.

Theorem 1 can be thought of as an extension of [13] from finite dimensional Lie groups to the loop group.

## 2.1 Orbits of the loop group.

The loop group $LG$ acts on $\widehat{L\mathbf{g}}$ by

$$\gamma \cdot (\xi, \mu) = (\gamma \cdot \xi - \mu\gamma'\gamma^{-1}, \mu).$$

For $\xi = 0$ the orbit is given by the based loop group $\Omega G$. More generally, we get $LG \cdot (\xi, \mu) \cong LG/Z_\xi$ where the isotropy subgroup $Z_\xi$ is described explicitly in the following proposition.

**Proposition 2.1 (Pressley and Segal)** *For $\pi_1 G = 0$ and $\mu \neq 0$ the orbits of $LG$ on $\widehat{L\mathbf{g}}$ correspond precisely to the conjugacy classes of $G$ under the map $(\xi, \mu) \mapsto M_\xi \in G$ where $M_\xi$ is obtained by solving the ordinary differential equation $h'h^{-1} = -\mu^{-1}\xi$ and noticing $h(\theta + 2\pi) = h(\theta)M_\xi$. The isotropy subgroup of $\xi$ is given by*

$$Z_\xi = \{\gamma \in LG | \gamma(0) \in C[M_\xi], \gamma(\theta) = h(\theta)\gamma(0)h(\theta)^{-1}\} \tag{3}$$

*where $C[M_\xi]$ is the centraliser of the conjugacy class of $M_\xi$ in $G$.*

Equivalently, the orbits are given by gauge equivalence classes of connections on a trivialised bundle over the circle of radius $1/\mu$. Each orbit is labeled by the underlying connection which is determined by its holonomy.

In the next section we will equip the orbit of the loop group with a complex structure.

## 2.2 Loop groups and flat connections.

Donaldson [5] re-interpreted elements of the loop group in terms of holomorphic bundles over the disk framed on the boundary, and the factorisation theorem in terms of flat connections on these bundles. He showed that each framed holomorphic bundle over the disk possesses a unique Hermitian-Yang-Mills (flat) connection.

**Theorem 2.2 (Donaldson)** *There is a $1-1$ correspondence between*

*(i) holomorphic bundles over $D$ framed over $\partial D$;*

*(ii) unitary Hermitian-Yang-Mills connections over $D$ on a bundle with a unitary framing over $\partial D$.*

Donaldson's argument generalises to parabolic bundles—holomorphic bundles over the disk with a flag specified over the origin [15]. In this case the flat connection must be singular at the origin.

**Proposition 2.3** *There is a* $1-1$ *correspondence between*

*(i) parabolic bundles over* $D$ *framed over* $\partial D$*;*

*(ii) unitary Hermitian-Yang-Mills connections over* $D-\{0\}$ *on a bundle with a unitary framing over* $\partial D$*. The singularity of the connection at* $0$ *encodes the flag at* $0$*.*

Following Donaldson, we can re-interpret this result in terms of a factorisation theorem for loop groups as follows.

A parabolic bundle over the disk has an underlying trivial holomorphic bundle and a trivialisation compared to the framing over the boundary produces a loop $\gamma \in LG^c$. Any other trivialisation that preserves the parabolic structure at $0 \in D$ changes $\gamma$ by an element of $L^+P$—those loops that are boundary values of holomorphic maps from the disk to $GL(n, \mathbf{C})$ with value at 0 lying in $P$. So (i) in the statement of Proposition 2.3 is equivalently to choosing an element of $LG^c/L^+P$.

A unitary Hermitian-Yang-Mills (or, equivalently, flat) connection over $D-\{0\}$ is determined uniquely by the parabolic structure at $0 \in D$. (This would not be true if there was more than one puncture.) With respect to the unitary framing over the boundary, the flat connection defines an element of the orbit $LG \cdot \xi \in \widehat{L\mathbf{g}}$. We saw in the previous section that the orbit is isomorphic to $LG/Z_\xi$. Thus we get the following restatement of Proposition 2.3.

**Corollary 2.4** *For any* $\xi \in L\mathbf{g}$ *we have*

$$LG^c/L^+P \cong LG/Z_\xi.$$

We could have proven the factorisation theorem in a different way. In the special case that $Z_\xi$ consists of only constant loops then Corollary 2.4 follows from the standard factorisation theorem for loop groups. In general, each orbit of $LG$ possesses a nice representative which simplifies the isotropy subgroup to consist only of constant loops so the general case follows from the special case.

The importance of the treatment here is that at the same time as establishing a complex structure on the orbit space, $\xi$ remains the natural base-point for the holomorphic map and we get an interpretation of the orbit space in terms of flat connections over the disk on a bundle framed over the boundary. In the next section we will see how a holomorphic map from $S^2$ into a space of flat connections is related to an instanton over an associated four-manifold.

# 3 Instantons and holomorphic maps into spaces of flat connections.

Atiyah showed that there is a one-to-one correspondence between instantons over the four-sphere and holomorphic maps from the two-sphere to the loop group [1]. The interpretation of elements of the loop group in terms of flat connections means that Atiyah's result can be viewed as a relationship between instantons and holomorphic maps from the two-sphere to a space of flat connections. This approach was exploited in [14]. Another result of this type was obtained by Dostoglou and Salamon [6] in their proof of the Atiyah-Floer conjecture. They showed that the instanton Floer homology associated to the three-manifold given by a mapping torus $S^1\tilde{\times}\Sigma$ is the same as the symplectic Floer homology of the space of flat connections over $\Sigma$.

The relationship between instantons and holomorphic maps into spaces of flat connections can be understood as follows. Suppose that locally a four-manifold is given by a product of two complex curves $U\times V$ equipped with the product metric. The anti-self-dual equations with respect to local coordinates $\{w\}\times\{z\}$ are given by:

$$\left.\begin{array}{l}[\partial^A_{\bar{w}},\partial^A_{\bar{z}}]=0\\ [\partial^A_{\bar{z}},\partial^A_{z}]=\rho(w,z)[\partial^A_{\bar{w}},\partial^A_{w}]\end{array}\right\}\tag{4}$$

where $\rho(w,z)$ depends on the metrics on $U$ and $V$.

Let $f:U\to\mathcal{M}_V$ be a holomorphic map from $U$ into the space of flat connections $\mathcal{M}_V$ over $V$. (The conformal structure on $V$ equips the space of flat connections with a natural complex structure.) Define a connection over $U\times V$ by

$$A=df+f(w)\tag{5}$$

where $df$ is a Lie algebra valued 1-form over $U\times V$ and $f(w)$ is a flat connection over $\{w\}\times V$. Then $A$ satisfies the following equations which resemble (4):

$$\left.\begin{array}{l}[\partial^A_{\bar{w}},\partial^A_{\bar{z}}]=0\\ [\partial^A_{\bar{z}},\partial^A_{z}]=0\end{array}\right\}\tag{6}$$

The first equation is equivalent to the holomorphic condition on the map $f$ and the second equation uses the fact that $f$ maps to a space of flat connections.

We can think of the second equation of each of (4) and (6) as a type of moment map. One can move from solutions of (6) to solutions of (4) using the Yang-Mills flow, as we do in this paper or, say, by using the implicit function theorem.

In order to apply this to periodic instantons we exploit the conformal invariance of the anti-self-dual equations. Let $\Sigma$ be the punctured disk $D^2-\{0\}$ equipped with the complete hyperbolic metric $|dz|^2/(|z|\ln|z|)^2$. There is a conformal equivalence:

$$S^1\times(\mathbf{R}^3-\{0\})\simeq S^2\times\Sigma,$$

where $S^1\times(\mathbf{R}^3-\{0\})$ is equipped with the flat metric and $S^2\times\Sigma$ is equipped with the product metric

$$ds^2=\frac{4d\bar{w}dw}{(1+|w|^2)^2}+\frac{d\bar{z}dz}{|z|^2(\ln|z|)^2}.\tag{7}$$

On $S^2 \times \Sigma$ the anti-self-dual equations are given by (4) with

$$\rho(w,z) = \left(\frac{1+|w|^2}{|z|\ln|z|}\right)^2 .$$

Our course is set. We have shown that a holomorphic map from $S^2$ to $LG\cdot(\xi,\mu)$ is the same as a holomorphic map from $S^2$ to a space of flat connections which gives an approximate instanton over $S^2 \times \Sigma$. In Section 4 we will use rather standard techniques to move from an approximate instanton to an exact one. Under the conformal equivalence described above, this instanton will correspond to a periodic instanton.

## 3.1 Approximate instantons.

Beginning with a holomorphic map from the two-sphere to an orbit of $LG$, we will construct an approximate instanton over $S^1 \times \mathbf{R}^3$. This will be an explicit realisation of (5).

The map $f : S^2 \to LG/Z_\xi$ is holomorphic when

$$f^{-1}\partial_{\bar{w}} f : S^2 \to L^+\mathbf{p}$$

where $L^+\mathbf{p} \subset L^+\mathbf{g}^c$ is given by those loops that extend to a holomorphic map of the disk whose value at the origin lies in $\mathbf{p}$.

Put $\eta$ equal to the holomorphic extension of $f^{-1}\partial_{\bar{w}} f$ to the disk. Over $S^2 \times \Sigma = \{(w,z)\}$, define a connection

$$A = \eta d\bar{w} - H_\xi^{-1}\eta^* H_\xi dw + i\xi dz/z \tag{8}$$

which is Hermitian with respect to the Hermitian metric

$$H_\xi = exp(i\xi \ln z)^* exp(i\xi \ln z) \tag{9}$$

and flat on each $\{w\} \times D$. Over $S^1 \times \mathbf{R}^3$ in a radially-free gauge we get:

$$(A,\Phi) = (\exp(i\xi r)\eta \exp(-i\xi r) d\bar{w} - \exp(-i\xi r)\eta^* \exp(i\xi r) dw, \xi)$$

Furthermore,

$$*F_A = d_A\Phi - \mu\partial_\theta A + (1+|w|^2)^2 F_{\bar{w}w} dr/r^2 \tag{10}$$

which resembles the periodic instanton equation, (1).

# 4 Construction

In this section we will use the Yang-Mills flow to move from the "approximate" periodic instanton (8) to an exact one. Instead of working directly with the connections, we will follow Donaldson [3] and work with a Hermitian metric on a holomorphic bundle which gives a Hermitian connection.

In fact, we will work with a pair $(H,\eta)$ consisting of a Hermitian metric $H$ on a holomorphic bundle and a map $\eta: S^2\times D^2 \to \mathbf{g}^c$ that is holomorphic in the second factor. A connection $A$ is obtained from the pair $(H,\eta)$ by:

$$A = H^{-1}\partial_z H dz + \eta(w,z)d\bar{w} + (H^{-1}\partial_w H - H^1\eta(w,z)H)dw. \tag{11}$$

Associate to the pair $(H,\eta)$ the Hermitian-Yang-Mills tensor

$$\begin{aligned} B(H,\eta) \quad &= \quad |z|^2(\ln|z|)^2\partial_{\bar{z}}(H^{-1}\partial_z H) + (1+|w|^2)^2\{\partial_{\bar{w}}(H^{-1}\partial_w H) \\ &\quad -\partial_{\bar{w}}(H^{-1}\eta^* H) - \partial_w\eta + [\eta, H^{-1}\partial_w H - H^{-1}\eta^* H]\}. \end{aligned}$$

When $B(H,\eta)\equiv 0$, the connection (11) is anti-self-dual.

Following Donaldson [3] we study the heat flow for the Hermitian metric $H$ in place of the Yang-Mills flow for the associated connection. Since the Hermitian metrics we deal with here are not bounded we need to extend Donaldson's results and their generalisations due to Simpson [23]. Essentially we need to understand properties of the Laplacian of the Kahler manifold $S^2\times\Sigma$ with metric (7) and properties of the initial Hermitian metric (9). Similar results specialised to other non-compact Kahler manifolds exist in [8, 14].

## 4.1 The heat flow.

Associate to a holomorphic map $f: S^2 \to LG/Z_\xi$ the map $\eta: S^2\times D^2 \to \mathbf{g}^c$ given by the holomorphic extension of $f^{-1}\partial_{\bar{w}} f$ to the disks in the second factor. We would like to construct a Hermitian metric $H$ that satisfies the equation $B(H,\eta)=0$. This would produce an anti-self-dual connection associated to the map $f$.

Consider the heat flow equation over $S^2\times\Sigma$

$$H^{-1}\partial H/\partial t = B(H,\eta),\ \ H(w,z,0) = H_\xi \tag{12}$$

where $H_\xi$ is defined in (9). A solution of (12) will converge to the required solution of $B(H,\eta)=0$ as $t\to\infty$. Instead of solving (12) we will work with a family of boundary value problems. Put

$$S^2\times\Sigma_{\epsilon,\delta} = \{(w,z)\in S^2\times\Sigma \mid \epsilon\le|z|\le\delta\}$$

so the $S^2\times\Sigma_{\epsilon,\delta}$ exhaust $S^2\times\Sigma$ as $\delta\to 1$ and $\epsilon\to 0$.

**Proposition 4.1** *Over each $S^2\times\Sigma_{\epsilon,\delta}$ there is a unique solution of the boundary value problem*

$$\left.\begin{array}{c} H^{-1}\partial H/\partial t = B(H,\eta) \\ H(w,z,0) = H_\xi \\ H|_{\partial S^2\times\Sigma_{\epsilon,\delta}} = H_\xi \end{array}\right\} \tag{13}$$

*given by $H^{\epsilon,\delta}(w,z,t)$ and converging to a smooth metric $H^{\epsilon,\delta}(w,z,\infty)$ that satisfies $B(H^{\epsilon,\delta}(w,z,\infty),\eta)=0$.*

*Proof.* Since we have fixed $S^2 \times \Sigma_{\epsilon,\delta}$ for the moment we will omit the superscript in $H^{\epsilon,\delta}(w,z,t)$ during this proof. Short-time existence of a solution of (13) is automatic since $B(H,\eta)$ is elliptic in $H$ and we have Dirichlet boundary conditions. In order to extend this to long-time existence we will take the approach given by Donaldson [3] and extended by Simpson [23] and show that a solution on $[0,T)$ gives a limit at $T$ which is a good initial condition to start the flow again. The lemmas we need to prove on the way use the details of our particular case and allow us to proceed with Donaldson's proof.

A Hermitian metric $H$ takes its values in the space $G^c/G$ which comes equipped with the complete metric $d$ given locally by $tr(H^{-1}\delta H)^2$. Following Donaldson, we will use both this metric and the convenient function $\sigma(H_1,H_2) = tr(H_1^{-1}H_2) + tr(H_1 H_2^{-1}) - 2n$ that satisfies $c_1 d^2 \leq \sigma \leq c_2 d^2$ for constants $c_1, c_2$. (Aside: if we take the loop group perspective described in [7], then a Hermitian metric takes its values in the space $LG^c/LG$. We have not checked that this is a complete metric space.)

**Lemma 4.2** *If $H_1$ and $H_2$ are two solutions of the heat equation then*

$$\partial_t \sigma + \Delta \sigma \leq 0 \tag{14}$$

*for $\sigma = \sigma(H_1,H_2)$.*

*Proof.* See [14]. $\square$

Apply (14) to $H(w,z,t)$ and $H(w,z,t+\tau)$, the flow at two times. Since they obey the same boundary conditions on $S^2 \times \Sigma_{\epsilon,\delta}$, $\sigma$ vanishes on the boundary. By the maximum principle $\sup_{S^2\times\Sigma_{\epsilon,\delta}} \sigma$ is a non-increasing function of $t$. By continuity, for any $\rho > 0$ there exists a $\tau$ small enough so that

$$\sup_{S^2\times\Sigma_{\epsilon,\delta}} \sigma(H(w,z,t),H(w,z,t')) < \rho$$

for $0 < t,t' < \tau$. It follows from the non-increasing property of $\sigma$ that

$$\sup_{S^2\times\Sigma_{\epsilon,\delta}} \sigma(H(w,z,t),H(w,z,t')) < \rho$$

for $T-\tau < t,t' < T$. Since $\rho$ can be made arbitrarily small, $H(w,z,t)$ is a Cauchy sequence in the $C^0$ norm as $t \to T$. The metrics take their values in a complete metric space (described below) and the function $\sigma$ acts like the metric so there is a *continuous* limit $H_T$ of the sequence. Notice also that (14) and the maximum principle show that this short-time solution to the heat flow equation is unique.

Using the heat equation and the metric on $G^c/G$, we have

$$d(H(w,z,t),H(w,z,0)) \leq \int_0^t |B(H(w,z,s),\eta)|ds$$

where $|B(H(w,z,s),\eta)|^2 = tr(B^*B)$ and the adjoint is taken with respect to the metric $H_s$. Notice that $B^* = B$ so $|B(H(w,z,s),\eta)|^2 = tr(B^2)$.

**Lemma 4.3** *If $H(w,z,t)$ is a solution of the heat equation then*

$$(d/dt + \Delta)|B(H(w,z,t),\eta)| \le 0 \text{ whenever } |B| > 0 \tag{15}$$

*Proof.* See [14]. $\square$

The next two lemmas use the particular features of the Kahler manifold $S^2 \times \Sigma$ together with the initial Hermitian metric $H_\xi$ to get $C^0$ control on $H(w,z,t)$ during the flow.

**Lemma 4.4** *When $\eta$ is the holomorphic extension of $f^{-1}\partial_{\bar{w}} f$, for a given holomorphic map $f : S^2 \to \Omega LG/Z_\xi$, there exists a constant $M$ such that $|B(H_\xi,\eta)| \le M(1-|z|)$ on $S^2 \times \Sigma$.*

*Proof.*

$$B(H_\xi,\eta) = -(1+|w|^2)^2(\partial_w \eta + \partial_{\bar{w}}(H_\xi^{-1}\eta^* H_\xi) + [\eta, H_\xi^{-1}\eta^* H_\xi])$$

and since $[\eta(0),\xi] = 0$, $|B(H_\xi,\eta)|$ is bounded near $z = 0$. Since $f$ takes its values in the unitary loop group and $H_\xi = I$ on $|z| = 1$, we can identify $B(H_\xi,\eta)$ with the curvature of a flat connection which is 0. Furthermore, $B(H_\xi,\eta)$ is continuous and differentiable up to $|z| = 1$ so it vanishes like $1-|z|$ there. $\square$

**Lemma 4.5** *There is a constant $C$ independent of $\epsilon$ and $\delta$ such that*

$$d(H^{\epsilon,\delta}(w,z,t), H_\xi) \le C\ln(1-\ln|z|)$$

*for all $(w,z,t) \in S^2 \times \Sigma_{\epsilon,\delta} \times \mathbf{R}$.*

*Proof.* It follows from (15) and the maximum principle that if there is a function $b(w,z,t)$ defined on $S^2 \times \Sigma_{\epsilon,\delta} \times \mathbf{R}$ that satisfies $(\partial_t + \Delta)b = 0$ and $|B(H_\xi,\eta)| \le b(w,z,0)$ then $|B(H(w,z,t),\eta)| \le b(w,z,t)$ for all $t$.

Put $b(w,z,0) = M(1-|z|)$. Notice that $b(w,z,0) = b(|z|)$, so we only need use the one-dimensional Laplacian and $b(w,z,t) = b(|z|,t)$. From the flow equation (13) we have

$$\begin{aligned} d(H(w,z,t), H_\xi(w,z)) &= \int_0^t B(H(w,z,\tau))d\tau \\ &\le \int_0^t b(w,z,\tau)d\tau \\ &\le \int_0^\infty b(w,z,\tau)d\tau \end{aligned} \tag{16}$$

Now, $b(|z|,t) = \int b(s,t)k(|z|,s,t)ds$ where $k$ is the one-dimensional heat kernel operator. Since $\int_0^\infty k(|z|,s,t)dt = G(|z|,s)$, the Green's operator, is finite, Fubini's theorem allows us to interchange the order of integration in (16). So

$$\begin{aligned} d(H(w,z,t), H_\xi(w,z)) &\leq M \int_0^\epsilon (1-s) G(|z|,s) ds \\ &\leq M \int_0^1 (1-s) G(|z|,s) ds \ . \end{aligned}$$

With respect to the Laplacian

$$\Delta = -(1+|w|)^2 \partial_{\bar{w}} \partial_w - 4|z|^2 (\ln|z|)^2 \partial_{\bar{z}} \partial_z = -(\ln|z|)^2 \partial^2_{\ln|z|}$$

reduced to one dimension, the Green's operator is given by

$$G(|z|,s) = \min\{-\ln|z|, -\ln s\}/s(\ln s)^2 \ .$$

Actually, this Green's operator is only valid for the entire interval ($\epsilon = 1$) and Fubini's theorem doesn't apply there. There is a monotone property of heat kernels which means that our choice of $G$ is simply an overestimate when $\epsilon < 1$ so the calculation is valid. Thus

$$\begin{aligned} d(H(w,z,t), H_\xi(w,z)) &\leq M \left( -\ln|z| \int_0^{|z|} \frac{(1-s)ds}{s(\ln s)^2} - \int_{|z|}^1 \frac{(1-s)ds}{s \ln s} \right) \\ &\leq C \ln(1 - \ln|z|) \end{aligned}$$

where the last inequality simply encodes the fact that the distance vanishes as $|z| \to 1$ and grows like $\ln(1 - \ln|z|)$ as $|z| \to 0$. □

The preceding lemmas have shown that there is a solution to the heat equation that satisfies $H(w,z,t) \to H(w,z,T)$ in $C^0$ and $H(w,z,t)$ is uniformly bounded with bound independent of $t$ (though depending on $\epsilon$). These are the conditions required to use Simpson's extension of Donaldson's result to show that $H(w,z,t)$ is bounded in $L^p_2$ uniformly in $t$. Hamilton's methods [9] then give control of all higher Sobolev norms. Thus we get a solution, $H(w,z,t)$, of (13) for all $t$ that converges to a smooth limit $H^{\epsilon,\delta}(w,z,\infty)$ defined on $S^2 \times \Sigma_{\epsilon,\delta}$ and satisfying $B(H^{\epsilon,\delta}(w,z,\infty),\eta) = 0$ and $H^{\epsilon,\delta}(w,z,\infty) = H_\xi$ on $\partial S^2 \times \Sigma_{\epsilon,\delta}$ so Proposition 4.1 is proven. □

**Proposition 4.6** *For each holomorphic map $f : S^2 \to LG/Z_\xi$ there is a periodic instanton $A_f$ on $S^1 \times \mathbf{R}^3$.*

*Proof.* We have proven the existence of a family of hermitian metrics $H^{\epsilon,\delta}$ respectively defined over $S^2 \times \Sigma_{\epsilon,\delta}$ and satisfying $B(H^{\epsilon,\delta},\eta) = 0$. Since $\sigma(H^{\epsilon,\delta}, H^{\epsilon',\delta'})$ is subharmonic its maximum occurs at the boundary of the set on which it is defined. For $\epsilon' \leq \epsilon \leq \delta \leq \delta'$, the common set is $S^2 \times \Sigma_{\epsilon,\delta}$. If we fix $\epsilon = \epsilon'$ and let $\delta \to 1$, then $\sigma = 0$ on $|z| = \epsilon$ and the maximum of $\sigma$ occurs on $|z| = \delta$. Since the metrics $\sigma$ and $d$ on $G^c/G$ are equivalent, the maximum value of $\sigma$ is bounded by a constant times $d(H^{\epsilon,\delta'}, H_\xi) \leq C \ln(1 - \ln \delta)$ using Lemma 4.5. This tends to 0 as $\delta \to 1$, thus we have a Cauchy sequence that converges uniformly to a Hermitian metric $H^\epsilon$ defined on $|z| \geq \epsilon$. The convergence can be improved to $L^p_2$ to ensure that $B(H^\epsilon, \eta) = 0$, [23].

In order to deal with $\epsilon \to 0$, notice that since $\ln |z|$ is harmonic on $S^2 \times \Sigma$, $\sigma + a \ln |z|$ is subharmonic for any $a$. Put $a = \sup_{|z|=\epsilon} \sigma / |\ln \epsilon|$. Then $\sigma + a \ln |z| \leq 0$ on $|z| = 1$ and $|z| = \epsilon$. Thus

$$\sigma \leq -\ln |z| \sup_{|z|=\epsilon} \sigma / |\ln \epsilon|. \tag{17}$$

By Lemma 4.5, $d(H^{\epsilon,\delta'}, H_\xi) \leq C \ln(1 - \ln \epsilon)$ so $\sigma = o(|\ln \epsilon|)$ as $\epsilon \to 0$. Thus the right hand side of (17) tends uniformly to 0 on compact sets away from $z = 0$. Again we conclude that the $\{H^\epsilon\}$ form a Cauchy sequence as $\epsilon \to 0$, converging uniformly on the complement of any neighbourhood of $S^2 \times \{0\}$ to a Hermitian metric $H$ that satisfies $B(H, \eta) = 0$ on $S^2 \times \Sigma$.

Using $S^1 \times (\mathbf{R}^3 - \{0\}) \cong S^2 \times \Sigma$ we see that the limit $H$ is smooth on $S^1 \times (\mathbf{R}^3 - \{0\})$ and continuous on all of $S^1 \times \mathbf{R}^3$, converging to $I$ on $S^1 \times \{0\}$. The connection $A$ obtained from $H$ via (11) is defined and anti-self-dual on $S^1 \times (\mathbf{R}^3 - \{0\})$. By the following lemma, $A$ has finite charge. Since codimension three singularities of finite charge anti-self-dual connections can be removed [22], $A$ is smooth on all of $S^1 \times \mathbf{R}^3$. □

**Lemma 4.7** *The curvature of the limiting connection $A$ has finite $L^2$ norm.*

*Proof.* The Yang-Mills flow decreases the $L^2$ norm of a connection, and any bubbling in the limit just decreases the $L^2$ norm further, so it is sufficent to show that the initial connection has finite $L^2$ norm.

For any connection $A$, we have

$$8\pi^2 \|F_A\|_2^2 = 2 \int |F_A^+|^2 - \int F_A \wedge F_A \tag{18}$$

where $F_A^+$ is the self-dual part of the curvature. We can calculate this explicitly for the initial connection defined in (8).

Notice that $F_A^+ = B(H_\xi, \eta)$ and by Lemma 4.4 we have $|B(H_\xi, \eta)| \leq M(1 - |z|)$. This is square-integrable over $S^2 \times \Sigma$ since $S^2$ is compact and $\Sigma$ has finite area near $z = 0$ and grows like $1/(1 - |z|)^2$ near $|z| = 1$.

As one might expect, the topological term in (18) will coincide with the topological degree of the map $f : S^2 \to LG/Z_\xi$.

$$k(E) = \frac{1}{8\pi^2} \int_{S^2 \times D} tr(F_A^2) = -\frac{1}{8\pi^2} \int_{S^2 \times D} tr(\partial_{\bar{z}} \eta^* \partial_z \eta) d\bar{z} dz d\bar{w} dw$$

since only the $F_{z\bar{w}}$ and $F_{\bar{z}w}$ terms contribute. Since $\eta$ is holomorphic in $z$, then on the disk $d\{tr(\eta^*\partial_z\eta)dz\} = tr(\partial_{\bar{z}}\eta^*\partial_z\eta)d\bar{z}dz$ so

$$\begin{aligned} k(E) &= -\frac{1}{8\pi^2}\int_{S^2}\int_{|z|=1} tr(\eta^*\partial_z\eta)dz d\bar{w}dw \\ &= \frac{1}{4\pi}\int_{S^2} \|f^{-1}\partial_{\bar{w}}f\|^2 \frac{d\bar{w}dw}{i} \end{aligned}$$

where $\|f^{-1}\partial_{\bar{w}f}\|^2$ uses the Kahler metric on $LG/Z_\xi$. This expression is the degree of $f$. $\square$

*Remark.* In the construction of this section we started with parabolic bundles over the disk. However, the reverse is not true that a periodic instanton gives rise to a family of parabolic bundles. By this we mean that the holomorphic structure defined on each punctured disk by the restriction of the periodic instanton does not extend to the entire disk. The curvature just fails to satisfy $F_A \in L^p$ for $p > 1$ as required in [2].

# 5 Injection

In this section we will show that the map produced in Section 4 is injective.

**Proposition 5.1** *Let $f : S^2 \to LG/Z_\xi$ and $g : S^2 \to LG/Z_\nu$ be two based holomorphic maps. Then the instantons $A_f$ and $A_g$ are gauge equivalent precisely when $\nu - \xi$ is in the root lattice and $g = f \cdot \exp(i(\nu - \xi)\ln z)$.*

*Proof.* The instanton $A_f$ is given by the expression (11) which depends on a pair $(H, \eta)$ consisting of a Hermitian metric, $H$, and the holomorphic extension of $f^{-1}\partial_{\bar{w}}f$ denoted by $\eta$ and likewise for $A_g$. These expressions are independent of the unitary gauge so $A_f \sim A_g$ only if $A_f = A_g$ or possibly if we have used different holomorphic trivialisations of the holomorphically trivial bundle restricted to each $\{w\} \times \Sigma$ for $A_f$ and $A_g$.

If $A_f = A_g$ then $f^{-1}\partial_{\bar{w}}f = \eta = g^{-1}\partial_{\bar{w}}g$, so $\partial_{\bar{w}}(gf^{-1}) = 0$ and this is global over $S^2$ thus $g = \gamma(z)f$ for some loop $\gamma(z)$ independent of $w$. The requirement that $f$ and $g$ map $\infty \in S^2$ to the constant loop $I$ forces $\gamma(z) = I$.

If $A_f \neq A_g$ and $A_f \sim A_g$ then $A_f$ uses the pair $(H, \eta)$ in (11) and $A_g$ uses the pair $(p^*Hp, p^{-1}\eta p + p^{-1}\partial_{\bar{w}}p)$ for a map $p : S^2 \times \Sigma \to G^c$ which is holomorphic on each $\{w\} \times \Sigma$ and unitary on its boundary. Note that this implies that $g = fp$ though since $p$ is not a priori in $L^+P$, the maps $f$ and $g$ can be distinct.

The proof of the proposition is completed by the following two lemmas that show that $g = fp$ together with the known growth of the Hermitian metrics associated to $f$ and $g$ forces $p$ to be constant or to be a standard holomorphic gauge change.

**Lemma 5.2** *If $\xi = \nu$ then $A_f \sim A_g$ only if $f = gu$ for $u \in P \cap G \cong Z_\xi$.*

*Proof.* We can apply Lemma 4.5 to the Hermitian-Yang Mills metric $H$ over all of $S^2 \times \Sigma$ even though it is only stated for $0 < \epsilon < \delta < 1$. Thus

$$d(H, H_\xi) + d(p^* H p, H_\xi) \leq C \ln(1 - \ln|z|)$$

for the initial metric $H_\xi$ defined in (9). Using the identity $d(p^* H p, H_\xi) = d((p^*)^{-1} H_\xi p^{-1}, H_\xi))$ and the triangle inequality we have

$$d(H, H_\xi) + d(p^* H p, H_\xi) \geq d((p^*)^{-1} H_\xi p^{-1}, H_\xi)) \tag{19}$$

and the right hand side is bounded by $C \ln(1 - \ln|z|)$ only if $p$ is bounded near $z = 0$ by $C \ln(1 - \ln|z|)$. Since it satisfies $\lim_{z \to 0} z p(z) \to 0$, $p$ extends across $z = 0$ and is holomorphic there. Furthermore we must have $p(0) \in P$ in order that the right hand side of (19) is bounded by $C \ln(1 - \ln|z|)$. Since $p$ is holomorphic on the disk and unitary on the boundary it must be unitary on the disk (by the maximum principle applied to the subharmonic function $tr(p^* p) + tr((p^* p)^{-1})$), and thus constant there, and moreover lie in $P \cap G$. □

**Lemma 5.3** *If $A_f \sim A_g$ then $\nu - \xi$ lies in the root lattice and*

$$g = f \exp(i(\nu - \xi) \ln z).$$

*Proof.* As described above, $g = fp$. Then $\lim_{z \to 0} z p^{-1} \partial_z p = \nu - \xi$. Since $z p^{-1} \partial_z p$ is bounded and holomorphic on the punctured disk, it extends to a holomorphic function of the disk. In fact $p^{-1} \partial_z p = q(z)/z$ so $p(z) = \exp(\int^z q(\zeta) d\zeta / \zeta)$ and $\nu - \xi = q(0)$ must lie in the integer lattice. Thus $p \cdot \exp(-i(\nu - \xi) \ln z)$ is holomorphic on the disk and unitary on the boundary and hence constant which we absorb in the unitary ambiguity of $f$. So $g = f \cdot \exp(i(\nu - \xi) \ln z)$. □

The proposition allowed for gauge transformations that have angular dependence at infinity (corresponding to $z = 0$). When we restrict the gauge transformations to have no angular dependence at infinity then the maps $f$ and $f \cdot \exp(i(\nu - \xi) \ln z)$ define inequivalent connections. Thus the map $f \mapsto A_f$ is injective.

# 6 Boundary conditions

There are natural boundary conditions that the periodic instantons constructed in this paper conjecturally satisfy: as $r \to \infty$

$$\begin{gathered} \|\Phi - \xi\| = O(1/r) \\ \partial \|\Phi - \xi\| / \partial \Omega = O(1/r^2) \\ \|\nabla(\Phi - \xi)\| = O(1/r^2) \end{gathered}$$

where $\xi$ is a given constant Higgs field, $r$ is the radial coordinate in $\mathbf{R}^3$, $\partial / \partial \Omega$ is an angular derivative, and the asymptotic constants are uniform in $\theta$.

In order to prove these conditions we would need to understand the precise elliptic constants for the Hermitian Yang-Mills Laplacian on $S^2 \times \Sigma$ near the puncture at $z = 0$. This would enable us to get estimates on the second derivatives of $H$ from the estimates on $H$ given in this paper and estimates on first derivatives of $H$ obtained from a maximum principle argument [5]. We hope to show this in future work.

Alternatively, one might prove the stronger conjecture that all finite energy periodic instantons satisfy these boundary conditions. Such a proof would again require a good understanding of the Laplacian on $S^2 \times \Sigma$ as in the special case of monopoles [11]. This stronger conjecture implies that the construction of this paper yields *all* periodic instantons. This can be proven by using a scattering argument to retrieve a holomorphic map from $S^2$ to an orbit of the loop group from a given periodic instanton.

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