Compactification of hyperbolic monopoles

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Abstract

We prove that the space of $SU(2)$ hyperbolic monopoles based at the centre of hyperbolic space is homeomorphic to the space of (un-based) rational maps of the two-sphere. The homeomorphism extends to a map of the natural compactifications of the two spaces. We also show that the scattering methods used in the study of monopoles apply to the configuration space for hyperbolic monopoles giving a homotopy equivalence of this space with the space of continuous self-maps of the two-sphere.

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1   Introduction.

It has long been known that moduli spaces of monopoles and holomorphic maps of the two-sphere are intimately related. In [1, 2] Atiyah introduced the space of hyperbolic monopoles showing that for integral mass the space of charge k $SU(2)$ monopoles based at infinity is isomorphic to the space of degree k based holomorphic self-maps of the two-sphere. His approach was to identify hyperbolic monopoles as instantons over the four-sphere invariant under a circle action. Sibner and Sibner [19] justified this identification using the natural decay conditions on monopoles suggested by Atiyah.

The similarity between the natural compactifications of the space of monopoles and the space of rational maps of the two-sphere suggests that Atiyah's map extends to the compactifications. This paper arose out of the study of that question. Unfortunately the framing condition obstructs the desired compactification since bubbling at the basepoint is forbidden. To get around this it is quite natural to put the basepoint, instead, at an interior point of hyperbolic space. Murray suggested the study of such spaces in [16]. Uniform bounds on the curvature of hyperbolic monopoles at interior points prevents bubbling there, allowing our study to proceed.

This novel choice of basepoint means that we cannot use Atiyah's results. In particular, we are considering a moduli space that has a bigger dimension. Still, we will prove there is a correspondence between monopoles and rational maps in this setting. The algebro-geometric methods employed by Atiyah seem to be limited in the study of the question of compactifying since the natural compactification associated to holomorphic bundles is the Gieseker compactification rather than the Uhlenbeck compactification of instantons which we require. Jun Li [15] showed how the Gieseker compactification of the space of holomorphic bundles is in some sense bigger than the Uhlenbeck compactification. In this paper we use the more suitable scattering methods used by Hitchin [12] in his study of Euclidean monopoles and suggested by Atiyah for hyperbolic monopoles.

The homotopy theory of the underlying configuration spaces respectively given by connections that do not satisfy the Bogomolny equations and continuous self-maps of the two-sphere is well-understood [4]. Using the holonomy of the connections or the uniqueness of classifying spaces the respective configuration spaces can be shown to be homotopy equivalent. One satisfying part of the scattering approach described here is that it produces the homotopy equivalence directly, avoiding the usual separate treatment.

We consider monopoles only with integer mass m. This is because the methods we use rely heavily on estimates supplied by working over the four-sphere. Still, one might expect that our results can be extended to arbitrary mass, especially considering the studies in [14, 17].

Define $\mathcal{B}_k^m$ to be the space of C^1 connections modulo gauge transformations on a framed $SU(2)$ bundle E over S^4 invariant under a $U(1)$-action where the weight of the $U(1)$-action is m and $2km = c_2(E)$. Equivalently,

$$\|\Phi\| \to m \text{ at } \infty \text{ and } \int_{H^3} F_A \wedge d_A \Phi = 4\pi m k .$$

We think of $S^4 - S^2 \sim H^3 \times S^1$ where $\sim$ denotes the conformal equivalence of metrics on the two spaces. The gauge transformations are required to be the identity at the basepoint and we choose the basepoint to lie off the fixed point sphere.

We will equivalently describe any element of $\mathcal{B}_k^m$ by a pair (A, Φ). Given such a pair, along radial geodesics beginning at $0 \in H^3$ consider *bounded* solutions of the scattering equation

$$\nabla_t s - i\Phi s = 0 \tag{1}$$

Each radial geodesic corresponds to a point $(u, v) \in S^2$. Define $\mathcal{F}(u, v) : \mathcal{B}_k^m \to \mathbf{CP}^1$ by

$$\mathcal{F}(u, v)(A, \Phi) = s_{(u,v)}(0) \in \mathbf{C}^2 \tag{2}$$

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where, up to a constant multiple, $s_{(u,v)}(t)$ is the unique bounded solution of (1) along the geodesic corresponding to (u, v). It is well-defined since any gauge transformation fixes the frame at 0. Let $\Omega_k^2 S^2$ be the space of continuous self-maps of S^2 of degree k equipped with the uniform topology. The following theorem says that for each pair (A, Φ), its image $\mathcal{F}(A, \Phi)$ is continuous in (u, v) and that as we vary (A, Φ) continuously in the C^1 sense, the image $\mathcal{F}(A, \Phi)$ varies uniformly in $\Omega_k^2 S^2$.

Theorem 1 *The map $\mathcal{F} : \mathcal{B}_k^m \to \Omega_k^2 S^2$ is continuous.*

This map restricted to a deformation retract of $\mathcal{B}_k^m$ realizes the homotopy equivalence proven by Gritsch [11]. The space of hyperbolic monopoles is defined by

$$\mathcal{M}_k^m = \{(A, \Phi) \in \mathcal{B}_k^m \mid d_A \Phi = *F_A\}/\mathcal{G}$$

where $\mathcal{G}$ is the group of gauge transformations that are the identity at 0 and the Hodge star is taken with respect to the hyperbolic metric. Define $Rat_k(S^2)$ to be the space of degree k rational self-maps of S^2.

Theorem 2 *The restriction of $\mathcal{F}$ to monopoles defines a homeomorphism*

$$\mathcal{F} : \mathcal{M}_k^m \to Rat_k(S^2) \ .$$

Note that there is no natural map between the space of based rational maps in Atiyah's theorem [1] and the space of unbased rational maps appearing here. The quotients of these spaces by $U(1)$ and $SU(2)$ respectively are both isomorphic to the space of monopoles on the unframed bundle however there is still no natural map between them.

The space $Rat_k(S^2)$ naturally compactifies to $\mathbf{CP}^{2k+1}$ [8] and $\mathcal{M}_k^m$ possesses the Uhlenbeck compactification [6].

Theorem 3 *The map $\mathcal{F}$ extends to a continuous map between the respective natural compactifications of the spaces of monopoles and rational maps.*

Atiyah and Hitchin [3] proved a weaker result for Euclidean monopoles to demonstrate the type of superposition properties of monopoles.

Let $\gamma : \mathbf{R} \to H^3$ be a (complete) geodesic. Define $V_\gamma \subset \mathcal{M}_k^m$ to be the set of monopoles that possess a non-trivial bounded solution of $(\nabla_t - i\Phi)s = 0$ along γ.

Theorem 4 *For each geodesic γ, V_γ is a codimension 2 submanifold of $\mathcal{M}_k^m$ which extends to the compactification.*

This is the analogue of Donaldson's μ-map [10]. The spectral curve of a monopole is a dual notion of the μ-map.

This paper is organized as follows. In Section 2 we describe hyperbolic space and its relationship with the four-sphere. Sections 3, 4, 5, 6 are devoted respectively to the proofs of Theorems 1, 2, 3, 4.

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2 Hyperbolic space

We will use spherical coordinates (u, v, r) or (u, v, t) on H^3 where (u, v) parametrises spheres of constant radius and $w = u + iv$ is the induced conformal structure. The coordinate t gives the hyperbolic distance from the centre of hyperbolic space while the alternative coordinate, r, gives the distance from the centre of hyperbolic space in terms of the round metric on S^4 where we think of $H^3 \subset S^4$.

The hyperbolic metric in (u, v, r) coordinates is

$$ds^2 = \frac{16r^2(du^2 + dv^2)}{(1 - r^2)^2(1 + u^2 + v^2)^2} + \frac{4dr^2}{(1 - r^2)^2}.$$

We may put this also in terms of t, using $r = \tanh(t/2)$, as well as $4r^2/(1 - r^2)^2 = \sinh^2(t)$, to get:

$$ds^2 = \frac{4\sinh^2(t)(du^2 + dv^2)}{(1 + u^2 + v^2)^2} + dt^2.$$

2.1 From S^4 to H^3.

As mentioned in the introduction, each $U(1)$-invariant connection defined over S^4 gives rise to a pair (A, Φ) defined over H^3. To go from H^3 to S^4, one uses the removable singularities theorem of Sibner and Sibner [19] that says that a pair (A, Φ) with integral mass gives rise to a connections over S^4. Furthermore, since the natural metrics on S^4 and $H^3 \times S^1$ are conformally equivalent, instantons give rise to monopoles. We work only with differentiable connections over S^4 which gives a restricted space of pairs (A, Φ) over H^3 since the theorem of Sibner and Sibner only produces a $W^{1,2}$ connection over S^4 in general.

It is worth remembering here that $U(1)$ invariance requires a lifting of the action to the bundle, so it follows that the connection and Higgs field must be abelian on the fixed $S^2 \subset S^4$. This fixed sphere manifests itself as the boundary sphere in the ball model of H^3. In contrast to the Euclidean case, where the connection on the 'sphere at infinity' is of a standard type (the curvature is just the volume form of the sphere multiplied by the charge k), there is an entire moduli of connections in this hyperbolic case, and indeed Austin and Braam [5] have shown that this 'boundary value' actually determines the whole monopole.

The following two lemmas show that connections over S^4 give scattering coefficients in (1) that decay fast enough for the appropriate analysis, and that we can choose a gauge so that the L^1 norm of the scattering coefficients in (1) is controlled by the C^1 norm of the connections on S^4.

Lemma 2.1

$$\nabla_t - i\Phi = d/dt - i\Phi(\infty) + O(e^{-2t}) \ .$$

Proof. Let $\tilde{A}$ be a smooth $U(1)$-invariant connection on S^4. In a $U(1)$-invariant gauge put $\tilde{A} = A + \Phi d\theta$ where (A, Φ) is defined over H^3. A natural coordinate system for H^3 considered as a bounded subset of S^4 is the spherical coordinates (u, v, r) where (u, v) gives local coordinates on S^2 and $r \in [0, 1)$. If we replace r by the hyperbolic distance from $0 \in H^3$, $t = \log\{(1 + r)/(1 - r)\}$ then since $dt = 2dr/(1 - r^2) = 2\cosh^2(t/2)dr$ we have

$$\nabla_t \Phi dt d\theta = F_{t\theta} dt d\theta = F_{r\theta} dr d\theta \;\Rightarrow\; \nabla_t \Phi = F_{r\theta}/2\cosh^2(t/2)$$

and similarly

$$A_t = A_r/2\cosh^2(t/2) \;.$$

From $\nabla_t \Phi = \partial\Phi/\partial t + [A_r, \Phi]/2\cosh^2(t/2)$ we get

$$\partial\Phi/\partial t = (F_{r\theta} - [A_r, \Phi])/2\cosh^2(t/2)$$

so

$$
\begin{aligned}
|\Phi(t) - \Phi(\infty)| &= |\int_t^\infty ([A_r, \Phi] - F_{r\theta})/2\cosh^2(t/2)dt| \\
&\leq \int_t^\infty M/2\cosh^2(t/2)dt = M/(e^{2t} + 1)
\end{aligned}
$$

where $|F_{r\theta} - [A_r, \Phi]| \leq M$. Thus

$$\nabla_t - i\Phi = d/dt - i\Phi(\infty) + O(e^{-2t})$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.2

$$\|\nabla_t^A - i\Phi - g \cdot (\nabla_t^B - i\Psi)\|_{L^1(\mathbf{R}^+)} \leq C\|\tilde{B} - \tilde{A}\|_{C^1(S^4)}$$

where C depends only on a neighbourhood of $\tilde{A}$.

Proof. If $\tilde{A}, \tilde{B}$ are $U(1)$-invariant connections on S^4 satisfying

$$\|\tilde{B} - \tilde{A}\|_{C^1(S^4)} < \epsilon$$

then on H^3 the Higgs field Ψ associated to $\tilde{B}$ satisfies

$$\Psi(t) - \Psi(\infty) = \int_t^\infty ([B_r, \Psi] - F_{r\theta}^B)/2\cosh^2(t/2)dt$$

and the similar expression for Φ associated to $\tilde{A}$ is given in the proof of the previous lemma. Let g be a $U(1)$-equivariant gauge transformation over S^4 that satisfies $g \cdot \Psi(\infty) = \Phi(\infty)$. Then

$$
\begin{aligned}
\Phi(t) - \Psi(t) &= \Phi(\infty) - \Psi(\infty) \\
&+ \int_t^\infty \frac{([A_r, \Phi] - F_{r\theta}^A - [B_r, \Psi] + F_{r\theta}^B)dt}{2\cosh^2(t/2)} \\
\Psi(t) - g \cdot \Psi(t) &= \Psi(\infty) - g \cdot \Psi(\infty) \\
&+ \int_t^\infty \frac{([B_r, \Psi] - F_{r\theta}^B - g \cdot [B_r, \Psi] + g \cdot F_{r\theta}^B)dt}{2\cosh^2(t/2)}
\end{aligned}
$$

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so if we choose g so that $\|g \cdot \tilde{B} - \tilde{B}\|_\infty < \epsilon \|\tilde{B}\|_\infty$ then

$$|\Phi(t) - g \cdot \Psi(t)| \leq C\epsilon e^{-2t}$$

for a constant C that depends only on the ϵ neighbourhood of $\tilde{A}$. Similarly $A_t - g \cdot B_t \leq C\epsilon e^{-2t}$. Thus

$$\|\nabla_t^A - i\Phi - g \cdot (\nabla_t^B - i\Psi)\|_{L^1(\mathbf{R}^+)} \leq C\|\tilde{B} - \tilde{A}\|_{C^1(S^4)}$$

where C depends only on a neighbourhood of $\tilde{A}$. □

3 Scattering.

In this section we will prove Theorem 1. The metric enters here only in the finite action condition. The Bogomolny equations are not needed until the next section. We will show that the solution of (1) varies continuously with a continuous change of the geodesic γ and the pair (A, Φ). It will follow that $\mathcal{F}(A, \Phi)$ is both continuous and varies continuously in (A, Φ) with respect to the uniform topology on $\Omega_k^2 S^2$.

3.1 Levinson's Theorem.

We begin by giving the proof of the following standard result on ordinary differential equations. This is a minor variation on the Euclidean monopole treatment [12] combined with the better estimates supplied from working over S^4.

Theorem 3.1 (Levinson [7]) *For $m_1 \geq 0 \geq m_2$, the solutions of the 'unperturbed' equation*

$$\frac{dz}{dt} + \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} z = 0 \qquad (3)$$

which are bounded as $t \to \infty$, are in bicontinuous 1-1 linear correspondence with those of the 'perturbed' equation

$$\frac{dx}{dt} + \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} x + f(t)x = 0 \qquad (4)$$

on $[0, \infty)$, where $|f(t)|$ is bounded and lies in $L_{[0,\infty)}^1$ Furthermore, if $x(t), z(t)$ are corresponding solutions then $|x(t) - z(t)| \to 0$ as $t \to \infty$.

Proof. Let P_1 be the projection of C^2 onto the subspace defined by the first basis vector and let P_2 be the (complementary) projection

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onto that spanned by the second basis vector. Notice that we have a 'fundamental matrix' solution

$$Z(t) = \begin{pmatrix} e^{-m_1 t} & 0 \\ 0 & e^{-m_2 t} \end{pmatrix},$$

and that the first diagonal entry gives a solution which is bounded as $t \to \infty$. Notice that

$$|Z(t)P_1 Z^{-1}(s)| \leq 1 \text{ for } t_0 \leq s \leq t,$$

$$|Z(t)P_2 Z^{-1}(s)| \leq 1 \text{ for } t_0 \leq t \leq s.$$

From the conditions on f, it follows that we can choose $t_1 > 0$ such that

$$\theta = \int_{t_1}^{\infty} |f(t)| < 1.$$

Let $x(t)$ be any continuous function with $\| x \| = \sup_{t \geq t_1} |x(t)| < \infty$, and define

$$Tx(t) = \int_t^{\infty} Z(t)P_2 Z^{-1}(s)f(s)x(s)ds - \int_{t_1}^t Z(t)P_1 Z^{-1}(s)f(s)x(s)ds.$$

This is well defined by the above properties, and also

$$\| Tx \| \leq \int_{t_1}^{\infty} |f(s)||x(s)|ds \leq \theta \| x \| .$$

In particular, the map $x \mapsto z + Tx$ for any bounded continuous vector function $z(t)$ is a contraction mapping, so there exists a unique fixed point, i.e. a bounded continuous map $x(t)$ such that

$$x(t) = z(t) + Tx(t). \tag{5}$$

But, $\frac{d}{dt}(Tx)(t) = -\begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}(Tx)(t) - f(t)x(t)$, so that if $z(t)$ satisfies the unperturbed equation (3), then $x(t)$ satisfies the perturbed equation (4), giving us our 1-1 correspondence. Also, equation (5) shows that this correspondence is linear. For bicontinuity, suppose that we have two correspondences, $x_i = z_i + Tx_i, i = 1, 2$, then:

$$\| z_1 - z_2 \| \leq \| x_1 - x_2 \| + \| Tx_1 - Tx_2 \| \leq (1 + \theta) \| x_1 - x_2 \|,$$
$$\| x_1 - x_2 \| \leq \| z_1 - z_2 \| + \| Tx_1 - Tx_2 \|,$$

so that

$$(1 + \theta)^{-1} \| z_1 - z_2 \| \leq \| x_1 - x_2 \| \leq (1 - \theta)^{-1} \| z_1 - z_2 \| \tag{6}$$

which gives us bicontinuity on the interval $[t_1, \infty)$. But since solutions are determined by their initial values, the same applies to $[0, \infty)$.

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Finally, then, notice that $|x - z|(t) = |Tx|(t)$, and given $\epsilon > 0$, we can find $t_2 > t_1$ such that

$$\int_{t_2}^{\infty} \| x \| f(s)ds < \epsilon.$$

Then, for $t > t_2$, we have:

$$\begin{aligned} |Tx(t)| &\leq \epsilon + Z(t)P_1 \int_{t_1}^{t_2} |Z(s)^{-1}f(s)| \, \| x \| \, ds \\ &\leq 2\epsilon, \end{aligned}$$

for t sufficiently large, since $Z(t)P_1 \to 0$ as $t \to \infty$. $\qquad\square$

We can now analyse $\mathcal{F}(A, \Phi) : S^2 \to S^2$ defined in (2) for a given $(A, \Phi) \in \mathcal{B}_k^m$.

Corollary 3.2 *The map $\mathcal{F}(A, \Phi)(u, v)$ is continuous in (u, v).*

Proof. We need to compare the solutions to (1) at nearby geodesics. Since the $U(1)$-invariant connection over S^4 corresponding to the pair (A, Φ) is uniformly continuous and has uniformly continuous first derivatives, the results of Section 2.1 apply to give us L^1 convergence of the scattering coefficients. Put $m_1 = m$ and $m_2 = -m$ into Theorem 3.1 and consider the solutions of

$$\frac{dx}{dt} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} x + f(t)x = 0,$$

$$\frac{dy}{dt} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} y + g(t)y = 0.$$

We wish to show that if $\|f - g\|_{L^1}$ is small enough then the complex lines determined by $x(0)$ and $y(0)$ are close. For the moment let's assume that $\|f\|_{L^1}$ is very small and $g \equiv 0$ so $y = z$, the bounded solution of (3). Then we can choose $t_1 = 0$ in the proof of Theorem 3.1 ($\|f\|_{L^1} < 1$). Thus

$$|x(0) - z(0)| = |Tx(0)| \leq \|f\|_{L^1}\|x\|_{\infty} \ .$$

Of course we need to normalize the vectors so that we are not simply showing that two small vectors have small norm. For this reason, choose $|z(0)| = 1$. Bicontinuity (6) gives us $\|x\|_{\infty} \leq 1/(1 - \|f\|_{L^1})$ so

$$|x(0) - z(0)| \leq \|f\|_{L^1}/(1 - \|f\|_{L^1})$$

and this tends to zero as $\|f\|_{L^1}$ tends to zero as required. More generally, when g is not identically zero, Theorem 3.1 gives the existence of a fundamental matrix solution Y satisfying

$$dY/dt + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} Y + g(t)Y = 0 \ .$$

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The unbounded solution comes from considering the equation

$$\frac{dy}{dt} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} y + f(t)y + mI = 0$$

when we set $m_1 = 2m$ and $m_2 = 0$ in the theorem. Put

$$S = Y \begin{pmatrix} e^{mt} & 0 \\ 0 & e^{-mt} \end{pmatrix}.$$

This is a bounded matrix and $z = S^{-1}y$ satisfies the unperturbed equation (3) so we are back to the previous case of $g \equiv 0$ by substituting x and f with $S^{-1}x$ and $S^{-1}(f - g)S$. Since S is bounded these two substitutions satisfy the requirements. $\qquad\square$

Corollary 3.3 *The map $\mathcal{F} : \mathcal{B}_k^m \to \Omega_k^2 S^2$ is continuous.*

Proof. We use the C^1 topology on $\mathcal{B}_k^m$ and the uniform topology on $\Omega_k^2 S^2$. The proof is just a uniform version of the previous corollary. We could have proven both at once, however this way makes it clear that there are two issues. Again Section 2.1 applies here. Given $(A, \Phi) \in \mathcal{B}_k^m$, we get solutions $y(t)$ along each geodesic that define the continuous self-map of S^2. Let $x(t)$ denote the solutions along geodesics corresponding to a nearby connection. Using the terminology of the proof of the previous corollary, if we choose S to be unitary at $t = 0$ and set $|y(0)| = 1$ we get

$$\begin{aligned} |x(0) - y(0)| &= |S^{-1}(x(0) - y(0))| \\ &\leq \|S^{-1}(f - g)S\|_{L^1}/(1 - \|S^{-1}(f - g)S\|_{L^1}) \\ &\leq M\|f - g\|_{L^1}/(1 - M\|f - g\|_{L^1}) \end{aligned}$$

for some M depending on g (and hence on (A, Φ)) and independent of f. Thus, connections that are C^1 close to (A, Φ) produce continuous maps that are C^0 close. Of course, the rate of convergence depends on $M(A, \Phi)$ as expected. $\qquad\square$

Corollary 3.4 *The map $\mathcal{F}(A, \Phi)(u, v)$ is differentiable in (u, v).*

Proof. In order not to confuse the parameter from the variable t we will notate the parameter value by a subscript. Suppose that in (4) $f_u(t)$ is differentiable in a parameter u, that $\|\partial f_u/\partial u(t)\|_{L^1}$ is uniformly bounded in u and further, for simplicity, that $\|f_u(t)\|_{L^1}$ is small for each u. In order to show that $\partial x_u(0)/\partial u$ exists at $u = 0$ we will study the following equation there.

$$\frac{\partial x_0}{\partial u}(t) = \int_0^\infty Z(t)P_1 Z^{-1}(s)\{\frac{\partial f_0}{\partial u}(s)x_0(s) + f_0(s)\frac{\partial x_0}{\partial u}(s)\}ds.$$

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We think of this as a contraction mapping for $\partial x_0 / \partial u(t)$ since

$$\int_0^\infty Z(t) P_1 Z^{-1}(s) \frac{\partial f_0}{\partial u}(s) x_0(s) ds$$

is bounded in t. We want to show that the unique fixed point of this equation is the actual derivative of x as suggested by the notation.

Lemma 3.5

$$\lim_{u \to 0} \left| \frac{x_u - x_0}{u} - \frac{\partial x_0}{\partial u} \right| (0) = 0 \ .$$

Proof.

$$\left| \frac{x_u - x_0}{u} - \frac{\partial x_0}{\partial u} \right| (0) = \int_0^\infty Z(t) P_1 Z^{-1}(s) [\frac{f_u(s) - f_0(s)}{u}(s) x_0(s)$$
$$- \frac{\partial f_0}{\partial u}(s) x_0(s) + f_0(s) \frac{x_u - x_0}{u} - f_0(s) \frac{\partial x_0}{\partial u}(s)] ds$$

This is a contraction mapping for $(x_u - x_0)/u - \partial x_0 / \partial u$ since

$$\int_0^\infty Z(t) P_1 Z^{-1}(s) [\frac{f_u(s) - f_0(s)}{u}(s) - \frac{\partial f_0}{\partial u}(s)] x_0(s) ds \qquad (7)$$

is bounded. In fact, by the mean value theorem on the derivative of the connection over S^4 (7) tends to 0 as $u \to 0$. Bicontinuity then shows us that $|(x_u - x_0)/u - \partial x_0 / \partial u|$ tends to zero as required. $\square$
As in the proof of Corollary 3.2, the simplifying assumption on the size of $\|f_u(t)\|_{L^1}$ can be removed by using a fundamental matrix solution Y to change the gauge. This completes the proof of the corollary. What we have seen is that the continuous derivative of the connection over S^4 gives both continuity and differentiability of the map on S^2. Still, the continuity is uniform which gives uniform convergence of maps while the limit for the derivative is only pointwise. $\square$

It is also true that when the pair (A, Φ) is smooth then $\mathcal{F}(A, \Phi)$ is smooth.

3.2   Homotopy equivalence.

We will now show that $\mathcal{F}$ is onto and that the degree of $\mathcal{F}(A, \Phi)$ corresponds to the charge of (A, Φ). This will follow from a discussion of the homotopy implications of the map $\mathcal{F}$.

It seems reasonable to conjecture that when restricted to the component $\mathcal{B}_k^m$, the map $\mathcal{F}$ is a homotopy equivalence. We have been unable to show this however there is a deformation retract of $\mathcal{B}_k^m$ where this is true. (After this deformation hyperbolic monopoles in $\mathcal{B}_k^m$ do not map to holomorphic maps in $Rat_k(S^2)$.)

First we will choose a gauge so that we may ignore the action of the gauge group. In each gauge orbit there is a unique connection that is radially trivial. The existence follows from parallel transport and the uniqueness from the fact that gauge transformations are required to be the identity at $0 \in H^3$. Now, define

$$\widetilde{\mathcal{A}}_k^m = \{(A, \Phi) \in \mathcal{B}_k^m | A_t = 0, \ \Phi = \tanh(t)\phi(u, v)\}.$$

This is easily seen to be a deformation retract of $\mathcal{B}_k^m$ using the retraction $\Phi(u, v, t) \mapsto \Phi(u, v, \lambda t) \tanh(t)/\tanh(\lambda t)$ as λ runs from 1 to ∞ and leaving A in radial gauge.

Proposition 3.6 *The restriction*

$$\mathcal{F} : \widetilde{\mathcal{A}}_k^m \to \Omega_k^2 S^2$$

is a homotopy equivalence.

Proof. In order to show that $\mathcal{F}$ is onto we will construct a map

$$\nu : \Omega_k^2 S^2 \to \widetilde{\mathcal{A}}_k^m$$

(defined only on the differentiable maps) such that $\mathcal{F} \circ \nu$ is the identity. Then we will check that the fibre in $\widetilde{\mathcal{A}}_k^m$ of a point in $\Omega_k^2 S^2$ is contractible.

Let $f : S^2 \to S^2$ be a C^1 map. Define $\phi_f : S^2 \to \mathbf{su}(2)$ by composing f with the embedding $S^2 \hookrightarrow \mathbf{su}(2)$ given by mapping S^2 to the sphere of radius m. It satisfies $\phi(w)f(w) = i \cdot m \cdot f(w)$ when we think of f as taking values in the space of non-zero complex lines in $\mathbf{C}^2$. Let $B^k(w)$ be a connection on the line bundle of Chern class k over S^2. Define

$$\nu(f) = (\tanh(t)B^k(w), \tanh(t)\phi_f(w)) \tag{8}$$

It is easy to see that $(\tanh(t)B^k(w), \tanh(t)\phi_f(w)) \in \widetilde{\mathcal{A}}_k^m$ since it extends to a connection over S^4 and

$$\int_{H^3} F_A \wedge d_A \Phi = \int_{S_\infty^2} \langle F_A, \Phi \rangle$$

$$= 2m \int_{S^2} iF_B = 4\pi m k .$$

To calculate $\mathcal{F}(\nu(f))$ we must solve the scattering equation (1). When (A, Φ) is given by (8) we have $\dot{s}(t) = i \tanh(t)\phi_f(w)s(t)$. By construction, $f(w)$ is an eigenvector of $\phi_f(w)$ with eigenvalue $i \cdot m$. Thus $(\cosh(t))^{-m}f(w)$ is the required solution and $\mathcal{F}(\nu(f)) = f$.

The fibre of $f \in \Omega_k^2 S^2$ is given by

$$\{(A, \Phi) \in \widetilde{\mathcal{A}}_k^m \mid \Phi = \tanh(t)m\phi_f\} \tag{9}$$

The fibre is the affine space of radially trivial connections with $U(1)$-reduction at S^2 determined by the Higgs field, which is contractible. $\square$

Corollary 3.7 *For $(A, \Phi) \in \mathcal{B}_k^m$ we have $deg\mathcal{F}(A, \Phi) = k$.*

Proof. Simply move (A, Φ) to an element of $\widetilde{\mathcal{A}}_k^m$ under the contraction. Then use the continuity of the degree, the charge and $\mathcal{F}$. $\qquad\square$

Corollary 3.8 *For each k and m, the space $\mathcal{B}_k^m$ is homotopy equivalent to $\Omega_k^2 S^2$.*

Proof. This just combines the homotopy equivalence defined by $\mathcal{F}$ and the retraction described at the beginning of this section. We have also used the fact that the differentiable self-maps of S^2 define a subspace homotopy equivalent to the space of all continuous maps. $\square$

4 Monopoles.

From now on we will study only those pairs (A, Φ) (with integral mass) that satisfy the Bogomolny equations

$$d_A \Phi = *F_A$$

on H^3. We will show that the monopoles correspond to rational maps and this correspondence is a diffeomorphism.

We write the Bogomolny equations $*F_A = d_A \Phi$ in these coordinates: put $F_A = F_{uv} du \wedge dv + F_{ut} du \wedge dt + F_{vt} dv \wedge dt$, and so we get three equations:

$$F_{ut} = -\nabla_v \Phi, F_{vt} = \nabla_u \Phi, F_{uv} = \frac{4\sinh^2(t)}{(1 + |w|^2)^2} \nabla_t \Phi.$$

We can re-express the first two of these equations in terms of w: if we set $\bar{\partial}_w^A = \frac{1}{2}(\nabla_u + i\nabla_v)$ then the two equations $[\nabla_u, \nabla_t] = -\nabla_v \Phi$ and $[\nabla_v, \nabla_t] = \nabla_u \Phi$ appear as the real and imaginary parts of the 'integrability condition':

$$[\bar{\partial}_w^A, \nabla_t - i\Phi] = 0.$$

It is this fact which enables us to prove analytically that when (A, Φ) is a hyperbolic monopole the scattering definition of a map $S^2 \to CP^1$ is indeed a *rational* map.

Proposition 4.1 *The restriction of $\mathcal{F}$ to monopoles defines a map*

$$\mathcal{F} : \mathcal{M}_k^m \to Rat_k(S^2) \ .$$

Proof. Let (A, Φ) satisfy the Bogomolny equations and consider the bounded solution of the scattering equation

$$(\nabla_t - i\Phi)s = 0.$$

Since $[\bar{\partial}_w^A, \nabla_t - i\Phi] = 0$ and $s(w, t)$ is differentiable in w by Lemma 3.4, then $\bar{\partial}_w^A s(w, t)$ is also a solution of the scattering equation. Moreover, $\bar{\partial}_w^A s(w, t)$ is bounded in t since the derivative is bounded by construction, and the connection matrix $A_{\bar{w}}$ is also bounded. Since there is exactly one bounded solution of the scattering equation we must have $\bar{\partial}_w^A s(w, t) = \lambda(w)s(w, t)$ for some function λ independent of t. At $t = 0$ the connection matrix $A_{\bar{w}}(w, t)$ vanishes due to the choice of coordinate system, thus

$$\bar{\partial}_w s(w, 0) = \lambda(w)s(w, 0)$$

which is exactly the condition that a family of complex lines be holomorphic. By Corollary 3.7 the map $s(w, 0)$ has degree k so it is a rational map. $\qquad\square$

4.1   Uniqueness of the rational map

In this section, we show that each rational map $CP^1 \to CP^1$ is associated to at most one monopole (up to framed gauge equivalence). This is fairly straightforward: we show that a monopole, and more generally any pair (A, Φ) satisfying 2/3 of the Bogomolny equations, is determined by a 'metric' H on a trivialized bundle over H^3. Furthermore, if (A, Φ) is a monopole then H satisfies a certain differential equation. There is also a relative version of this: if H_1, H_2 are two monopoles, then $h = H_1^{-1}H_2$ satisfies a similar differential equation. Also, there is a measure $\sigma(h) = tr(h) - 2$ of the size of $h - I$, and hence the distance between H_1 and H_2, which satisfies a differential inequality $D(\sigma) \leq 0$.

Finally, we notice that if two monopoles have the same rational map, then $\sigma(h)$ is *bounded* on H^3, and that the only such solutions of $D(\sigma) \leq 0$ are constants. Since $h(0) = I$, it follows that σ must be identically zero, and that the two monopoles must be framed gauge equivalent.

First of all, then, suppose we have a pair (A, Φ) satisfying the integrability condition

$$[\bar{\partial}_w^A, \nabla_t - i\Phi] = 0.$$

Suppose also that it is framed at the origin. Then we simply solve the equation

$$(\nabla_t - i\Phi)g = 0$$

for $g \in \text{End}(E)$, starting with $g(0) = I$, given by the framing. It follows that the equation $\bar{\partial}_w^A g = 0$ is then automatically satisfied for $t > 0$. Set $H = g^* g$. This is the metric in the gauge defined by g, which depends only on the framing. It is straightforward to calculate A and Φ in terms of H in this new gauge: $A^{0,1} = A_t - i\Phi = 0$ by definition, and since the connection is unitary, we get $A^{1,0} = H^{-1} \partial_w H$, $A_t + i\Phi = H^{-1} \partial_t H$. (For example, we can transform to a unitary gauge by g^{-1}, where the (0,1) part of the connection must be $-\bar{\partial}_w g . g^{-1}$. Thus by unitarity the w part of the connection form must be $-\bar{\partial}_w g g^{-1} + g^{*-1} \partial g^*$, and transforming back by g, we get $g^{-1}(-\bar{\partial}_w g g^{-1} + g^{*-1} \partial_w g^*) g + g^{-1}(\partial_w + \bar{\partial}_w) g = h^{-1} \partial_w h$, as claimed).

By differentiating these formulae, we get formulae for the curvature tensor, the Bogomolny tensor $B(A, \Phi) = *F_A - d_A \Phi$:

$$
B(A, \Phi) \quad = \quad \frac{idt}{2} \left(\frac{(1 + |w|^2)^2}{\sinh^2(t)} \bar{\partial}_w (H^{-1} \partial_w H) + \partial_t (H^{-1} \partial_t H) \right).
$$

Next, we have the relative version of this: if H_1, H_2 are two metrics obtained as above, then write $(A_1, \Phi_1), (A_2, \Phi_2)$ for the two pairs on the trivialized bundles (which we can thus identify if we wish). Set $h = H_1^{-1} H_2 \in \text{End}(E)$. Then we have

$$
\partial_w^{A_1} + h^{-1} \partial_w^{A_1} h = \partial_w^{A_1} + h^{-1} \partial_w h + h^{-1}[H_1^{-1} \partial_w H_1, h] = \partial_w^{A_2},
$$

and so by the same calculation as before, or by replacing H with h, and ∂ by ∂^{A_1} we get the gauge-invariant formula:

$$
\begin{aligned}
B(A_2, \Phi_2) \quad = \quad & B(A_1, \Phi_1) \\
& + \frac{idt}{2} \left(\frac{(1 + |w|^2)^2}{\sinh^2(t)} \bar{\partial}_w^{A_1} (h^{-1} \partial_w^{A_1} h) + \partial_t^{A_1} (h^{-1} \partial_t^{A_1} h) \right).
\end{aligned}
$$

Thus, if (A_1, Φ_1) and (A_2, Φ_2) are two monopoles, then we get a quantity h satisfying the (almost elliptic) equation

$$
(1 + |w|^2)^2 \bar{\partial}_w^{A_1} (h^{-1} \partial_w^{A_1} h) + \sinh^2(t) \partial_t^{A_1} (h^{-1} \partial_t^{A_1} h) = 0.
$$

Define
$$
D = (1 + |w|^2) \bar{\partial}_w \partial_w + \sinh^2(t) \partial_t^2.
$$

If we multiply the last equation for h on the left by h, and take traces, we are able to deduce:

Lemma 4.2 *With $h = H_1^{-1} H_2$ as above coming from two monopoles,*

$$
D(tr(h)) \geq 0
$$

on $H^3 - \{0\}$.

Remarks: (i) We have excepted the origin here since our coordinate system and D blow up there. However, $h = I$ there, so it will not concern us.

(ii) This lemma, and indeed the whole line of attack, draws inspiration from [9].

Proof. Since $\partial_w(tr(h)) = tr(\partial_w^A h)$, and so on, we have

$$D(tr(h)) = (1 + |w|^2)^2 tr\left(\bar{\partial}_w^{A_1} h.h^{-1}\partial_w^{A_1} h\right) + \sinh^2(t) tr(\partial_t^{A_1} h.h^{-1}\partial_t^{A_1} h).$$

Near any point, we can find an H_1-orthogonal basis in which h is diagonal. In such a basis, $\partial_t^{A_1} h$ and $\partial_w^{A_1} h$ will be Hermitian-adjoint to $\partial_t^{A_1} h$ and $\bar{\partial}_w^{A_1} h$ respectively. It follows that the right-hand side is positive. (The quantities involved are clearly gauge-invariant and real). $\qquad\square$

The crucial ingredient now is:

Proposition 4.3 *If (A_1, Φ_1) and (A_2, Φ_2) are framed monopoles that have the same rational map, then $tr(h)$ is uniformly bounded on H^3.*

Then we use

Lemma 4.4 *If σ is a bounded real-valued function on H^3 which satisfies $D(\sigma) > 0$ whenever $\sigma > 0$, and $\sigma(0) = 0$, then in fact $\sigma \leq 0$ on all H^3.*

Remarks:(i) This is supposed to be just the maximum principle for this Laplacian-like operator.

(ii) Setting $\sigma = tr(h) - 2$, it follows immediately that at most one monopole can give rise to any particular rational map, since $\sigma = 0$ implies that $h = I$.

Proof of Lemma 4.4. First of all notice that $D(a+bt) = 0$ for any $a, b \in R$. Hence, for any $a, b > 0$, $\tau = \sigma - a - bt$ satisfies $D(\tau) \geq 0$, and $\tau(0) < 0, \tau(w, t) < 0$ for all $t \gg 0$. It suffices to prove that $\tau \leq 0$ everywhere, for letting $a, b \to 0$, the lemma follows. So suppose not for a contradiction: then there is some *finite* point x where τ is positive and has a maximum. Also, $\tau < 0$ for $t < t_0$ and $t > t_1$, say. On the complement of this set, D is uniformly elliptic, and so τ cannot have an interior maximum by the maximum principle. Thus $\tau < 0$ everywhere, as required. $\qquad\square$

Proof of Proposition 4.3. Suppose that the rational map is represented by the family of complex lines $\alpha(w) = (a(w), b(w))$ where $|\alpha(w)$. Then there is some bounded function $\lambda(w)$ on CP^1 such that the section satisfying $(\nabla_t - i\Phi)s = 0$ and $s(0) = \alpha(w)$ looks like $\lambda(w)e^{-mt}e_1$ for some unit vector e_1 as $t \to \infty$.

We can find $\beta(w)$ such that $|\beta(w)| = 1$ and the two vectors $\alpha(w)$ and $\beta(w)$ form the columns of a unitary matrix $F^{-1}(w)$, say. If we

solve $(\nabla_t - i\Phi)s = 0$ starting at $p\alpha(w) + q\beta(w)$, with $q \neq 0$, then s looks like $q\lambda(w)^{-1}e^{mt}e_2$ for a unit vector e_2 orthogonal to e_1. Thus starting with $g(0) = F^{-1}$, $g(t)^*g(t)$ looks like

$$\Lambda = \begin{pmatrix} e^{-mt}\lambda(1 + o(1)) & o(1) \\ o(1) & e^{mt}\lambda^{-1}(1 + o(1)) \end{pmatrix}$$

as $t \to \infty$.

And so, starting instead with $g(0) = I$, the term $g(t)^*g(t)$ looks like $F(w)^*\Lambda F(w)$ as $t \to \infty$. But this is of course just our H. Thus the asymptotic behaviour is determined by F, which depends only on the rational map, and a bounded function λ. (A different choice of F will only change the $o(1)$ terms, or λ).

Finally, then, suppose we have two metrics H_i, $i = 1, 2$, coming from monopoles with the same rational map. Then $h = H_1^{-1}H_2$ looks like $F^{-1}\Lambda_1^{-1}\Lambda_2 F$ as $t \to \infty$. Here Λ_i are as above, depending on perhaps different functions λ_i. An easy check shows that $tr(h)$ is indeed bounded as $t \to \infty$ (indeed the limit is determined only by λ_1/λ_2), as required. $\qquad \square$

4.2 A linearized version of uniqueness

In this section, we shall describe the linearized map from the moduli space of monopoles to the moduli space of rational maps. That is, we describe the derivative of our map, as a map on tangent spaces. We will show that the derivative is an isomorphism at each point. This will allow us to employ the inverse function theorem and deduce that the image of $\mathcal{F}$ is open.

First of all, we need to describe what the tangent spaces are. Let R_k denote the space of rational maps from CP^1 to itself of degree k. Then clearly

$$\begin{aligned} T_f(R_k) &= H^0(CP^1, f^*(T(CP^1))) \\ &= H^0(CP^1, f^*(\mathcal{O}(2))) \cong H^0(CP^1, \mathcal{O}(2k)), \end{aligned}$$

which has dimension $4k + 2$.

Linearising the Bogomolny equations, we get the following condition on a tangent vector (a, ϕ) to the space of pairs at (A, Φ):

$$*d_A a = d_A \phi + [a, \Phi].$$

Braam [6] described the tangent space of the moduli space of hyperbolic monopoles in detail. He represented the pairs (a, ϕ) as $U(1)$-invariant elements of the kernel of a Dirac operator on S^4. The dimension he gets, from an equivariant index calculation, is $4k - 1$. Since

we have an extra factor of $\mathbf{su}(2)$ coming from infinitesimal change of the frame we get a $4k + 2$-dimensional tangent space as expected.

Note. One might impose the condition of orthogonality to compactly-supported gauge transformations, to give a second equation:

$$*d_A * a + [\Phi, \phi] = 0.$$

Over $\mathbf{R}^3$, Taubes [21] has shown that the resulting space has dimension 4k, and includes the special vector $d_A\Phi$, corresponding to a non-compactly-supported gauge transformation. David Stuart [20] has shown that the vectors (a, ϕ) in this 4k-dimensional space are fairly well-behaved: they're all in L^2, and indeed when split into the usual 'longitudinal' and 'transverse' components, the former decay like r^{-2}, and the latter decay exponentially. As pointed out by Austin and Braam [5], the different values of the curvature of hyperbolic monopoles on the sphere at infinity prevent a similar result from holding over H^3.

Lemma 4.5 *For any smooth pair (a, ϕ) satisfying the linearized Bogomolny equations at a monopole (A, Φ), one can define $\eta \in \Omega^0(sl_2)$ by $\eta(0) = 0$ and*
$$(\nabla_t - i\Phi)\eta = -(a_r - i\phi)$$
along rays out of the origin. Then
$$(1 + |w|^2)^2 \bar{\partial}_w^A \partial_w^A(\eta + \eta^*) + \sinh^2(t)(\partial_t^A - i\Phi)(\partial_t^A + i\Phi)(\eta + \eta^*) = 0.$$

Remark: If (a, ϕ) comes from a sequence (A_t, Φ_t) of monopoles, then we get a sequence g_t satisfying $(\nabla_t^{A_t} - i\Phi_t)g_t = 0$, and η will be $\partial_t g \cdot g^{-1}$, evaluated at $t = 0$. Also, setting $H_t = g_t^* g_t$, $g^{-1}(\eta + \eta^*)g = H^{-1}\partial_t H$. We don't need to assume this sequence, though, and this remark merely serves as motivation.

Proof. We differentiate the defining equation for η with respect to w, to give:
$$\bar{\partial}_w^A(\partial_t^A - i\Phi)\eta = -\bar{\partial}_w^A(a_t - i\Phi).$$

The Bogomolny equations tell us that $[\partial_t^A - i\Phi, \bar{\partial}_w^A] = 0$, and so the linearized equations (just differentiate this) tell us that

$$(\partial_t^A - i\Phi)(a_{\bar{w}}) = \bar{\partial}_w^A(a_t - i\Phi),$$

where here we have written $a_{\bar{w}}$ for the $d\bar{w}$ component of a. Hence

$$(\partial_t^A - i\Phi)(\bar{\partial}_w^A\eta) = -(\partial_t^A - i\Phi)(a_{\bar{w}}).$$

But $a_{\bar{w}} = \bar{\partial}_w^A\eta = 0$ at $t = 0$, so we deduce that $a_{\bar{w}} = \bar{\partial}_w^A\eta$ for all (r, w). We can take the adjoints of these equations to give us:

$$(\partial_t^A + i\Phi)\eta^* = (a_t + i\phi), \quad \partial_w^A\eta^* = a_w.$$

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We now use the third of the linearized Bogomolny equations, namely

$$(\partial_t^A - i\Phi)(a_r + i\phi) - (\partial_t^A + i\Phi)(a_r - i\phi) = \frac{-(1+|w|^2)^2}{\sinh^2(t)}(\bar{\partial}_w^A a_w - \partial_w^A a_{\bar{w}}).$$

Putting all this in terms of η, then, we get:

$$\left((\partial_t^A - i\Phi)(\partial_t^A + i\Phi) + \frac{(1+|w|^2)^2}{\sinh^2(t)}\bar{\partial}_w^A \partial_w^A \right) (\eta + \eta^*)$$
$$= \left[\frac{i(1+|w|^2)^2}{2\sinh^2(t)} F_{w\bar{w}} + \partial_t \Phi, \eta \right],$$

and the right hand side is zero since (A, Φ) satisfies the Bogomolny equations. $\qquad\square$

Corollary 4.6 *With η as in lemma 4.5,*

$$D(|\eta + \eta|^2) \geq 0.$$

Proof. We have $|\eta + \eta^*|^2 = tr((\eta + \eta^*)^2)$. Writing $\langle B, C \rangle = tr(BC)$, it is easy to see that $\bar{\partial}_w \partial_w \langle B, B \rangle = 2\langle \bar{\partial}_w^A \partial_w^A B, B \rangle + 2\langle \partial_w^A B, \bar{\partial}_w^A B \rangle$. With $B = (\eta + \eta^*)$, $B = B^*$, and so $\langle \partial_w^A B, \bar{\partial}_w^A B \rangle \geq 0$. A similar calculation with $(\partial_w^A, \bar{\partial}_w^A)$ replaced by $(\partial_t + i\Phi, \partial_t - i\Phi)$, together with the previous lemma, then proves the claim. $\qquad\square$

With this corollary and our previous knowledge of D, we see that if η is bounded, then $|\eta + \eta^*|$ must be constant (and so zero). Notice that if $\eta = -\eta^*$, then

$$a = -d_A\eta, \quad \phi = -[\Phi, \eta],$$

that is, η corresponds to an infinitesimal gauge transformation. We shall show that η determines a tangent vector to the space of rational maps, and that if this tangent vector is zero then η is bounded on H^3.

Proposition 4.7 *If the image of (a, ϕ) in $T_f(R_k)$ is zero, then η, as defined above, is bounded on H^3, and so by the preceding corollary, $\eta = -\eta^*$.*

Proof. Over each ray $w = $ constant, consider sections s_τ which satisfy $(\partial_r^{A_\tau} - i\Phi_\tau)s_\tau = 0$, and which decay as $t \to \infty$, for a (smooth) path of pairs (A_τ, Φ_τ). We proved in Section 3 that s_τ depends differentiably on τ. The value of $f_\tau(w) \in CP^1$ is then represented by $s_\tau(0) \in C^2$. Suppose that $s_0(0) = (a, b)$ and $\partial_\tau s_\tau(0)|_{\tau=0} = (c, d)$. The value of $\partial_\tau(f_\tau)(w)$ is proportional to $ad - bc$.

In terms of the tangent vector $(a, \phi) = (\partial_\tau(A_\tau), \partial_\tau(\Phi_\tau))$, then, we have to consider the differential equation

$$(\nabla_t - i\Phi)\sigma = -(a_t - i\Phi)s,$$

where s is a solution to $(\nabla_t - i\Phi)s = 0$ which decays as $t \to \infty$ along each line. This will have a one-dimensional space of solutions which decay as $t \to \infty$ (we are using some nice decay properties of (a, ϕ) here), each element differing by a multiple of s, and in fact decaying at least as fast as e^{-mt} as $t \to \infty$. Thus setting $s = (a, b)$ and $\sigma = (c, d)$ as before, the quantity $ad - bc$ will depend only on s, and whether it is zero or not will depend only on (a, ϕ).

So we have to deal with the situation where σ as defined above is such that $\sigma(0)$ is proportional to $s(0)$. First of all, notice that

$$(\nabla_t - i\Phi)(\eta g \cdot s(0)) = -(a_t - i\Phi)s.$$

Here we let ∇_t and Φ act on $\eta.g(s(0))$ by multiplication on the left (rather than the adjoint action that we were using for η before). This equation follows by using $(\nabla_t - i\Phi) = g \circ \partial_t \circ g^{-1}$, and $s = g \cdot s(0)$.

Since $\eta(0) = 0$, $\eta g \cdot s(0)$ will be σ plus a multiple of s in the case we are considering. Thus we have proved that $\eta \cdot s$ decays like e^{-mt} as $t \to \infty$. The proof is thus completed by the following lemma. $\square$

Lemma 4.8 *With s, η as above, if $\eta \cdot s$ decays like e^{-mt} as $t \to \infty$ then η is bounded on H^3.*

Remark: Since η is continuous, it suffices to show that it is bounded as $t \to \infty$ along each line.

Proof. Applying Levinson's theorem to the adjoint representation, one gets a (complex) basis e_i for sl_2 such that $g(e_1)g^{-1}$ tends to a constant, $g(e_2)g^{-1}$ grows, and $g(e_3)g^{-1}$ decays as $t \to \infty$. Indeed one can find $\lambda \in SL_2(C)$ such that $e_i = Ad(\lambda)(f_i)$, where f_i is the following basis for sl_2:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Put another way, we can set $\hat{g}(t) = \begin{pmatrix} e^{-mt} & 0 \\ 0 & e^{mt} \end{pmatrix}$, and then we can find λ such that $Ad(g\lambda\hat{g}^{-1})(f_i)$ tends to a (non-zero) constant as $t \to \infty$ for each i. Hence $g\lambda\hat{g}^{-1}$ itself tends to a constant (in $SL_2(C)$) as $t \to \infty$.

Thus we can write $\eta = \sum_i \eta_i(t)Ad(g\lambda)(f_i)$, and we have $s = g\lambda(1, 0)$, up to a constant which we can ignore. So $\eta \cdot s = g\lambda(\eta_1, \eta_2)$, and since $g\lambda\hat{g}^{-1}$ tends to a constant, we deduce that $e^{-mt}\eta_1$ and $e^{mt}\eta_2$

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decay as $t \to \infty$. We thus deduce that η_2 must decay, at least as fast as e^{-2mt}. (Notice that $Ad(g)(e_2)$ grows like e^{2mt}, so this tells us that the second component of η is $O(1)$, as we require).

To get the desired bound on the remaining components of η is more straightforward: We have $\eta_i(0) = 0$, and $(\partial_t^A - i\Phi)\eta = -(a_t - i\Phi)$ becomes

$$\sum_i \partial_t(\eta_i) Ad(g\lambda)(e_i) = Ad(g\lambda\hat{g}^{-1})(O(t^{-2})),$$

so we get

$$\eta_1 = \int_0^t O(r^{-2})dr, \; \eta_2 = \int_0^t O(r^{-2}) \cdot e^{-2mr}dr, \; \eta_3 = \int_0^t O(r^{-2}) \cdot e^{2mr}dr.$$

So we are able to deduce that η_1, η_2 tend to constants as $t \to \infty$, and see that η_3 grows no faster than e^{2mt}. Hence $\eta_i Ad(g)(e_i)$ is bounded for $i = 1, 3$. Together with the previous result for $i = 2$, we are done.□

4.3 Surjection.

Here we complete the proof of Theorem 2 by showing that the image of $\mathcal{F}$ is closed and combining this with the result of the previous section that the image of $\mathcal{F}$ is open.

Proof of Theorem 2. Since we know something about the compactification of the space of monopoles we can show that the image of $\mathcal{F}$ is closed. Let $\{f_n\} \subset \text{image}\mathcal{F}$ be a sequence that is Cauchy in the C^0 topology in $Rat_k(S^2)$. It converges to $f \in Rat_k(S^2)$ and the question is whether f lies in the image of $\mathcal{F}$. Let $\{(A_n, \Phi_n)\}$ be a sequence of monopoles mapping under $\mathcal{F}$ respectively to f_n. There is a subsequence of this sequence of monopoles that converges to an ideal monopole (A, Φ). The continuity argument in Corollary 3.2 extended to the compactification then implies that $\mathcal{F}(A, \Phi) = f$ as required (and in particular that (A, Φ) is a true monopole). An essentially equivalent way of looking at this is to leave the closedness issue until after the compactification result ($\mathcal{F}$ extends) so closedness of the image simply follows from the continuity of $\mathcal{F}$.

Now, the C^0 norm gives a Banach manifold structure to the space of holomorphic maps. We have shown that with respect to this norm the map

$$\mathcal{F} : \mathcal{M}_k^m \to Rat_k(S^2)$$

is differentiable. The linearized uniqueness proof together with the fact that the dimensions of the respective tangent spaces are $4k + 2$ shows us that at each point $D\mathcal{F}$ is an isomorphism. Thus we can invoke the inverse function theorem so $\mathcal{F}$ is locally a homeomorphism and in particular its image is open.

Since the space of rational maps of a fixed degree is connected it remains to show that the image of $\mathcal{F}$ is non-empty in each component so then an open and closed non-empty set must consist of the entire component and $\mathcal{F}$ is onto. Atiyah [1] proved the existence of hyperbolic monopoles of integral mass and arbitrary charge. The techniques used in [13] for Euclidean monopoles adapt to the hyperbolic case to give another proof of existence for arbitrary mass as well as an alternative proof that $\mathcal{F}$ is onto. The global uniqueness then completes the proof of Theorem 2. □

5   Compactification.

In this section we will study the compactification of the space of hyperbolic monopoles. As usual we will think of a hyperbolic monopole as an instanton on S^4 invariant under the circle action.

Theorem 5.1 (Uhlenbeck) *There is a constant $\epsilon > 0$ such that if $\widetilde{A}$ is an anti-self-dual connection over B^4 with $\|F_{\widetilde{A}}\|_{L^2(B^4)} < \epsilon$ then we can put $\widetilde{A} = d + a$ in Coulomb gauge with*

$$\|a\|_{W^{l,2}(B^4_{1/2})} \leq c_l \|F_{\widetilde{A}}\|_{L^2(B^4)} \ .$$

The following lemma shows that we can apply Uhlenbeck's theorem at each point of $H^3 \times S^1$.

Lemma 5.2 *Let $\widetilde{A}$ be a $U(1)$-invariant instanton on $H^3 \times S^1$ of charge $8\pi^2 mk$. Given ϵ as in Theorem 5.1 and $x \in H^3 \times S^1$ then the ball B around x of radius $\epsilon \mathrm{sech}^2(t/2)/16\pi mk$, where t is the hyperbolic distance of the component of x in H^3 from $0 \in H^3$ satisfies $\|F_{\widetilde{A}}\|_{L^2(B)} < \epsilon$.*

Proof. The proof follows from the rather trivial observation that the radius of the circle through x is $\mathrm{sech}^2(t/2)/2$ so there are $\epsilon/8\pi^2 mk$ disjoint balls each of which contributes the same to the charge by $U(1)$-invariance. Since the total charge is $8\pi^2 mk$ a ball can contribute at most ϵ. □

On the fixed $S^2 \subset S^4$ we can use Uhlenbeck's theorem except at finitely many points. This follows from the usual convergence of measures argument. Thus, we can put any family of monopoles in Coulomb gauge and get uniform bounds on the 1-forms and their derivatives. By the Arzela-Ascoli theorem there is a uniformly (in fact smoothly) convergent subsequence and the limiting 1-form represents an anti-self-dual connection. We can see that the limiting instanton is a monopole, i.e. it is $U(1)$-invariant, in two ways. Braam [6] showed

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that the sequence of gauges chosen by Uhlenbeck can be chosen to be $U(1)$-equivariant so that the limiting connection is $U(1)$-invariant. He also showed that the singularity is removed with a $U(1)$-equivariant gauge transformation. Alternatively, we can use the estimates obtained in four dimensions on the connection in three dimensions together with uniform estimates on the Higgs field to get convergence on H^3. The uniform estimates on the Higgs field follow easily from the fact that it satisfies the maximum principle. Braam also showed that a sequence of monopoles loses charge in multiples of m.

Thus, the moduli space of hyperbolic monopoles can be compactified by considering ideal connections

$$\overline{\mathcal{M}}_k^m = \cup_l \mathcal{M}_l^m \times S^{k-l}(S^2)$$

where $S^{k-l}(S^2)$ is the $(k-l)$-fold symmetric product of the two-sphere. Similarly we can define a compactification of $Rat_k(S^2)$ by

$$\overline{Rat}_k(S^2) = \cup_l Rat_l(S^2) \times S^{k-l}(S^2) \ .$$

These spaces are topologized using limits of sequences. We have $\overline{Rat}_k(S^2) \cong \mathbf{CP}^{2k+1}$ [8].

Away from the singular points on S^2 we have uniform convergence so the results of Section 3 apply to show that locally the rational maps converge to the expected rational map. Thus, the limiting monopole of lower charge corresponds, as expected, to the limiting rational map of lower charge. It remains to show that the points in $S^{k-l}(S^2)$ correspond.

Lemma 5.3 *Given $x \in S^2$ and a sequence of monopoles $\{(A_n, \Phi_n)\}$ with corresponding rational maps $\{\mathcal{F}(A_n, \Phi_n)\}$ the monopoles bubble off at x if and only if the rational maps bubble off at x.*

Proof. By the results of Section 3 if $\{(A_n, \Phi_n)\}$ does not bubble off at x then there is a uniformly convergent subsequence in a neighbourhood of the geodesic γ joining x to $0 \in H^3$ and thus the solutions of $(\nabla_t^n - i\Phi_n)s_n = 0$ along γ converge to a non-trivial solution of $(\nabla_t^\infty - i\Phi_\infty)s_\infty = 0$. For the rational map to bubble off $s_\infty(0)$ must vanish. Thus $s_\infty(t)$ would vanish identically, contradicting the non-triviality of the limit.

In the other direction, if the sequence of rational maps does not bubble off at x then we can define a sequence of good gauges in a neighbourhood of the geodesic γ as in Section 4.1 that converge to a good gauge. With respect to these good gauges the monopoles are smooth. $\qquad\qquad\square$

Proposition 5.4 *Let* $\{(A_n, \Phi_n)\}$ *be a sequence of monopoles that converges to the ideal monopole* $(A_\infty, \Phi_\infty, P)$ *where* $(A_\infty, \Phi_\infty) \in \mathcal{M}_l^m$ *and* $P \in S^{k-l}(S^2)$. *Then* $\{\mathcal{F}(A_n, \Phi_n)\}$ *converges to the ideal rational map* $(\mathcal{F}(A_\infty, \Phi_\infty), P)$.

Proof. We will solve the scattering equation (1) for (A_∞, Φ_∞) in the gauge supplied by Uhlenbeck's compactness theorem *before* removing the singularity with a gauge transformation. Suppose the rational map loses charge l_j at $x_j \in S^2$. For z in a neighbourhood of x_j the solution $s_z(t)$ of (1) along the geodesic defined by z looks like

$$((z - x_j)^{l_j} p(z), (z - x_j)^{l_j} q(z))$$

when evaluated at $t = 0$. Choose the unitary gauge transformation g in this neighbourhood defined by multiplying the one-dimensional subspace generated by $s_z(t)$ by $\exp(-il_j\theta)$ and multiplying the orthogonal subspace by $(\exp il_j\theta)$. It is easy to see that this has removed the singularity of the rational map. By Lemma 5.3 this new gauge is non-singular. Kronheimer and Mrowka [18] show that the winding number of g at x_j times the mass exactly measures charge of the monopole that is lost. Thus the result follows. $\qquad\square$

We can use the topology on $\mathbf{CP}^{2k+1}$ and the map $\mathcal{F}$ to induce a good topology on the compactification of the space of monopoles. What we have shown in this section is that this topology agrees with the C^1 topology away from ideal connections. We have proven Theorem 3, i.e.

$$\mathcal{F} : \mathcal{M}_k^m \to Rat_k(S^2)$$

extends to a continuous map between their compactifications.

6 The μ-map on geodesics.

In this section we will show how to associate a codimension two submanifold in the moduli space to a geodesic in H^3. This is the analogue of the μ-map of Donaldson.

Let $\gamma : \mathbf{R} \to H^3$ be a geodesic. Define

$$\mathcal{G}_\gamma : \mathcal{M}_k^m \times \mathbf{P}W^{1,2}(\mathbf{R}) \to L^2(\mathbf{R})$$

by $\mathcal{G}_\gamma(A, \Phi, s) = \gamma^*(\nabla^A - i\Phi)s$. By $\mathbf{P}W^{1,2}(\mathbf{R})$ we mean the projective space $(W^{1,2}(\mathbf{R})\backslash\{0\})/\mathbf{C}^*$. Put $V_\gamma = \pi(\mathcal{G}_\gamma^{-1}(0))$ where the map

$$\pi : \mathcal{M}_k^m \times \mathbf{P}W^{1,2}(\mathbf{R}) \to \mathcal{M}_k^m$$

gives projection onto the first factor. So V_γ is the space of monopoles that have γ as a 'spectral line'.

Lemma 6.1 *The subspace V_γ is a submanifold of codimension* 2.

Proof. This lemma follows from the fact that along each geodesic the scattering operator is an index 0 Fredholm operator with kernel at most 1-dimensional. We omit some details since the lemma will also follow from the symmetry argument in Lemma 6.3. The map $\mathcal{G}_\gamma$ is the section of a bundle over $\mathcal{M}_k^m \times \mathbf{P}W^{1,2}(\mathbf{R})$ with linearisation given by

$$D_{(A,\Phi,s)}\mathcal{G}_\gamma(a,\phi,\sigma) = (a_t - i\phi)s + (\nabla_t^A - i\Phi)\sigma$$

for $(a,\phi,\sigma) \in T_{(A,\Phi)}\mathcal{M}_k^m \times T_s W^{1,2}(\mathbf{R})$. This is a surjective Fredholm operator so $\mathcal{G}_\gamma$ is transverse to the zero section and its zero set is a submanifold. The restriction of the projection π to this zero set defines a homeomorphism to V_γ. The dimension of V_γ follows from the calculation of the index of $D\mathcal{G}_\gamma$. Alternatively, one can use $D\mathcal{G}_\gamma$ to construct a determinant line bundle to see the (real) codimension 2 property more easily. $\qquad\square$

Lemma 6.2 *For any two geodesics γ and γ' there is an isomorphism $V_\gamma \cong V_{\gamma'}$.*

Proof. First notice that $SL(2,\mathbf{C})$, the isometries of H^3, acts transitively on the space of geodesics in H^3. Since the submanifold V_γ does not depend on the framing of the bundle the induced $SL(2,\mathbf{C})$ action on $\mathcal{M}_k^m/SU(2)$ gives the result. $\qquad\square$

Proposition 6.3 *The submanifold V_γ extends to the compactification of $\mathcal{M}_k^m$.*

Proof. By Lemma 6.2 we can choose γ to contain $0 \in H^3$. Let $(A,\Phi) \in V_\gamma$. Then there exists $s \in W^{1,2}(\mathbf{R})$ with

$$(\nabla_t - i\Phi)s = 0 \tag{10}$$

where t parametrizes γ. For $t > 0$ this is just the scattering equation (1) so $s(0)$ gives $\mathcal{F}(A,\Phi)$ evaluated at the point z corresponding to γ. Put $\tau = -t$ for $t < 0$. Then (10) becomes $(\nabla_\tau + i\Phi)s = 0$ along the geodesic corresponding to $\hat{z}$, the antipodal point of z. Let $r \in W^{1,2}(\mathbf{R}^+)$ satisfy $(\nabla_\tau - i\Phi)r = 0$ so $r(0)$ gives $\mathcal{F}(A,\Phi)$ evaluated at $\hat{z}$. Then

$$
\begin{aligned}
0 &= \int_0^\infty \langle (\nabla_\tau + i\Phi)s, r \rangle d\tau \\
&= \int_0^\infty d\langle s, r \rangle/d\tau - \int_0^\infty \langle s, (\nabla_\tau - i\Phi)r \rangle d\tau \\
&= -\langle s(0), r(0) \rangle
\end{aligned}
$$

so (in affine coordinates)

$$\mathcal{F}(A, \Phi)(z)\overline{\mathcal{F}(A, \Phi)(\hat{z})} = -1 \qquad (11)$$

The converse is also true: if (A, Φ) satisfies (11) then there exists $s \in W^{1,2}(\mathbf{R})$ satisfying (10) along γ.

This is easily seen to be a codimension 2 submanifold of the space of rational maps and hence of the space of monopoles. Furthermore, this description makes it clear that V_γ extends to a closed submanifold in the compactification since (11) makes sense in $\mathbf{CP}^{2k+1}$. $\qquad\square$

Since $SL(2, \mathbf{C})$ is the group of isometries of H^3 it acts on the space of (unframed) monopoles. If we choose a trivialisation of the bundle over H^3 then the action lifts to an action on the space of framed monopoles. This induces an action of $SL(2, \mathbf{C})$ on $Rat_k(S^2)$ which is a perturbation of the natural action of $SL(2, \mathbf{C})$ on $Rat_k(S^2)$ coming from acting on the domain S^2. The two actions coincide when restricted to the subgroup $SU(2)$. In order to see V_γ or the spectral curve of a monopole more explicitly we would need to better understand the perturbed action of $SL(2, \mathbf{C})$ on $Rat_k(S^2)$ or similarly the complex structure on the moduli space. The action of $SL(2, \mathbf{C})$ is intimately related to the issue of choosing a different base point in H^3 and understanding the relationship between the different rational maps obtained each way. A limit case of this issue would bring in the relationship between the rational map we have obtained and the based rational map obtained by Atiyah [1].

References

[1] M.F. Atiyah. Magnetic monopoles in hyperbolic space. In *Proceedings of the International Colloquium on Vector Bundles, Tata Institute, Bombay*, 1984.

[2] M.F. Atiyah. Instantons in two and four dimensions. *Comm. Math. Phys.*, **93** (1984), 437-451.

[3] M.F. Atiyah and N.J. Hitchin. *The geometry and dynamics of magnetic monopoles.* Princeton University Press, Princeton, 1988.

[4] M.F. Atiyah and J.D.S. Jones. Topological aspects of Yang-Mills theory. *Comm. Math. Phys.*, **61** (1978), 97-118.

[5] David M. Austin and Peter J. Braam. Boundary values of hyperbolic monopoles. *Nonlinearity*, **3** (1990), 809-823.

[6] Peter J. Braam. Magnetic monopoles on three-manifolds. *Journal of Differential Geometry*, **30** (1989), 425-464.

[7] Earl A. Coddington and Norman Levinson. *Theory of ordinary differential equations*. McGraw-Hill, New York, 1955.

[8] R.L. Cohen, J.D.S. Jones, and G.B. Segal. Floer's infinite dimensional Morse theory and homotopy theory. *Floer Memorial Volume*, 1993.

[9] S.K. Donaldson. Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. *Proc. Lond. Math. Soc.*, **30** (1985), 1-26.

[10] S.K. Donaldson and P.B. Kronheimer. *The geometry of four-manifolds*. Oxford University Press, 1990.

[11] U. Gritsch. Morse theory for equivariant Yang-Mills on 4-manifolds and equivariant homotopy theory. *Phd Thesis, Stanford University*, 1996.

[12] N.J. Hitchin. Monopoles and geodesics. *Comm. Math. Phys.*, **83** (1982), 579-602.

[13] Stuart Jarvis. Monopoles to rational maps via radial scattering. *Preprint*, (1996).

[14] Stuart Jarvis and Paul Norbury. Zero and infinite curvature limits of hyperbolic monopoles. *To appear in Bull. LMS*, (1997).

[15] Jun Li. Algebraic geometric interpretation of Donaldson's polynomial invariants. *Journal of Differential Geometry*, **37** (1993), 417-466.

[16] M.K. Murray. Phd thesis. (1984).

[17] M.K. Murray and M. Singer. Spectral curves of non-integral hyperbolic monopoles. *Nonlinearity*, **9** (1996), 973-997.

[18] P.B.Kronheimer and T.S.Mrowka. Gauge theory for embedded surfaces I. *Topology*, **32** (1993), 773-826.

[19] L.M. Sibner and R.J. Sibner. Classification of singular Sobolev connections by their holonomy. *Comm. Math. Phys.*, **144** (1992), 337-350.

[20] David Stuart. The geodesic approximation for the Yang-Mills-Higgs equations. *Comm. Math. Phys.*, **166** (1994), 149-190.

[21] C.H. Taubes. Stability in Yang-Mills theory. *Comm. Math. Phys.*, **91** (1983), 235-263.