ENUMERATIVE GEOMETRY VIA THE MODULI SPACE OF SUPER RIEMANN SURFACES

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Abstract. In this paper we relate volumes of moduli spaces of super Riemann surfaces to integrals over the moduli space of stable Riemann surfaces $\overline{\mathcal{M}}_{g,n}$. This allows us to use a recursion between the super volumes recently proven by Stanford and Witten to deduce recursion relations of a natural collection of cohomology classes $\Theta_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n})$. We give a new proof that a generating function for the intersection numbers of $\Theta_{g,n}$ with tautological classes on $\overline{\mathcal{M}}_{g,n}$ is a KdV tau function. This is an analogue of the Kontsevich-Witten theorem where $\Theta_{g,n}$ is replaced by the unit class $1 \in H^*(\overline{\mathcal{M}}_{g,n})$. The proof is analogous to Mirzakhani's proof of the Kontsevich-Witten theorem replacing volumes of moduli spaces of hyperbolic surfaces with volumes of moduli spaces of super hyperbolic surfaces.

Contents

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1. Introduction

Mumford initiated a systematic approach to calculating intersection numbers of tautological classes on the moduli space of stable Riemann surfaces $\overline{\mathcal{M}}_{g,n}$ in [45]. Witten conjectured a recursive structure on a collection of these intersection numbers [62] and Kontsevich proved the conjecture in [33], now known as the Kontsevich-Witten theorem. Other proofs followed [32, 43, 49]. The proof by Mirzakhani [43] deduced the Kontsevich-Witten theorem by proving recursion relations between Weil-Petersson volumes of moduli spaces of smooth hyperbolic surfaces. Wolpert had proven earlier in [63, 64] that the Weil-Petersson symplectic form ω^{WP} extends to the moduli space of stable curves, and related it to a tautological class, κ_1, which was studied by Mumford in [45]. This enabled Mirzakhani to relate integrals over $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$.

Stanford and Witten [57] proved recursive relations between volumes of moduli spaces of smooth super hyperbolic surfaces using methods analogous to those of Mirzakhani. In this paper we prove an analogue of Wolpert's results which relates super volumes to integrals over the moduli space of stable curves. This uses classes $\Theta_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n})$ previously studied by the author [48].

Super Riemann surfaces have been studied over the last thirty years [8, 22, 34, 53, 57, 61]. Underlying any super Riemann surface is a Riemann surface equipped with a spin structure. In this paper we consider Riemann surfaces of finite type $\Sigma = \overline{\Sigma} - D$ where $\overline{\Sigma}$ is a compact curve with boundary divisor $D = \{p_1, ..., p_n\} \subset \overline{\Sigma}$, and equivalently study pairs $(\overline{\Sigma}, D)$. The moduli space of super Riemann surfaces can be defined algebraically, complex analytically and using hyperbolic geometry, building on the same approaches to the moduli space of Riemann surfaces. The last of these approaches regards a super Riemann surface as a super hyperbolic surface, which is a quotient of super hyperbolic space $\widehat{\mathbb{H}}$ defined in (45).

A Riemann surface equipped with a spin structure, or spin surface, has a well-defined square root bundle, $T_\Sigma^{\frac{1}{2}}$, of the tangent bundle, so $T_\Sigma^{\frac{1}{2}} \otimes T_\Sigma^{\frac{1}{2}} \cong T_\Sigma$, which is also a real subbundle of the rank two bundle of spinors $T_\Sigma^{\frac{1}{2}} \otimes_{\mathbb{R}} \mathbb{C} \cong S_\Sigma$. It is a flat $SL(2,\mathbb{R})$-bundle, and the flat structure defines the sheaf of locally constant sections of $T_\Sigma^{\frac{1}{2}}$ with sheaf cohomology $H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}})$. The sheaf cohomology can be calculated via the cohomology of the twisted de Rham complex defined by the flat connection associated to the flat bundle $T_\Sigma^{\frac{1}{2}}$, justifying the subscript. The deformation theory of a super Riemann surface with underlying spin surface Σ defines a bundle

$$E_{g,n} \to \mathcal{M}_{g,n}^{\text{spin}}$$

with fibres $E_{g,n}|_{(\Sigma,\theta)} = H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}})$ over the moduli space of smooth genus g spin Riemann surfaces $\Sigma = \overline{\Sigma} - D$ with $|D| = n$. The total space of $E_{g,n}$ gives the underlying smooth moduli space of super Riemann surfaces.[1]

One new contribution of this paper to the study of the moduli space of super Riemann surfaces is the extension of the bundle $E_{g,n}$ to the moduli space of stable spin curves $\overline{\mathcal{M}}_{g,n}^{\text{spin}}$. A stable spin curve is a stable orbifold curve with n labeled

[1]Donagi and Witten proved in [14] that the moduli space of super Riemann surfaces cannot be represented as the total space of a holomorphic bundle over the moduli space of super Riemann surfaces. Here we consider the moduli space of super Riemann surfaces as a smooth supermanifold/orbifold which can always be represented as the total space of a smooth bundle.

points $(\mathcal{C}, D)$, equipped with a spin structure θ satisfying $\theta^2 = \omega_{\mathcal{C}}^{\log} = \omega_{\mathcal{C}}(D)$. The points of $D = \{p_1, ..., p_n\}$, and nodal points of $\mathcal{C}$ are orbifold points with isotropy group $\mathbb{Z}_2$—see Section 2. There is a map from $\mathcal{C}$ to its underlying coarse curve which forgets the orbifold structure.

When $\mathcal{C}$ is smooth, $\mathcal{C} - D = \Sigma$ is a Riemann surface and $\theta^\vee|_\Sigma = T_\Sigma^{\frac{1}{2}}$. Using a theorem of Simpson [55, 56] we prove in Section 3.3 a canonical isomorphism for $\mathcal{C}$ smooth:

$$H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}}) \cong H^1(\mathcal{C}, \theta^\vee)^\vee. \tag{1}$$

The cohomology groups $H^1(\mathcal{C}, \theta^\vee)$ are well-defined on any stable spin curve $(\mathcal{C}, \theta)$ and $\dim H^1(\mathcal{C}, \theta^\vee)$ is locally constant on $\overline{\mathcal{M}}_{g,n}^{\text{spin}}$, hence the bundle $E_{g,n} \to \mathcal{M}_{g,n}^{\text{spin}}$ is the restriction of a bundle $\widehat{E}_{g,n} \to \overline{\mathcal{M}}_{g,n}^{\text{spin}}$ with fibres $H^1(\mathcal{C}, \theta^\vee)^\vee$. The total space of $\widehat{E}_{g,n}$ gives the compactification of the moduli space of super Riemann surfaces studied by Witten in [61, Section 6]. The extension of the bundle $E_{g,n}$ to a compactification is a crucial ingredient for enumerative methods such as the relation of intersection numbers to the KdV hierarchy in Corollary 2 below.

The isomorphism (1) is non-trivial even in the case $D = \varnothing$ where $\theta^\vee \cong T_\Sigma^{\frac{1}{2}}$. The left hand side of (1) uses the sheaf of locally constant sections while the right hand side uses the sheaf of locally holomorphic sections, and we take the sheaf cohomology in both cases. The difference between the sheaf structures on each side of (1) is demonstrated clearly in the following case. As a bundle $\theta^\vee|_\Sigma \cong T_\Sigma^{\frac{1}{2}}$, however the sheaf of locally holomorphic sections of $\theta^\vee|_\Sigma$ is trivial when $n > 0$, whereas the sheaf of locally constant sections of $T_\Sigma^{\frac{1}{2}}$ is non-trivial, since $H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}}) \neq 0$. One particularly satisfying aspect of applying Simpson's parabolic Higgs bundles techniques to the pair $(\overline{\Sigma}, D)$ is that it naturally gives rise to the orbifold curve $(\mathcal{C}, D) \to (\overline{\Sigma}, D)$. Parabolic bundles over the coarse curve $\overline{\Sigma}$ correspond to the push-forward of bundles over $\mathcal{C}$. In the Neveu-Schwarz case, given by Definition 3.1, the push-forward of $\theta^\vee$ to $\overline{\Sigma}$ is $T_{\overline{\Sigma}}^{\frac{1}{2}}(-D)$ which embeds in a parabolic bundle, as described in 3.3.5. In particular, we can express (1) in terms of the coarse curve $(\mathcal{C}, D) \to (\overline{\Sigma}, D)$ via $H^1(\mathcal{C}, \theta^\vee) \cong H^1(\overline{\Sigma}, T_{\overline{\Sigma}}^{\frac{1}{2}}(-D))$.

Under the forgetful map $p : \overline{\mathcal{M}}_{g,n}^{\text{spin}} \to \overline{\mathcal{M}}_{g,n}$, define the push-forward classes

$$\Theta_{g,n} := 2^{g-1+n} p_* c_{2g-2+n}(\widehat{E}_{g,n}) \in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$$

for $g \geq 0$, $n \geq 0$ and $2g - 2 + n > 0$. These classes are shown in [48] to pull back naturally under the gluing maps

$$\overline{\mathcal{M}}_{g-1,n+2} \xrightarrow{\phi_{\text{irr}}} \overline{\mathcal{M}}_{g,n}, \qquad \overline{\mathcal{M}}_{h,|I|+1} \times \overline{\mathcal{M}}_{g-h,|J|+1} \xrightarrow{\phi_{h,I}} \overline{\mathcal{M}}_{g,n}, \quad I \sqcup J = \{1, ..., n\}$$

and the forgetful map $\overline{\mathcal{M}}_{g,n+1} \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}$ as follows.

$$\phi_{\text{irr}}^* \Theta_{g,n} = \Theta_{g-1,n+2}, \quad \phi_{h,I}^* \Theta_{g,n} = \Theta_{h,|I|+1} \otimes \Theta_{g-h,|J|+1}, \tag{2}$$

$$\Theta_{g,n+1} = \psi_{n+1} \cdot \pi^* \Theta_{g,n} \tag{3}$$

where $\psi_{n+1} \in H^2(\overline{\mathcal{M}}_{g,n+1}, \mathbb{Q})$ is a natural class, defined in (9) in Section 2. Properties (2), (3) and a single calculation $\int_{\overline{\mathcal{M}}_{1,1}} \Theta_{1,1} = \frac{1}{8}$ are enough to uniquely determine the intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}\prod_{j=1}^N\kappa_j^{\ell_j}$$

via a reduction argument—see Section 2 for details. In particular, we restrict to the case of only κ_1 classes below.

The volume of the moduli space of super hyperbolic surfaces is shown in [57] from general considerations to coincide with the integral

$$\widehat{V}_{g,n}^{WP}=\int_{\mathcal{M}_{g,n}^{\rm spin}}e(E_{g,n})\exp\omega^{WP} \tag{4}$$

where $e(E_{g,n})$ is a differential form representing the the Euler class of the bundle $E_{g,n}$. Wolpert [63, 64] proved that ω^{WP} extends from $\mathcal{M}_{g,n}$ to $\tilde{\omega}^{WP}$ defined on $\overline{\mathcal{M}}_{g,n}$, with cohomology class $[\tilde{\omega}^{WP}]=2\pi^2\kappa_1\in H^2(\overline{\mathcal{M}}_{g,n})$. More generally, over the moduli space $\mathcal{M}_{g,n}(L_1,...,L_n)$ of hyperbolic surfaces with geodesic boundary components of lengths $L_1,...,L_n$, the extension has cohomology class

$$[\tilde{\omega}^{WP}]=2\pi^2\kappa_1+\frac{1}{2}\sum_{i=1}^nL_i^2\psi_i.$$

In particular, the Weil-Petersson volumes are intersection numbers:

$$V_{g,n}^{WP}(L_1,...,L_n)=\int_{\mathcal{M}_{g,n}(L_1,...,L_n)}\exp\omega^{WP}=\int_{\overline{\mathcal{M}}_{g,n}}\exp(2\pi^2\kappa_1+\frac{1}{2}\sum_{i=1}^nL_i^2\psi_i).$$

Define the polynomials

$$V_{g,n}^{\Theta}(L_1,...,L_n):=\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\exp\left\{2\pi^2\kappa_1+\frac{1}{2}\sum_{i=1}^nL_i^2\psi_i\right\}. \tag{5}$$

Combining Wolpert's result with the extension of $E_{g,n}$ to $\overline{\mathcal{M}}_{g,n}^{\rm spin}$, the volume of the moduli space of super hyperbolic surfaces with geodesic boundary components of lengths $L_1,...,L_n$ is

$$\widehat{V}_{g,n}^{WP}(L_1,...,L_n)=(-1)^n2^{1-g-n}V_{g,n}^{\Theta}(L_1,...,L_n).$$

Recursive relations between volumes of moduli spaces of super hyperbolic surfaces produce recursive relations between intersections numbers over $\overline{\mathcal{M}}_{g,n}$ involving the classes $\Theta_{g,n}$ and the tautological classes κ_1,ψ_i.

Allow the symmetric polynomials $V_{g,n}^{\Theta}$ to have arguments given by (unordered) sets of variables such as $L_A=\{L_i\mid i\in A\}$ for any set of integers A. The following theorem gives recursion relations satisfied by the polynomials $V_{g,n}^{\Theta}(L_1,...,L_n)$. Introduce the kernel

$$H(x,y)=\frac{1}{4\pi}\left(\frac{1}{\cosh\frac{x-y}{4}}-\frac{1}{\cosh\frac{x+y}{4}}\right)$$

and the associated kernels

$$D(x,y,z)=H(x,y+z),\quad R(x,y,z)=\frac{1}{2}H(x+y,z)+\frac{1}{2}H(x-y,z). \tag{6}$$

Theorem 1.

$$L_1 V^{\Theta}_{g,n}(L_1,L_K) = \frac{1}{2}\int_0^\infty\int_0^\infty xyD(L_1,x,y)P_{g,n}(x,y,L_K)dxdy + \sum_{j=2}^n \int_0^\infty xR(L_1,L_j,x)V^{\Theta}_{g,n-1}(x,L_{K\setminus\{j\}})dx \tag{7}$$

where $K=\{2,...,n\}$ and

$$P_{g,n}(x,y,L_K) = V^{\Theta}_{g-1,n+1}(x,y,L_K) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=K}}^{\text{stable}} V^{\Theta}_{g_1,|I|+1}(x,L_I)V^{\Theta}_{g_2,|J|+1}(y,L_J).$$

The main tools in the proof of Theorem 1 are the recursive relations proven by Stanford and Witten [57] for super volumes and the relation of volumes over the smooth moduli space to integrals over the compactified moduli space via (1).

The polynomial $V^{\Theta}_{g,n}(L_1,...,L_n)$ is of degree $2g-2$ and its top degree terms store the intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}$ involving only ψ_i classes with $\Theta_{g,n}$. Theorem 1 produces a recursion among these intersection numbers which is used to give a new proof of the following relationship of the intersection numbers with the KdV hierarchy.

Corollary 2 ([48]).

$$Z^{\Theta}(t_0,t_1,...) = \exp\sum\frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_j^{k_i}t_{k_i}$$

is a tau function of the KdV hierarchy.

In particular, $U=\partial_{t_0}^2\log Z^{\Theta}$ is a solution of the KdV hierarchy which is uniquely determined by $U(t_0,0,0,...)=\frac{1}{8(1-t_0)^2}$. But this coincides with the initial condition of the Brézin-Gross-Witten tau function of the KdV hierarchy which comes from a $U(n)$ matrix model [5, 26]. Hence the tau functions coincide:

$$Z^{\Theta}(t_0,t_1,...) = Z^{BGW}(t_0,t_1,...).$$

See Section 6 for more details.

If we replace the classes $\Theta_{g,n}\in H^*(\overline{\mathcal{M}}_{g,n})$ in Corollary 2 by the unit class $1\in H^*(\overline{\mathcal{M}}_{g,n})$ then the analogous statement is the theorem conjectured by Witten and proven by Kontsevich.

Theorem 3 (Kontsevich-Witten 1992).

$$Z^{KW}(t_0,t_1,...) = \exp\sum_{g,n,\vec{k}}\frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^n\psi_i^{k_i}t_{k_i}$$

is a tau function of the KdV hierarchy.

Theorem 1 is analogous to Mirzakhani's recursion for volumes of hyperbolic surfaces, described in Section 4, which she used to prove Theorem 3. The proof of Corollary 2 from Theorem 1 is analogous to Mirzakhani's proof of Theorem 3. In fact, Theorem 1 and Corollary 2 are equivalent—see Theorem 7.11.

Corollary 2 is a special case of a more general tau function of the KdV hierarchy involving all of the classes κ_j, $j=1,2,...$ which is analogous to the higher

Weil-Petersson volumes in the case that $\Theta_{g,n}$ is replaced by 1. This appears as Theorem 6.5 in Section 6.

Theorem 1 enables one to calculate $V^{\Theta}_{g,n}$ for $n>0$ whereas the definition (5) makes sense also for $n=0$ and $g>1$. The $n=0$ volumes can be calculated from the $n=1$ polynomial as follows. For $g>1$,

$$V^{\Theta}_{g,0}=\frac{1}{2g-2}V^{\Theta}_{g,1}(2\pi i).$$

Note that although the volumes require $L_i\geq 0$, the polynomial allows any complex argument. The formula for $V^{\Theta}_{g,0}$ is a special case of the following more general relation which is proven in Section 6.2.

$$V^{\Theta}_{g,n+1}(2\pi i,L_1,...,L_n)=(2g-2+n)V^{\Theta}_{g,n}(L_1,...,L_n). \tag{8}$$

Eynard and Orantin [19] proved that Mirzakhani's volume recursion, given by(37) in Section 4, can be neatly expressed in terms of topological recursion, defined in Section 7, applied to the spectral curve

$$x=\frac{1}{2}z^2,\quad y=\frac{\sin(2\pi z)}{2\pi}.$$

The following theorem describes a similar spectral curve on which topological recursion is equivalent to the recursion (7) in Theorem 1. Essentially the spectral curve efficiently encodes the kernels $D(x,y,z)$ and $R(x,y,z)$ defined in (6). Let

$$\mathcal{L}\{V^{\Theta}_{g,n}(L_1,...,L_n)\}=\int_0^\infty ... \int_0^\infty V^{\Theta}_{g,n}(L_1,...,L_n)\prod_{i=1}^n\exp(-z_iL_i)dL_i$$

denote the Laplace transform.

Theorem 4. *Topological recursion applied to the spectral curve*

$$x=\frac{1}{2}z^2,\quad y=\frac{\cos(2\pi z)}{z}$$

produces correlators

$$\omega_{g,n}=\frac{\partial}{\partial z_1}...\frac{\partial}{\partial z_n}\mathcal{L}\{V^{\Theta}_{g,n}(L_1,...,L_n)\}dz_1...dz_n.$$

Outline: In Section 2 we define the classes $\Theta_{g,n}$ required for the definition of the polynomials $V^{\Theta}_{g,n}$. Section 3 contains the proof that the bundle $E_{g,n}\to\mathcal{M}^{\rm spin}_{g,n}$ naturally extends to $\overline{\mathcal{M}}^{\rm spin}_{g,n}$. In Section 3, spin structures on hyperbolic surfaces are studied from a gauge theoretic viewpoint. This viewpoint brings in Higgs bundles techniques which achieves a number of goals: it relates the sheaf cohomologies arising from a flat structure and a holomorphic structure on a bundle, such as (1); it relates hyperbolic metrics on a non-compact Riemann surface $\Sigma=\overline{\Sigma}-D$ to bundles on the compact pair $(\overline{\Sigma},D)$; it naturally produces bundles on the orbifold curve $(\mathcal{C},D)\to(\overline{\Sigma},D)$ which makes a connection with the construction in Section 2. In Section 4 we give details of Mirzakhani's techniques which are need in the proofs of Theorem 1 and Corollary 2. Section 5 contains the recursion of Stanford and Witten between volumes of moduli spaces of super hyperbolic surfaces analogous to Mirzakhani's recursions between volumes of moduli spaces of hyperbolic surfaces which gives the proof of Theorem 1. The proof of Corollary 2 is presented in Section 6. The relationship to topological recursion is given in Section 7.

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2. The classes $\Theta_{g,n}\in H^*(\overline{\mathcal{M}}_{g,n})$.

In this section we define the cohomology classes $\Theta_{g,n}\in H^*(\overline{\mathcal{M}}_{g,n})$ via a construction over the moduli space of stable twisted spin curves $\overline{\mathcal{M}}_{g,n}^{\rm spin}$. We define $\Theta_{g,n}$ to be a multiple of the push-forward of the top Chern class of a bundle $\widehat{E}_{g,n}\to\overline{\mathcal{M}}_{g,n}^{\rm spin}$ with fibre $H^1(\theta^\vee)^\vee\cong\mathbb{C}^{2g-2+n}$. We show that the volume polynomials $V^\Theta_{g,n}(L_1,...,L_n)$ defined in (5) and the partition function $Z^\Theta(t_0,t_1,...)$ defined in Corollary 2 depend only on the characterisation (2), (3) of $\Theta_{g,n}$ and $\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}=\frac{1}{8}$. In other words, $V^\Theta_{g,n}(L_1,...,L_n)$ and $Z^\Theta(t_0,t_1,...)$ can be characterised purely in terms of $\overline{\mathcal{M}}_{g,n}$ without reference to $\overline{\mathcal{M}}_{g,n}^{\rm spin}$.

Define

$$\overline{\mathcal{M}}_{g,n}^{\rm spin}=\{(\mathcal{C},\theta,p_1,...,p_n,\phi)\mid\phi:\theta^2\stackrel{\cong}{\longrightarrow}\omega_{\mathcal{C}}^{\rm log}\}$$

where $\mathcal{C}$ is a stable twisted curve, or stack, with group $\mathbb{Z}_2$ such that generic points have trivial isotropy group and non-trivial orbifold points have isotropy group $\mathbb{Z}_2$, [1]. The stable twisted curve $\mathcal{C}$ is equipped with a map which forgets the orbifold structure $\rho:\mathcal{C}\to C$ where C is a stable curve known as the coarse curve of $\mathcal{C}$. Each nodal point of $\mathcal{C}$ (corresponding to a nodal point of C) has non-trivial isotropy group and all other points of $\mathcal{C}$ with non-trivial isotropy group are labeled points of $\mathcal{C}$. The map ρ induces a map

$$p:\overline{\mathcal{M}}_{g,n}^{\rm spin}\to\overline{\mathcal{M}}_{g,n}$$

where $\overline{\mathcal{M}}_{g,n}$ is the moduli space of genus g stable curves—curves with only nodal singularities and finite automorphism group—with n labeled points disjoint from nodes. In fact, the map p is a composition of ρ with the 2^{2g} to 1 map to the moduli space of twisted curves $\overline{\mathcal{M}}_{g,n}^{\rm spin}\to\overline{\mathcal{M}}_{g,n}^{(2)}$, where the latter moduli space is defined as above without the spin structure, consisting of twisted curves $\{(\mathcal{C},p_1,...,p_n)\}$. There are 2^{2g} choices of (θ,ϕ) for each point of $\overline{\mathcal{M}}_{g,n}^{(2)}$.

The bundles $\omega_{\mathcal{C}}^{\rm log}$ and θ are line bundles over $\mathcal{C}$, i.e. locally equivariant bundles over the local charts such that at each nodal point there is an equivariant isomorphism of fibres. On each fibre over an orbifold point p the equivariant isomorphism associates a representation of $\mathbb{Z}_2$ which is either trivial or the unique non-trivial representation. The equivariant isomorphism at nodes guarantees that the representations agree on each local irreducible component at the node, known as the balanced condition. The representation associated to $\omega_{\mathcal{C}}^{\rm log}$ at p_i and nodal points is trivial since locally $dz/z\stackrel{z\mapsto -z}{\longrightarrow}dz/z$. The representations associated to θ at each p_i are chosen to be non-trivial, whereas at nodal points p, both types—trivial and non-trivial representations can occur. Among the 2^{2g} different spin structures on a twisted curve $\mathcal{C}$, some will have trivial representations at the nodes, and some will have non-trivial representations. See [20] for further details.

We have $\deg\omega_{\mathcal{C}}^{\log} = 2g-2+n$ and $\deg\theta = g-1+\frac{1}{2}n$ which may be a half-integer since the orbifold points allows for such a possibility. In particular $\deg\theta^\vee = 1-g-\frac{1}{2}n<0$, and for any irreducible component $\deg\theta^\vee|_{\mathcal{C}'}<0$ since $\mathcal{C}'$ is stable so its log canonical bundle has negative degree. Thus $H^0(\theta^\vee)=0$ so $H^1(\theta^\vee)$ has constant dimension and defines a vector bundle $\widehat{E}_{g,n}\to\overline{\mathcal{M}}_{g,n}^{\rm spin}$. By the Riemann-Roch theorem $H^1(\theta^\vee)\cong\mathbb{C}^{2g-2+n}$. More formally, denote by $\mathcal{E}$ the universal spin structure over $\overline{\mathcal{M}}_{g,n}^{\rm spin}$.

Definition 2.1. Define the bundle $\widehat{E}_{g,n} = (-R\pi_*\mathcal{E}^\vee)^\vee$ over $\overline{\mathcal{M}}_{g,n}^{\rm spin}$ with fibre $H^1(\theta^\vee)^\vee$.

Definition 2.2. $\Theta_{g,n} = 2^{g-1+n}p_*c_{2g-2+n}(\widehat{E}_{g,n})\in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$

It is proven in [48] that $\Theta_{g,n}$ satisfies the pull-back properties (2) and (3) and $\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1}=\frac{1}{8}$ where

$$\psi_i = c_1(L_i)\in H^2(\overline{\mathcal{M}}_{g,n},\mathbb{Q}) \tag{9}$$

is the first Chern class of the line bundle $L_i\to\overline{\mathcal{M}}_{g,n}$ with fibre above $[(C,p_1,...,p_n)]$ given by $T^*_{p_i}C$.

Using the forgetful map $\overline{\mathcal{M}}_{g,n+1}\xrightarrow{\pi}\overline{\mathcal{M}}_{g,n}$, define

$$\kappa_m := \pi_*\psi_{n+1}^{m+1}\in H^{2m}(\overline{\mathcal{M}}_{g,n},\mathbb{Q}). \tag{10}$$

The following proposition states that the intersection numbers of $\Theta_{g,n}$ with ψ classes and κ classes, such as those used to construct $V^\Theta_{g,n}(L_1,...,L_n)$ and $Z^\Theta(t_0,t_1,...)$, can be characterised purely in terms of $\overline{\mathcal{M}}_{g,n}$ without reference to $\overline{\mathcal{M}}_{g,n}^{\rm spin}$.

Proposition 2.3 ([48])**.** *For any collection $\Theta_{g,n}\in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$ satisfying the pull-back properties (2) and (3), the intersection numbers*

$$\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}\prod_{j=1}^N\kappa_{\ell_j} \tag{11}$$

are uniquely determined from the initial condition $\Theta_{1,1}=\lambda\psi_1$ for $\lambda\in\mathbb{C}$.

Sketch of proof. For $n>0$, since $\psi_n\psi_k=\psi_n\pi^*\psi_k$ for $k<n$ and $\Theta_{g,n}=\psi_n\cdot\pi^*\Theta_{g,n-1}$ then

$$\Theta_{g,n}\psi_k=\Theta_{g,n}\pi^*\psi_k,\quad k<n.$$

When there are no κ classes.

$$\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}=\int_{\overline{\mathcal{M}}_{g,n}}\pi^*\Big(\Theta_{g,n-1}\prod_{i=1}^{n-1}\psi_i^{m_i}\Big)\psi_n^{m_n+1}=\int_{\overline{\mathcal{M}}_{g,n-1}}\Theta_{g,n-1}\prod_{i=1}^{n-1}\psi_i^{m_i}\kappa_{m_n}$$

so we have reduced an intersection number over $\overline{\mathcal{M}}_{g,n}$ to an intersection number over $\overline{\mathcal{M}}_{g,n-1}$. In the presence of κ classes, replace κ_{ℓ_j} by $\kappa_{\ell_j}=\pi^*\kappa_{\ell_j}+\psi_n^{\ell_j}$ and repeat the push-forward as above on all summands. By induction, we see that

$$\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{m_i}\prod_{j=1}^N\kappa_{\ell_j}=\int_{\overline{\mathcal{M}}_g}\Theta_g\cdot p(\kappa_1,\kappa_2,...,\kappa_{3g-3})$$

i.e. the intersection number (11) reduces to an intersection number over $\overline{\mathcal{M}}_g$ of Θ_g times a polynomial in the κ classes. Since $\deg\Theta_g = 2g-2$ we may assume the polynomial p consists only of terms of homogeneous degree $g-1$. Any homogeneous degree $g-1$ monomial in the κ classes is equal in cohomology to the sum of boundary terms, [35, 51]. By (2) the pull-back of Θ_g to these boundary terms is $\Theta_{g',n'}$ for $g' < g$ so we have expressed (11) as a sum of integrals of $\theta_{g',n'}$ against ψ and κ classes. By induction, one can reduce to the integral $\int_{\overline{\mathcal{M}}_{1,1}}\Theta_{1,1} = \frac{\lambda}{24}$ and the proposition is proven. □

2.1. **Cohomological field theories.** The classes $\Theta_{g,n}$ pair with any cohomological field theory, such as Gromov-Witten invariants, to give rise to new invariants. Recall that a *cohomological field theory* is a pair (V,η) composed of a finite-dimensional complex vector space V equipped with a nondegenerate, bilinear, symmetric form η which we call a metric (although it is not positive-definite) and for $n\geq 0$ a sequence of S_n-equivariant maps.

$$\Omega_{g,n} : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$$

that satisfy pull-back properties with respect to the gluing maps defined in the introduction, that generalise (2).

$$\phi_{\text{irr}}^*\Omega_{g,n}(v_1\otimes ...\otimes v_n) = \Omega_{g-1,n+2}(v_1\otimes ...\otimes v_n\otimes\Delta) \tag{12}$$

$$\phi_{h,I}^*\Omega_{g,n}(v_1\otimes ...\otimes v_n) = \Omega_{h,|I|+1}\otimes\Omega_{g-h,|J|+1}\big(\bigotimes_{i\in I}v_i\otimes\Delta\otimes\bigotimes_{j\in J}v_j\big) \tag{13}$$

where $\Delta\in V\otimes V$ is dual to the metric $\eta\in V^*\otimes V^*$.

There exists a vector $1\!\!1\in V$ satisfying

$$\Omega_{0,3}(v_1\otimes v_2\otimes 1\!\!1) = \eta(v_1,v_2) \tag{14}$$

which is essentially a non-degeneracy condition. A CohFT defines a product $\cdot$ on V using the non-degeneracy of η by

$$\eta(v_1\cdot v_2, v_3) = \Omega_{0,3}(v_1,v_2,v_3). \tag{15}$$

and $1\!\!1$ is a unit for the product. Such CohFTs were classified by Teleman [60]. We will also consider sequences of S_n-equivariant maps $\Omega_{g,n}$ that satisfy (12) and (13), but do not satisfy (14) which we call *degenerate* CohFTs.

The CohFT is said to have *flat unit* if

$$\Omega_{g,n+1}(1\!\!1\otimes v_1\otimes ...\otimes v_n) = \pi^*\Omega_{g,n}(v_1\otimes ...\otimes v_n) \tag{16}$$

for $2g-2+n>0$. An alternative to (3) satisfied by degenerate CohFTs

$$\Omega_{g,n+1}(1\!\!1\otimes v_1\otimes ...\otimes v_n) = \psi_{n+1}\pi^*\Omega_{g,n}(v_1\otimes ...\otimes v_n). \tag{17}$$

The product (15) is *semisimple* if it is diagonal $V\cong\mathbb{C}\oplus\mathbb{C}\oplus...\oplus\mathbb{C}$, i.e. there is a canonical basis $\{u_1,...,u_N\}\subset V$ such that $u_i\cdot u_j = \delta_{ij}u_i$. The metric is then necessarily diagonal with respect to the same basis, $\eta(u_i,u_j) = \delta_{ij}\eta_i$ for some $\eta_i\in\mathbb{C}\setminus\{0\}$, $i=1,...,N$.

For a one-dimensional CohFT, i.e. $\dim V = 1$, identify $\Omega_{g,n}$ with the image $\Omega_{g,n}(1\!\!1^{\otimes n})$, so we write $\Omega_{g,n}\in H^*(\overline{\mathcal{M}}_{g,n})$. An example of a one-dimensional CohFT is

$$\Omega_{g,n} = \exp(2\pi^2\kappa_1).$$

The classes $\Theta_{g,n}$ define a one-dimensional degenerate CohFT.

The partition function of a CohFT $\Omega = \{\Omega_{g,n}\}$ is defined by:

$$Z_\Omega(\hbar, \{t_k^\alpha\}) = \exp \sum_{g,n,\vec{k}} \frac{\hbar^{g-1}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(e_{\alpha_1} \otimes ... \otimes e_{\alpha_n}) \cdot \prod_{j=1}^n \psi_j^{k_j} \prod t_{k_j}^{\alpha_j} \tag{18}$$

where $\{e_1, ..., e_N\}$ is a basis of V, $\alpha_i \in \{1, ..., N\}$ and $k_j \in \mathbb{N}$.

For any CohFT Ω on (V, η) define $\Omega^\Theta = \{\Omega_{g,n}^\Theta\}$ to be the degenerate CohFT $\Omega_{g,n}^\Theta : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$ given by $\Omega_{g,n}^\Theta(v_1 \otimes ... \otimes v_n) = \Theta_{g,n} \cdot \Omega_{g,n}(v_1 \otimes ... \otimes v_n)$.

Apply this to the example above to get $\Omega_{g,n}^\Theta = \Theta_{g,n} \cdot \exp(2\pi^2\kappa_1)$ which has a partition function that stores all of the volume polynomials

$$Z_{\Omega^\Theta}(\hbar, \{t_k\}) = \exp \sum_{g,n} \frac{\hbar^{g-1}}{n!} V_{g,n}^\Theta(L_1, ..., L_n)|_{\{L_i^{2k} = 2^k k! t_k\}}.$$

Note that the substitution $L_i^{2k} = 2^k k! t_k$ requires one to take the highest power of L_i in each monomial, and importantly, to substitute $L_i^0 = t_0$ when L_i is missing from a monomial of $V_{g,n}^\Theta(L_1, ..., L_n)$. See 7.0.4 for further details.

3. Hyperbolic geometry and spin structures

In this section we construct the bundle $E_{g,n} \to \mathcal{M}_{g,n}^{\text{spin}}$ via hyperbolic geometry and prove that it coincides with the restriction of the bundle $\widehat{E}_{g,n} \to \overline{\mathcal{M}}_{g,n}^{\text{spin}}$ defined in Definition 2.1. We begin with a description of spin structures via Fuchsian representations into $SL(2, \mathbb{R})$. On a spin hyperbolic surface Σ the representation produces the associated flat $SL(2, \mathbb{R})$-bundle $T_\Sigma^{\frac{1}{2}}$ which is used to construct the bundle $E_{g,n}$ from the cohomology of the locally constant sheaf of sections of $T_\Sigma^{\frac{1}{2}}$. Using Higgs bundles we prove a canonical isomorphism between fibres of $E_{g,n}$ and fibres of $\widehat{E}_{g,n}$ over smooth Σ. When Σ is non-compact this requires parabolic Higgs bundles on $(\overline{\Sigma}, D)$ where $\overline{\Sigma} - D = \Sigma$.

3.1. **Fuchsian representations.** Given any smooth curve with marked points $(\overline{\Sigma}, p_1, ..., p_n)$ define the underlying real, oriented surface $\Sigma = \overline{\Sigma} - \{p_1, ..., p_n\}$. The surface Σ possesses a unique representative, in its conformal class, by a complete finite area hyperbolic surface with cusps at the points p_i. More generally, for any n lengths $L_1, ..., L_n > 0$ there exists a unique representative, in the same conformal class, by a hyperbolic surface which is the interior of a compact hyperbolic surface with geodesic boundary components of lengths $L_1, ..., L_n$. We abuse notation and also denote this compact surface with boundary by Σ. The hyperbolic structure is defined via a Fuchsian representation

$$\overline{\rho} : \pi_1\Sigma \to PSL(2, \mathbb{R})$$

which uniformises Σ, meaning $\Sigma \simeq \mathbb{H}/\overline{\rho}(\pi_1\Sigma)$ where

$$\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$$

is hyperbolic space and $\simeq$ denotes conformal equivalence.

A *boundary class* $\gamma \subset \Sigma$ represents a homotopy class of simple, closed, separating curves such that one component of $\Sigma - \gamma$ is an annulus. It determines a class $[\gamma] \in H_1(\Sigma)$ which we also call a boundary class. Boundary classes with parabolic, respectively hyperbolic, images under $\rho : \pi_1\Sigma \to PSL(2, \mathbb{R})$ correspond to cusps, respectively geodesic boundary components. We write $\overline{\rho}$ because we will instead consider representations

$$\rho:\pi_1\Sigma\to SL(2,\mathbb{R})$$

such that the composition $\overline{\rho}$ of ρ with the map $SL(2,\mathbb{R})\to PSL(2,\mathbb{R})$ is Fuchsian. Any closed curve $\gamma\subset\Sigma$ corresponds to a conjugacy class in $\pi_1\Sigma$ and we write $[\gamma]\in\pi_1\Sigma$ for any representative of the conjugacy class associated to γ. A Fuchsian representation satisfies the property that $|\mathrm{tr}\,\rho([\gamma])|\geq 2$ for all simple closed curves $\gamma\subset\Sigma$ and it equals 2 only when $[\gamma]$ is a boundary class. The geometric meaning of the Fuchsian property uses the fact that for any closed curve $\gamma\subset\Sigma$ there exists a unique closed geodesic g_γ in its free homotopy class and $|\mathrm{tr}\,\rho([\gamma])|=2\cosh(\ell(g_\gamma)/2)$ determines its hyperbolic length $\ell(g_\gamma)$. The Fuchsian property of $\overline{\rho}:\pi_1\Sigma\to PSL(2,\mathbb{R})$ can be determined via its circle bundle over Σ defined via the action of $PSL(2,\mathbb{R})$ on the circle at infinity $S^1\cong\partial\mathbb{H}$. If the Euler class of this circle bundle is equal to $\pm(2g-2+n)$ then $\overline{\rho}$ is a Fuchsian representation, [24, 27].

3.1.1. A Riemannian metric, in particular the hyperbolic metric, on an orientable surface Σ determines a principal $SO(2)$ bundle $P_{SO}(\Sigma)$ given by the orthonormal frame bundle of Σ. A *spin structure* on a Riemannian surface Σ is a principal $SO(2)$ bundle $P_{\mathrm{Spin}}(\Sigma)\to\Sigma$ that is a double cover of the orthonormal frame bundle $P_{\mathrm{Spin}}(\Sigma)\to P_{SO}(\Sigma)$ which restricts to a non-trivial double cover on each $SO(2)$ fibre. Any spin structure is naturally identified with an element of $H^1(P_{SO}(\Sigma);\mathbb{Z}_2)=\mathrm{Hom}(\pi_1(P_{SO}(\Sigma)),\mathbb{Z}_2)$. The non-trivial double-cover condition on each $SO(2)$ fibre is captured by the exact sequence in cohomology

$$0\to H^1(\Sigma;\mathbb{Z}_2)\to H^1(P_{SO}(\Sigma);\mathbb{Z}_2)\stackrel{r}{\to} H^1(SO(2);\mathbb{Z}_2)\to 0$$

by requiring that r is non-zero, [40]. The rightmost arrow is defined by the vanishing second Stiefel-Whitney class which take values in $H^2(\Sigma;\mathbb{Z}_2)$ and guarantees the existence of a spin structure. The exact sequence shows that the set of spin structure on Σ is an $H^1(\Sigma,\mathbb{Z}_2)$ affine space.

3.1.2. The bundle of spinors $S_\Sigma\to\Sigma$ is the associated bundle

$$S_\Sigma=P_{\mathrm{Spin}}(\Sigma)\times_{SO(2)}\mathbb{C}^2$$

where $SO(2)$ acts by the natural representation on $\mathbb{C}^2$ (which is the unique irreducible representation of the complexified Clifford algebra $\mathrm{Spin}(2)\subset Cl_2\otimes\mathbb{C}=M(2,\mathbb{C})$). The represention of $SO(2)$ decomposes into irreducible representations of weights $\chi=e^{i\alpha}$ and $\chi^{-1}=e^{-i\alpha}$ so the spinor bundle decomposes into complex line bundles $S_\Sigma=T_\Sigma^{\frac12}\oplus T_\Sigma^{-\frac12}$ where $T_\Sigma^{\frac12}=P_{\mathrm{Spin}}(\Sigma)\times_{\mathrm{Spin}(2)}\mathbb{C}_\chi$. Since the weight of the tangent bundle T_Σ is χ^2,

$$T_\Sigma^{\frac12}\otimes T_\Sigma^{\frac12}=P_{\mathrm{Spin}}(\Sigma)\times_{\mathrm{Spin}(2)}\mathbb{C}_{\chi^2}=P_{SO}(\Sigma)\times_{SO(2)}\mathbb{C}_{\chi^2}=T_\Sigma$$

is holomorphic hence $T_\Sigma^{\frac12}$ and $T_\Sigma^{-\frac12}$ are holomorphic.

3.1.3. The orthonormal frame bundle $P_{SO}(\Sigma)$ and any spin structure of a hyperbolic surface Σ arise naturally via representations of $\pi_1\Sigma$ as follows. The group $PSL(2,\mathbb{R})$ acts freely and transitively on $P_{SO}(\mathbb{H})$, the orthonormal frame bundle of $\mathbb{H}$, hence the two are naturally identified:

$$P_{SO}(\mathbb{H})\cong PSL(2,\mathbb{R})\to\mathbb{H}.$$

The double cover $SL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$ is a non-trivial double cover on each $SO(2)$ fibre since a path from I to $-I$ in $SL(2,\mathbb{R})$ lives above the fibre $SO(2) \subset PSL(2,\mathbb{R})$. Hence $SL(2,\mathbb{R}) \cong P_{\rm Spin}(\mathbb{H})$ is the unique spin structure. When $\Sigma = \mathbb{H}/\overline{\rho}(\pi_1\Sigma)$ is hyperbolic, $PSL(2,\mathbb{R})$ descends to the orthonormal frame bundle of Σ:

$$P_{SO}(\Sigma) \cong PSL(2,\mathbb{R})/\overline{\rho}(\pi_1\Sigma) \to \Sigma.$$

A representation $\rho : \pi_1\Sigma \to SL(2,\mathbb{R})$ that lives above $\overline{\rho}$ produces a double cover

$$SL(2,\mathbb{R})/\rho(\pi_1\Sigma) \to P_{SO}(\Sigma)$$

which is a non-trivial double cover on each $SO(2)$ fibre since it locally resembles $SL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$. Hence ρ defines a spin structure on Σ.

There is an action of $H^1(\Sigma;\mathbb{Z}_2)$ on representations ρ living above a given representation $\overline{\rho}$ obtained by multiplying any representation by the representation $\epsilon : \pi_1\Sigma \to \{\pm I\}$ associated to an element of $H^1(\Sigma;\mathbb{Z}_2)$. Since the set of spin structure on Σ is an $H^1(\Sigma;\mathbb{Z}_2)$ affine space, this shows that all spin structures on Σ arise via representations $\rho : \pi_1\Sigma \to SL(2,\mathbb{R})$ once we know that at least one lift ρ of $\overline{\rho}$ exists.

For a given representation $\overline{\rho} : \pi_1\Sigma \to PSL(2,\mathbb{R})$, the existence of a lift $\rho : \pi_1\Sigma \to SL(2,\mathbb{R})$ is elementary in the case that Σ is non-compact. Choose a presentation

$$\pi_1\Sigma = \{a_1, a_2, ..., a_g, b_1, ..., b_g, c_1, ..., c_n \mid \prod_{i=1}^{g}[a_i, b_i]\prod_{j=1}^{n} c_j = \mathbb{1}\}.$$

Choose any lifts of $\overline{\rho}(a_i)$, $\overline{\rho}(b_i)$ and $\overline{\rho}(c_j)$ in $PSL(2,\mathbb{R})$ to $\rho(a_i)$, $\rho(b_i)$ and $\rho(c_j)$ in $SL(2,\mathbb{R})$, for $i = 1, ..., g$ and $j = 1, ..., n$. Then $\prod_{i=1}^{g}[\rho(a_i), \rho(b_i)]\prod_{j=1}^{n}\rho(c_j) = \pm\mathbb{1}$ which is the fibre over $\mathbb{1}$. Since $n > 0$, by possibly replacing $\rho(c_n) \to -\rho(c_n)$ we get the existence of a single lift. When Σ is compact, cut it into two pieces $\Sigma = \Sigma_1 \cup_\gamma \Sigma_2$ along a simple closed curve γ containing the basepoint used to define $\pi_1\Sigma$, say a genus 1 piece and a genus $g-1$ piece (Σ is hyperbolic so $g > 1$). Now $\overline{\rho} : \pi_1\Sigma \to PSL(2,\mathbb{R})$ induces representations $\overline{\rho}_i : \pi_1(\Sigma_i) \to PSL(2,\mathbb{R})$, for $i = 1, 2$. As above choose lifts of ρ_i of $\overline{\rho}_i$. The lifts ρ_1 and ρ_2 necessarily agree on their respective boundary components because they come from $\overline{\rho}$ and both traces are negative by a homological argument given by Corollary 3.4 in 3.1.6. Hence we can glue to get a lift ρ.

3.1.4. The disk D^2 possesses a unique spin structure. Its bundle of frames is trivial, i.e. $P_{SO}(D^2) \cong D^2 \times S^1$, for any Riemannian metric on D^2. Hence a spin structure over a disk is unique and given by the non-trivial double cover of $D^2 \times S^1$ or equivalently the non-trivial element $\eta \in H^1(D^2 \times S^1;\mathbb{Z}_2) \cong \mathbb{Z}_2$. An annulus $\mathbb{A}$, possesses two spin structures corresponding to the trivial and non-trivial double covers of $\mathbb{A} \times S^1$. One of these spin structures extends to the disk and one does not.

Definition 3.1. Given a spin structure over Σ, a boundary class $\gamma \subset \Sigma$ is said to be *Neveu-Schwarz* if the restriction of the spin structure to γ is non-trivial, or equivalently if the spin structure extends to a disk glued along γ. The boundary class γ is *Ramond* if the restriction of the spin structure to γ is trivial.

Hence, on a surface $\Sigma = \overline{\Sigma} - \{p_1, ..., p_n\}$, the boundary class ∂D_{p_i} is Neveu-Schwarz exactly when the spin structure extends over the completion $\Sigma \cup p_i$ at p_i. It is Ramond if the spin structure does not extend over the completion there.

3.1.5. A quadratic form q on $H_1(\Sigma;\mathbb{Z}_2)$ is a map $q : H_1(\Sigma;\mathbb{Z}_2) \to \mathbb{Z}_2$ satisfying

$$q(a+b) = q(a) + q(b) + (a,b)$$

where (a,b) is the mod 2 intersection form on $H_1(\Sigma;\mathbb{Z}_2)$. Quadratic forms are called *Arf functions* in [16, 46]. The set of quadratic forms is clearly an $H^1(\Sigma;\mathbb{Z}_2)$ affine space. A quadratic form naturally associated to any spin structure due to Johnson [29] is defined as follows. Represent $[C] \in H_1(\Sigma;\mathbb{Z}_2)$ by a finite sum of disjoint, embedded, oriented closed curves $C = \sum_{i=1}^n C_i$ and define a map

$$\ell : H_1(\Sigma;\mathbb{Z}_2) \to H_1(P_{SO}(\Sigma);\mathbb{Z}_2)$$

by $\ell([C]) = n\sigma + \sum_{i=1}^n \tilde{C}_i$ where σ is the image of the generator of $H_1(SO(2);\mathbb{Z}_2)$ in $H_1(P_{SO}(\Sigma);\mathbb{Z}_2)$ under the natural inclusion of the fibre, and $\tilde{C}_i$ is the lift of C_i to $P_{SO}(\Sigma)$ using its tangential framing. The map ℓ is well-defined on homology since it is invariant under isotopy, trivial on the boundary of a disk which lifts via its tangential framing to σ, and invariant under replacement of crossings by locally embedded curves. Identify a given spin structure with an element $\eta \in H^1(P_{SO}(\Sigma);\mathbb{Z}_2)$ satisfying $\eta(\sigma) = 1$, and define

$$q_\eta = \eta \circ \ell.$$

It is routine to check that q_η is a quadratic form, and that $\eta \mapsto q_\eta$ defines an isomorphism of $H^1(\Sigma;\mathbb{Z}_2)$ affine spaces between spin structures and quadratic forms.

Neveu-Schwarz and Ramond boundary classes of a spin structure defined in Definition 3.1 can be stated efficiently in terms of the quadratic form of a spin structure. Equip the disk D with its unique spin structure. The tangential framing of the boundary ∂D has winding number 1 with respect to the trivialisation hence its lift $\ell(\partial D)$ to $D^2 \times S^1$ satisfies $\eta(\ell(\partial D)) = 1$. Thus the quadratic form is given by $q(\partial D) = \eta(\ell(\partial D)) + 1 = 1 + 1 = 0$.

Definition 3.1*. Given a spin structure over Σ with associated quadratic form q, a boundary class $[\gamma] \in H_1(\Sigma)$ is said to be *Neveu-Schwarz* if $q([\gamma]) = 0$ and *Ramond* if $q([\gamma]) = 1$.

The boundary type $\vec{\epsilon} \in \mathbb{Z}_2^n$ of a spin structure consists of the quadratic form applied to each of the n boundary classes, hence 0, respectively 1, for Neveu-Schwarz, respectively Ramond, boundary classes. Since a quadratic form is a homological invariant, the number of Ramond boundary classes is necessarily even. Thus there are 2^{n-1} boundary types $\vec{\epsilon}$ for a given topological surface $\Sigma = \overline{\Sigma} - D$, $D = \{p_1, ..., p_n\}$. The Teichmüller space of hyperbolic spin surfaces is the same as usual Teichmüller space despite the extra data of a spin structure. It is the action of the mapping class group that differs which is explained as follows. Fix a topological type of a spin structure, i.e. its boundary type $\vec{\epsilon}$ and its Arf invariant. Given any point of Teichmüller space, equip it with a spin structure of the given topological type. This choice determines a spin structure, of the same topological type, on any other point in Teichmüller space, by continuity and discreteness of the choice. Thus, the same Teichmüller space is used when the hyperbolic surfaces are equipped with spin structures and its quotient by the mapping class group defines the moduli space of hyperbolic spin surfaces.

Definition 3.2. Define

$$\mathcal{M}^{\rm spin}_{g,n,\vec{\epsilon}}=\Big\{(\Sigma,\eta)\mid\Sigma=\overline{\Sigma}-\{p_1,...,p_n\}\text{ genus }g\text{ hyperbolic surface, }\ \gamma_i=\partial D_{p_i}$$
$$\text{spin structure }\eta\in H^1(P_{SO}(\Sigma);\mathbb{Z}_2),\ q_\eta(\gamma_i)=\epsilon_i\Big\}/\sim$$

and put $\mathcal{M}^{\rm spin}_{g,n}:=\mathcal{M}^{\rm spin}_{g,n,\vec{0}}$.

Similarly, define $\mathcal{M}^{\rm spin}_{g,n,\vec{\epsilon}}(\vec{L})\subset\mathcal{M}^{\rm spin}_{g,n,\vec{\epsilon}}$ to consist of those elements with hyperbolic boundary classes, or equivalently geodesic boundary components, of lengths $\ell(c_i)=L_i$ for $\vec{L}\in\mathbb{R}^n_+$.

The Mayer Vietoris sequence for $\Sigma\cup D=\overline{\Sigma}$ where D is a union of disks around $\{p_i\}\subset\overline{\Sigma}$ gives the exact sequence $H_1(\Sigma\cap D;\mathbb{Z}_2)\to H_1(\Sigma;\mathbb{Z}_2)\to H_1(\overline{\Sigma};\mathbb{Z}_2)$. When all boundary classes of a spin structure are Neveu-Schwarz, the associated quadratic form $q:H_1(\Sigma;\mathbb{Z}_2)\to\mathbb{Z}_2$ vanishes on $H_1(\Sigma\cap D;\mathbb{Z}_2)$ hence it is the pull-back of a quadratic form defined on the symplectic vector space $H_1(\overline{\Sigma};\mathbb{Z}_2)$, which reflects the fact that the spin structure extends to $\overline{\Sigma}$. The *Arf invariant* of a quadratic form q defined on a symplectic vector space over $\mathbb{Z}_2$ is a $\mathbb{Z}_2$-valued invariant defined by

$$\mathrm{Arf}(q)=\sum_{i=1}^g q(\alpha_i)q(\beta_i)$$

for any standard symplectic basis $\{\alpha_1,\beta_1,...,\alpha_g,\beta_g\}$ of $H_1(\overline{\Sigma},\mathbb{Z}_2)$, so $(\alpha_i,\beta_j)=\delta_{ij}$, $(\alpha_i,\alpha_j)=0=(\beta_i,\beta_j)$. (More generally, the intersection form $(\cdot,\cdot)$ is replaced by the symplectic form.) This is independent of the choice of $\{\alpha_i,\beta_i\}$. A spin structure is *even* if its quadratic form has even Arf invariant and *odd* if its quadratic form has odd Arf invariant. Of the 2^{2g} spin structures with only Neveu-Schwarz boundary classes, the number of even, respectively odd, spin structures is given by $2^{g-1}(2^g+1)$, respectively $2^{g-1}(2^g-1)$. In particular both odd and even spin structures exist for $g>0$.

By analysing the action on spin structures of the mapping class group of a genus g surface $\Sigma=\overline{\Sigma}-\{p_1,...,p_n\}$ (consisting of isotopy classes of homeomorphisms that fix each p_i), it is proven in [46] that the monodromy of the $H^1(\overline{\Sigma};\mathbb{Z}_2)$ bundle $\mathcal{M}^{\rm spin}_{g,n}(\vec{\epsilon})\to\mathcal{M}_{g,n}$ acts transitively, except in the case of only Neveu-Schwarz boundary classes where there are exactly two orbits. This uses the symplectic action of the mapping class group on $H^1(\overline{\Sigma};\mathbb{Z}_2)$. To see this, equivalently consider the action of the mapping class group on quadratic forms. The idea is that one can choose a basis $\{a_1,b_2,...,a_g,b_g,c_1,...,c_{n-1}\}$ of $H_1(\Sigma;\mathbb{Z}_2)$, where $a_i\cdot b_j=\delta_{ij}$ and c_i are boundary classes, with the following prescribed values of the given quadratic form q. One can arrange $q(a_i)=0=q(b_i)$ for $i>1$ and $q(c_i)=\epsilon_i$. Finally, $q(a_1)=q(b_1)=$ the Arf invariant of q which is set to be zero if $\vec{\epsilon}\neq 0$. This is achieved first algebraically, then geometrically. It is perhaps best understood in the following example. Suppose $g=n=1$, which necessarily has Neveu-Schwarz boundary value. Consider two distinct quadratic forms q_1 and q_2, both with Arf invariant zero, defined on a basis a_1,b_1 of $H_1(\Sigma;\mathbb{Z}_2)$ by $q_1(a_1)=1$, $q_1(b_1)=0$ and $q_2(a_1)=0$, $q_2(b_1)=0$. Consider a second basis $a_1'=a_1+b_1,b_1'=b_1$. Then $q_1(a_1')=0=q_1(b_1')$. Hence an element of the mapping class group that sends $a_1\to a_1'$ and $b_1\to b_1'$ pulls back q_1 to q_2.

Since the set of spin structures with fixed boundary type is an affine $H^1(\overline{\Sigma};\mathbb{Z}_2)$ space, this proves connectedness of components with given boundary type and Arf invariant. Each boundary type determines a connected component of the moduli space of Fuchsian representations $\rho : \pi_1\Sigma \to SL(2,\mathbb{R})$, except in one case—when all boundary classes are Neveu-Schwarz there are two connected components distinguished by the Arf invariant.

3.1.6. The quadratic form $q_\rho : H_1(\Sigma;\mathbb{Z}_2) \to \mathbb{Z}_2$ associated to a spin structure defined by a Fuchsian representation $\rho : \pi_1\Sigma \to SL(2,\mathbb{R})$ has a convenient description. We have renamed $q_{\eta_\rho} =: q_\rho$ where $\eta_\rho \in H^1(PSL(2,\mathbb{R})/\overline{\rho}(\pi_1\Sigma);\mathbb{Z}_2)$ is the cohomology class defined by the spin structure of ρ. By the decomposition of homology classes into simple closed curves used in the definition of $q_\eta = \eta \circ \ell$ above, it is enough to consider the quadratic form evaluated only on simple closed curves. We say that $[\gamma] \in \pi_1\Sigma$ is *simple* if it can be represented by a simple closed curve in Σ.

Lemma 3.3. *Given a Fuchsian representation $\rho : \pi_1\Sigma \to SL(2,\mathbb{R})$, and any simple $[\gamma] \in \pi_1\Sigma$*

$$(-1)^{q_\rho(\lfloor\gamma\rfloor)} = -\operatorname{sgn} tr\,\rho([\gamma]). \tag{19}$$

where $\lfloor\gamma\rfloor \in H_1(\Sigma;\mathbb{Z}_2)$ is the image of $[\gamma]$ under $\pi_1\Sigma \to H_1(\Sigma;\mathbb{Z}_2)$.

Proof. Note that the right hand side of (19) depends only on the homology class $\lfloor\gamma\rfloor \in H_1(\Sigma;\mathbb{Z}_2)$ since $\lfloor\gamma\rfloor$ uniquely determines $[\gamma]$ up to conjugation and trace is conjugation invariant.

Evaluation of the quadratic form q_ρ depends only on a neighbourhood of a simple loop in Σ representing $[\gamma]$ since it uses only the tangential lift. By continuity, the discrete-valued quadratic form does not change in a continuous family. The sign of the trace separates the hyperbolic elements of $SL(2,\mathbb{R})$ into two components hence it does not change in a continuous family. To prove (19), we may first deform the representation $\rho : \pi_1\Sigma \to SL(2,\mathbb{R})$ to any Fuchsian representation in the same connected component. Moreover, we can use deformations of the representation defined only in a neighbourhood of a simple closed geodesic, that do not necessarily extend to Σ.

The dependence on a neighbourhood of a simple closed geodesic and deformation invariance of both sides of (19) reduces the lemma to a single calculation. We can take any simple closed geodesic in any hyperbolic surface. The geodesic boundary of a one-holed torus Σ is a well-studied example. Given a Fuchsian representation $\overline{\rho} : \pi_1\Sigma \to PSL(2,\mathbb{R})$ and $A, B \in PSL(2,\mathbb{R})$ the image of the generators of $\pi_1\Sigma$, the trace of the commutator $ABA^{-1}B^{-1}$ is well-defined independently of the lift of $\overline{\rho}$ to ρ. The following explicit calculation shows that $\operatorname{tr}(ABA^{-1}B^{-1}) < 0$. Conjugate A and B so that A is diagonal:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The invariant geodesic of A is given by $x = 0$ in $\mathbb{H} = \{x+iy \mid y > 0\}$. The invariant geodesics of A and B must meet since they lift from generators of π_1 of the torus. The two fixed points of B are the roots z_1 and z_2 of $cz^2 + (d-a)z - b = 0$, hence $z_1 z_2 = -b/c$. They must lie on either side of 0 on the real axis, hence their product is negative so $bc > 0$. By direct calculation, $\operatorname{tr}(ABA^{-1}B^{-1}) = 1-(\lambda^2+\lambda^{-2}-1)bc < 1$ since $bc > 0$. By assumption, Σ is hyperbolic, so $|\operatorname{tr}(ABA^{-1}B^{-1})| \geq 2$, hence we must have $\operatorname{tr}(ABA^{-1}B^{-1}) \leq -2 < 0$.

The homology class $\lfloor\gamma\rfloor$ represented by $\rho([\gamma]) = ABA^{-1}B^{-1}$ is trivial hence $q(\lfloor\gamma\rfloor) = 0$ and we have just shown $\mathrm{tr}\,(\rho([\gamma])) < 0$ which agrees with (19). Actually it proves (19) since an element $\eta \in H^1(\Sigma;\mathbb{Z}_2)$ that is non-trivial on a homology class, say $\eta([C]) = 1$, sends $q(C) \mapsto q(C) + 1$ and $\rho(C) \mapsto -\rho(C) \in SL(2,\mathbb{R})$ which flips the sign of the trace, proving the equivalence of the negative and positive trace cases of (19). Although a general element of a fundamental group is not a commutator, the neighbourhood of any simple closed geodesic is canonical hence behaves as in the calculated example and the lemma is proven.

The reduction of (19) to the single calculation above is convenient, but one can also see the relationship to the sign of the trace directly as follows. Since q_ρ depends only on a neighbourhood of a simple loop we may assume that $\pi_1\Sigma = \mathbb{Z}$ and $\Sigma = \mathbb{H}/\mathbb{Z}$ is a hyperbolic annulus with a unique simple closed geodesic $C \subset \Sigma$. The spin structure is the double cover $SL(2,\mathbb{R})/\mathbb{Z} \to PSL(2,\mathbb{R})/\mathbb{Z}$. We may deform the generator $g \in SL(2,\mathbb{R})$ of $\mathbb{Z} \cong \langle g\rangle$ to any given element, for example a diagonal element, with trace of the same sign. The tangential lift $\tilde{C}$ of the simple closed geodesic C defines an element of $\pi_1(PSL(2,\mathbb{R})/\mathbb{Z})$. If we start upstairs at $I \in SL(2,\mathbb{R})/\mathbb{Z}$ and move around the loop downstairs, then the lift of the loop is again a loop in $SL(2,\mathbb{R})/\mathbb{Z}$ precisely when $\mathrm{sgn}\,\mathrm{tr}\,(g) > 0$ because g can be deformed to I. In other words $\eta_\rho(\tilde{C}) = 0$. The holonomy is non-trivial when $\mathrm{sgn}\,\mathrm{tr}\,(g) < 0$, or $\eta_\rho(\tilde{C}) = 1$. Since $\ell(\lfloor\gamma\rfloor) = \sigma + [\tilde{C}]$ then we have $q_{\eta_\rho}(\lfloor\gamma\rfloor) = \eta_\rho \circ \ell(\lfloor\gamma\rfloor) = \eta_\rho(\sigma) + \eta_\rho([\tilde{C}]) = 1$ when $\mathrm{sgn}\,\mathrm{tr}\,(g) > 0$ and $q_{\eta_\rho}(\lfloor\gamma\rfloor) = 0$ when $\mathrm{sgn}\,\mathrm{tr}\,(g) < 0$ as required. □

The set of hyperbolic and parabolic elements of $SL(2,\mathbb{R})$ satisfy $|\mathrm{tr}\,\rho([\gamma])| \geq 2$, hence it has two components determined by the sign of the trace. Given a Fuchsian representation $\rho : \pi_1\Sigma \to SL(2,\mathbb{R})$, Definition 3.1 and Lemma 3.3 show that a boundary class $[\gamma]$ is Neveu-Schwarz if $\mathrm{tr}\,\rho([\gamma]) < 0$ and Ramond if $\mathrm{tr}\,\rho([\gamma]) > 0$.

A consequence of Lemma 3.3 and the homological nature of the quadratic form is the following property.

Corollary 3.4. *Let Σ be a surface with boundary classes $\gamma_1, ...\gamma_n$. Any Fuchsian representation $\rho : \pi_1\Sigma \to SL(2,\mathbb{R})$ satisfies*

$$(-1)^n \prod_{i=1}^{n} tr(\rho([\gamma_i])) > 0.$$

This property of the product of traces of Fuchsian representations into $SL(2,\mathbb{R})$ has been studied particularly in the 2-generator free group case—as the negative trace theorem in [37]—proving that for the pair of pants and the once-punctured torus, the product of the traces of the boundary classes is negative.

3.2. **Flat bundles.** In this section we describe how the spinor bundle $S_\Sigma \to \Sigma$ of a hyperbolic surface equipped with a spin structure is a flat bundle. Equivalently, there exists a flat connection on S_Σ, which must differ from the lift of the Levi-Civita connection by cohomological considerations—see Remark 3.5. The flat structure is visible via representations of $\pi_1\Sigma$ into $SL(2,\mathbb{R})$.

3.2.1. The right action of $\mathrm{Spin}(2) = SO(2)$ on $P_{\mathrm{Spin}}(\Sigma) \cong SL(2,\mathbb{R})/\rho(\pi_1\Sigma)$ ($\rho(\pi_1\Sigma)$ acts on the left of $SL(2,\mathbb{R})$) is used to define the associated spinor bundle

$$S_\Sigma = P_{\mathrm{Spin}}(\Sigma) \times_{SO(2)} \mathbb{C}^2 \cong \left(\mathbb{H} \times \mathbb{C}^2\right)/\rho(\pi_1\Sigma). \tag{20}$$

The flat real bundle $T_\Sigma^{\frac{1}{2}}$ is obtained by replacing $\mathbb{C}^2$ with $\mathbb{R}^2$ in (20). The right hand side of (20) defines a flat bundle over Σ associated to the representation $\rho:\pi_1\Sigma\to SL(2,\mathbb{R})$ where the action is given by $g\cdot(z,v)=(g\cdot z,g\cdot v)$. The map $SL(2,\mathbb{R})\times\mathbb{C}^2\ni(g,u)\mapsto(g\cdot i,gu)\in\mathbb{H}\times\mathbb{C}^2$ defines the isomorphism in (20). It is well-defined on orbits (gk^{-1},ku), $k\in SO(2)$ and descends to the quotient by $\rho(\pi_1\Sigma)$ on both sides.

The spinor bundle S_Σ is flat hence holomorphic. We show below that $T_\Sigma^{\frac{1}{2}}$ is a subbundle of S_Σ in two different ways, compatible with the flat, respectively holomorphic, structure of S_Σ. It is the underlying flat real bundle $T_\Sigma^{\frac{1}{2}}\stackrel{r}{\to}S_\Sigma$ which is the fixed point set of the real involution on S_Σ. It is also a holomorphic subbundle $T_\Sigma^{\frac{1}{2}}\stackrel{h}{\to}S_\Sigma$ which is an eigenspace of the action of $SO(2)$. The images of r and h intersect trivially.

The weights $\chi^{\pm1}$, defined in 3.1.1, of the $SO(2)$ representation of $\mathbb{C}^2=\mathbb{C}_\chi\oplus\mathbb{C}_{\chi^{-1}}$ defines a decomposition of S_Σ into holomorphic line bundles $S_\Sigma=T_\Sigma^{\frac{1}{2}}\oplus T_\Sigma^{-\frac{1}{2}}$. With respect to this decomposition, $SL(2,\mathbb{R})$ acts via $SU(1,1)$, i.e. the matrix of any $g\in SL(2,\mathbb{R})$, with respect to a basis of eigenvectors of $\chi^{\pm1}$, lives in $SU(1,1)$. With respect to the decomposition $\mathbb{C}^2=\mathbb{C}_\chi\oplus\mathbb{C}_{\chi^{-1}}$, the real structure σ on $\mathbb{C}^2$ (which is complex conjugation with respect to a complex structure different to that on $\mathbb{C}_\chi\oplus\mathbb{C}_{\chi^{-1}}$) is given by $(u,v)\mapsto(\overline{v},\overline{u})$. The real structure commutes with the actions of the structure groups of the bundle, $SO(2)$ on the left hand side of (20) and $SL(2,\mathbb{R})$ on the right hand side of (20). (Note that $SL(2,\mathbb{R})$ commutes with complex conjugation and $SU(1,1)$ commutes with $\sigma(u,v)=(\overline{v},\overline{u})$ which is the same group action and real structure with respect to different bases.) Hence the bundle S_Σ is equipped with a real structure σ with fixed point set the underlying flat real bundle $T_\Sigma^{\frac{1}{2}}$, obtained by replacing $\mathbb{C}^2$ with $\mathbb{R}^2$ on both sides of (20). In 3.2.3 the real structure on S_Σ will involve the Hermitian metric used to reduce the structure group to $SO(2)$.

Remark 3.5. Note that the flat bundle $T_\Sigma^{\frac{1}{2}}$ has non-zero Euler class. The Euler class can be obtained via a metric connection on $T_\Sigma^{\frac{1}{2}}$ as described in 3.4.1, so in particular if the metric connection were flat, the Euler class would vanish. There is no contradiction here because $\mathbb{R}^2$ admits no metric invariant under $SL(2,\mathbb{R})$, so we cannot find a metric on $T_\Sigma^{\frac{1}{2}}$ which is preserved by its flat connection. This example is discussed by Milnor and Stasheff in [41, p.312].

3.2.2. A Hermitian metric h on a line bundle $L\to\Sigma$ defines an isomorphism $\overline{L}\stackrel{\cong}{\to}L^\vee$ by $\ell\mapsto h(\overline{\ell},\cdot)$, where $\overline{L}$ is the conjugate bundle, defined via conjugation of transition functions. For example, a metric on a Riemann surface compatible with its conformal structure is equivalent to a Hermitian metric h^2 on T_Σ, and in fact a Hermitian metric on any power $K_\Sigma^{\otimes n}$ such as a choice of spin structure $K_\Sigma^{1/2}$. Hence

$$\overline{K_\Sigma^{\otimes n}}\stackrel{h^*}{\cong}\left(K_\Sigma^{-1}\right)^{\otimes n}$$

where the isomorphism h^* depends on the Hermitian metric on $K_\Sigma^{\otimes n}$ via $\ell\mapsto h(\overline{\ell},\cdot)^{2n}$.

3.2.3. The real structure σ defined on the spinor bundle $S_\Sigma = T_\Sigma^{\frac{1}{2}} \oplus T_\Sigma^{-\frac{1}{2}}$ in 3.2.1 is induced by the isomorphism $\overline{T}_\Sigma^{\frac{1}{2}} \overset{h^*}{\cong} T_\Sigma^{-\frac{1}{2}}$, from the Hermitian metric h on $T_\Sigma^{\frac{1}{2}}$ which is the square root of the hyperbolic metric on Σ. It is defined on local sections by

$$\sigma(u, v) = (h^{-1}\overline{v}, h\overline{u}).$$

The underlying real bundle $T_\Sigma^{\frac{1}{2}}$ is the subbundle of fixed points of σ which is locally given by $(u, h\overline{u})$. In particular $u \mapsto (u, h\overline{u})$ defines a natural isomorphism between the flat real subbundle and the holomorphic subbundle given by an eigenspace of the action of $SO(2)$, both isomorphic to $T_\Sigma^{\frac{1}{2}}$.

3.2.4. A flat bundle E over a surface Σ defines a locally constant sheaf given by its sheaf of locally flat sections which we also denote by E. We denote its sheaf cohomology by $H^i_{dR}(\Sigma, E)$. We will apply this to the spinor bundle $E = S_\Sigma$ and its underlying real bundle $E = T_\Sigma^{\frac{1}{2}}$. The cohomology can be calculated in different ways, and the label dR for de Rham, following Simpson [55], refers to its calculation via the following complex which uses the covariant derivative d_A defined by the flat connection on E:

$$A^0_\Sigma(E) \overset{d_A}{\to} A^1_\Sigma(E) \overset{d_A}{\to} A^2_\Sigma(E). \tag{21}$$

Here $A^k_\Sigma(E) := \Gamma(\Sigma, \Lambda^k(T^*\Sigma) \otimes E)$ denotes global C^∞ differential k-forms with coefficients in E. It defines a complex because $d_A \circ d_A = F^A \in \Omega^2(\mathrm{End}E)$ is given by the curvature which vanishes in this case. Define $H^i_{dR}(\Sigma, E)$ for $i = 0, 1, 2$ to be the cohomology of the complex. We will use the complex (21) again only in **??** and instead mainly use Čech cohomology to calculate $H^i_{dR}(\Sigma, E)$.

3.2.5. The sheaf cohomology $H^i_{dR}(\Sigma, E)$ can be calculated using Čech cohomology applied to an open cover of Σ obtained from a triangulation. A triangulation of Σ is a simplicial complex $\mathcal{C} = \bigcup_{k=0}^{2} \mathcal{C}_k$ where $\mathcal{C}_k$ denotes k-simplices $\sigma : \Delta_k \to \Sigma$, and we further require the regularity condition that each 2-simplex is a homeomorphism onto its image. The regularity condition ensures that 2-simplices incident at an edge or vertex are distinct. We identify simplices with their images in Σ and refer to them as faces, edge and vertices of the triangulation. To each simplex σ of the triangulation associate the open set $U_\sigma \subset \Sigma$ given by the union of the interiors of all simplices whose closure contains σ. Hence, to each vertex of the triangulation $v \in \mathcal{C}_0$, associate the open set $U_v \subset \Sigma$ given by the union of the interiors of all simplices whose closure meets v, as in Figure 1, so it includes the vertex v, no other vertices, and the interiors of all incident edges and faces.

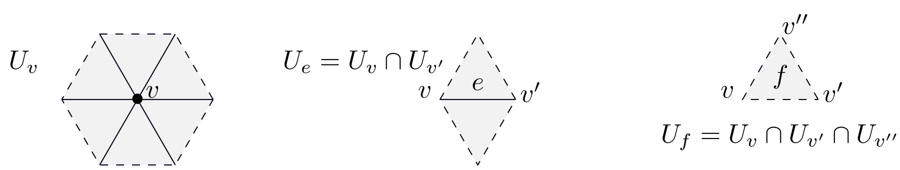

FIGURE 1. Open cover associated to triangulation

This produces an open cover:

$$\Sigma = \bigcup_{\sigma \in \mathcal{C}} U_\sigma. \tag{22}$$

We allow more general cell decompositions where faces of the triangulation can be polygons, not only triangles. For v and v' vertices of an edge e, and v, v' and v'' vertices of a face f we have

$$U_e = U_v \cap U_{v'}, \quad U_f = U_v \cap U_{v'} \cap U_{v''}.$$

Note that $U_v \cap U_{v'}$ or $U_v \cap U_{v'} \cap U_{v''}$ is empty if there is no edge containing v and v', or face containing v, v' and v''. For example, given a triangulation, where faces are indeed triangles, for more than three distinct vertices $\{v_i\}$ the intersection is empty $\bigcap_i U_{v_i} = \varnothing$. On a compact surface, one can define the open cover using only the vertices $\Sigma = \bigcup_{v\in\mathcal{C}_0} U_v$ so that the sets associated to edges and faces are not part of the cover, and instead arise as intersections. This results in fewer coboundary maps in the construction of 3.2.6.

We allow a generalisation of triangulations, where some of the vertices are missing (from both Σ and the triangulation) which is particularly useful for non-compact Σ. In this case, the regularity condition on a face is required only in its domain which is a 2-simplex with some vertices removed. Hence U_e and U_f may not arise as intersections of U_v for $v \in \mathcal{C}_0$ justifying the open cover (22). The set of vertices may be empty, as is the case for ideal triangulations, in which case there are no open sets U_v.

3.2.6. The Čech cochains with respect to the open cover (22) of the sheaf of locally constant sections of E are defined by

$$C^k(\Sigma, E) = \bigoplus_{\sigma\in\mathcal{C}_k} \Gamma(U_\sigma, E), \quad k = 0, 1, 2.$$

The coboundary map δ is given by restriction and Čech cohomology $H^\bullet_{dR}(\Sigma, E)$ is defined to be the cohomology of the complex

$$0 \to \bigoplus_{v\in\mathcal{C}_0} \Gamma(U_v, E) \xrightarrow{\delta} \bigoplus_{e\in\mathcal{C}_1} \Gamma(U_e, E) \xrightarrow{\delta} \bigoplus_{f\in\mathcal{C}_2} \Gamma(U_f, E) \to 0. \tag{23}$$

Note that $C^k(E) = 0$ for $k > 2$ since these correspond to empty intersections. If we allow more general cell decompositions where faces of the triangulation can be polygons, not only triangles, then there are non-trivial $C^k(E)$ for $k > 2$, but still $H^k_{dR}(\Sigma, E) = 0$ for $k > 2$.

Since the cohomology of (23) defines the sheaf cohomology $H^k_{dR}(\Sigma, E)$ it is independent of the choice of cell decomposition of Σ. It follows that duality of triangulations gives duality of cohomology groups.

3.2.7. Čech cohomology was calculated in 3.2.6 using a *good open cover*, meaning that intersections of open sets in the cover are contractible, which is achieved from the regularity condition on triangulations.

If we relax the regularity condition in 3.2.5 on a triangulation $\mathcal{C} = \bigcup_{k=0}^{2} \mathcal{C}_k$ of Σ so that a 2-simplex is not necessarily one-to-one onto its image, we describe a construction, used in [57], of the sheaf cohomology of E as follows. It coincides with the dual of the construction in 3.2.6 when the triangulation satisfies the regularity condition.

For $\sigma \in \mathcal{C}$, let $\mathcal{V}_\sigma = H^0(\sigma, E)$ denote the covariant constant sections $s|_\sigma$ of E over σ. Here we identify σ with its image. Define

$$C_k(\Sigma, E) = \bigoplus_{\sigma \in \mathcal{C}_k} \mathcal{V}_\sigma$$

and boundary maps

$$\begin{array}{rcl} C_{k+1}(\Sigma, E) & \stackrel{\partial}{\to} & C_k(\Sigma, E) \\ s|_\sigma & \mapsto & s|_{\partial\sigma} = \bigoplus (-1)^{\epsilon_i} s|_{\sigma_i} \end{array}$$

where $\partial\sigma = \bigcup_i (-1)^{\epsilon_i}\sigma_i$ as oriented simplices. A section $s|_\sigma$ is well-defined on the pull-back of E to the cell, but possible multiply-defined on the boundary of σ, and we use the extension from the interior in the definition of ∂. This ambiguity arises precisely due to the relaxation of the regularity condition in 3.2.5.

It is clear that $\partial^2 = 0$ since the contribution at any vertex of a 2-cell essentially gives the covariant constant section extended to the vertex, appearing with opposite sign due to orientations, or vanishing of the square of the usual boundary map on simplices. The same argument applies to higher dimensional simplices and their codimension two cells. One can approach the vertex along two edges, and the vanishing then reflects the trivial local holonomy of the flat connection.

Denote by $H_k(\Sigma, E)$ the homology of the complex

$$C_2(\Sigma, E) \stackrel{\partial}{\to} C_1(\Sigma, E) \stackrel{\partial}{\to} C_0(\Sigma, E).$$

3.2.8. There is a natural symplectic structure on S_Σ and $T_\Sigma^{\frac{1}{2}}$ arising from the symplectic form on $\mathbb{C}^2$ and $\mathbb{R}^2$ preserved by the $SL(2,\mathbb{R})$ action. Hence there is a natural isomorphism $C_k(\Sigma, S_\Sigma) \cong C_k(\Sigma, S_\Sigma^\vee) \cong C_k(\Sigma, S_\Sigma)^\vee$ which gives a natural isomorphism

$$C_k(\Sigma, S_\Sigma)^\vee \cong C^k(\Sigma, S_\Sigma).$$

Moreover, $(\partial\eta, f) = (\eta, \delta f)$ since both sides use the symplectic form applied to the extension of η and f or η and the restriction of f which is the same. Thus we see that

$$H_k(\Sigma, S_\Sigma)^\vee \cong H^k_{dR}(\Sigma, S_\Sigma)$$

and the same isomorphism holds for $T_\Sigma^{\frac{1}{2}}$.

When the triangulation is regular, the isomorphism between cohomology and homology is visible via the cochains in 3.2.6 and the chains in 3.2.7 coinciding, $C^k(\Sigma, S_\Sigma) = C_k(\Sigma, S_\Sigma)$, while the maps δ and ∂ go in opposite directions. In terms of the open sets U_σ defined in 3.2.5, δ are restriction maps while ∂ are extension maps.

3.2.9. An ideal triangulation of a non-compact surface Σ is a triangulation with no vertices, and all faces triangles. The number of faces and edges is $4g - 4 + 2n$, respectively $6g - 6 + 3n$ for $\Sigma = \overline{\Sigma} - \{p_1, ..., p_n\}$ of genus g. Dual to an ideal triangulation is a trivalent fatgraph $\Gamma = V(\Gamma) \cup E(\Gamma)$ which is a triangulation of a retract of Σ with only vertices $V(\Gamma)$ and edges $E(\Gamma)$, and no faces.

With respect to an ideal triangulation, $H^k_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}})$ is conveniently calculated using the dual fatgraph. The complex is rather simple since there are only 2-cochains and 1-cochains. Or dually, using the fatgraph Γ there are only 0-chains and 1-chains. We can equally work with the restriction of the flat bundle $T_\Sigma^{\frac{1}{2}}|_\Gamma$

which we also denote by $T_\Sigma^{\frac{1}{2}}$. Following 3.2.7, for $e \in E(\Gamma)$, let $\mathcal{V}_e$ denote the covariant constant sections $s|_e$ of $T_\Sigma^{\frac{1}{2}}$ over e, and for $v \in V(\Gamma)$, let $\mathcal{V}_v$ denote the covariant constant sections $s|_v$ of $T_\Sigma^{\frac{1}{2}}$ over v. Define

$$C_0(\Gamma, T_\Sigma^{\frac{1}{2}}) = \bigoplus_{v\in V(\Gamma)} \mathcal{V}_v, \quad C_1(\Gamma, T_\Sigma^{\frac{1}{2}}) = \bigoplus_{e\in E(\Gamma)} \mathcal{V}_e$$

and boundary maps

$$\begin{array}{rcl} C_1(\Gamma, T_\Sigma^{\frac{1}{2}}) & \stackrel{\partial}{\to} & C_0(\Gamma, T_\Sigma^{\frac{1}{2}}) \\ s|_e & \mapsto & s|_{\partial e} = s|_{e_+} - s|_{e_-} \end{array}$$

where $e_\pm \in V(\Gamma)$ are the vertices bounding the oriented edge e.

The sheaf cohomology $H^k_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}})$ is given by the homology of the complex

$$C_1(\Gamma, T_\Sigma^{\frac{1}{2}}) \stackrel{\partial}{\to} C_0(\Gamma, T_\Sigma^{\frac{1}{2}}).$$

We have $H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}}) \cong H_1(\Gamma, T_\Sigma^{\frac{1}{2}}) = \ker\partial$ and $H^0_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}}) \cong H_0(\Gamma, T_\Sigma^{\frac{1}{2}}) = 0$. The vanishing of $H^0_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}})$ uses the ideal triangulation so in particular there are no 0-cochains.

Theorem 3.6. *For any hyperbolic spin surface Σ with Neveu-Schwarz boundary*

$$H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}}) \cong \mathbb{R}^{4g-4+2n}$$

and this defines a vector bundle

$$E_{g,n} \to \mathcal{M}_{g,n}^{spin}$$

with fibres $H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}})$.

Proof. First consider the case when Σ is non-compact hence admits an ideal triangulation. A hyperbolic spin surface is equivalent to a flat $SL(2,\mathbb{R})$ connection over the dual fatgraph Γ of an ideal triangulation of Σ. Arbitrarily orient each edge of Γ. The flat connection is equivalent to associating an element $g_e \in SL(2,\mathbb{R})$ to each oriented edge e of Γ. The holonomy around any oriented loop $\gamma \subset \Gamma$ is the product $g_\gamma = \prod g_e^{\pm 1}$ of the elements along edges of the loop with ± 1 determined by whether the orientation of the edge agrees with the orientation of the loop. The holonomy around any oriented loop satisfies $|\mathrm{tr}\, g_\gamma| \geq 2$.

An element of $\ker\delta \cong H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}})$ is a collection of vectors $v_e \in \mathbb{R}^2$ assigned to each oriented edge, satisfying a condition at each vertex. We choose the convention that the trivialisation of $T_\Sigma^{\frac{1}{2}}$ over an oriented edge e is induced from the trivialisation of $T_\Sigma^{\frac{1}{2}}$ over its source vertex e_-. Hence

$$\partial v_e|_{e_+} = g_e v_e, \quad \partial v_e|_{e_-} = -v_e.$$

The condition at a vertex is the vanishing of the sum of contributions from the three oriented edges adjacent to the given vertex, such as $\sum g_e v_e = 0$ for a vertex with only incoming edges, or more generally each summand is $g_e v_e$ or $-v_e$.

Choose an ideal triangulation of Σ with dual fatgraph Γ that admits a *dimer* covering $D \subset E(\Gamma)$ which is a collection of $2g-2+n$ edges such that each vertex of Γ is the boundary of a unique edge in the dimer. Such an ideal triangulation always exists, for example one can always choose an ideal triangulation with bipartite dual fatgraph [52], then any perfect matching is a dimer covering. We will prove that for all edges e of D the vectors $v_e \in \mathbb{R}^2$ can be arbitrarily and independently assigned, and they uniquely determine the vectors on all other edges, hence they produce a basis of $2(2g-2+n)$ vectors for $H_1(\Gamma, T_\Sigma^{\frac{1}{2}})$. In Remark 3.7 below we show how to produce a basis of $2(2g-2+n)$ vectors for $H_1(\Gamma, T_\Sigma^{\frac{1}{2}})$ for any dual fatgraph Γ, not necessarily admitting a dimer covering.

Given $e_0 \in D$, choose an arbitrary non-zero $v_{e_0} \in \mathbb{R}^2$ and set $v_e = 0$ for all other dimer edges $e \in D \backslash \{e_0\}$. Since Γ is trivalent, $\Gamma \backslash D$ is a collection of embedded loops. Along an oriented loop $\gamma \subset \Gamma \backslash D$, the vertex condition on elements of $\ker \delta$ uniquely determines each vector v_e on an edge $e \in \gamma$ from the preceding edge. For example, if the orientation on each edge agrees with the orientation on γ, then $ge_i = e_{i+1}$ where e_i and e_{i+1} are consecutive oriented edges in γ.

If a loop $\gamma \subset \Gamma \backslash D$ avoids e_0, then we must have $v_e = g_\gamma v_e$ where e is an edge of γ and g_γ is the holonomy around the loop starting from e. But $g_\gamma - I$ is invertible, or equivalently g_γ does not have eigenvalue 1, since non-boundary loops satisfy $|\operatorname{tr} g_\gamma| > 2$ and boundary loops satisfy $\operatorname{tr} g_\gamma \leq -2$ by the Neveu-Schwarz boundary condition and Lemma 3.3. Hence $v_e = 0$ for all edges $e \in \gamma$.

If a loop $\gamma \subset \Gamma \backslash D$ meets e_0, then we now have

$$(g_\gamma - I)v_e - v_{e_0} = 0$$

(or $(g_\gamma - I)v_e + g_{e_0} v_{e_0} = 0$) and since $g_\gamma - I$ is invertible this uniquely determines $v_e \in \mathbb{R}^2$ and all vectors along γ.

Hence a choice of non-zero $v_{e_0} \in \mathbb{R}^2$ uniquely determines a vector in $\ker \delta$. Clearly elements of $\ker \delta$ associated to different dimer edges are linearly independent because each vanishes on the other dimer edges. We also see that if an element of $\ker \delta$ vanishes on all dimer edges then it vanishes identically. Hence each edge $e \in D$ determines two independent vectors in $H_1(\Gamma, T_\Sigma^{\frac{1}{2}})$, and the union over the $2g-2+n$ edges in D produces a basis of $2(2g-2+n)$ vectors for $H_1(\Gamma, T_\Sigma^{\frac{1}{2}})$.

We have proved $H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}}) \cong \mathbb{R}^{4g-4+2n}$ which is the first part of the Theorem. In fact we have a canonical isomorphism between $H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}})$ and $(\mathbb{R}^2)^D$, for $D \subset E(\Gamma)$ a dimer covering. But this gives a local trivialisation over the moduli space $\mathcal{M}_{g,n}^{\text{spin}}$ since a choice of ideal triangulation defines the Teichmüller space of the moduli space. A choice of $D \subset E(\Gamma)$ is well-defined on the Teichmüller space producing a trivial bundle $(\mathbb{R}^2)^D$, from which we get a local trivialisation over the moduli space.

When Σ is compact it has genus $g > 1$, and we choose a decomposition $\Sigma = \Sigma_1 \cup \Sigma_2$ into genus $g-1$ and genus 1 surfaces glued along boundary annuli. We have $H^k_{dR}(\Sigma_1 \cap \Sigma_2, T_\Sigma^{\frac{1}{2}}) = 0$ for $k = 0, 1$ by hyperbolicity of the holonomy as follows. For $U \cup V = \Sigma_1 \cap \Sigma_2$, the sequence (23) becomes

$$0 \to \Gamma(U, T_\Sigma^{\frac{1}{2}}) \xrightarrow{\delta} \Gamma(V, T_\Sigma^{\frac{1}{2}}) \to 0$$

with boundary map $\delta = g_\gamma - I$ where g_γ is the holonomy around a loop $\gamma \subset \Sigma_1 \cap \Sigma_2$. But g_γ is hyperbolic so it satisfies $|\operatorname{tr} g_\gamma| > 2$ and in particular $g_\gamma - I$ is invertible, and the cohomology groups $H^k_{dR}(\Sigma_1 \cap \Sigma_2, T_\Sigma^{\frac{1}{2}}) = 0$ vanish.

Hence the Mayer-Vietoris sequence gives

$$0 \to H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}}) \to H^1_{dR}(\Sigma_1, T_\Sigma^{\frac{1}{2}}) \oplus H^1_{dR}(\Sigma_2, T_\Sigma^{\frac{1}{2}}) \to 0.$$

We have shown above that $H^1_{dR}(\Sigma_1, T_\Sigma^{\frac{1}{2}}) \cong \mathbb{R}^{4g-6}$ and $H^1_{dR}(\Sigma_2, T_\Sigma^{\frac{1}{2}}) \cong \mathbb{R}^2$ and they define local trivialisations over the respective moduli spaces of bundles $E_{g-1,1}$ and $E_{1,1}$. This gives a local decomposition $E_g \cong E_{g-1,1} \oplus E_{1,1}$ proving that E_g is indeed a vector bundle. The decomposition $\Sigma = \Sigma_1 \cup \Sigma_2$ does not make sense over the moduli space since the mapping class group does not preserve the decomposition, and is only well-defined over Teichmüller space. Nevertheless, it does make sense locally which is enough to prove that E_g is a rank $4g-4$ vector bundle.

□

Remark 3.7. In Theorem 3.6, one can drop the assumption that the dual fatgraph Γ of the ideal triangulation of Σ must admit a dimer covering. On any dual fatgraph Γ, there exists a collection $C \subset E(\Gamma)$ of $2g-2+n$ edges of Γ on which the vectors $v_e \in \mathbb{R}^2$ can be independently assigned, and which uniquely determine the vectors on all other edges. We call such a collection C a *base* of edges of Γ. Each edge $e \in C$ determines two independent vectors in $H_1(\Gamma, T_\Sigma^{\frac{1}{2}})$, and the union over the $2g-2+n$ edges in C produces a basis of $2(2g-2+n)$ vectors for $H_1(\Gamma, T_\Sigma^{\frac{1}{2}})$.

To prove the existence of a base of edges, begin with a bipartite dual fatgraph, which always admits a dimer covering. Any ideal triangulation of Σ can be transformed by Whitehead moves to an ideal triangulation with bipartite dual [52].

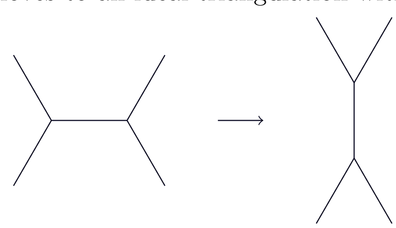

Figure 2. Whitehead move

Under a Whitehead move, neither the bipartite property nor the existence of a dimer covering is preserved. However, there is a natural bijection of edges under Whitehead moves, and a base of edges is sent to a base of edges under this bijection. Since we compute cohomology of Σ, which is independent of the choice of Γ, there is a natural isomorphism $H_1(\Gamma, T_\Sigma^{\frac{1}{2}}) \cong H_1(\Gamma', T_\Sigma^{\frac{1}{2}})$ when Γ and Γ' are related by a Whitehead move. In particular, the image $C' \subset E(\Gamma')$ of a base of edges $C \subset E(\Gamma)$ under the Whitehead move inherits the following two properties of C—for $e \in C$ the vectors $v_e \in \mathbb{R}^2$ can be independently assigned, and uniquely determine the vectors on all edges in $\Gamma \backslash C$—and thus is also a base of edges.

Stanford and Witten [57] used the torsion on $H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}})$ to define a measure on the bundle $E_{g,n} \to \mathcal{M}^{\text{spin}}_{g,n}$ used to define the super volume measure.

3.3. **Higgs bundles.** In this section we will prove that the restriction of the bundle $\widehat{E}_{g,n} \to \overline{\mathcal{M}}^{\text{spin}}_{g,n}(\vec{0})$ defined in Section 2 to the smooth moduli space gives the bundle $E_{g,n} \to \mathcal{M}^{\text{spin}}_{g,n}$ defined in Section 3. The constructions of the bundles $\widehat{E}_{g,n}$ and $E_{g,n}$ over the moduli spaces of stable and smooth spin curves respectively use the cohomology of different sheaves. We will prove that over smooth spin curves $\Sigma=\overline{\Sigma}-D$ the sheaf cohomology groups are isomorphic

$$H^1_{dR}(\Sigma,T_\Sigma^{\frac12})\cong H^1(\overline{\Sigma},T_{\overline{\Sigma}}^{\frac12}(-D)). \tag{24}$$

The natural way to prove the isomorphism (24) relating flat and holomorphic structures on bundles over Σ uses Higgs bundles. More precisely, there is a natural identification of any flat structure on a bundle $E\to\Sigma$, with an extension of E to $\overline{\Sigma}$ equipped with a holomorphic structure, Higgs field and parabolic structure. Applied to the spinor bundle $E=S_\Sigma$, this gives a natural way to realise uniformisation of Σ which naturally associates a unique hyperbolic metric on Σ in the conformal class defined by Σ. Furthermore, it gives an isomorphism between the respective moduli spaces. We will see that the sheaves on both sides of (24) arise naturally from this proof of uniformisation.

The use of Higgs bundles achieves two goals. It relates the sheaf cohomologies arising from a flat structure and a holomorphic structure on a bundle. It also relates cohomological constructions on a non-compact Riemann surface $\Sigma=\overline{\Sigma}-D$ and on the compact pair $(\overline{\Sigma},D)$. We will start with the case when Σ is compact, i.e. $D=\varnothing$. This will simplify the exposition and focus only on the first goal. Then we will consider the general case, which requires parabolic structures on bundles over $(\overline{\Sigma},D)$. The general proof essentially follows the proof in the compact case with some technical adjustments.

3.3.1. Higgs bundles over a compact Riemann surface Σ with canonical bundle K_Σ were defined by Hitchin in [27] as follows.

Definition 3.8. A Higgs bundle over a compact Riemann surface Σ is a pair (E,ϕ) where E is a holomorphic vector bundle over Σ and $\phi\in H^0(\mathrm{End}(E)\otimes K_\Sigma)$.

The pair (E,ϕ) is *stable* if for any ϕ-invariant subbundle $F\subset E$, i.e. $\phi(F)\subset F\otimes K_\Sigma$, we have $\frac{c_1(F)}{\mathrm{rank}\ F}<\frac{c_1(E)}{\mathrm{rank}\ E}$. When $\phi=0$, every subbundle is ϕ-invariant and the definition of stable reduces to the usual definition of stable for a holomorphic bundle E.

A Hermitian structure on E is a Hermitian metric H defined on E with respect to its complex structure. It defines a reduction of the structure group of E from $GL(n,\mathbb{C})$ to $U(n)$. The holomorphic structure and Hermitian metric H on E together define a unitary connection A on E via $d_A=\overline{\partial}+\overline{\partial}^*$, where $\overline{\partial}_A=\overline{\partial}$ is the natural operator on E and ∂_A is the adjoint of $\overline{\partial}_A$ with respect to H. The curvature of a unitary connection A on E is a unitary endomorphism valued two-form F_A. Since $[\phi,\phi^*]$ is also a unitary endomorphism valued two-form, they can be compared. The connection A (or equivalently the Hermitian metric H) is said to satisfy the Higgs bundle equations if

$$F_A+[\phi,\phi^*]=0 \tag{25}$$

Importantly, (25) is equivalent to the connection $A+\phi+\phi^*$ being a flat $SL(2,\mathbb{C})$ connection. This relation between holomorphic and flat structures will be used to relate those structures on $T_\Sigma^{\frac12}$.

One can consider a broader class of sections ϕ, allowing them to be smooth endomorphism valued one-forms and add to (25) the equation

$$\overline{\partial}_A\phi=0$$

which is the condition that ϕ is holomorphic. This makes the invariance of the equations under the unitary gauge group clear but now $\overline{\partial}\mapsto\overline{\partial}_A$. Note that constant unitary gauge transformations are both holomorphic gauge transformations and smooth gauge transformations, and in particular they preserve $\overline{\partial}$.

Theorem 3.9 (Hitchin [27])**.** *A stable Higgs bundle (E,ϕ) of degree zero admits a unique unitary connection A satisfying (25). Conversely a Higgs bundle (E,ϕ) which admits a connection A satisfying (25) is of degree zero and stable.*

3.3.2. Apply Theorem 3.9 to the spinor bundle $E=S_\Sigma=T_\Sigma^{\frac12}\oplus T_\Sigma^{-\frac12}$ with Higgs field

$$\phi=\frac12\left(\begin{array}{cc}0&1\\0&0\end{array}\right)\in H^0(\mathrm{End}(E)\otimes K_\Sigma) \tag{26}$$

where 1 is the natural section of $\mathcal{O}_\Sigma\cong T_\Sigma^{\frac12}\otimes T_\Sigma^{\frac12}\otimes K_\Sigma$ which gives a linear map $T_\Sigma^{-\frac12}\to T_\Sigma^{\frac12}\otimes K_\Sigma$. The only ϕ-invariant subbundle of S_Σ is $T_\Sigma^{\frac12}$ and for $g>1$ we have $1-g=c_1(T_\Sigma^{\frac12})<\frac12 c_1(S_\Sigma)=0$, so the pair (S_Σ,ϕ) is stable. (More generally, one can choose (S_Σ,ϕ) for $\phi=\frac12\left(\begin{array}{cc}0&1\\q&0\end{array}\right)$ for $q\in H^0(K_\Sigma^2)$, a quadratic differential. We will not consider this here.)

Hitchin [27], showed that the two sides of Theorem 3.9 applied to (S_Σ,ϕ) naturally correspond to a hyperbolic metric and a conformal structure, leading to a proof of uniformisation as follows. The key idea is to show that A is reducible so the associated Hermitian metric on S_Σ is also reducible and defines a Hermitian metric on $T_\Sigma^{\frac12}$. Theorem 3.9 produces a unique unitary connection A on S_Σ. For a constant $\alpha\in\mathbb{R}$, $(A,e^{i\alpha}\phi)$ also satisfies (25). We can act by a constant unitary gauge transformation, which preserves (25) and holomorphicity of ϕ, to get

$$u_\alpha\cdot(A,e^{i\alpha}\phi)=(u_\alpha\cdot A,e^{i\alpha}u_\alpha\cdot\phi)=(u_\alpha\cdot A,\phi),\quad u_\alpha=\left(\begin{array}{cc}e^{-i\alpha/2}&0\\0&e^{i\alpha/2}\end{array}\right).$$

Since (A,ϕ) and $(u_\alpha\cdot A,\phi)$ satisfy (25), by the uniqueness of A we must have $u_\alpha\cdot A=A$ for each $\alpha\in\mathbb{R}$ so the connection A is reducible.

Corresponding to the reducible connection A is a reducible Hermitian metric $H=h\oplus h^{-1}$ on S_Σ where h is defined on $T_\Sigma^{\frac12}$ so h^2 defines a Hermitian metric on Σ with real part a Riemannian metric. Write $h^2=h_0^2dz\otimes d\bar z$ where $h_0=h_0(z,\bar z)$ is a locally defined real-valued function. The curvature of the connection on $T_\Sigma^{\frac12}$, is given by $(\partial_{\bar z}\partial_z\log h_0)d\bar z\wedge dz$ and satisfies (25). This yields

$$\partial_{\bar z}\partial_z\log h_0 d\bar z\wedge dz+\frac14 h_0^2dz\wedge d\bar z=0$$

or $\partial_{\bar z}\partial_z\log h_0=\frac14 h_0^2$. Hence the Gaussian curvature of the Riemannian metric is

$$K=-\frac{2}{h^2}\frac{\partial^2}{\partial z\partial\bar z}\log h_0^2=-1$$

which proves uniformisation for a compact Riemann surface Σ—it possesses a hyperbolic metric in its conformal class. The $SL(2,\mathbb{R})$ holonomy of the flat connection $A+\phi+\phi^*$ lives above the $PSL(2,\mathbb{R})$ holonomy of the developing map of the hyperbolic metric on Σ.

3.3.3. We are now in a position to compare $H^k(\Sigma,T_\Sigma^{\frac12})$ and $H^k_{dR}(\Sigma,T_\Sigma^{\frac12})$. The flat connection $A^\phi=A+\phi+\phi^*$ on S_Σ coming out of Theorem 3.9 is given in terms of its $(1,0)$ and $(0,1)$ parts by

$$\partial_{A^\phi}=\begin{pmatrix}\partial+h^{-1}\partial h & \frac12\\ 0 & \partial-h^{-1}\partial h\end{pmatrix},\quad \overline{\partial}_{A^\phi}=\begin{pmatrix}\overline{\partial} & 0\\ \frac12 h^2 & \overline{\partial}\end{pmatrix}$$

where, as above, the upper right term is a linear map $T_\Sigma^{-\frac12}\to T_\Sigma^{\frac12}\otimes K_\Sigma$ and the lower left term is its adjoint $T_\Sigma^{\frac12}\to T_\Sigma^{-\frac12}\otimes\overline{K}_\Sigma$. Note that ϕ^* is an $\mathrm{End}(S_\Sigma)$-valued $(0,1)$ form, so a Hermitian metric $\frac12h^2=\frac12h_0^2dz\otimes d\bar z$ naturally lives in the lower left position, rather than a quadratic differential which would yield an $\mathrm{End}(S_\Sigma)$-valued $(1,0)$ form.

The connection A^ϕ is compatible with the real structure σ

$$d_{A^\phi}\circ\sigma=\sigma\circ d_{A^\phi}$$

and it is enough to prove $\partial_{A^\phi}\circ\sigma=\sigma\circ\overline{\partial}_{A^\phi}$:

$$\begin{aligned}\partial_{A^\phi}\circ\sigma\begin{pmatrix}u\\ v\end{pmatrix}&=\partial_{A^\phi}\begin{pmatrix}h^{-1}\overline{v}\\ h\overline{u}\end{pmatrix}=\begin{pmatrix}\frac12h\overline{u}+h^{-1}\partial\overline{v}\\ h\partial\overline{u}\end{pmatrix}\\ &=\sigma\begin{pmatrix}\overline{\partial}u\\ \frac12h^2u+\overline{\partial}v\end{pmatrix}=\sigma\circ\overline{\partial}_{A^\phi}\begin{pmatrix}u\\ v\end{pmatrix}.\end{aligned}$$

Hence it defines a flat $SU(1,1)\cong SL(2,\mathbb{R})$ connection on the bundle S_Σ.

3.3.4. The Higgs field defines a complex

$$0\to\Omega^0_\Sigma(S_\Sigma)\stackrel{\phi\cdot}{\to}\Omega^1_\Sigma(S_\Sigma)\to0.$$

Simpson [55] defined the Dolbeault cohomology of S_Σ to be the hypercohomology of this complex $H^k_{\mathrm{Dol}}(\Sigma,S_\Sigma):=\mathbb{H}^k([\Omega^0_\Sigma(S_\Sigma)\to\Omega^1_\Sigma(S_\Sigma)])$ and proved the following relation with the sheaf cohomology of the flat bundle S_Σ.

Theorem 3.10 (Simpson [55]). *When Σ is compact, there is a canonical isomorphism*

$$H^k_{dR}(\Sigma,S_\Sigma)\cong H^k_{Dol}(\Sigma,S_\Sigma).$$

An application of this theorem is the following crucial canonical isomorphism.

Theorem 3.11. *When Σ is compact, there is a canonical isomorphism*

$$H^k(\Sigma,T_\Sigma^{\frac12})^\vee\stackrel{\cong}{\to}H^k_{dR}(\Sigma,T_\Sigma^{\frac12}),\quad k=0,1,2. \tag{27}$$

Proof. The first step is to evaluate the hypercohomology in Simpson's theorem. Hypercohomology is an invariant of the quasi-isomorphism class of a complex of sheaves. For ϕ given by (26), the map $T_\Sigma^{\frac12}\oplus T_\Sigma^{-\frac12}\stackrel{\phi\cdot}{\to}(T_\Sigma^{\frac12}\oplus T_\Sigma^{-\frac12})\otimes K_\Sigma$ defines an isomorphism $T_\Sigma^{-\frac12}\stackrel{\cong}{\to}T_\Sigma^{\frac12}\otimes K_\Sigma$ and has kernel $T_\Sigma^{\frac12}$ and cokernel $T_\Sigma^{-\frac12}\otimes K_\Sigma$. Hence the natural inclusions given by the vertical arrows below define a quasi-isomorphism:

$$\begin{array}{ccc}\Omega^0_\Sigma(T_\Sigma^{\frac12}) & \stackrel{0\cdot}{\to} & \Omega^1_\Sigma(T_\Sigma^{-\frac12})\\ \downarrow & & \downarrow\\ \Omega^0_\Sigma(T_\Sigma^{\frac12}\oplus T_\Sigma^{-\frac12}) & \stackrel{\phi\cdot}{\to} & \Omega^1_\Sigma(T_\Sigma^{\frac12}\oplus T_\Sigma^{-\frac12}).\end{array}$$

Thus $H^k_{\rm Dol}(\Sigma,S_\Sigma)=\mathbb{H}^k(C^\bullet)$ where $C^\bullet=[\Omega^0_\Sigma(T_\Sigma^{\frac12})\to\Omega^1_\Sigma(T_\Sigma^{-\frac12})]$ and the arrow is the zero map. The hypercohomology can be calculated from a long exact sequence

$$..\to H^{k-1}(\Omega^1_\Sigma(T_\Sigma^{-\frac12}))\to\mathbb{H}^k(C^\bullet)\to H^k(\Sigma,T_\Sigma^{\frac12})\to H^k(\Omega^1_\Sigma(T_\Sigma^{-\frac12}))\to\mathbb{H}^{k+1}(C^\bullet)\to..$$

Thus

$$\mathbb{H}^0(C^\bullet)\cong H^0(\Sigma,T_\Sigma^{\frac12})=0$$

for $g>1$ since $\deg T_\Sigma^{\frac12}=1-g<0$ and

$$\mathbb{H}^2(C^\bullet)\cong H^2(\Sigma,T_\Sigma^{\frac12})=0$$

for $g>1$ since $H^1(\Omega^1_\Sigma(T_\Sigma^{-\frac12}))\cong H^0(\Sigma,T_\Sigma^{\frac12})^\vee=0$. We see that (27) is proven for $k=0$ and 2 by Theorem 3.10 and the injection $H^k_{dR}(\Sigma,T_\Sigma^{\frac12})\hookrightarrow H^k_{dR}(\Sigma,S_\Sigma)=0$.

It remains to prove the $k=1$ case. The sequence

$$0\to H^0(\Omega^1_\Sigma(T_\Sigma^{-\frac12}))\to\mathbb{H}^1(C^\bullet)\to H^1(\Sigma,T_\Sigma^{\frac12})\to 0$$

splits giving

$$\mathbb{H}^1(C^\bullet)\cong H^1(\Sigma,T_\Sigma^{\frac12})\oplus H^1(\Sigma,T_\Sigma^{\frac12})^\vee$$

which uses the isomorphism $H^1(\Sigma,T_\Sigma^{\frac12})^\vee\cong H^0(\Sigma,K_\Sigma\otimes T_\Sigma^{-\frac12})$. The complex vector space $H^1(\Sigma,T_\Sigma^{\frac12})$ is equipped with a Hermitian metric induced from the Hermitian metric on $T_\Sigma^{\frac12}$—see 3.4.2. Hence its dual vector space is isomorphic to its complex conjugate. Equivalently

$$\mathbb{H}^1(C^\bullet)\cong H^1(\Sigma,T_\Sigma^{\frac12})\otimes_{\mathbb{R}}\mathbb{C}$$

which completes the calculation of the hypercohomology.

We have $H^1_{dR}(\Sigma,S_\Sigma)=H^1_{dR}(\Sigma,T_\Sigma^{\frac12})\otimes\mathbb{C}$ by construction. So Simpson's theorem proves that there is a canonical isomorphism

$$H^1(\Sigma,T_\Sigma^{\frac12})\otimes_{\mathbb{R}}\mathbb{C}\cong H^1_{dR}(\Sigma,T_\Sigma^{\frac12})\otimes_{\mathbb{R}}\mathbb{C}.$$

To see the real structure of the isomorphism, we need to understand the proof of the canonical isomorphism in [55] which uses a quasi-isomorphism between the complexes

$$A^0_\Sigma(S_\Sigma)\stackrel{D_i}{\to}A^1_\Sigma(S_\Sigma)\stackrel{D_i}{\to}A^2_\Sigma(S_\Sigma)$$

for $D_1=\overline{\partial}_A$ and $D_2=d_A+\phi+\phi^*$ and the identity map on $A^k_\Sigma(S_\Sigma)$. The kernel of D_1 naturally produces representatives in $H^1(\Sigma,T_\Sigma^{\frac12})\oplus H^0(\Omega^1_\Sigma(T_\Sigma^{-\frac12}))$ since A is diagonal and when $H^0(T_\Sigma^{-\frac12})\neq 0$, the sequence is

$$H^0(T_\Sigma^{-\frac12})\to H^0(T_\Sigma^{-\frac12})\oplus H^1(K_\Sigma\otimes T_\Sigma^{\frac12})\to H^1(K_\Sigma\otimes T_\Sigma^{\frac12})$$

which has vanishing cohomology. The map to the kernel of D_2 is described as follows. Given a $T_\Sigma^{-\frac12}$-valued holomorphic 1-form $\eta\in H^0(\Sigma,K_\Sigma\otimes T_\Sigma^{-\frac12})\subset A^1_\Sigma(T_\Sigma^{-\frac12})$ then $(h^{-1}\overline{\eta},\eta)\in A^1_\Sigma(T_\Sigma^{\frac12}\oplus T_\Sigma^{-\frac12})=A^1_\Sigma(S_\Sigma)$ and in fact takes its values in the real part $A^1_\Sigma(T_\Sigma^{\frac12})$ (using the antidiagonal embedding $T_\Sigma^{\frac12}\to S_\Sigma$ which differs from the first factor embedding—see 3.2.1).

For $\eta \in H^0(\Sigma, K_\Sigma \otimes T_\Sigma^{-\frac{1}{2}})$,

$$d_{A^\phi}\begin{pmatrix} h^{-1}\overline{\eta} \\ \eta \end{pmatrix} = \partial_{A^\phi}\begin{pmatrix} h^{-1}\overline{\eta} \\ 0 \end{pmatrix} + \overline{\partial}_{A^\phi}\begin{pmatrix} 0 \\ \eta \end{pmatrix} = \begin{pmatrix} h^{-1}\partial\overline{\eta} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \overline{\partial}\eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{28}$$

where the first equality uses the fact that η is a $(1,0)$ form and the second equality uses $\overline{\partial}_A = \overline{\partial}$ and $\partial_A = \partial + h^{-1}\partial h$. The final equality uses the holomorphicity of η. Hence $(h^{-1}\overline{\eta}, \eta)$ is a cocycle in $A^1_\Sigma(T_\Sigma^{\frac{1}{2}})$.

Thus we have defined a natural map

$$\begin{array}{ccc} H^1(\Sigma, T_\Sigma^{\frac{1}{2}})^\vee & \to & H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}}) \\ \eta & \mapsto & (h^{-1}\overline{\eta}, \eta) \end{array} \tag{29}$$

which indeed defines an isomorphism by the following lemma.[2]

Lemma 3.12. *Given a cocycle $\alpha \in A^1_\Sigma(S_\Sigma)$ so $d_{A^\phi}\alpha = 0$, there exists a unique $\beta \in A^0_\Sigma(S_\Sigma)$ such that*

$$\alpha - d_{A^\phi}\beta = \begin{pmatrix} 0 \\ * \end{pmatrix} dz + \begin{pmatrix} * \\ 0 \end{pmatrix} d\bar{z}. \tag{30}$$

Proof. Let $\beta = \begin{pmatrix} w \\ h\overline{w} \end{pmatrix}$ and decompose α into its $(1,0)$ and $(0,1)$ parts.

$$\alpha = \alpha' + \alpha'' = \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} h^{-1}\overline{v} \\ h\overline{u} \end{pmatrix}$$

It is enough to solve $\alpha' - \partial_{A^\phi}\beta = \begin{pmatrix} 0 \\ * \end{pmatrix}$ since $\overline{\partial}_{A^\phi}$ sends β to a $(0,1)$-form. Hence

$$Pw := \partial w + (h^{-1}\partial h)w + \tfrac{1}{2}h\overline{w} = u.$$

Here P is a real linear elliptic operator acting on a rank 2 real vector bundle. It has trivial kernel because if $Pw = 0$ then its complex conjugate equation is $\frac{1}{2}h^2 w + \overline{\partial}(h\overline{w}) = 0$ hence

$$Pw = 0 \ \Rightarrow\ 0 = \overline{\partial}_A(Pw) = \overline{\partial}_A\partial_A w + \tfrac{1}{2}\overline{\partial}_A(h\overline{w}) = (\overline{\partial}_A\partial_A - \tfrac{1}{4}h^2)w \ \Rightarrow\ w = 0$$

where the second implication uses the fact that the operator $\overline{\partial}_A\partial_A - \frac{1}{4}h^2$ is negative definite which follows from the following standard argument that the operator $\overline{\partial}_A\partial_A$ is negative semi-definite.

$$\begin{aligned} \int_\Sigma \langle \overline{\partial}_A\partial_A s, s\rangle &= -\int_\Sigma \langle \partial_A s, \partial_A s\rangle + \int_\Sigma \partial\langle \partial_A s, s\rangle \\ &= -\int_\Sigma \langle \partial_A s, \partial_A s\rangle + \int_\Sigma d\langle \partial_A s, s\rangle = -\int_\Sigma \langle \partial_A s, \partial_A s\rangle \leq 0. \end{aligned}$$

The replacement of ∂ by d in the second equality, which leads to vanishing of the integral, uses the three facts: $d = \partial + \overline{\partial}$, $\langle \partial_A s, s\rangle$ is a $(0,1)$ form, and the space of $(0,2)$ forms is zero. Hence P is invertible, and we can solve $Pw = u$ uniquely.

By the reality condition, the vanishing of the first coefficient of dz guarantees the vanishing of the second coefficient of $d\bar{z}$ as required. □

[2]The author is grateful to Edward Witten for explaining the proof of this lemma.

Lemma 3.12 shows that we may assume any cocycle in $A^1_\Sigma(T_\Sigma^{\frac{1}{2}})$ is of the form in the right hand side of (30) hence we can use (28), which only needs the given $(1,0)$ and $(0,1)$ decomposition of the right hand side of (30), to deduce that the dz part is holomorphic, i.e. lives in $H^0(\Sigma, K_\Sigma\otimes T_\Sigma^{-\frac{1}{2}})$. By the reality condition the cocycle lives in the image of (29). Thus the map in (29) is surjective onto equivalence classes of cocycles representing classes in $H^1_{dR}(\Sigma,T_\Sigma^{\frac{1}{2}})$. It is injective since if $(h^{-1}\overline{\eta},\eta)=d_{A^\phi}\beta$ is exact, by the invertibility of the elliptic operator P, i.e. the uniqueness statement in Lemma 3.12, $\beta=0$.

Hence we have proven

$$H^1(\Sigma,T_\Sigma^{\frac{1}{2}})^\vee\cong H^1_{dR}(\Sigma,T_\Sigma^{\frac{1}{2}}).$$

□

We have proved that the fibres of the bundles $\widehat{E}_g\to\overline{\mathcal{M}}_g^{\rm spin}$ defined in Definition 2.1 and $E_g\to\mathcal{M}_g^{\rm spin}$ defined in Theorem 3.6 over a point represented by a smooth compact hyperbolic surface are canonically isomorphic. The importance of the canonical isomorphism is that the *bundles* are isomorphic over the moduli space of smooth spin curves. An analogous canonical isomorphism exists for the usual moduli space using $H^1(\Sigma,T_\Sigma)$ and $H^1_{dR}(\Sigma,\mathbf{g}_\rho)$ where $\mathbf{g}_\rho$ is the flat $\mathbf{sl}(2,\mathbb{R})$-bundle associated to a representation $\rho:\pi_1\Sigma\to SL(2,\mathbb{R})$.

3.3.5. We now consider general $\Sigma=\overline{\Sigma}-D$, dropping the earlier assumption that Σ is compact. The arguments in 3.3.1, 3.3.2, 3.3.3, 3.3.4 and **??** generalise. When Σ is not compact, the bundle S_Σ can have different extensions to $\overline{\Sigma}$. We will use the extension of S_Σ given by

$$E\cong T_{\overline{\Sigma}}^{\frac{1}{2}}(-D)\oplus T_{\overline{\Sigma}}^{-\frac{1}{2}}.$$

The bundle E naturally possesses a parabolic structure which we now define, following Mehta and Seshadri [39].

Definition 3.13. Let $(\overline{\Sigma},D)$ be a compact surface containing $D=\sum p_i$ and E a holomorphic vector bundle over $\overline{\Sigma}$. A parabolic structure on E is a flag at each point p_i, $E_{p_i}=F_1^i\supset F_2^i\supset ...\supset F_{r_i}^i$, with attached weights $0\le\alpha_1^i<\alpha_2^i<...<\alpha_{r_i}^i<1$.

Define the multiplicity of α_j^i to be $k_j^i=\dim F_j^i-\dim F_{j+1}^i$, $j=1,...,r_i-1$ and $k_{r_i}^i=\dim F_{r_i}^i$. The parabolic degree of E is defined to be

$$\operatorname{pardeg}E=\deg E+\sum_{i,j}k_j^i\alpha_j^i.$$

A parabolic Higgs bundle generalises Definition 3.8 where the Higgs field has poles on D and preserves the flag structure.

Definition 3.14. A parabolic Higgs bundle over $(\overline{\Sigma},D)$ is a pair (E,ϕ) where E is a holomorphic vector bundle over $(\overline{\Sigma},D)$ equipped with a parabolic structure $\{F_j^i,\alpha_j^i\}$ and $\phi\in H^0(\operatorname{End}(E)\otimes K_{\overline{\Sigma}}(D))$ which satisfies $\operatorname{Res}_{p_i}\phi F_j^i\subset F_j^i$.

Note that some authors also write $K_{\overline{\Sigma}}(\log D))=K_{\overline{\Sigma}}(D)$ where the two coincide over a curve $\overline{\Sigma}$ but differ on higher dimensional varieties.

The following pair is a parabolic Higgs bundle generalising the construction in 3.3.2.

$$E \cong T_{\overline{\Sigma}}^{\frac{1}{2}}(-D) \oplus T_{\overline{\Sigma}}^{-\frac{1}{2}}, \quad \phi = \frac{1}{2}\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right) \in H^0(\mathrm{End}(E) \otimes K_{\overline{\Sigma}}(D)).$$

Following [3], at each point p_i of D, E_{p_i} is equipped with the trivial flag E_{p_i} of weight $1/2$. Note that ϕ does indeed have a pole at each point p_i of D and we take its residue to test for stability. We see the pole in the upper right element of ϕ which gives a map $T_{\overline{\Sigma}}^{-\frac{1}{2}} \to T_{\overline{\Sigma}}^{\frac{1}{2}}(-D) \otimes K_{\overline{\Sigma}}(D)$, or an element of

$$\mathcal{O}_{\overline{\Sigma}} \cong T_{\overline{\Sigma}}^{\frac{1}{2}} \otimes T_{\overline{\Sigma}}^{\frac{1}{2}}(-D) \otimes K_{\overline{\Sigma}}(D).$$

Locally, the upper right element of ϕ produces $z/dz : T_{\overline{\Sigma}}^{-\frac{1}{2}} \to T_{\overline{\Sigma}}^{\frac{1}{2}}(-D)$ which is the residue of $1 = z/dz \cdot dz/z$. For the same reason as described in 3.3.2, the pair (E, ϕ) is stable, which now means that for any ϕ-invariant sub-parabolic bundle $F \subset E$, we have $\frac{\mathrm{pardeg}\,(F)}{\mathrm{rank}\,F} < \frac{\mathrm{pardeg}\,(E)}{\mathrm{rank}\,E}$.

Theorem 3.15 (Simpson [56])**.** *A stable parabolic Higgs bundle (E, ϕ) of parabolic degree zero admits a unique unitary connection A with regular singularities satisfying (25). Conversely a parabolic Higgs bundle (E, ϕ) which admits a connection A with regular singularities satisfying (25) is of parabolic degree zero and stable.*

The connection must preserve the weight spaces of the parabolic structure on the bundle. This condition is automatic for our application since the weight space is the entire fibre. A regular singularity means a pole of order 1 of an algebraic connection—see [56, p.724] for details. Biswas, Gastesi and Govindarajan [3] applied Theorem 3.15 to the stable parabolic bundle $E \cong T_{\overline{\Sigma}}^{\frac{1}{2}}(-D) \oplus T_{\overline{\Sigma}}^{-\frac{1}{2}}$ to prove uniformisation of Σ by a complete hyperbolic metric analogous to the argument of Hitchin presented in 3.3.2.

Simpson proved in [56] that there is a natural quasi-isomorphism between the de Rham complex of forms with coefficients in the flat bundle, and the Dolbeault complex with coefficients in the corresponding Higgs bundle. A consequence is the equality of cohomology groups.

Theorem 3.16 ([13, 56])**.** *There is a canonical isomorphism*

$$H^k_{dR}(\Sigma, S_\Sigma) \cong H^k_{Dol}(\overline{\Sigma}, T_{\overline{\Sigma}}^{\frac{1}{2}}(-D) \oplus T_{\overline{\Sigma}}^{-\frac{1}{2}})^\vee.$$

Remark 3.17. We have an isomorphism

$$H^k_{\mathrm{Dol}}(\overline{\Sigma}, T_{\overline{\Sigma}}^{\frac{1}{2}}(-D) \oplus T_{\overline{\Sigma}}^{-\frac{1}{2}}) \cong H^k(\mathcal{C}, \theta^\vee \oplus \theta)$$

where $\mathcal{C}$ is an orbifold curve as described in Section 2 with non-trivial isotropy group $\mathbb{Z}_2$ at D, $\theta^2 = \omega_{\mathcal{C}}(D)$ and its coarse curve is $p : (\mathcal{C}, D) \to (\overline{\Sigma}, D)$. The push-forward of a bundle over $\mathcal{C}$ to the coarse curve $\overline{\Sigma}$ is a bundle on $\overline{\Sigma}$ equipped with a parabolic structure [4, 23]. We find that

$$p_*(\theta^\vee \oplus \theta) = T_{\overline{\Sigma}}^{\frac{1}{2}}(-D) \oplus T_{\overline{\Sigma}}^{-\frac{1}{2}}$$

equipped with the trivial flag of weight $1/2$ at each point of D.

Theorem 3.16 allows us to drop the assumption that Σ is compact in Theorem 3.11.

Theorem 3.18. *There is a canonical isomorphism*

$$H^k((\overline{\Sigma},T_{\overline{\Sigma}}^{\frac{1}{2}}(-D))^\vee \xrightarrow{\cong} H^k_{dR}(\Sigma,T_{\Sigma}^{\frac{1}{2}}),\quad k=0,1,2. \tag{31}$$

The proof is the same as the proof of Theorem 3.11. The direct argument of Lemma 3.12 goes through when we replace cohomology with cohomology with compact supports.

3.3.6. In 3.3.5 the sheaf cohomology of a flat bundle over non-compact Σ was related to the sheaf cohomology of a bundle over a compactification $\overline{\Sigma}$ of Σ. A conformal structure on a punctured surface can compactify in different ways and we show here that it naturally compactifies to an orbifold curve $\mathcal{C}$ with $\mathbb{Z}/2$ orbifold structure at $D=\mathcal{C}-\Sigma$. This is important to relate to the bundle $E_{g,n}$ constructed in Section 2

As in Remark 3.17, we push forward bundles over $\mathcal{C}$ using the map $p:(\mathcal{C},D)\to(\overline{\Sigma},D)$ that forgets the orbifold structure at D. We have

$$p_*\theta^\vee=T_{\overline{\Sigma}}^{\frac{1}{2}}(-D)$$

and in particular

$$H^1(\theta^\vee)=H^1((\overline{\Sigma},T_{\overline{\Sigma}}^{\frac{1}{2}}(-D)).$$

Hence by Theorem 3.18, over a smooth hyperbolic surface Σ there is a canonical isomorphism $H^1(\theta^\vee)^\vee\cong H^1_{dR}(\Sigma,T_{\Sigma}^{\frac{1}{2}})$ which allows us to prove the following.

Corollary 3.19. *The bundles defined in Definition 2.1 and Theorem 3.6 are isomorphic on the smooth part of the moduli space:*

$$\widehat{E}_{g,n}|_{\mathcal{M}^{spin}_{g,n}}\cong E_{g,n}. \tag{32}$$

3.4. **Euler class of $\widehat{E}_{g,n}$.** The Euler class of $\widehat{E}_{g,n}$ is used in Section 5 to calculate the volume of the moduli space of super hyperbolic surfaces.

3.4.1. The Euler class of a real oriented bundle $E\to M$ of rank N can be represented by a differential form $e(E)\in\Omega^N(M)$ as follows. Equip the bundle E with a Riemannian metric $\langle\cdot,\cdot\rangle$. Choose any metric connection A on E, meaning that $d\langle s_1,s_2\rangle=\langle\nabla^A s_1,s_2\rangle+\langle s_1,\nabla^A s_2\rangle$ for sections s_1 and s_2 of E. The curvature of the connection is an endomorphism-valued 2-form $F_A\in\Omega^2(M,\mathrm{End}(E))$. The endomorphism preserves the metric $\langle\cdot,\cdot\rangle$ hence F_A is locally $so(N)$-valued. The Pfaffian defines a map $\mathrm{pf}:so(N)\to\mathbb{R}$ rather like the determinant. It vanishes for N odd and for N even is defined using (but independent of the choice of) an orthonormal basis $\{e_1,...,e_N\}$ by $\frac{1}{(N/2)!}A\wedge A\wedge...\wedge A=\mathrm{pf}(A)e_1\wedge...\wedge e_N$ where $A\in\wedge^2\mathbb{R}^N\cong so(N)$. It satisfies $\mathrm{pf}(A)^2=\det(A)$. It is invariant under conjugation by $O(N)$, $\mathrm{pf}(uAu^{-1})=\mathrm{pf}(A)$ and hence makes sense on the associated $\mathbf{so}(N)$ bundle, and in particular on F_A. The Euler class can be expressed as a polynomial in the curvature F_A using the Pfaffian [50]

$$e(E)=\left(\frac{1}{4\pi}\right)^N\mathrm{pf}(F_A). \tag{33}$$

A complex bundle $E\to M$ equipped with a Hermitian metric is naturally a real oriented bundle of even rank with a Riemannian metric, hence its Euler class can be expressed via (33). Furthermore, if E is holomorphic then it comes equipped with a unique natural Hermitian connection compatible with both the holomorphic structure and the Hermitian metric, and this is a metric connection with respect to the underlying Riemannian metric on E. In this case, since $\det(iu) = \mathrm{pf}(u^{\mathbb{R}})$, where $u^{\mathbb{R}}$ is the image of $u \in \mathbf{u}(N/2)$ in $\mathbf{so}(N)$, then (33) coincides with the Chern-Weil construction of the top Chern class of E realising $e(E) = c_{N/2}(E)$.

3.4.2. There is a natural Hermitian metric and connection on $\widehat{E}_{g,n}$ defined as follows. As described in 3.2.2 a Hermitian metric h on a line bundle $L \to \Sigma$ defines an isomorphim $\overline{L} \stackrel{\cong}{\to} L^\vee$, by $\ell \mapsto h(\overline{\ell}, \cdot)$. This induces a Hermitian metric on $H^1(\overline{\Sigma}, \tilde{L})$, for $\tilde{L}$ any extension of L to $\overline{\Sigma}$, via

$$\langle \xi_1, \xi_2 \rangle = \int_{\overline{\Sigma}} \overline{\xi}_1 \wedge \xi_2$$

where $\overline{\xi}_1 \wedge \xi_2$ is a global $(1,1)$ form on $\overline{\Sigma}$ defined by $\overline{\xi}_1 \wedge \xi_2 = h(\ell_1(z), \ell_2(z)) dz \wedge d\bar{z}$ locally for $\xi_i = \ell_i(z) d\bar{z} \in H^1(\overline{\Sigma}, \tilde{L})$ its Dolbeault representative.

When $L = T_\Sigma$ the hyperbolic metric on Σ representing its conformal class is Kähler and produces a Hermitian metric on T_Σ. The induced Hermitian metric on $H^1(\overline{\Sigma}, T_{\overline{\Sigma}}(-D))$ is $\langle \xi_1, \xi_2 \rangle = \int_\Sigma (\overline{\xi}_1, \xi_2) d\mathrm{vol}$ where $d$vol is the volume form induced by the hyperbolic metric, and $(\overline{\xi}_1, \xi_2)$ uses the natural pairing between tangent and cotangent vectors given locally by $(\overline{\xi}_1, \xi_2) = \overline{a_1(z)} a_2(z)$ for $\xi_i = a_i(z) \frac{\partial}{\partial z} \otimes d\bar{z}$. The The tangent bundle of the moduli space $\mathcal{M}_{g,n}$ of curves can be identified with the log-tangent bundle of $\overline{\mathcal{M}}_{g,n}$ which has fibres $H^1(\overline{\Sigma}, T_{\overline{\Sigma}}(-D))$. The metric above defines the Weil-Petersson metric.

When $L = T_\Sigma^{\frac{1}{2}}$, the hyperbolic metric on Σ representing its conformal class also produces a Hermitian metric on $T_\Sigma^{\frac{1}{2}}$. This induces a Hermitian metric on $H^1(\overline{\Sigma}, T_{\overline{\Sigma}}^{\frac{1}{2}}(-D))$ which defines a Hermitian metric on $\widehat{E}_{g,n}$. The Hermitian metric and Weil-Petersson form should combine to give the super symplectic form defined by Penner and Zeitlin [52]. Since $\widehat{E}_{g,n}$ is holomorphic the Hermitian metric produces a connection and curvature which builds the Euler class.

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4. Mirzakhani's volume recursion

In this section we describe Mirzakhani's recursive relations between volumes of moduli spaces of hyperbolic surfaces and her proof of the Kontsevich-Witten theorem. This will be useful for the proofs of Theorem 1 and Corollary 2 which follow Mirzakhani's arguments closely.

4.1. **Hyperbolic geometry.** Define the moduli space of complete oriented hyperbolic surfaces

$$\mathcal{M}_{g,n}^{\mathrm{hyp}} = \{\Sigma \mid \Sigma = \text{genus } g \text{ oriented hyperbolic surface with } n \text{ labeled cusps}\}/\sim$$

where the quotient is by isometries preserving each cusp. Note that (generically) a hyperbolic surface appears twice in $\mathcal{M}_{g,n}^{\mathrm{hyp}}$ equipped with each of its two orientations. Define the moduli space of oriented hyperbolic surfaces with fixed length geodesic boundary components by

$$\mathcal{M}_{g,n}^{\mathrm{hyp}}(L_1, ..., L_n) = \Big\{(\Sigma, \beta_1, ..., \beta_n) \mid \Sigma \text{ genus } g \text{ oriented hyperbolic surface,}$$
$$\partial\Sigma = \sqcup \beta_i \text{ are geodesic}, L_i = \ell(\beta_i)\Big\}/\sim$$

where again the quotient is by isometries preserving each β_i. Any non-trivial isometry must rotate each β_i non-trivially. The moduli spaces are all diffeomorphic $\mathcal{M}_{g,n}^{\rm hyp} \cong \mathcal{M}_{g,n}^{\rm hyp}(L_1,...,L_n)$ and we will see below that they give a family of deformations of a natural symplectic structure on $\mathcal{M}_{g,n}^{\rm hyp}$.

4.1.1. The hyperbolic metric on Σ defines a Hermitian metric h on T_Σ since it is compatible with the complex structure J on Σ, meaning$h(Jv_1,Jv_2)=h(v_1,v_2)$. Recall from 3.2.2 that any Hermitian metric on T_Σ defines a Hermitian metric on $H^1(T_\Sigma)$, hence on $T_{[\Sigma]}\mathcal{M}_{g,n}$. The Weil-Petersson symplectic form ω^{WP} on $\mathcal{M}_{g,n}$ is the imaginary part of the Hermitian metric on $T_{[\Sigma]}\mathcal{M}_{g,n}$ defined by the hyperbolic metric. It defines a volume form on $\mathcal{M}_{g,n}$ with finite integral known as the Weil-Petersson volume of $\mathcal{M}_{g,n}$:

$$V_{g,n}^{WP} := \int_{\mathcal{M}_{g,n}} \exp\left\{\omega^{WP}\right\}.$$

4.1.2. Teichmüller space gives a way to realise ω^{WP} via local coordinates on $\mathcal{M}_{g,n}$. Fix a smooth genus g oriented surface $\Sigma_{g,n}=\overline{\Sigma}_{g,n}-\{q_1,...,q_n\}$. A *marking* of a genus g hyperbolic surface $\Sigma=\overline{\Sigma}-\{p_1,...,p_n\}$ is an orientation preserving homeomorphism $f:\Sigma_{g,n}\stackrel{\cong}{\to}\Sigma$. Define the Teichmüller space of marked hyperbolic surfaces (Σ,f) of type (g,n) to be

$$\mathcal{T}_{g,n}=\{(\Sigma,f)\}/\sim$$

where the equivalence is given by $(\Sigma,f)\sim(T,g)$ if $g\circ f^{-1}:\Sigma\to T$ is isotopic to an isometry. The mapping class group $\mathrm{Mod}_{g,n}$ of isotopy classes of orientation preserving diffeomorphisms of the surface that preserve boundary components acts on $\mathcal{T}_{g,n}$ by its action on markings. The quotient of Teichmüller space by this action produces the moduli space

$$\mathcal{M}_{g,n}=\mathcal{T}_{g,n}/\mathrm{Mod}_{g,n}.$$

4.1.3. Global coordinates for Teichmüller space, known as Fenchel-Nielsen coordinates, are defined as follows. Choose a maximal set of disjoint embedded isotopically inequivalent simple closed curves on the topological surface $\Sigma_{g,n}$. The complement of this collection is a union of pairs of pants known as a pants decomposition of the surface $\Sigma_{g,n}$. Each pair of pants contributes Euler characteristic -1, so there are $2g-2+n=-\chi(\Sigma)$ pairs of pants in the decomposition, and hence $3g-3+n$ closed geodesics (not counting the boundary classes.) A marking $f:\Sigma_{g,n}\to\Sigma$ of a hyperbolic surface with n cusps Σ induces a pants decomposition on Σ from $\Sigma_{g,n}$. The isotopy classes of embedded closed curves can be represented by a collection $\{\gamma_1,...,\gamma_{3g-3+n}\}$ of disjoint embedded simple closed geodesics which cuts Σ into hyperbolic pairs of pants with geodesic and cusp boundary components. Their lengths $\ell_1,...,\ell_{3g-3+n}$ give half the Fenchel-Nielsen coordinates, and the other half are the twist parameters $\theta_1,...,\theta_{3g-3+n}$ which we now define. Any hyperbolic pair of pants contains three geodesic arcs giving the shortest paths between boundary components, or horocycles around cusps. The simple closed geodesic γ_i intersects the geodesic arcs on the pair of pants on one side of γ_i at a pair of (metrically opposite) points on γ_i, and similarly γ_i intersects the geodesic arcs on the pair of pants on the other side of γ_i at a pair of (metrically opposite) points on γ_i. The oriented distance between these points lies in $[0,\ell_i/2]$ and after a choice that fixes the ambiguity arising from choosing one out of a pair of points the oriented distance lies in $[0,\ell_i]$ which defines $\theta_i(\text{mod }\ell_i)$. A further lift $\theta_i\in\mathbb{R}$ is obtained by continuous paths in $\mathcal{T}_{g,n}$ which amount to rotations around γ_i. The coordinates (ℓ_j,θ_j) for $j=1,2,...,3g-3+n$ give rise to an isomorphism

$$\mathcal{T}_{g,n}\cong(\mathbb{R}^+\times\mathbb{R})^{3g-3+n}.$$

4.1.4. The Fenchel-Nielsen decomposition induces an action of S^1 along each simple closed geodesic γ_i by rotation. In local coordinates $\theta_i\mapsto\theta_i+\phi$ for $\phi\in\mathbb{R}/\ell_i\mathbb{Z}\cong S^1$. This action defines a vector field, given locally by $\partial/\partial\theta_i$. Wolpert proved that $\partial/\partial\theta_i$ is a Hamiltonian vector field with respect to ω^{WP} with Hamiltonian given by ℓ_i. In other words $(\ell_1,...,\ell_{3g-3+n},\theta_1,...,\theta_{3g-3+n})$ are Darboux coordinates for ω^{WP}. This is summarised in the following theorem.

Theorem 4.1 (Wolpert [64])**.**

$$\omega^{WP}=\sum d\ell_j\wedge d\theta_j. \tag{34}$$

Since ω^{WP} is defined over $\mathcal{M}_{g,n}$ it follows that this expression for ω^{WP} is invariant under the action of the mapping class group $\text{Mod}_{g,n}$. There are a finite number of pants decompositions up to the action of the mapping class group, each class consisting of infinitely many geometrically different types. Thus once a topological pants decomposition of the surface is chosen a given hyperbolic surface has infinitely many geometrically different pants decompositions equivalent under $\text{Mod}_{g,n}$. Each different decomposition associates different lengths and twist parameters, hence different coordinates, to the same hyperbolic surface.

Wolpert proved that the Weil-Petersson symplectic form ω^{WP} extends from $\mathcal{M}_g$ to $\overline{\mathcal{M}}_g$ and coincides with $2\pi^2\kappa_1$ defined in (10). His proof extends to $\mathcal{M}_{g,n}$ and importantly gives

$$V^{WP}_{g,n}=\int_{\mathcal{M}_{g,n}}\exp\left\{\omega^{WP}\right\}=\int_{\overline{\mathcal{M}}_{g,n}}\exp\left\{2\pi^2\kappa_1\right\}.$$

4.1.5. Wolpert's local formula (34) generalises below in (35) to define a symplectic form $\omega^{WP}(L_1,...,L_n)$ on $\mathcal{M}_{g,n}(L_1,...,L_n)$ which pulls back under the isomorphism

$$\mathcal{M}_{g,n}\cong\mathcal{M}_{g,n}(L_1,...,L_n)$$

to define a family of deformations of the Weil-Petersson symplectic form, depending on the parameters $(L_1,...,L_n)$. The pairs of pants decomposition of an oriented hyperbolic surface with cusps naturally generalises to an oriented hyperbolic surface with geodesic boundary components. The lengths and twist parameters of the $3g-3+n$ interior geodesics gives rise to Fenchel-Nielsen coordinates $(\ell_1,...,\ell_{3g-3+n},\theta_1,...,\theta_{3g-3+n})$ on the Teichmüller space

$$\mathcal{T}^{\text{hyp}}_{g,n}(L_1,...,L_n)=\{(\Sigma,f)\}/\sim$$

of marked genus g oriented hyperbolic surfaces with geodesic boundary components of lengths $(L_1,...,L_n)$ and an isomorphsim $\mathcal{T}^{\text{hyp}}_{g,n}(L_1,...,L_n)\cong(\mathbb{R}^+\times\mathbb{R})^{3g-3+n}$. Wolpert's local formula (34) can be used to define a symplectic form

$$\omega^{WP}(L_1,...,L_n)=\sum d\ell_j\wedge d\theta_j \tag{35}$$

again known as the Weil-Petersson symplectic form, on $\mathcal{T}^{\rm hyp}_{g,n}(L_1,...,L_n)$. It is invariant under the mapping class group and descends to the moduli space

$$\mathcal{M}^{\rm hyp}_{g,n}(L_1,...,L_n)=\mathcal{T}_{g,n}(L_1,...,L_n)/{\rm Mod}_{g,n}.$$

Wolpert's result [63] generalises to show that $\omega^{WP}(L_1,...,L_n)$ extends to $\overline{\mathcal{M}}_{g,n}$.

Mirzakhani [43] proved that $\mathcal{M}_{g,n}(L_1,...,L_n)$ arises as a symplectic quotient of a symplectic manifold with T^n action and moment map $(\frac12 L_1^2,...,\frac12 L_n^2)$. Each level set of the moment map or equivalently each choice of $(L_1,...,L_n)$ gives a symplectic quotient. Quite generally, the symplectic form on the quotient is a deformation by first Chern classes of line bundles related to the T^n action. In this case it is $\omega^{WP}+\sum\frac12 L_i^2\psi_i$ where $\psi_i=c_1(\mathcal{L}_i)\in H^2(\overline{\mathcal{M}}_{g,n})$ are defined in 9 which produces:

$$V^{WP}_{g,n}(L_1,...,L_n)=\int_{\mathcal{M}_{g,n}(L_1,...,L_n)}\exp\left\{\omega^{WP}(L_1,...,L_n)\right\}$$
$$=\int_{\overline{\mathcal{M}}_{g,n}}\exp\left\{2\pi^2\kappa_1+\frac12\sum_{i=1}^n L_i^2\psi_i\right\}. \tag{36}$$

The extension of $\omega^{WP}(L_1,...,L_n)$ to $\overline{\mathcal{M}}_{g,n}$ uses Wolpert's theorem together with the extensions of the classes ψ_i from $\mathcal{M}_{g,n}$ to $\overline{\mathcal{M}}_{g,n}$. In particular the volumes depend non-trivially on L_i proving that $\omega^{WP}(L_1,...,L_n)$ is a non-trivial deformation of ω^{WP}.

4.2. **Volume recursion.** Mirzakhani proved the following recursive relations between the volumes $V^{WP}_{g,n}(L_1,...,L_n)$.

Theorem 4.2 (Mirzakhani [42])**.**

$$L_1V^{WP}_{g,n}(L_1,...,L_n)=\frac12\int_0^\infty\int_0^\infty xyD^M(L_1,x,y)P_{g,n}(x,y,L_2,..,L_n)dxdy$$
$$+\sum_{j=2}^n\int_0^\infty xR^M(L_1,L_j,x)V^{WP}_{g,n-1}(x,L_2,..,\hat{L}_j,..,L_n)dx \tag{37}$$

where $P_{g,n}(x,y,L_K)=V^{WP}_{g-1,n+1}(x,y,L_K)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=K}}V^{WP}_{g_1,|I|+1}(x,L_I)V^{WP}_{g_2,|J|+1}(y,L_J)$

for $K=\{2,...,n\}$.

The kernels in (37) are defined by

$$H^M(x,y)=1+\tfrac12\tanh\tfrac{x-y}{4}-\tfrac12\tanh\tfrac{x+y}{4}$$

which uniquely determine $D^M(x,y,z)$ and $R^M(x,y,z)$ via

$$\frac{\partial}{\partial x}D^M(x,y,z)=H^M(x,y+z),\ \frac{\partial}{\partial x}R^M(x,y,z)=\frac12\big(H^M(z,x+y)+H^M(z,x-y)\big)$$

and the initial conditions $D^M(0,y,z)=0=R^M(0,y,z)$. Explicitly

$$R^M(x,y,z)=x-\log\left(\frac{\cosh\frac{y}{2}+\cosh\frac{x+z}{2}}{\cosh\frac{y}{2}+\cosh\frac{x-z}{2}}\right) \tag{38}$$

and $D^M(x,y,z)$ is given by the relation

$$D^M(x,y,z)=R^M(x,y,z)+R^M(x,z,y)-x \tag{39}$$

which follows from

$$2H^M(x,y+z)=H^M(z,x+y)+H^M(z,x-y)+H^M(y,x+z)+H^M(y,x-z)-2. \tag{40}$$

The relations (37) uniquely determine $V^{WP}_{g,n}(L_1,...,L_n)$ from

$$V^{WP}_{0,3}=1,\quad V^{WP}_{1,1}=\frac{1}{48}(4\pi^2+L^2).$$

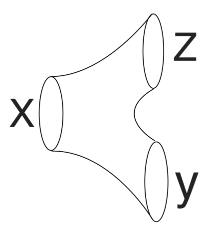
The first two calculations are

$$V^{WP}_{0,4}=\frac{1}{2}(4\pi^2+\sum L_i^2),\qquad V^{WP}_{1,2}=\frac{1}{384}(4\pi^2+\sum L_i^2)(12\pi^2+\sum L_i^2).$$

Mirzakhani used the recursion (37) to prove that the top coefficients of the polynomial $V^{WP}_{g,n}(L_1,...,L_n)$ satisfy Virasoro constraints which proves Theorem 3 of Witten-Kontsevich. See the Proof of Theorem 6.1 in Section 6.

The proof of Theorem 4.2 uses an unfolding of the volume integral to an integral over associated moduli spaces. This allows the integral to be related to volumes over simpler moduli spaces. A non-trivial decomposition of the constant function on the moduli space is used to achieve the unfolding. This is explained in this section, particularly because the same ideas are required in the super moduli space case.

4.2.1. The functions $D^M(x,y,z)$, $R^M(x,y,z)$ and the identity (39) have the following geometric interpretation. Given $x>0, y>0, z>0$ there exists a unique hyperbolic pair of pants with geodesic boundary components β_1, β_2 and β_3 of respective lengths x, y and z.



Consider geodesics orthogonal to the boundary component β_1. Travel along any such geodesic beginning at β_1 and stop if the geodesic meets itself or a boundary component. Such geodesics have four types of behaviour and their initial points partition $\beta_1=I_1\sqcup I_2\sqcup I_3\sqcup I_4$.

(i) The geodesic meets itself, or β_1 for a second time;
(ii) the geodesic meets β_2;
(iii) the geodesic meets β_3;
(iv) the geodesic remains embedded for all time.

The initial points of geodesics of types (i), (ii), (iii) and (iv) lie in $I_1\subset\beta_1$, respectively $I_2\subset\beta_1$, respectively $I_3\subset\beta_1$, respectively $I_4\subset\beta_1$. The subset I_1 is a disjoint union of two open intervals while each of I_2 and I_3 is a single open interval. The subset I_4 given by initial points of geodesics of types (iv) consist of the four points given by the intersection of the closures of I_1, I_2 and I_3.

The kernels $D^M(x,y,z)$ and $R^M(x,y,z)$ arise from this partition of β_1. We have $D^M(x,y,z)=\ell(I_1)$ where $\ell(I_1)$ is the length of I_1 using the hyperbolic metric, and $R^M(x,y,z)=\ell(I_1\cup I_2)$. Hence $R^M(x,z,y)=\ell(I_1\cup I_3)$ so in particular

$$R^M(x,y,z)+R^M(x,z,y)=\ell(I_1)+\ell(I_2)+\ell(I_1)+\ell(I_3)=\ell(I_1)+x=D^M(x,y,z)+x$$

which is (39).

4.2.2. Mirzakhani [42] proved the following non-trivial sum of functions of lengths of geodesics on a hyperbolic surface, known as a McShane identity because it generalises an identity of McShane [38]. Given a hyperbolic surface Σ with n geodesic boundary components $\beta_1,...,\beta_n$, define $\mathcal{P}_i$, respectively $\mathcal{P}_{ij}$, to be the set of isometric embeddings $P\to\Sigma$ of hyperbolic pairs of pants with geodesic boundary, which meet the boundary of Σ precisely at β_i, respectively at β_i and β_j. Denote by $\ell_{\partial_i P}$ the length of the ith geodesic boundary component of P. Define $R^M(P)=R^M(\ell_{\partial_1 P}=L_1,\ell_{\partial_2 P}=L_j,\ell_{\partial_3 P})$ for R^M defined in (38), and $D^M(P)=D^M(\ell_{\partial_1 P}=L_1,\ell_{\partial_2 P},\ell_{\partial_3 P})$ for D^M defined in (39).

Theorem 4.3 (Mirzakhani [42])**.** *Given a genus g hyperbolic surface Σ with n geodesic boundary components β_1,..., β_n of lengths $L_1,...,L_n$ we have:*

$$L_1=\sum_{P\in\mathcal{P}_1}D^M(P)+\sum_{j=2}^n\sum_{P\in\mathcal{P}_{1j}}R^M(P). \tag{41}$$

The proof of Theorem 4.3 partitions β_1 into a countable collection of disjoint interval associated to embedded pairs of pants $P\subset\Sigma$, together with a measure zero subset, using geodesics perpendicular to β_1. The length of each interval is determined by a pair of pants, as in 4.2.2. The identity (41) sums these lengths to get $L_1=\ell(\beta_1)$.

The sum over pairs of pants is topological, so it depends only on the topology of Σ, since an isometrically embedded pair of pants in Σ is uniquely determined by a topological embedding of a pair of pants into Σ. The left hand side of (41) is independent of the hyperbolic metric on Σ, whereas each summand on the right hand side dependends on the hyperbolic metric of Σ. The importance of (41) is that it allows one to integrate the constant function L_1 over the moduli space.

4.2.3. Mirzakhani used the identity (41) to integrate functions of a particular form over the moduli space [42]. Applied to the constant function, this yields the volume of the moduli space. Given a closed curve $\gamma_0\subset\Sigma_{g,n}$ in a topological surface surface $\Sigma_{g,n}$, its mapping class group orbit $\mathrm{Mod}_{g,n}\cdot\gamma_0$ gives a well-defined collection of closed geodesics in any hyperbolic surface $\Sigma\in\mathcal{M}_{g,n}(L_1,...,L_n)$. Define a function over $\mathcal{M}_{g,n}(L_1,...,L_n)$ of the form

$$F(\Sigma)=\sum_{\gamma\in\mathrm{Mod}_{g,n}\cdot\gamma_0}f(l_\gamma^\Sigma)$$

where f is an arbitrary function and the length of the geodesic l_γ^Σ shows the dependence on the hyperbolic surface $\Sigma\in\mathcal{M}_{g,n}(L_1,...,L_n)$. When f decays fast enough the sum is well-defined on the moduli space. More generally, one can consider an arbitrary (decaying) function on collections of geodesics and sum over orbits of the mapping class group acting on the collection. Mirzakhani unfolded the integral of F to an integral over a moduli space $\widetilde{\mathcal{M}}_{g,n}(L_1,...,L_n)$ of pairs (Σ,γ) consisting of a hyperbolic surface Σ and a collection of geodesics $\gamma \subset \Sigma$.

$$\begin{array}{c}\mathcal{T}_{g,n}(L_1,...,L_n)\\ \downarrow\\ \widetilde{\mathcal{M}}_{g,n}(L_1,...,L_n)\\ \downarrow\\ \mathcal{M}_{g,n}(L_1,...,L_n)\end{array}$$

The unfolded integral

$$\int_{\mathcal{M}_{g,n}(L_1,...,L_n)} F\cdot d\text{vol} = \int_{\widetilde{\mathcal{M}}_{g,n}(L_1,...,L_n)} f(l_\gamma)\cdot d\text{vol}$$

can be expressed in terms of an integral over the simpler moduli space obtained by cutting Σ along the geodesic γ.

The identity (41) is exactly of the right form for Mirzakhani's scheme since it expresses the constant function $F = L_1$ as a sum of functions of lengths over orbits of the mapping class group. In this case,

$$L_1 V^{WP}_{g,n}(L_1,...,L_n) = \int_{\mathcal{M}_{g,n}(L_1,...,L_n)} F\cdot d\text{vol} = \int_{\widetilde{\mathcal{M}}_{g,n}(L_1,...,L_n)} f(l_{\gamma_1},l_{\gamma_2})\cdot d\text{vol}$$

expresses the volume $V^{WP}_{g,n}(L_1,...,L_n)$ recursively in terms of the simpler volumes $V^{WP}_{g',n'}(L_1,...,L_{n'})$ where $2g'-2+n' < 2g-2+n$ which gives Theorem 4.2.

The polynomiality of $V^{WP}_{g,n}(L_1,...,L_n)$ is immediate from its identification with intersection numbers on $\overline{\mathcal{M}}_{g,n}$ via (36). Polynomiality also follows from the following property of the kernel proven in [42]. Define

$$F^M_{2k+1}(t) = \int_0^\infty x^{2k+1}H^M(x,t)dx.$$

Then

$$\frac{F^M_{2k+1}(t)}{(2k+1)!} = \sum_{i=0}^{k+1}\zeta(2i)(2^{2i+1}-4)\frac{t^{2k+2-2i}}{(2k+2-2i)!}$$

so $F^M_{2k+1}(t)$ is a degree $2k+2$ polynomial in t with leading coefficient $t^{2k+2}/(2k+2)$. We prove analogous properties in Section 5.4 for kernels arising out of super hyperbolic surfaces which we will need when proving the Virasoro constraints in Section 6. Polynomiality of the double integrals uses the same result. By the change of coordinates $x = u+v$, $y = u-v$ one can prove

$$\int_0^\infty\int_0^\infty x^{2i+1}y^{2j+1}H^M(x+y,t)dxdy = \frac{(2i+1)!(2j+1)!}{(2i+2j+3)!}F^M_{2i+2j+3}(t). \tag{42}$$

5. Moduli space of super hyperbolic surfaces

The main result of the section is the recursion (49) between volumes of moduli spaces of super hyperbolic surfaces due to Stanford and Witten [57]. There are a number of approaches to defining a super Riemann surface and its moduli space. Above any Riemann surface equipped with a spin structure is a vector space of super Riemann surfaces identified with $H^1(\Sigma,S)$ where S is the spinor bundle of Σ. Different descriptions of Σ as an algebraic curve, a Riemann surface or a hyperbolic surface, lead to different descriptions of $H^1(\Sigma,S)$ and the corresponding super Riemann surface. The main approach we take here is via hyperbolic geometry.

5.1. **Supermanifolds.**

5.1.1. Define $\Lambda_N = \Lambda_N(\mathbb{K})$ to be the Grassman algebra over a field $\mathbb{K}$ with generators $\{1, e_1, e_2, ..., e_N\}$. We will consider only $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. An element $a \in \Lambda$ is a sum of monomials

$$a = a^{\#} + \sum_i a_i e_i + \sum_{i<j} a_{ij} e_i \wedge e_j + \sum_{i<j<k} a_{ij} e_i \wedge e_j \wedge e_k + ...$$

in the 2^N dimensional vector space Λ_N. The element $a^{\#} \in \mathbb{K}$ is the *body* of a. Define $\Lambda = \lim_{N\to\infty} \Lambda_N$. The Grassman algebra decomposes into even polynomials Λ^0, and odd polynomials Λ^1:

$$\Lambda = \Lambda^0 \oplus \Lambda^1$$

also known as the *bosonic* (even) and *fermionic* (odd) parts.

5.1.2. Define $(m|n)$ dimensional superspace, which is a Λ^0-module, by:

$$\mathbb{K}^{(m|n)} = \{(z_1, z_2, ..., z_m | \theta_1, ..., \theta_n) \mid z_i \in \Lambda^0(\mathbb{K}),\ \theta_j \in \Lambda^1(\mathbb{K})\}.$$

A super manifold of dimension $(m|n)$ is locally modeled on $\mathbb{K}^{(m|n)}$. The symmetries of $\mathbb{K}^{(m|n)}$ consist of $(m+n) \times (m+n)$ matrices

$$G = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

with even (meaning the entries are in Λ^0) $m \times m$ blocks and $n \times n$ blocks A and D, and odd $m \times n$ and $n \times m$ blocks B and C. The super transpose G^{st} of $G \in M(m|n)$ is defined by:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)^{st} = \left(\begin{array}{c|c} A^t & C^t \\ \hline -B^t & D^t \end{array}\right)$$

and the *Berezinian*, a generalisation of the determinant is defined by:

$$\mathrm{Ber}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \frac{\det(A - BD^{-1}C)}{\det(D)}$$

which is invariant under the super transpose due to oddness of B and C .

5.1.3. Endomorphisms of $\mathbb{R}^{(2|1)}$ are given by

$$M(2|1) = \left\{ \left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \beta \\ \hline \gamma & \delta & e \end{array}\right) \middle|\ a, b, c, d, e \in \Lambda_0,\ \alpha, \beta, \gamma, \delta \in \Lambda_1 \right\}. \tag{43}$$

Define $\mathrm{OSp}(1|2) \subset M(2|1)$ (the label $(2|1)$ has switched) to be those elements of Berezinian equal to one that preserve the following bilinear form J:

$$\mathrm{OSp}(1|2) = \{G \in M(2|1) \mid G^{st} J G = J,\ \mathrm{Ber}(G) = 1\}, \quad J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The condition $G^{st} J G = J$ constrains γ and δ in (43) to be linear combinations of α and β, and also $ad - bc - 1$ to be a multiple of $\alpha\beta$. These constraints, together with the condition $\mathrm{Ber}(G) = 1$ leads to the following form of any element $G \in \mathrm{OSp}(1|2)$.

$$G = \left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \beta \\ \hline a\beta - c\alpha & b\beta - d\alpha & 1 - \alpha\beta \end{array}\right) = \left(\begin{array}{c|c} g & v \\ \hline v^{st} g & 1 + \frac{1}{2} v^{st} v \end{array}\right) \in \mathrm{OSp}(1|2) \tag{44}$$

where $ad - bc = 1 + \alpha\beta$ and $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}^{st} = \begin{pmatrix} \beta & -\alpha \end{pmatrix}$. In particular, the dimension of OSp(1|2) is (3|2). Its Lie algebra is given by

$$\mathrm{osp}(1|2) = \{\xi \in M(2|1) \mid \xi^{st}J + J\xi = 0\} = (\Lambda_0 \cdot e) \oplus (\Lambda_0 \cdot f) \oplus (\Lambda_0 \cdot h) \oplus (\Lambda_1 \cdot q_1) \oplus (\Lambda_1 \cdot q_2)$$

which is spanned by the bosonic generators:

$$e = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad f = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad h = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

and fermionic generators:

$$q_1 = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & -1 & 0 \end{array}\right), \quad q_2 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \end{array}\right).$$

5.2. **Super hyperbolic surfaces.** Define the super upper half plane by

$$\widehat{\mathbb{H}} = \{(z|\theta) \in \mathbb{C}^{(1|1)} \mid \mathrm{Im}\, z^\# > 0\}. \tag{45}$$

The group of conformal transformations of $\mathbb{H}$, which is $PSL(2,\mathbb{R})$ with action $z \mapsto \frac{az+b}{cz+d}$, lifts to an action of OSp(1|2) on $\widehat{\mathbb{H}}$ given by:

$$(z|\theta) \mapsto \left(\frac{az+b}{cz+d} + \theta \frac{\alpha z + \beta}{(cz+d)^2} \,\middle|\, \frac{\alpha z + \beta}{cz+d} + \theta \frac{1 - \frac{1}{2}\alpha\beta}{cz+d} \right).$$

A discrete subgroup of OSp(1|2) is Fuchsian if its image is Fuchsian under the map OSp(1|2) $\to SL(2,\mathbb{R})$ defined by

$$g \mapsto f^\# \begin{pmatrix} a^\# & b^\# \\ c^\# & d^\# \end{pmatrix}.$$

The quotient of $\widehat{\mathbb{H}}$ by a Fuchsian subgroup is a super hyperbolic surface which is an example of a super Riemann surface which we now define.

A *super Riemann surface* is a complex supermanifold $\hat{\Sigma}$ of dimension (1|1) with a dimension (0|1) subbundle $\mathcal{D} \subset T_{\hat{\Sigma}}$ that is everywhere non-integrable hence $\mathcal{D}$ and $\{\mathcal{D}, \mathcal{D}\} = \mathcal{D}^2$ are linearly independent or $T_{\hat{\Sigma}}/\mathcal{D} \cong \mathcal{D}^2$. Equivalently, the transition functions are superconformal transformations of $\mathbb{C}^{(1|1)}$ locally given by:

$$\hat{z} = u(z) + \theta\eta(z)\sqrt{u'(z)}, \quad \hat{\theta} = \eta(z) + \theta\sqrt{u'(z) + \eta(z)\eta'(z)}.$$

The action by OSp(1|2) on $\widehat{\mathbb{H}}$ is of this form. The dimension (0|1) subbundle is locally generated by the super vector field D given in superconformal coordinates by

$$D = \theta\frac{\partial}{\partial z} + \frac{\partial}{\partial \theta}.$$

A vector field v generates a superconformal transformation if the Lie derivative with respect to v of D preserves D, i.e. $[v, D] = \lambda D$ where $[.,.]$ is the commutator on even elements and anti-commutator on odd elements. For example,

$$v = z\frac{\partial}{\partial z} + \frac{1}{2}\theta\frac{\partial}{\partial\theta}$$

satisfies $[v, D] = \frac{1}{2}D$ and generates the scaling $(z|\theta) \mapsto (\lambda z|\lambda^{1/2}\theta)$ for $\lambda \in \mathbb{C}^*$.

5.2.1. The Teichmüller space of super hyperbolic surfaces has analogous constructions to those of usual Teichmüller space. Coordinates on the Teichmüller space of super hyperbolic surfaces are constructed via: representations—Crane-Rabin [8] and Natanzon [46], ideal triangulations—Penner and Zeitlin [52], and pairs of pants decompositions—Stanford and Witten [57]. The bosonic part of the Teichmüller space is the same as usual Teichmüller space despite the extra data of a spin structure as explained in 3.1.5. The quotient of the Teichmüller space of super hyperbolic surfaces by the mapping class group of the underlying hyperbolic surface gives rise to a well-defined moduli space.

Definition 5.1. Define $\widehat{\mathcal{M}}_{g,n,\vec{\epsilon}}$ to be the moduli space of genus g super hyperbolic surfaces with underlying spin structure of boundary type $\vec{\epsilon}$ and put $\widehat{\mathcal{M}}_{g,n} := \widehat{\mathcal{M}}_{g,n,\vec{0}}$.

The tangent bundle of a super hyperbolic surface $\hat{\Sigma}$ can be identified with the pull-back of $T_\Sigma \oplus T_\Sigma^{\frac{1}{2}}$ under $\hat{\Sigma} \to \Sigma$, where the second factor gives fermionic directions. Analogous to the deformation theory of the moduli space of hyperbolic surfaces, the tangent space to the moduli space of super hyperbolic surfaces is given by the cohomology group of the log-tangent bundle

$$H^1(\overline{\Sigma}, \left(T_{\overline{\Sigma}} \oplus T_{\overline{\Sigma}}^{\frac{1}{2}}\right) \otimes \mathcal{O}(-D)) = H^1(\Sigma, T_{\overline{\Sigma}}(-D)) \oplus H^1(\Sigma, T_{\overline{\Sigma}}^{\frac{1}{2}}(-D))$$

for $D = \overline{\Sigma} - \Sigma$. The component $H^1(\Sigma, T_{\overline{\Sigma}}(-D))$ is tangent along the bosonic directions which is isomorphic to the tangent space of the usual moduli space and $H^1(\Sigma, T_{\overline{\Sigma}}^{\frac{1}{2}}(-D))$ is tangent along the fermionic directions—see [22, 34, 61]. More generally it is shown in [53] that for any holomorphic line bundle $L \to \overline{\Sigma}$, $H^0(\overline{\Sigma}, L) \oplus H^0(\overline{\Sigma}, L \otimes T_\Sigma^{-\frac{1}{2}})$ is naturally a superspace with $H^0(\overline{\Sigma}, L)$ its even part and $H^0(\overline{\Sigma}, L \otimes T_{\overline{\Sigma}}^{-\frac{1}{2}})$ its odd part, and similarly for H^1, which can be identified with the cohomology of a holomorphic line bundle over a super Riemann surface.

One can realise the odd directions via tensor product with $\Lambda^1 = \lim_{N\to\infty} \Lambda^{odd}(\mathbb{R}^N)$. On the hyperbolic surface, replace the $\pi_1\Sigma$-module $\mathbb{R}^2$ by the $\pi_1\Sigma$-module $(\Lambda^1)^2$ to produce a flat bundle $T_\Sigma^{\frac{1}{2}} \otimes_{\mathbb{R}} \Lambda^1 \to \Sigma$. The cohomology groups $H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}})$ are the fibres of the bundle $E_{g,n} \to \mathcal{M}_{g,n}^{\text{spin}}$, and similarly the cohomology groups $H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}} \otimes_{\mathbb{R}} \Lambda^1) \cong H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}}) \otimes_{\mathbb{R}} \Lambda^1$ produce the bundle $E_{g,n} \otimes_{\mathbb{R}} \Lambda^1 \to \mathcal{M}_{g,n}^{\text{spin}}$ which is written by some authors as $\Pi E_{g,n}$.

5.2.2. The symplectic group of $(\mathbb{R}^2, dx \wedge dy)$ is $SL(2,\mathbb{R})$. The *affine* symplectic group is the semi-direct product

$$ASL(2,\mathbb{R}) = SL(2,\mathbb{R}) \ltimes \mathbb{R}^2.$$

which is the product equipped with multiplication

$$(g_1, v_1) \cdot (g_2, v_2) = (g_1 g_2, v_1 + g_1 v_2), \quad (g_i, v_i) \in SL(2,\mathbb{R}) \times \mathbb{R}^2.$$

There is an exact sequence

$$0 \to \mathbb{R}^2 \to ASL(2,\mathbb{R}) \to SL(2,\mathbb{R}) \to 1 \tag{46}$$

which is equivariant with respect to the natural action of $SL(2,\mathbb{R})$—the action on $ASL(2,\mathbb{R})$ is by conjugation, i.e. $h \in SL(2,\mathbb{R})$ acts by

$$h \cdot (g, v) = (h, 0)(g, v)(h^{-1}, 0) = (hgh^{-1}, hv).$$

An affine transformation of $\mathbb{R}^2$ can be realised as a linear transformation of $\mathbb{R}^3$ that leaves invariant an affine plane, leading to a 3×3 matrix representation. Then any $G\in ASL(2,\mathbb{R})$ can be represented by the matrix

$$G=\left(\begin{array}{c|c} g & v \\ \hline 0 & 1\end{array}\right),\quad (g,v)\in SL(2,\mathbb{R})\times\mathbb{R}^2.$$

A representation $\rho:\pi_1\Sigma\to SL(2,\mathbb{R})$ defines a $\pi_1\Sigma$-module structure on $\mathbb{R}^2$ which gives rise to group cohomology $H^k(\pi_1\Sigma,\mathbb{R}^2)$. The bar construction for group cohomology and homology is equivalent to the construction of homology with local coefficients, [58], since Σ is an Eilenberg-Maclane space $K(\pi_1\Sigma,1)$ which is a classifying space for its fundamental group—a good reference is [2, Proposition 3.5]. Hence

$$H^k_{dR}(\Sigma,T_\Sigma^{\frac12})\cong H^k(\pi_1\Sigma,\mathbb{R}^2).$$

5.2.3. Using the $\pi_1\Sigma$-module $(\Lambda^1)^2$ described in 5.2.1, the group cohomology $H^k(\pi_1\Sigma,(\Lambda^1)^2)\cong H^k(\pi_1\Sigma,\mathbb{R}^2)\otimes_{\mathbb{R}}\Lambda^1$ isomorphic to the de Rham cohomology $H^k_{dR}(\Sigma,T_\Sigma^{\frac12}\otimes_{\mathbb{R}}\Lambda^1)$. Define

$$\hat{A}SL(2,\mathbb{R})=SL(2,\mathbb{R})\ltimes(\Lambda^1)^2$$

with elements

$$G=\left(\begin{array}{c|c} g & v \\ \hline 0 & 1\end{array}\right),\quad (g,v)\in SL(2,\mathbb{R})\times(\Lambda^1)^2$$

and an exact sequence

$$0\to(\Lambda^1)^2\to\hat{A}SL(2,\mathbb{R})\to SL(2,\mathbb{R})\to 1 \tag{47}$$

There is a group homomorphism

$$\mathrm{OSp}(1|2)\to\hat{A}SL(2,\mathbb{R})\to SL(2,\mathbb{R})$$

given by

$$\left(\begin{array}{c|c} g & v \\ \hline v^{st}g & 1+\frac12 v^{st}v\end{array}\right)\mapsto\left(\begin{array}{c|c} \overline{g} & \overline{v} \\ \hline 0 & 1\end{array}\right)$$

where $\overline{g}$ and $\overline{v}$ are the images of g and v in the quotient of the Grassman algebra by terms of degree greater than one $\Lambda/\Lambda^{>1}$.

Lemma 5.2. *For any representation $\rho:\pi_1\Sigma\to SL(2,\mathbb{R})$, there is a one-to-one correspondence between cohomology classes $\eta\in H^1(\pi_1\Sigma,\mathbb{R}^2)$ and lifts of ρ to $\tilde{\rho}:\pi_1\Sigma\to ASL(2,\mathbb{R})$ well-defined up to conjugation by $\mathbb{R}^2\subset ASL(2,\mathbb{R})$.*

Proof. Given a surface Σ and a representation $\rho:\pi_1\Sigma\to SL(2,\mathbb{R})$, a lift of ρ to $\tilde{\rho}:\pi_1\Sigma\to ASL(2,\mathbb{R})$ is defined by a 1-cochain $\eta\in C^1(\pi_1\Sigma,\mathbb{R}^2)$. The cocycle property of $\eta:\pi_1\Sigma\to\mathbb{R}^2$ is easily seen via multiplication

$$(\rho(g_1),\eta(g_1))\cdot(\rho(g_2),\eta(g_2))=(\rho(g_1)\rho(g_2),\eta(g_1)+\rho(g_1)\eta(g_2))=(\rho(g_1g_2),\eta(g_1g_2))$$

where the last equality is equivalent to $d\eta(g_1,g_2)=\eta(g_1)-\eta(g_1g_2)+\rho(g_1)\eta(g_2)=0$.

Conjugate the representation $\tilde{\rho}=(\rho,\eta)$ by $(1,v)\in ASL(2,\mathbb{R})$ to get

$$(1,v)\cdot(\rho(g),\eta(g))\cdot(1,-v)=(\rho(g),\eta(g)+v)\cdot(1,-v)=(\rho(g),\eta(g)+v-\rho(g)v)$$

hence the cocycle is transformed by $\eta(g)\mapsto\eta(g)+v-\rho(g)v$. But any 0-cochain is given by a vector $v\in\mathbb{R}^2$ and an exact cocycle is of the form $dv(g)=g\cdot v-v$. Thus we see that the cocycle is transformed by $\eta(g)\mapsto\eta(g)-dv(g)$. Hence representations conjugate under the $\mathbb{R}^2$ action give rise to cohomologous cocycles, and conversely, cohomologous cocycles differ by $dv(g) = g \cdot v - v$ which corresponds to conjugation by v. □

The proof of the lemma immediately gives a one-to-one correspondence between cohomology classes $\eta \in H^1(\pi_1\Sigma, (\Lambda^1)^2)$ and lifts of ρ to $\tilde{\rho} : \pi_1\Sigma \to \hat{A}SL(2,\mathbb{R})$ well-defined up to conjugation by $(\Lambda^1)^2$. In particular, any representation $\rho : \pi_1(\Sigma) \to OSp(1|2)$ determines an element of $H^1(\Sigma, T_\Sigma^{\frac{1}{2}}) \otimes \Lambda^1$. We expect that there exists a unique lift of $\tilde{\rho} : \pi_1\Sigma \to \hat{A}SL(2,\mathbb{R})$ to $\rho : \pi_1(\Sigma) \to OSp(1|2)$ by [30] which shows that an element of $H^1(\Sigma, T_\Sigma^{\frac{1}{2}})$ and the hyperbolic metric and spin structure uniquely determine the super hyperbolic surface, we expect this map be one-to-one.

5.3. **Recursion for super volumes.** Stanford and Witten [57] proved a generalisation of Mirzakhani's volume recursion using a generalisation of the identity (41) to super hyperbolic surfaces which we now describe. Given a super hyperbolic surface Σ with n geodesic boundary components $\beta_1, ..., \beta_n$, define $\mathcal{P}_i$, respectively $\mathcal{P}_{ij}$, to be the set of isometric embeddings $P \to \Sigma$ of super hyperbolic pairs of pants with geodesic boundary, which meet the boundary of Σ precisely at β_i, respectively at β_i and β_j. A pair of pants $P(x, y, z|\alpha, \beta)$ now depends on three boundary lengths x, y, z and two odd moduli α, β. As before $\ell_{\partial_i P}$ is the length of the ith geodesic boundary component of P, and α_P, β_P are its odd moduli. Using a similar argument to the derivation of D and R in 4.2.1, Stanford and Witten derived

$$\widehat{R}(x, y, z|\alpha, \beta) = x - \log\left(\frac{\cosh\frac{y}{2} + \cosh\frac{x+z}{2} - \frac{1}{2}\alpha\beta(e^{\frac{x+z}{2}} + 1)}{\cosh\frac{y}{2} + \cosh\frac{x-z}{2} - \frac{1}{2}\alpha\beta(e^{\frac{x}{2}} + e^{\frac{z}{2}})}\right)$$

which restricts to (38) when $\alpha = 0 = \beta$. Using $\alpha^2 = 0 = \beta^2$, we can expand to get:

$$\widehat{R}(x, y, z|\alpha, \beta) = R^M(x, y, z) - \alpha\beta\frac{2\pi e^{\frac{x+z}{4}}}{\cosh(\frac{y}{4})}R(x, y, z) \tag{48}$$

and

$$\int_{\widehat{\mathcal{M}}_{0,3}(x,y,z)} \widehat{R}(x, y, z|\alpha, \beta)d\mu = R(x, y, z)$$

where the moduli space $\widehat{\mathcal{M}}_{0,3}(x, y, z)$ the vector space spanned by the two odd moduli α, β. Integration is over the measure $d\mu$ which includes the odd variables α, β and a factor $\frac{1}{2\pi}\cosh(\frac{y}{4})e^{-\frac{x+z}{4}}$ from the torsion of the circle as described in [57]. This gives a geometric meaning to the kernel

$$R(x, y, z) = \frac{1}{2}H(x + y, z) + \frac{1}{2}H(x - y, z)$$

for $H(x, y) = \frac{1}{4\pi}\left(\frac{1}{\cosh((x-y)/4)} - \frac{1}{\cosh((x+y)/4)}\right)$ defined in (6). If we instead write $H(x, y)$ as

$$H(x, y) = \frac{1}{2\pi}\left(\frac{e^{\frac{-x+y}{4}}}{1 + e^{\frac{-x+y}{2}}} + \frac{e^{\frac{x+y}{4}}}{1 + e^{\frac{x+y}{2}}}\right)$$

then it emphasises its similarities with Mirzakhani's kernel:

$$H^M(x, y) = \frac{1}{1 + \exp\frac{x+y}{2}} + \frac{1}{1 + \exp\frac{x-y}{2}}$$

and hence the resemblance of $D(x,y,z)$ and $R(x,y,z)$ with Mirzakhani's kernels $D^M(x,y,z)$ and $R^M(x,y,z)$.

Define $\widehat{D}(x,y,z|\alpha,\beta)=\widehat{R}(x,y,z|\alpha,\beta)+\widehat{R}(x,z,y|\alpha',\beta')-x$ where (α',β') is an unspecified transformation of (α,β) which is unimportant after integration over the odd variables:

$$\int_{\widehat{\mathcal{M}}_{0,3}(x,y,z)}\widehat{D}(x,y,z|\alpha,\beta)d\mu=D(x,y,z).$$

For P a super pair of pants, define $\widehat{R}(P)=\widehat{R}(\ell_{\partial_1 P}=L_1,\ell_{\partial_2 P}=L_j,\ell_{\partial_3 P}|\alpha_P,\beta_P)$ and $\widehat{D}(P)=\widehat{D}(\ell_{\partial_1 P}=L_1,\ell_{\partial_2 P},\ell_{\partial_3 P}|\alpha_P,\beta_P)$.

Theorem 5.3 ([57])**.** *For any super hyperbolic surface Σ with n geodesic boundary components of lengths $L_1,...,L_n$*

$$L_1=\sum_{P\in\mathcal{P}_1}\widehat{D}(P)+\sum_{j=2}^{n}\sum_{P\in\mathcal{P}_{1j}}\widehat{R}(P).$$

See also [28] where the super McShane identity is proven in the case $(g,n)=(1,1)$ in a different way using a generalisation of Penner coordinates.

Following Mirzakhani's methods, Stanford and Witten applied Theorem 5.3 to produce the following recursion using the kernels $D(x,y,z)$ and $R(x,y,z)$ defined in (6).

Theorem 5.4 ([57])**.**

$$L_1\widehat{V}^{WP}_{g,n}(L_1,L_K)=-\frac{1}{2}\int_0^\infty\int_0^\infty xyD(L_1,x,y)P_{g,n}(x,y,L_K)dxdy-\frac{1}{2}\sum_{j=2}^{n}\int_0^\infty xR(L_1,L_j,x)\widehat{V}^{WP}_{g,n-1}(x,L_{K\setminus\{j\}})dx \tag{49}$$

where $K=\{2,...,n\}$ and

$$P_{g,n}(x,y,L_K)=\widehat{V}^{WP}_{g-1,n+1}(x,y,L_K)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=K}}\widehat{V}^{WP}_{g_1,|I|+1}(x,L_I)\widehat{V}^{WP}_{g_2,|J|+1}(y,L_J).$$

Note that Stanford and Witten use a different normalisation $V^{SW}_{g,n}$ of the volume in [57]:

$$V^{SW}_{g,n}(L_1,...,L_n)=2^n\widehat{V}^{WP}_{g,n}(L_1,...,L_n)=(-1)^n2^{1-g}V^{\Theta}_{g,n}(L_1,...,L_n).$$

Multiply (49) by 2^n and absorb this into each volume, which replaces the coefficients $-\frac{1}{2}$ and $-\frac{1}{2}$ of the D and R terms by $-\frac{1}{4}$ and -1, so that (49) now agrees with [57, (5.42)].

Proof of Theorem 1. From Corollary 3.19 we have

$$\widehat{E}_{g,n}|_{\mathcal{M}^{\rm spin}_{g,n}}\cong E_{g,n}.$$

Choose a connection A on $\widehat{E}_{g,n}\to\overline{\mathcal{M}}^{\rm spin}_{g,n}$ which has curvature F_A with Pfaffian a differential form representing the Euler class of $\widehat{E}_{g,n}$ which coincides with $c_{2g-2+n}(\widehat{E}_{g,n})$. Restrict this to $\mathcal{M}^{\rm spin}_{g,n}$ to produce a differential form $e(E_{g,n})$ so that

$$\widehat{V}^{WP}_{g,n}(L_1,...,L_n)=\int_{\mathcal{M}^{\rm spin}_{g,n}}e(E_{g,n})\exp\omega^{WP}=(-1)^n2^{1-g-n}V^{\Theta}_{g,n}(L_1,...,L_n)$$

The recursion (7) is now a consequence of (49) which we multiply by $(-1)^n 2^{g-1+n}$ and absorb this into each volume, which replaces the coefficients $-\frac{1}{2}$ and $-\frac{1}{2}$ of the D and R terms by $\frac{1}{2}$ and 1 to get (7). □

Remark 5.5. For a cusped surface corresponding to $L_1 = 0$, replace the recursion (49) by the limit $L_1 \to 0$ of $1/L_1 \times$ (49) which replaces the kernels by the limits:

$$\lim_{x\to 0}\frac{1}{x}D(x,y,z) = \frac{1}{8\pi}\frac{\sinh\frac{y+z}{4}}{\cosh^2\frac{y+z}{4}}$$

$$\lim_{x\to 0}\frac{1}{x}R(x,y,z) = \frac{1}{16\pi}\left(-\frac{\sinh\frac{y-z}{4}}{\cosh^2\frac{y-z}{4}} + \frac{\sinh\frac{y+z}{4}}{\cosh^2\frac{y+z}{4}}\right).$$

5.4. **Calculations.** We demonstrate here how to use the recursion (7). It is clear from its definition (5) that the function $V^{\Theta}_{g,n}(L_1,...,L_n)$ is a degree $2g-2$ polynomial in L_i (and degree $g-1$ polynomial in L_i^2). We prove here that this polynomial behaviour also follows from the recursion (7) and elegant properties of the kernels $D(x,y,z)$ and $R(x,y,z)$, using arguments parallel to those of Mirzakahni [42]. Define

$$F_{2k+1}(t) = \int_0^\infty x^{2k+1}H(x,t)dx$$

Lemma 5.6. *$F_{2k+1}(t)$ is a degree $2k+1$ monic polynomial in t.*

Proof.

$$\begin{aligned}
F_{2k+1}(t) &= \frac{1}{4\pi}\int_0^\infty x^{2k+1}\left(\frac{1}{\cosh((x-t)/4)} - \frac{1}{\cosh((x+t)/4)}\right)dx \\
&= \frac{1}{4\pi}\int_{-t}^\infty \frac{(x+t)^{2k+1}}{\cosh x/4}dx - \frac{1}{4\pi}\int_t^\infty \frac{(x-t)^{2k+1}}{\cosh x/4}dx \\
&= \frac{1}{4\pi}\int_0^\infty \frac{(x+t)^{2k+1}-(x-t)^{2k+1}}{\cosh x/4}dx + \frac{1}{4\pi}\int_{-t}^0 \frac{(x+t)^{2k+1}}{\cosh x/4}dx \\
&\quad + \frac{1}{4\pi}\int_0^t \frac{(x-t)^{2k+1}}{\cosh x/4}dx \\
&= \frac{1}{4\pi}\int_0^\infty \frac{(x+t)^{2k+1}-(x-t)^{2k+1}}{\cosh x/4}dx \\
&= \frac{1}{2\pi}\sum_{i=0}^k t^{2i+1}\binom{2k+1}{2i+1}\int_0^\infty \frac{x^{2k-2i}}{\cosh x/4}dx \\
&= \sum_{i=0}^k t^{2i+1}\binom{2k+1}{2i+1}a_{k-i} \\
&= t^{2k+1} + O(t^{2k})
\end{aligned}$$

where a_n is defined by $\dfrac{1}{\cos(2\pi x)} = \displaystyle\sum_{n=0}^\infty a_n \frac{x^{2n}}{(2n)!}$. In particular $a_0 = 1$ giving the final equality above. □

Analogous to (42), by the change of coordinates $x = u + v$, $y = u - v$, we have the following identity:

$$\int_0^\infty \int_0^\infty x^{2i+1} y^{2j+1} H(t, x+y) dx dy = \frac{(2i+1)!(2j+1)!}{(2i+2j+3)!} F_{2i+2j+3}(t).$$

Since $D(x,y,z) = H(x, y+z)$ and $R(x,y,z) = \frac{1}{2}H(x+y,z) + \frac{1}{2}H(x-y,z)$ we have

$$\int_0^\infty\int_0^\infty x^{2i+1} y^{2j+1} D(L_1, x, y) dx dy = \frac{(2i+1)!(2j+1)!}{(2i+2j+3)!} L_1^{2i+2j+3} + O(L_1^{2i+2j+2}) \tag{50}$$

and

$$\int_0^\infty x^{2k+1} R(L_1, L_j, x) dx = \frac{1}{2}(L_1 + L_j)^{2k+1} + \frac{1}{2}(L_1 - L_j)^{2k+1} + O(L^{2k}) \tag{51}$$

where the right hand sides of (50) and (51) are polynomial and $O(L^{2k})$ means the top degree terms are homogeneous of degree $2k$ in L_1 and L_j. We see that the recursion (7) (and (49)) produces polynomials since the initial condition is a polynomial and it sends polynomials to polynomials. So, for example,

$$\int_0^\infty \int_0^\infty yz D(x,y,z) dy dz = \frac{x^3}{6} + 2\pi^2 x$$

and

$$\int_0^\infty z R(x,y,z) dx = x, \quad \int_0^\infty z^3 R(x,y,z) dx = x(x^2 + 3y^2 + 12\pi^2).$$

5.4.1. The recursion (7) leads to the following small genus calculations. The 1-point genus one volume can be calculated using an integral closely related to (7).

$$2LV_{1,1}^{\Theta}(L) = \int_0^\infty x D(L, x, x) dx = \int_0^\infty x H(L, 2x) dx = \frac{1}{4} F_1(L) = \frac{1}{4} L \tag{52}$$

Using (7) we calculate:

$$\begin{aligned} V_{1,n}^{\Theta}(L_1, ..., L_n) &= \frac{(n-1)!}{8} \\ V_{2,n}^{\Theta}(L_1, ..., L_n) &= \frac{3(n+1)!}{128} \left((n+2)\pi^2 + \frac{1}{4} \sum_{i=1}^n L_i^2 \right) \\ V_{3,n}^{\Theta}(L_1, ..., L_n) &= \frac{(n+3)!}{2^{16} \cdot 5} \Big(16(n+4)(42n+185)\pi^4 + 336(n+4)\pi^2 \sum_{i=1}^n L_i^2 \\ &\qquad + 25 \sum_{i=1}^n L_i^4 + 84 \sum_{i \neq j}^n L_i^2 L_j^2 \Big). \end{aligned}$$

6. KdV tau functions

A tau function $Z(\hbar, t_0, t_1, ...)$ of the KdV hierarchy (equivalently the KP hierarchy in odd times $p_{2k+1} = t_k/(2k+1)!!$) gives rise to a solution $U = \frac{\partial^2}{\partial t_0^2} \log Z$ of the KdV hierarchy

$$U_{t_1} = UU_{t_0} + \frac{\hbar}{12} U_{t_0 t_0 t_0}, \quad U(t_0, 0, 0, ...) = f(t_0). \tag{53}$$

The first equation in the hierarchy is the KdV equation (53), and later equations $U_{t_k} = P_k(U, U_{t_0}, U_{t_0 t_0}, ...)$ for $k > 1$ determine U uniquely from $U(t_0, 0, 0, ...)$, [44].

6.0.2. The Brézin-Gross-Witten solution $U^{\rm BGW}=\hbar\partial_{t_0}^2\log Z^{\rm BGW}$ of the KdV hierarchy arises out of a unitary matrix model studied in [5, 26]. It is defined by the initial condition
$$U^{\rm BGW}(t_0,0,0,...)=\frac{\hbar}{8(1-t_0)^2}.$$
The first few terms of $\log Z^{\rm BGW}$ are

(54)
$$\begin{aligned}\log Z^{\rm BGW}&=-\frac{1}{8}\log(1-t_0)+\frac{3\hbar}{128}\frac{t_1}{(1-t_0)^3}+\frac{15\hbar^2}{1024}\frac{t_2}{(1-t_0)^5}+\frac{63\hbar^2}{1024}\frac{t_1^2}{(1-t_0)^6}+...\\&=\frac{1}{8}t_0+\frac{1}{16}t_0^2+\frac{1}{24}t_0^3+\hbar\frac{3}{128}t_1+\hbar\frac{9}{128}t_0t_1+\hbar^2\frac{15}{1024}t_2+\hbar^2\frac{63}{1024}t_1^2+...\end{aligned}$$

6.0.3. The Kontsevich-Witten tau function $Z^{\rm KW}$ given in Theorem 3 is defined by the initial condition
$$U^{\rm KW}(t_0,0,0,...)=t_0$$
for $U^{\rm KW}=\hbar\partial_{t_0}^2\log Z^{\rm KW}$. The low genus terms of $\log Z^{\rm KW}$ are
$$\log Z^{\rm KW}(\hbar,t_0,t_1,...)=\hbar^{-1}(\frac{t_0^3}{3!}+\frac{t_0^3t_1}{3!}+\frac{t_0^4t_2}{4!}+...)+\frac{t_1}{24}+...$$
For each integer $m\geq -1$, define the differential operator

$$\widehat{\mathcal{L}}_m=\frac{\hbar}{2}\sum_{i+j=m-1}(2i+1)!!(2j+1)!!\frac{\partial^2}{\partial t_i\partial t_j}+\sum_{i=0}^{\infty}\frac{(2i+2m+1)!!}{(2i-1)!!}t_i\frac{\partial}{\partial t_{i+m}}+\frac{1}{8}\delta_{m,0}+\frac{1}{2}\frac{t_0^2}{\hbar}\delta_{m,-1}\tag{55}$$

where the sum over $i+j=m-1$ is empty when $m=0$ or -1 and $\frac{\partial}{\partial t_{-1}}$ is the zero operator. The Brézin-Gross-Witten and Kontsevich-Witten tau functions satisfy the following equations [9, 25, 33].
$$(2k+1)!!\frac{\partial}{\partial t_k}Z^{BGW}(\hbar,t_0,t_1,t_2,...)=\widehat{\mathcal{L}}_kZ^{BGW}(\hbar,t_0,t_1,t_2,...),\quad k=0,1,2,...$$
$$(2k+3)!!\frac{\partial}{\partial t_{k+1}}Z^{KW}(\hbar,t_0,t_1,t_2,...)=\widehat{\mathcal{L}}_kZ^{KW}(\hbar,t_0,t_1,t_2,...),\quad k=-1,0,1,...$$
These are known as Virasoro constraints when we write them instead as
$$\mathcal{L}_mZ^{\rm BGW}(\hbar,t_0,t_1,t_2,...)=0,\quad m=0,1,2,...\tag{56}$$
and
$$\mathcal{L}'_mZ^{\rm KW}(\hbar,t_0,t_1,t_2,...)=0,\quad m=-1,0,1,...\tag{57}$$
for
$$\mathcal{L}_m=-\tfrac{1}{2}(2m+1)!!\frac{\partial}{\partial t_m}+\tfrac{1}{2}\widehat{\mathcal{L}}_m,\qquad \mathcal{L}'_m=-\tfrac{1}{2}(2m+3)!!\frac{\partial}{\partial t_{m+1}}+\tfrac{1}{2}\widehat{\mathcal{L}}_m.\tag{58}$$

The set of operators $\{\mathcal{L}_0,\mathcal{L}_1,\mathcal{L}_2,\dots\}$ satisfy the Virasoro commutation relations
$$[\mathcal{L}_m,\mathcal{L}_n]=(m-n)\mathcal{L}_{m+n},\quad \text{for } m,n\geq 0.$$
Similarly $\{\mathcal{L}'_{-1},\mathcal{L}'_0,\mathcal{L}'_1,\dots\}$ satisfy $[\mathcal{L}'_m,\mathcal{L}'_n]=(m-n)\mathcal{L}'_{m+n},\quad$ for $m,n\geq -1$.

6.1. **Intersection numbers.** Kontsevich proved the conjecture of Witten that the KdV tau function $Z^{\rm KW}$ stores the intersection numbers of ψ classes in the following generating function:

$$Z^{\rm KW}(\hbar,t_0,t_1,...)=\exp\sum_{g,n,\vec{k}}\frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^n\psi_i^{k_i}t_{k_i}.$$

Weil-Petersson volumes satisfy the recursion (37) and arise as intersection numbers over the moduli space of stable curves

$$V_{g,n}^{WP}(L_1,...,L_n)=\int_{\overline{\mathcal{M}}_{g,n}}\exp\left\{2\pi^2\kappa_1+\frac{1}{2}\sum_{i=1}^n L_i^2\psi_i\right\}.$$

Together these imply relations among intersection numbers over the moduli space of stable curves equivalent to Kontsevich's theorem which we state here in its Virasoro form.

Theorem 6.1 (Kontsevich [33])**.**

$$\mathcal{L}'_m\left(\exp\sum_{g,n,\vec{k}}\frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^n\psi_i^{k_i}t_{k_i}\right)=0,\ \ m\geq -1.$$

We only sketch the proof due to Mirzakhani [42] using Weil-Petersson volumes since we will give the similar proof of the analogous result given by Corollary 2 in detail.

Proof. The top degree terms $\mathcal{V}_{g,n}(\mathbf{L})$ of $V_{g,n}^{\rm WP}(\mathbf{L})$ satisfy the homogeneous recursion:

(59)

$$\begin{aligned}\frac{\partial}{\partial L_1}\left(L_1\mathcal{V}_g(L_1,\boldsymbol{L}_K)\right)=\sum_{j=2}^n L_j\Bigg[\int_0^{L_1-L_j}dx\cdot x(L_1-x)\mathcal{V}_g(x,\boldsymbol{L}_{K\setminus\{j\}})\\+\frac{1}{2}\int_{L_1-L_j}^{L_1+L_j}x(L_1+L_j-x)\mathcal{V}_g(x,\boldsymbol{L}_{K\setminus\{j\}})\Bigg]\\+\frac{1}{2}\int_0^{L_1}\int_0^{L_1-x}dxdy\cdot xy(L_1-x-y)\Bigg[\mathcal{V}_{g-1}^{\rm WP}(x,y,\boldsymbol{L}_K)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=K}}\mathcal{V}_{g_1}^{\rm WP}(x,\boldsymbol{L}_I)\,\mathcal{V}_{g_2}^{\rm WP}(y,\boldsymbol{L}_J)\Bigg]\end{aligned}$$

We skip the proof of this since it is similar to the proof of Proposition 6.2 below.

Write $\langle\prod_{i=1}^n\tau_{k_i}\rangle:=\hbar^{g-1}\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^n\psi_i^{k_i}$ where g is intrinsic on the left hand side via $3g-3+n=\sum_{i=1}^n k_i$. Then (59) implies

$$\begin{aligned}(2k_1+1)!!\langle\prod_{i=1}^n\tau_{k_i}\rangle=\tfrac{\hbar}{2}\sum_{i+j=k_1-2}(2i+1)!!(2j+1)!!\Big(\langle\tau_i\tau_j\tau_K\rangle+\sum_{I\sqcup J=K}\langle\tau_i\tau_I\rangle\langle\tau_j\tau_J\rangle\Big)\\+\sum_{j=2}^n\frac{(2k_1+2k_j-1)!!}{(2k_j-1)!!}\langle\tau_{k_1+k_j-1}\tau_{K\setminus\{j\}}\rangle\end{aligned}$$

which is equivalent to:

$$(2k+1)!!\frac{\partial}{\partial t_k}Z(t_0,t_1,t_2,...)=\widehat{\mathcal{L}}_{k-1}Z(t_0,t_1,t_2,...),\quad k=0,1,2,...$$

This coincides with the Virasoro contraints satisfied by $Z^{\mathrm{KW}}(t_0,t_1,t_2,...)$ and they have the same initial condition $Z(t_0,0,0,...)=t_0^3/3!$ so coincide. ☐

We now prove Corollary 2, the analogue of the Kontsevich-Witten theorem, which instead relates Z^{BGW} to intersection numbers on $\overline{\mathcal{M}}_{g,n}$. Following Mirzakhani's method, we prove that the top degree terms of $V^{\Theta}_{g,n}(\mathbf{L})$ satisfy Virasoro relations. This is equivalent to the recursion (60) below which first appeared in [11].

Proposition 6.2. *The top degree terms $\mathcal{V}_g(\mathbf{L})$ of $V^{\Theta}_{g,n}(\mathbf{L})$ satisfy the homogeneous recursion:*

$$L_1\mathcal{V}_g(L_1,\boldsymbol{L}_K)=\frac{1}{2}\sum_{j=2}^{n}\Big[(L_j+L_1)\mathcal{V}_g(L_j+L_1,\boldsymbol{L}_{K\setminus\{j\}}) - (L_j-L_1)\mathcal{V}_g(L_j-L_1,\boldsymbol{L}_{K\setminus\{j\}})\Big]$$
$$+\frac{1}{2}\int_0^{L_1}dx\cdot x(L_1-x)\Big[\mathcal{V}_{g-1}(x,L_1-x,\boldsymbol{L}_K)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=K}}\mathcal{V}_{g_1}(x,\boldsymbol{L}_I)\,\mathcal{V}_{g_2}(L_1-x,\boldsymbol{L}_J)\Big]. \tag{60}$$

Proof. From the properties (50) and (51), the top degree terms $\mathcal{V}_{g,n}$ of $V^{\Theta}_{g,n}(\mathbf{L})$ only depend on the top degree terms $\mathcal{V}_{g',n'}$ of $V^{\Theta}_{g',n'}(\mathbf{L})$ in the recursion (49). Moreover,

$$\int_0^\infty xR(L_1,L_j,x)\mathcal{V}_g(x,L_{K\setminus\{j\}})dx=\frac{1}{2}(L_j+L_1)\mathcal{V}_g(L_j+L_1,\boldsymbol{L}_{K\setminus\{j\}}) -\frac{1}{2}(L_j-L_1)\mathcal{V}_g(L_j-L_1,\boldsymbol{L}_{K\setminus\{j\}})+\text{ lower order terms.}$$

By (50), the double integral in (49) is a linear operator with input monomials $x^{2i+1}y^{2j+1}$ of $P_{g,n}(x,y,L_K)$ and output $\frac{(2i+1)!(2j+1)!}{(2i+2j+3)!}L_1^{2i+2j+3}$. This linear operator can be realised via the following integral for input x^my^n:

$$\int_0^L x^m(L-x)^n dx=\frac{m!n!}{(m+n+1)!}L^{m+n+1}$$

which is immediate when $n=0$ and proven by induction for $n>0$ via differentiation of both sides by L. Hence

$$\frac{1}{2}\int_0^\infty\int_0^\infty xyD(L_1,x,y)P_{g,n}(x,y,L_K)dxdy=\int_0^{L_1}dx\cdot x(L_1-x)P_{g,n}(x,L_1-x,\boldsymbol{L}_K) + \text{lower order terms}$$

and the proposition is proven. ☐

Corollary 6.3.

$$Z^{\Theta}(t_0,t_1,...)=\exp\sum\frac{1}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\cdot\prod_{i=1}^n\psi_j^{k_i}t_{k_i}$$

is the Brézin-Gross-Witten tau function of the KdV hierarchy.

Proof. We see that $(2k_1+1)!$ times the coefficient of $L_1\prod_{i=1}^n L_i^{2k_i}$ in (60) is given by:

$$(2k_1+1)!!\langle\Theta\cdot\prod_{i=1}^n\tau_{k_i}\rangle=\sum_{j=2}^n\frac{(2k_1+2k_j+1)!!}{(2k_j-1)!!}\langle\Theta\cdot\tau_{k_1+k_j-1}\tau_{K\backslash\{j\}}\rangle$$
$$+\tfrac{\hbar}{2}\sum_{i+j=k_1-1}(2i+1)!!(2j+1)!!\Big(\langle\Theta\cdot\tau_i\tau_j\tau_K\rangle+\sum_{I\sqcup J=K}\langle\Theta\cdot\tau_i\tau_I\rangle\langle\Theta\cdot\tau_j\tau_J\rangle\Big) \tag{61}$$

for $\langle\Theta\cdot\prod_{i=1}^n\tau_{k_i}\rangle:=\hbar^{g-1}\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^n\psi_i^{k_i}$. The recursion (61) for $k_1=0,1,2,...$ is equivalent to

$$(2k+1)!!\frac{\partial}{\partial t_k}Z^\Theta(t_0,t_1,t_2,...)=\widehat{\mathcal{L}}_kZ^\Theta(t_0,t_1,t_2,...),\quad k=0,1,2,...$$

which coincides with the Virasoro constraints satisfied by $Z^{\rm BGW}(t_0,t_1,t_2,...)$. Furthermore, $\langle\Theta\cdot\tau_0\rangle=\frac18$ produces the initial condition

$$\log Z^\Theta(t_0,0,0,...)=-\frac18\log(1-t_0)$$

via $\mathcal{L}_0Z^\Theta(t_0,0,0,...)=0$. Hence $\partial_{t_0}^2\log Z^\Theta(t_0,0,0,...)=\frac{1}{8(1-t_0)^2}$ and

$$Z^\Theta(t_0,t_1,t_2,...)=Z^{\rm BGW}(t_0,t_1,t_2,...).$$

which coincides with the Virasoro contraints satisfied by $Z^{\rm BGW}(t_0,t_1,t_2,...)$ □

6.1.1. *Translation.* The partition function

$$Z_{\kappa_1}(\hbar,\vec{t},s)=\exp\left(\sum_{g,n}\frac{\hbar^{g-1}}{n!}\sum_{\vec{k}\in\mathbb{N}^n}\int_{\overline{\mathcal{M}}_{g,n}}\exp(s\kappa_1)\prod_{i=1}^n\psi_i^{k_i}t_{k_i}\right)$$

is built out of the Weil-Petersson volumes

$$Z_{\kappa_1}(\hbar,\vec{t},2\pi^2)=\exp\sum_{g,n}\frac{\hbar^{g-1}}{n!}V_{g,n}(L_1,...,L_n)|_{\{L_i^{2k}=2^kk!t_k\}}$$

and was proven by Manin and Zograf [36] to be related to the Kontsevich-Witten tau function via translation

$$Z_{\kappa_1}(\hbar,\vec{t},s)=Z^{\rm KW}(\hbar,t_0,t_1,t_2+s,t_3-s^2/2,...,t_k+(-1)^k\frac{s^{k-1}}{(k-1)!},...). \tag{62}$$

Similarly, the Weil-Petersson super-volumes build a partition function

$$Z^\Theta_{\kappa_1}(\hbar,\vec{t},2\pi^2)=\exp\sum_{g,n}\frac{\hbar^{g-1}}{n!}V^\Theta_{g,n}(L_1,...,L_n)|_{\{L_i^{2k}=2^kk!t_k\}}$$

which is a translation of the Brézin-Gross-Witten tau function.

$$Z^\Theta_{\kappa_1}(\hbar,\vec{t},s)=\exp\left(\sum_{g,n}\frac{\hbar^{g-1}}{n!}\sum_{\vec{k}\in\mathbb{N}^n}\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\exp(s\kappa_1)\prod_{i=1}^n\psi_i^{k_i}t_{k_i}\right)$$
$$=Z^{\rm BGW}(\hbar,t_0,t_1+s,t_2-s^2/2,...,t_k-(-1)^k\frac{s^k}{(k)!},...). \tag{63}$$

Note that the translation in (63) is shifted by 1 compared to the translation in (62). We will prove (63) as a special case of a more general result involving all kappa classes.

6.1.2. *Higher Weil-Petersson volumes.* Define the generating function

$$Z_\kappa(\hbar,\vec{t},\vec{s})=\exp\sum_{g,n}\frac{\hbar^{g-1}}{n!}\sum_{\vec{k}\in\mathbb{N}^n}\int_{\overline{\mathcal{M}}_{g,n}}\prod_{i=1}^n\psi_i^{k_i}t_{k_i}\prod_{j=1}^\infty\kappa_j^{m_j}\frac{s_j^{m_j}}{m_j!}.$$

for integrals involving all kappa classes, known as higher Weil-Petersson volumes. Define the weighted homogeneous polynomials p_j of degree j by

$$1-\exp\left(-\sum_{i=1}^\infty s_iz^i\right)=\sum_{j=1}^\infty p_j(s_1,...,s_j)z^j.$$

Theorem 6.4 ([36])**.**

$$Z_\kappa(\hbar,\vec{t},\vec{s})=Z^{KW}(\hbar,t_0,t_1,t_2+p_1(\vec{s}),...,t_j+p_{j-1}(\vec{s}),....)$$

The KdV hierarchy is invariant under translations, so an immediate consequence of Theorem 6.4 is that Z_κ is a tau function of the KdV hierarchy in the t_i variables, and the same is true of Z_κ^Θ defined analogously by

$$Z_\kappa^\Theta(\hbar,\vec{t},\vec{s})=\exp\sum_{g,n}\frac{1}{n!}\sum_{\vec{k}\in\mathbb{N}^n}\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\cdot\prod_{i=1}^n\psi_i^{k_i}t_{k_i}\prod_{j=1}^\infty\kappa_j^{m_j}\frac{s_j^{m_j}}{m_j!}.$$

Theorem 6.5.

$$Z_\kappa^\Theta(\hbar,\vec{t},\vec{s})=Z^{BGW}(\hbar,t_0,t_1+p_1(\vec{s}),...,t_j+p_j(\vec{s}),....)$$

Proof. The class $\Theta_{g,n}\in H^*(\overline{\mathcal{M}}_{g,n})$ pulls back under the forgetful map by

$$\Theta_{g,n+1}=\psi_{n+1}\cdot\pi^*\Theta_{g,n}$$

which gives push-forward relations

$$\pi_*(\Theta_{g,n+1}\psi_{n+1}^m)=\pi_*(\psi_{n+1}^{m+1}\cdot\pi^*\Theta_{g,n})=\Theta_{g,n}\kappa_m.$$

This is a shift by 1 of the usual pushforward relation $\pi_*(\psi_{n+1}^{m+1})=\kappa_m$.

We will first prove the case $s_i=0$ for $i>1$, which is (63). The proof in [36] of (62) uses the following push-forward relation from [31] for κ_1^m involving a sum over ordered partitions of m.

$$\frac{\kappa_1^m}{m!}\prod_{j=1}^n\psi_j^{k_j}=\pi_*\left(\sum_{\mu\vdash m}\frac{(-1)^{m+\ell(\mu)}}{\ell(\mu)!}\prod_{j=n+1}^{n+\ell(\mu)}\frac{\psi_j^{\mu_j+1}}{\mu_j!}\prod_{j=1}^n\psi_j^{k_j}\right)\tag{64}$$

where $\mu\vdash m$ is an ordered partition of m of length $\ell(\mu)$ and $\pi_*:\overline{\mathcal{M}}_{g,n+\ell(\mu)}\to\overline{\mathcal{M}}_{g,n}$. The factor $\prod_{j=1}^n\psi_j^{k_j}$ in (64) essentially does not participate since it can be replaced by its pull-back in the right hand side of (64), using $\psi_{n+1}\cdot\prod_{j=1}^n\psi_j^{k_j}=\psi_{n+1}\cdot\pi^*\prod_{j=1}^n\psi_j^{k_j}$, and then brought outside of the push-forward.

Integrate (64) to get

$$\int_{\overline{\mathcal{M}}_{g,n}} \frac{\kappa_1^m}{m!} \prod_{j=1}^n \psi_j^{k_j} = \sum_{\mu \vdash m} \frac{(-1)^{m+\ell(\mu)}}{\ell(\mu)!} \int_{\overline{\mathcal{M}}_{g,n+\ell(\mu)}} \prod_{j=n+1}^{n+\ell(\mu)} \frac{\psi_j^{\mu_j+1}}{\mu_j!} \prod_{j=1}^n \psi_j^{k_j}$$

which is easily seen to be equivalent to the translation (62) on generating functions. Notice that $\mu_j + 1 \geq 2$ hence the first variable that is translated is t_2.

When $\Theta_{g,n}$ is present, there is a shift by 1 in the pushforward relations, hence $\psi_j^{\mu_j+1}$ in the right hand side of (64) is replaced by $\psi_j^{\mu_j}$

$$\Theta_{g,n} \frac{\kappa_1^m}{m!} \prod_{j=1}^n \psi_j^{k_j} = \pi_* \left(\Theta_{g,n+\ell(\mu)} \sum_{\mu \vdash m} \frac{(-1)^{m+\ell(\mu)}}{\ell(\mu)!} \prod_{j=n+1}^{n+\ell(\mu)} \frac{\psi_j^{\mu_j}}{\mu_j!} \prod_{j=1}^n \psi_j^{k_j} \right)$$

which leads to the translation (63) on generating functions. Notice now that $\mu_j \geq 1$ and the first variable that is translated is t_1. This also explains the shift by 1 between the translations (62) and (63).

We have proven that via translation, one can remove the term $\exp(2\pi^2\kappa_1)$ from $Z^{\Theta}_{\kappa_1}$, but this corresponds to using only the top degree terms of $V^{\Theta}_{g,n}(L_1, ..., L_n)$ in the generating function. This generating function has been shown to satisfy the KdV hierarchy in Corollary 6.3 and moreover it coincides with the Brézin-Gross-Witten tau function Z^{BGW}. Thus $Z^{\Theta}_{\kappa_1}$ is indeed translation of Z^{BGW}.

The proof of the general case, when all s_i are present, is similar, albeit more technical. The following relation is proven in [31].

$$\frac{\kappa_1^{m_1} ... \kappa_N^{m_N}}{m_1! ... m_N!} \prod_{j=1}^n \psi_j^{k_j} = \pi_* \left(\sum_{k=1}^{|m|} \frac{(-1)^{|\mathbf{m}|+\mathbf{k}}}{k!} \sum_{\mu \vdash_k \mathbf{m}} \prod_{j=n+1}^{n+k} \frac{\psi_j^{|\mu^{(j)}|+1}}{\mu^{(j)}!} \prod_{j=1}^n \psi_j^{k_j} \right) \tag{65}$$

where $\pi_* : \overline{\mathcal{M}}_{g,n+N} \to \overline{\mathcal{M}}_{g,n}$, $\mathbf{m} = (m_1, ..., m_N) \in \mathbb{Z}^N$, and $\mu \vdash_k \mathbf{m}$ is a partition into k parts, i.e. $\mu^{(1)} + ... + \mu^{(k)} = \mathbf{m}$, $\mu^{(j)} \neq 0$, $\mu^{(j)} \in \mathbb{Z}^N$, $|\mu^{(j)}| = \sum_i \mu_i^{(j)}$, $\mu^{(j)}! = \prod_i \mu_i^{(j)}!$. As in the special case above, on the level of generating functions (65) leads to the translation in Theorem 6.4

Again, when $\Theta_{g,n}$ is present, there is a shift by 1 in the pushforward relations, hence $\psi_j^{|\mu^{(j)}|+1}$ in the right hand side of (64) is replaced by $\psi_j^{|\mu^{(j)}|}$

$$\Theta_{g,n} \frac{\kappa_1^{m_1} ... \kappa_N^{m_N}}{m_1! ... m_N!} \prod_{j=1}^n \psi_j^{k_j} = \pi_* \left(\Theta_{g,n+N} \sum_{k=1}^{|m|} \frac{(-1)^{|\mathbf{m}|+\mathbf{k}}}{k!} \sum_{\mu \vdash_k \mathbf{m}} \prod_{j=n+1}^{n+k} \frac{\psi_j^{|\mu^{(j)}|}}{\mu^{(j)}!} \prod_{j=1}^n \psi_j^{k_j} \right)$$

which has the effect of translation shifted by 1 given in the statement of the Theorem. By the proof of the case $s_i = 0$ for $i > 1$ we see that it is translation of the Brézin-Gross-Witten tau function Z^{BGW}. □

6.2. **Hyperbolic cone angles.** One can relax the hyperbolic condition on a representation $\rho : \pi_1\Sigma \to SL(2, \mathbb{R})$ and allow the image of boundary classes to be elliptic. The trace of an elliptic element is $\operatorname{tr} h = 2\cos(\phi/2) \in (-2, 2)$, hence such a boundary class corresponds to a cone of angle ϕ. A hyperbolic element with trace $\operatorname{tr} g = 2\cosh(L/2))$ corresponds to a closed geodesic of length L. Since

$2\cos(\phi/2) = 2\cosh(i\phi/2)$, one can interpret a point with cone angle in terms of an imaginary length boundary component, and some formulae generalise by replacing positive real parameters with imaginary parameters. Explicitly, a cone angle ϕ appears by substituting the length $i\phi$ in the volume polynomial. Mirzakhani's recursion uses a generalised McShane formula [38] on hyperbolic surfaces, which was adapted in [59] to allow a cone angle ϕ that ends up appearing as a length $i\phi$ in such a formula, and hence in the volume polynomial. The importance of hyperbolic monodromy g is that it gives invertibility of $g - I$ used, for example, in the calculation of the cohomology groups H^k_{dR} of the representation. Perhaps this condition is required only on the interior and not on the boundary classes. Regardless of the mechanism of the proofs when cone angles are present, one can evaluate the volume polynomials at imaginary values, and find good behaviour.

Theorem 6.6.

$$V^{\Theta}_{g,n+1}(2\pi i, L_1, ..., L_n) = (2g-2+n)V^{\Theta}_{g,n}(L_1, ..., L_n) \tag{66}$$

Proof. Using

$$V^{\Theta}_{g,n}(L_1, ..., L_n) = \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \cdot \exp\left\{2\pi^2\kappa_1 + \frac{1}{2}\sum_{i=1}^{n} L_i^2\psi_i\right\}$$

the coefficient of $L_1^{2\alpha_1}...L_n^{2\alpha_n}$ in $V^{\Theta}_{g,n+1}(2\pi i, L_1, ..., L_n)$ is

$$\begin{aligned}
\sum_{j=0}^{m} \frac{(2\pi i)^{2j} 2^{-|\alpha|-j}}{\alpha! j! (m-j)!} & \int_{\overline{\mathcal{M}}_{g,n+1}} \Theta_{g,n+1}\psi^{\alpha}\psi^{j}_{n+1}(2\pi^2\kappa_1)^{m-j} \\
&= \int_{\overline{\mathcal{M}}_{g,n+1}} \Theta_{g,n+1}\frac{\psi^{\alpha}}{\alpha!}\frac{2^{-|\alpha|}}{m!}\sum_{j=0}^{m}\binom{m}{j}(-1)^j(2\pi^2\psi_{n+1})^j(2\pi^2\kappa_1)^{m-j} \\
&= \int_{\overline{\mathcal{M}}_{g,n+1}} \Theta_{g,n+1}\frac{\psi^{\alpha}}{\alpha!}\frac{2^{-|\alpha|}}{m!}(2\pi^2\kappa_1 - 2\pi^2\psi_{n+1})^m \\
&= \int_{\overline{\mathcal{M}}_{g,n+1}} \Theta_{g,n+1}\frac{\psi^{\alpha}}{\alpha!}\frac{2^{-|\alpha|}}{m!}(2\pi^2\pi^*\kappa_1)^m \\
&= \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1}2^{-|\alpha|}\pi^*\big(\Theta_{g,n}\frac{\psi^{\alpha}}{\alpha!}\frac{(2\pi^2\kappa_1)^m}{m!}\big) \\
&= (2g-2+n)2^{-|\alpha|}\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n}\frac{\psi^{\alpha}}{\alpha!}\frac{(2\pi^2\kappa_1)^m}{m!}
\end{aligned}$$

which is exactly $2g-2+n$ times the coefficient of $L_1^{2\alpha_1}...L_n^{2\alpha_n}$ in $V^{\Theta}_{g,n}$. □

For $g > 1$, the integrals

$$V^{\Theta}_{g,0} = \int_{\overline{\mathcal{M}}_{g}} \Theta_g \cdot \exp\left\{2\pi^2\kappa_1\right\}$$

which give the super volumes

$$V^{SW}_{g,0} = 2^{1-g}V^{\Theta}_{g,0}$$

do not arise out of the recursion (7). Nevertheless, setting $n=0$ in (66) allows one to calculate these integrals from $V^{\Theta}_{g,1}(L)$ which do arise out of the recursion (7)

$$V^{\Theta}_{g,1}(2\pi i)=(2g-2)V^{\Theta}_{g,0}.$$

Analogous results were proven in [12] for the Weil-Petersson volumes.

Theorem 6.7 ([12]). *For* $\mathbf{L}=(L_1,...,L_n)$

$$V^{WP}_{g,n+1}(\mathbf{L},2\pi i)=\sum_{k=1}^{n}\int_0^{L_k}L_kV^{WP}_{g,n}(\mathbf{L})dL_k$$

and

$$\frac{\partial V^{WP}_{g,n+1}}{\partial L_{n+1}}(\mathbf{L},2\pi i)=2\pi i(2g-2+n)V^{WP}_{g,n}(\mathbf{L}).$$

It is interesting that the evaluation of the derivative of the Weil-Petersson volume at $2\pi i$ gives the closest analogue to (66) which does not require a derivative. This feature resembles the relations between the kernels for recursions between super volumes $D(x,y,z)=H(x,y+z)$, and between Weil-Petersson volumes $\frac{\partial}{\partial x}D^M(x,y,z)=H^M(x,y+z)$, and similarly for $R(x,y,z)$ and $R^M(x,y,z)$, where the Weil-Petersson volumes again require a derivative.

6.2.1. For a given genus g, $V^{\Theta}_{g,g-1}(L_1,...,L_{g-1})$ determines all the polynomials $V^{\Theta}_{g,n}(L_1,...,L_n)$ as follows. When $n<g-1$ use (66) to produce $V^{\Theta}_{g,n}(L_1,...,L_n)$ from $V^{\Theta}_{g,g-1}(L_1,...,L_{g-1})$. When $n\geq g$, $V^{\Theta}_{g,n}(L_1,...,L_n)$, which is a degree $g-1$ symmetric polynomial in $L_1^2,...,L_n^2$, is uniquely determined by evaluation at $L_n=2\pi i$, and this is determined by $V_{g,n-1}(L_1,...,L_{n-1})$ via Theorem 68. This follows from the elementary fact that a symmetric polynomial $f(x_1,...,x_n)$ of degree less than n is uniquely determined by evaluation of one variable at any $a\in\mathbb{C}$, $f(x_1,...,x_{n=1},a)$. To see this, suppose otherwise. Any symmetric $g(x_1,...,x_n)$ of degree less than n that evaluates at a as f does, satisfies

$$\begin{aligned}f(x_1,...,x_{n-1},a)=g(x_1,...,x_{n-1},a)&=(x_n-a)P(x_1,...,x_n)\\&=Q(x_1,...,x_n)\prod_{j=1}^{n}(x_j-a)\end{aligned}$$

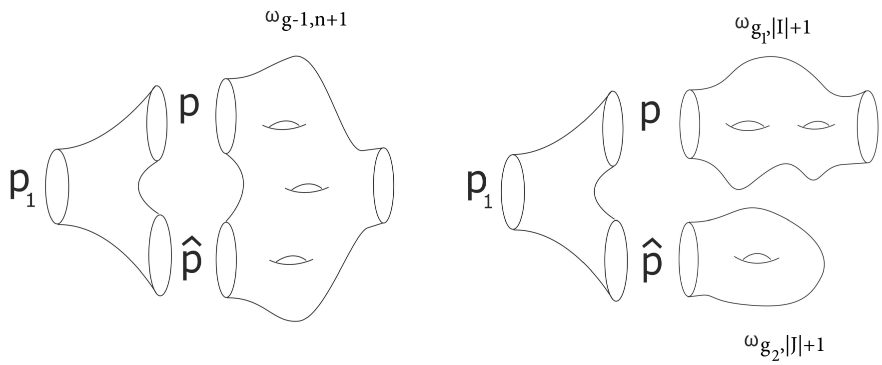
but the degree is less than n so the difference is identically 0.

7. Topological recursion

Topological recursion produces a collection of correlators $\omega_{g,n}(p_1,\dots,p_n)$, for $p_i\in C$, from a *spectral curve* (C,B,x,y) consisting of a compact Riemann surface C, a bidifferential B on C, and meromorphic functions $x,y:C\to\mathbb{C}$. It arose out of loop equations satisfied by matrix models and was developed by Chekhov, Eynard and Orantin [6, 18]. A technical requirement is that the zeros of dx are simple and disjoint from the zeros of dy [18]. In many cases the bidifferential B is taken to be the fundamental normalised differential of the second kind on C, [21], and given by the Cauchy kernel $B=\frac{dz_1dz_2}{(z_1-z_2)^2}$ when C is rational with global rational parameter z.

The correlators $\omega_{g,n}(p_1,...,p_n)$ are a collection of symmetric tensor products of meromorphic 1-forms defined on C^n where $p_i\in C$, for integers $g\geq 0$ and $n\geq 1$. They are defined recursively from $\omega_{g',n'}(p_1,...,p_{n'})$ for (g',n') satisfying

$2g' - 2 + n' < 2g - 2 + n$. The recursion can be represented pictorially via different ways of decomposing a genus g surface with n labeled boundary components into a pair of pants containing the first boundary component and simpler surfaces.



For $2g - 2 + n > 0$ and $L = \{2, \ldots, n\}$, define

$$\omega_{g,n}(p_1, p_L) = \sum_{\alpha} \operatorname*{Res}_{p=\alpha} K(p_1, p) \Big[\omega_{g-1,n+1}(p, \hat{p}, p_L) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = L}}^{\circ} \omega_{g_1,|I|+1}(p, p_I)\, \omega_{g_2,|J|+1}(\hat{p}, p_J) \Big] \tag{67}$$

where the outer summation is over the zeros α of dx and the $\circ$ over the inner summation means that we exclude terms that involve ω_1^0. The point $\hat{p} \in C$ is defined to be the unique point $\hat{p} \neq p$ close to α such that $x(\hat{p}) = x(p)$. It is unique since each zero α of dx is assumed to be simple, and (67) needs only consider $p \in C$ close to α. The recursion takes as input the unstable cases

$$\omega_{0,1} = -y(p_1)\, dx(p_1) \qquad \text{and} \qquad \omega_{0,2} = B(p_1, p_2).$$

The kernel K is defined by

$$K(p_1, p) = \frac{-\int_{\hat{p}}^{p} \omega_2^0(p_1, p')}{2[y(p) - y(\hat{p})]\, dx(p)}$$

which is well-defined in a neighbourhood of each zero of dx. Note that the quotient of a differential by the differential $dx(p)$ is a meromorphic function. For $2g-2+n > 0$, the correlator $\omega_{g,n}$ is symmetric, with poles only at the zeros of dx and vanishing residues.

The poles of the correlator $\omega_{g,n}$ occur at the zeros of dx. A zero α of dx is *regular*, respectively *irregular*, if y is regular, respectively has a simple pole, at α. A spectral curve is regular if all zeros of dx are regular and irregular otherwise.

The order of the pole in each variable of $\omega_{g,n}$ at a regular, respectively irregular, zero of dx is $6g-4+2n$, respectively $2g$, [10, 18].

Two cases of interest in this paper use $x=\frac{1}{2}z^2$, B is the Cauchy kernel and $y=\frac{\sin(2\pi z)}{2\pi}$, respectively $y=\frac{\cos(2\pi z)}{z}$. The recursion (67) allows for functions y that are not algebraic as in these two examples. Moreover, the recursive definition of $\omega_{g,n}(p_1,\dots,p_n)$ uses only local information of x, y and B around zeros of dx. In particular, y and B need to be only defined in a neighbourhood of the zeros of dx and topological recursion generalises to *local* curves in which C is an open subset of a compact Riemann surface [17].

7.0.2. In many examples $\omega_{g,n}(p_1,p_2,...,p_n)$ gives the coefficients in the large N expansion of expected values of multiresolvents in a matrix model

$$\left\langle \mathrm{Tr}\left(\frac{1}{x(p_1)-A}\right)...\mathrm{Tr}\left(\frac{1}{x(p_n)-A}\right)\right\rangle_c$$

where N is the size of the matrix and g indexes the order in the $1/N$ expansion. The subscript c means cumulant, or the connected part in a graphical expansion. In such cases, topological recursion follows from the loop equations satisfied by the resolvents. Saad, Shenker and Stanford [54] introduced a matrix model corresponding to the spectral curve $x=\frac{1}{2}z^2$, $y=\frac{\sin(2\pi z)}{2\pi}$. Stanford and Witten [57] used these ideas to produce the spectral curve $x=\frac{1}{2}z^2$, $y=\frac{\cos(2\pi z)}{z}$.

7.0.3. Define $\Phi(p)$ up to an additive constant by $d\Phi(p)=y(p)dx(p)$. For $2g-2+n>0$, the correlators $\omega_{g,n}$ satisfy the dilaton equation [18]

$$\sum_\alpha \operatorname*{Res}_{p=\alpha}\Phi(p)\,\omega_{g,n+1}(p,p_1,\dots,p_n)=(2-2g-n)\,\omega_{g,n}(p_1,\dots,p_n), \tag{68}$$

where the summation is over the zeros α of dx. The relation (68) is invariant under $\Phi\mapsto\Phi+c$ where c is a constant, since the poles of $\omega_{g,n+1}(p,p_1,\dots,p_n)$ are residueless. The dilaton equation enables the definition of the so-called *symplectic invariants*

$$\omega_{g,0}=\sum_\alpha \operatorname*{Res}_{p=\alpha}\Phi(p)\,\omega_{g,1}(p).$$

7.0.4. The correlators $\omega_{g,n}$ are normalised differentials of the second kind in each variable—they have zero $\mathcal{A}$-periods, and poles only at the zeros $\mathcal{P}_i$ of dx of zero residue. Their principal parts are skew-invariant under the local involution $p\mapsto\hat{p}$. The correlators $\omega_{g,n}$ are polynomials in a basis $V^i_k(p)$ of normalised differentials of the second kind, which have poles only at the zeros of dx with skew-invariant principal part, constructed from x and B as follows.

Definition 7.1. For a Riemann surface equipped with a meromorphic function (Σ,x) we define evaluation of any meromorphic differential ω at a simple zero $\mathcal{P}$ of dx by

$$\omega(\mathcal{P})^2:=\operatorname*{Res}_{p=\mathcal{P}}\frac{\omega(p)\otimes\omega(p)}{dx(p)}\in\mathbb{C}$$

and we choose a square root of $\omega(\mathcal{P})^2$ to remove the ±1 ambiguity.

Definition 7.2. For a Riemann surface C equipped with a meromorphic function $x : C \to \mathbb{C}$ and bidifferential $B(p_1, p_2)$ define the auxiliary differentials on C as follows. For each zero $\mathcal{P}_i$ of dx, define

$$\xi_0^i(p) = B(\mathcal{P}_i, p), \quad \xi_{k+1}^i(p) = -d\left(\frac{\xi_k^i(p)}{dx(p)}\right), \; i = 1, ..., N, \quad k = 0, 1, 2, ... \tag{69}$$

where evaluation $B(\mathcal{P}_i, p)$ at $\mathcal{P}_i$ is given in Definition 7.1.

From any spectral curve S, one can define a partition function Z^S by assembling the polynomials built out of the correlators $\omega_{g,n}$ [15, 17].

Definition 7.3.

$$Z^S(\hbar, \{u_k^\alpha\}) := \exp \sum_{g,n} \frac{\hbar^{g-1}}{n!} \omega_{g,n}^S \Big|_{\xi_k^\alpha(p_i) = u_k^\alpha}.$$

Theorem 7.4 ([15])**.** *Given any semisimple CohFT Ω with flat unit, there exists a local spectral curve S whose topological recursion partition function coincides with the partition function of the CohFT:*

$$Z^S(\hbar, \{u_k^\alpha\}) = Z_\Omega(\hbar, \{t_k^\alpha\})$$

for $\{u_k^\alpha\}$ linearly related to $\{t_k^\alpha\}$.

The following partial converse of Theorem 7.4 allows for degenerate CohFT, and in particular a CohFT is not required to have flat unit.

Theorem 7.5 ([7])**.** *Consider a spectral curve $S = (\Sigma, B, x, y)$ with possibly irregular zeros of dx. There exist a possibly degenerate CohFT Ω such that operators $\hat{R}$, $\hat{T}$ and $\hat{\Delta}$ determined explicitly by (Σ, B, x, y) such that*

$$Z^S(\hbar, \{u_k^\alpha\}) = Z_\Omega(\hbar, \{t_k^\alpha\}).$$

Theorem 7.5 is a consequence of the following more technical result from [7]. Given a spectral curve $S = (\Sigma, B, x, y)$ with m irregular zeros of dx at which y has simple poles, and $D - m$ regular zeros, there exist operators $\hat{R}$, $\hat{T}$ and $\hat{\Delta}$ determined explicitly by (Σ, B, x, y) such that the following holds:

$$Z^S = \hat{R}\hat{T}\hat{\Delta} Z^{\mathrm{BGW}}(\hbar, \{v^{k,1}\}) ... Z^{\mathrm{BGW}}(\hbar, \{v^{k,m}\}) Z^{\mathrm{KW}}(\hbar, \{v^{k,m+1}\}) ... Z^{\mathrm{KW}}(\hbar, \{v^{k,D}\}). \tag{70}$$

The operators $\hat{R}$, $\hat{T}$ and $\hat{\Delta}$ can be used to construct a CohFT with partition function given by the right hand side of (70).

The CohFT $\Omega_{g,n} = \exp(2\pi^2\kappa_1)$ which has partition function

$$Z_\Omega(\hbar, \{t_k\}) = \exp \sum_{g,n,\vec{k}} \frac{\hbar^{g-1}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \exp(2\pi^2\kappa_1) \cdot \prod_{j=1}^n \psi_j^{k_j} \prod t_{k_j}$$
$$= \exp \sum_{g,n} \frac{\hbar^{g-1}}{n!} V_{g,n}(L_1, ..., L_n)|_{\{L_i^{2k} = 2^k k! t_k\}}.$$

does not have flat identity. Its relation to topological recursion, given in the following theorem, was proven by Eynard and Orantin. Theorem 7.5 applies to the spectral curve produced by the following theorem.

Theorem 7.6 ([19])**.** *Topological recursion applied to the spectral curve*

$$S_{EO}=\left(\mathbb{C},x=\frac{1}{2}z^2,y=\frac{\sin(2\pi z)}{2\pi},B=\frac{dzdz'}{(z-z')^2}\right)$$

has partition function

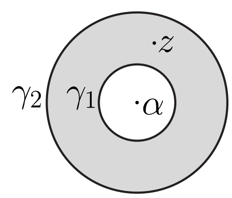
$$Z_{S_{EO}}(\hbar,\{t_k\})=\exp\sum_{g,n}\frac{\hbar^{g-1}}{n!}V^{WP}_{g,n}(L_1,...,L_n)|_{\{L_i^{2k}=2^kk!t_k\}}.$$

We now prove the main result of this section which is an analogue of Theorem 7.6 and is equivalent to Theorem 4. The proof requires the following property of the principal part. The principal part of a rational function $r(z)$ at a point $\alpha\in\mathbb{C}$, denoted by $[r(z)]_\alpha$, is the negative part of the Laurent series of $r(z)$ at α. It has the integral expression

$$[r(z)]_{z=\alpha}=\operatorname*{Res}_{w=\alpha}\frac{r(w)dw}{z-w}$$

since the right hand side is analytic for $z\in\mathbb{C}\backslash\{\alpha\}$ and

$$r(z)=-\operatorname*{Res}_{w=z}\frac{r(w)dw}{z-w}=\frac{1}{2\pi i}\int_{\gamma_1-\gamma_2}\frac{r(w)dw}{z-w}=[r(z)]_{z=\alpha}-\frac{1}{2\pi i}\int_{\gamma_2}\frac{r(w)dw}{z-w}$$



so that $r(z)-[r(z)]_{z=\alpha}$ is analytic in the region enclosed by γ_2 in the diagram. For $\alpha=0$, the even and odd parts of the principal part under $z\mapsto -z$ are denoted by $[r(z)]^+_{z=0}$, respectively $[r(z)]^-_{z=0}$.

Theorem 7.7. *The recursion* (7) *satisfied by* $V^\Theta_{g,n}(L_1,...,L_n)$ *is equivalent to topological recursion applied to the spectral curve*

$$S=\left(\mathbb{C},x=\frac{1}{2}z^2,y=\frac{\cos(2\pi z)}{z},B=\frac{dzdz'}{(z-z')^2}\right).$$

In particular, S *has correlators*

$$\omega_{g,n}=\frac{\partial}{\partial z_1}...\frac{\partial}{\partial z_n}\mathcal{L}\{V^\Theta_{g,n}(L_1,...,L_n)\}dz_1...dz_n$$

and partition function

$$Z_S(\hbar,\{t_k\})=\exp\sum_{g,n}\frac{\hbar^{g-1}}{n!}V^\Theta_{g,n}(L_1,...,L_n)|_{\{L_i^{2k}=2^kk!t_k\}}.$$

Proof. The proof is analogous to the proof of Theorem 7.6 by Eynard and Orantin in [19]. It is rather technical so we will give the key idea here. Topological recursion applied to the spectral curve S is related to the recursion (7) by the Laplace transform, and in particular there is a one-to-one correspondence between terms in each of the two recursions. Lemmas 7.8 and 7.9 are the main new ideas in the proof, enabling the calculation of the Laplace transform of the recursion (7), while the last part of the proof uses techniques which have arisen previously to relate topological recursion to a variety of recursive structures in geometry.

The Laplace transform of a polynomial $P(x_1, ..., x_n)$ which is defined by

$$\mathcal{L}\{P\}(z_1, ..., z_n) = \int_0^\infty ... \int_0^\infty e^{-(z_1x_1+...+z_nx_n)} P(x_1, ..., x_n)dx_1...dx_n$$

for $Re(z_i) > 0$, is a polynomial in z_i^{-1} hence it extends to a meromorphic function on $\mathbb{C}^n$ with a poles along the divisors $z_i = 0$.

The recursion (7) involves the following two linear transformations

$$P(x, y) \mapsto \int_0^\infty\int_0^\infty D(z, x, y)P(x, y)dxdy, \quad P(z) \mapsto \int_0^\infty R(x, y, z)P(z)dz$$

from the spaces of odd (in each variable) polynomials in one and two variables to the spaces of polynomials in two and one variable. These linear transformations induce linear transformations of the Laplace transforms. The following two lemmas calculate the Laplace transform of these linear transformations.

Lemma 7.8. *For $P(x, y)$ an odd polynomial in x and y:*

$$\mathcal{L}\left\{\int_0^\infty\int_0^\infty dxdyD(L, x, y)P(x, y)\right\} = \left[\frac{1}{\cos(2\pi z)}\mathcal{L}\{P\}(z, z)\right]_{z=0}$$

Proof. By linearity we may choose $P = \frac{x^{2i+1}y^{2j+1}}{(2i+1)!(2j+1)!}$ which has Laplace transform $\mathcal{L}\{P\}(z_1, z_2) = \frac{1}{z_1^{2i+2}z_2^{2j+2}}$. Recall from Section 5.4 that

$$F_{2k+1}(t) = \int_0^\infty x^{2k+1}H(x, t)dx = \sum_{i=0}^{k} t^{2i+1}\binom{2k+1}{2i+1}a_{k-i}$$

where a_n is defined by $\dfrac{1}{\cos(2\pi z)} = \displaystyle\sum_{n=0}^\infty a_n\frac{z^{2n}}{(2n)!}$. Then $D(x, y, z) = H(x, y + z)$ and a change of coordinates gives:

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{x^{2i+1}y^{2j+1}}{(2i+1)!(2j+1)!}D(L, x, y)dxdy &= \frac{F_{2i+2j+3}(L)}{(2i+2j+3)!}\\ &= \sum_{m=0}^{i+j+1}\frac{L^{2m+1}}{(2m+1)!}\frac{a_{i+j+1-m}}{(2i+2j+2-2m)!}.\end{aligned}$$

Hence its Laplace transform is

$$\mathcal{L}\left\{\int_0^\infty\int_0^\infty \frac{x^{2i+1}y^{2j+1}}{(2i+1)!(2j+1)!}D(L, x, y)dxdy\right\} = \sum_{m=0}^{i+j+1}\frac{1}{z^{2m+2}}\frac{a_{i+j+1-m}}{(2i+2j+2-2m)!}$$

which coincides with the even principal part of

$$\frac{1}{\cos(2\pi z)}\mathcal{L}\{P\}(z, z) \sim \sum_{n=0}^\infty a_n\frac{z^{2n}}{(2n)!}\frac{1}{z^{2i+2j+4}}$$

where $\sim$ means the Laurent series at $z = 0$. Note that the principal part is even so we can replace $\left[\frac{1}{\cos(2\pi z)}\mathcal{L}\{P\}(z, z)\right]_{z=0}$ by $\left[\frac{1}{\cos(2\pi z)}\mathcal{L}\{P\}(z, z)\right]_{z=0}^+$ in the statement. □

Lemma 7.9. *For $P(x)$ an odd polynomial:*

$$\mathcal{L}\left\{\int_0^\infty dxR(L_1, L_2, x)P(x)\right\} = \left[\frac{1}{\cos(2\pi z_1)}\frac{\mathcal{L}\{P\}(z_1)}{(z_2 - z_1)}\right]_{z_1=0}^+$$

Proof. Recall that $R(x,y,z)=\frac{1}{2}H(x+y,z)+\frac{1}{2}H(x-y,z)$ and choose $P=x^{2k+1}$. Hence

$$\begin{aligned}\int_0^\infty dxR(L_1,L_2,x)x^{2k+1}&=\frac{1}{2}F_{2k+1}(L_1+L_2)+\frac{1}{2}F_{2k+1}(L_1-L_2)\\&=\sum_{\epsilon=\pm1}\frac{1}{2}\sum_{m=0}^{k}(L_1+\epsilon L_2)^{2m+1}\binom{2k+1}{2m+1}a_{k-m}\\&=(2k+1)!\sum_{m=0}^{k}\sum_{\substack{j\text{ even}\\ i+j=2m+1}}\frac{L_1^iL_2^j}{i!j!}\frac{a_{k-m}}{(2k-2m)!}.\end{aligned}$$

Hence its Laplace transform is:

$$\mathcal{L}\left\{\int_0^\infty dxR(L_1,L_2,x)x^{2k+1}\right\}=(2k+1)!\sum_{m=0}^{k}\sum_{\substack{j\text{ even}\\ i+j=2m+1}}\frac{1}{z_1^{i+1}z_2^{j+1}}\frac{a_{k-m}}{(2k-2m)!}$$

which coincides with the even principal part in z_1 of

$$\frac{1}{\cos(2\pi z_1)}\frac{\mathcal{L}\{x^{2k+1}\}(z_1)}{(z_2-z_1)}\sim\sum_{n=0}^{\infty}a_n\frac{z_1^{2n}}{(2n)!}\sum_{j=0}^{\infty}\frac{z_1^j}{z_2^{j+1}}\frac{(2k+1)!}{z_1^{2k+2}}$$

where $\sim$ means the Laurent series at $z_1=0$ for fixed z_2, hence $|z_1|<|z_2|$. □

Apply Lemmas 7.8 and 7.9 to the recursion (7).

(71)

$$\begin{aligned}\mathcal{L}\left\{L_1V^{\Theta}_{g,n}(L_1,L_K)\right\}=&\frac{1}{2}\mathcal{L}\left\{\int_0^\infty\int_0^\infty xyD(L_1,x,y)P_{g,n}(x,y,L_K)dxdy\right.\\&\left.+\sum_{j=2}^{n}\int_0^\infty xR(L_1,L_j,x)V^{\Theta}_{g,n-1}(x,L_{K\setminus\{j\}})dx\right\}\\=&\frac{1}{2}\Big[\frac{1}{\cos(2\pi z_1)}\mathcal{L}\{xyV^{\Theta}_{g-1,n+1}\}(z_1,z_1,z_K)\\&+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=K}}\mathcal{L}\{xV^{\Theta}_{g_1,|I|+1}\}(z_1,z_I)\mathcal{L}\{yV^{\Theta}_{g_2,|J|+1}\}(z_1,z_J)\Big]^{+}_{z_1=0}\\&+\sum_{j=2}^{n}\int_0^\infty\left[\frac{1}{\cos(2\pi z_1)}\frac{\mathcal{L}\{xV^{\Theta}_{g,n-1}\}(z_1,z_{K\setminus\{j\}})}{z_j-z_1}\right]^{+}_{z_1=0}.\end{aligned}$$

The principal part of the term involving D coincides with its even principal part, as explained in the note at the end of the proof of Lemma 7.8, so we have written it as the even part.

Define $\Omega_{g,n}=(-1)^n\frac{\partial}{\partial z_1}\cdots\frac{\partial}{\partial z_n}\mathcal{L}\{V^{\Theta}_{g,n}(L_1,...,L_n)\}dz_1...dz_n$. (We will prove the equality $\Omega_{g,n}=\omega^S_{g,n}$.) Take $(-1)^{n-1}\frac{\partial}{\partial z_2}\cdots\frac{\partial}{\partial z_n}\Big[(71)\Big]dz_1...dz_n$, noting that $-\frac{\partial}{\partial z_1}$ is already present since $\mathcal{L}\{L_1P(L_1)\} = -\frac{\partial}{\partial z_1}\mathcal{L}\{P(z_1)\}$, to get

$$\begin{aligned}\Omega_{g,n}(z_1,z_K) =& \frac{1}{2}\left[\frac{1}{\cos(2\pi z_1)}\Omega_{g-1,n+1}(z_1,z_1,z_K)\right]^-_{z_1=0}\\ &+\frac{1}{2}\Big[\frac{1}{\cos(2\pi z_1)}\sum_{\substack{g_1+g_2=g\\ I\sqcup J=K}}\Omega_{g_1,|I|+1}(z_1,z_I)\Omega_{g_2,|J|+1}(z_1,z_J)\Big]^-_{z_1=0}\\ &+\sum_{j=2}^n\int_0^\infty\left[\frac{1}{\cos(2\pi z_1)}\frac{\Omega_{g,n-1}(z_1,z_{K\setminus\{j\}})}{(z_j-z_1)^2}\right]^-_{z_1=0}.\end{aligned}$$

The even part of the principal part becomes the odd part $[\cdot]^+ \to [\cdot]^-$ due to the factor of dz_1. The factors xy, x and y on the right hand side of (71) supply derivatives such as $\mathcal{L}\{xyV^\Theta_{g-1,n+1}\}(z_1,z_1,z_K) = \frac{\partial^2}{\partial w\partial z}\mathcal{L}\{V^\Theta_{g-1,n+1}\}(w{=}z_1,z{=}z_1,z_K)$.

Topological recursion for the spectral curve S is

$$\begin{aligned}\omega_{g,n}(z_1,z_K) =& \operatorname*{Res}_{z=0} K(z_1,z)\mathcal{F}(\{\omega_{g',n'}(z,z_K)\})dzdzdz_K\\ =& -\frac{1}{2}\operatorname*{Res}_{z=0}\left(\frac{dz_1}{z_1-z}-\frac{dz_1}{z_1+z}\right)\frac{1}{2\cos(2\pi z)}\mathcal{F}(\{\omega_{g',n'}(z,z_K)\})dzdz_K\\ =& -\frac{1}{2}\left[\frac{1}{\cos(2\pi z_1)}\mathcal{F}(\{\omega_{g',n'}(z_1,z_K)\})dz_1dz_K\right]^-_{z_1=0}\end{aligned}$$

where $\mathcal{F}(z_1,z_K)$ is a rational function given explicitly in (67) by

$$\begin{aligned}\mathcal{F}(z_1,z_K)dz_1^2dz_K =& \omega_{g-1,n+1}(z,-z,p_L)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=L}}^{\text{stable}}\omega_{g_1,|I|+1}(z,z_I)\,\omega_{g_2,|J|+1}(-z,z_J)\\ &+\sum_{j=2}^n\big(\omega_{0,2}(z,z_j)\,\omega_{g,n-1}(-z,z_{K\setminus\{j\}})+\omega_{0,2}(-z,z_j)\,\omega_{g,n-1}(z,z_{K\setminus\{j\}})\big)\\ =& -\omega_{g-1,n+1}(z,z,p_L)-\sum_{\substack{g_1+g_2=g\\ I\sqcup J=L}}^{\text{stable}}\omega_{g_1,|I|+1}(z,z_I)\,\omega_{g_2,|J|+1}(z,z_J)\\ &-\sum_{j=2}^n\big(\omega_{0,2}(z,z_j)-\omega_{0,2}(-z,z_j)\big)\omega_{g,n-1}(z,z_{K\setminus\{j\}})\end{aligned}$$

where we have used skew-symmetry of $\omega_{g,n}$ under $z_i\mapsto -z_i$, except for $\omega_{0,2}$. Hence

$$\begin{aligned}\omega_{g,n}(z_1,z_K) =& \frac{1}{2}\left[\frac{1}{\cos(2\pi z_1)}\omega_{g-1,n+1}(z_1,z_1,z_K)\right]^-_{z_1=0}\\ &+\frac{1}{2}\Big[\frac{1}{\cos(2\pi z_1)}\sum_{\substack{g_1+g_2=g\\ I\sqcup J=K}}^{\text{stable}}\omega_{g_1,|I|+1}(z_1,z_I)\omega_{g_2,|J|+1}(z_1,z_J)\Big]^-_{z_1=0}\\ &+\sum_{j=2}^n\int_0^\infty\left[\frac{1}{\cos(2\pi z_1)}\frac{\omega_{g,n-1}(z_1,z_{K\setminus\{j\}})}{(z_j-z_1)^2}\right]^-_{z_1=0}.\end{aligned}$$

where we have used $[\omega_{0,2}(-z,z_j)\eta(z)]^-_{z=0} = -[\omega_{0,2}(z,z_j)\eta(z)]^-_{z=0}$ for $\eta(z)$ odd.

The rational differentials $\Omega_{g,n}$ and $\omega_{g,n}$ are uniquely determined by their respective recursions and the initial value

$$\Omega_{1,1}(z_1)=-\frac{\partial}{\partial z_1}\mathcal{L}\{V^{\Theta}_{1,1}(L_1)\}dz_1=-\frac{\partial}{\partial z_1}\mathcal{L}\{\tfrac{1}{8}\}dz_1=\frac{dz}{8z^2}=\omega_{1,1}(z_1)$$

which both coincide, hence $\Omega_{g,n}=\omega_{g,n}$ as required.

The partition function uses $\xi_k=(2k-1)!!\frac{dz}{z^{2k}}$ defined in (69)

$$Z^S(\hbar,\{t_k\})=\exp\sum_{g,n}\frac{\hbar^{g-1}}{n!}\omega^S_{g,n}\Big|_{\xi_k(z_i)=t_k}=\exp\sum_{g,n}\frac{\hbar^{g-1}}{n!}V^{\Theta}_{g,n}(L_1,...,L_n)|_{\{L_i^{2k}=2^kk!t_k\}}.$$

□

Remark 7.10. Rewrite the expression for $F^M_{2k+1}(t)=\int_0^\infty x^{2k+1}H^M(x,t)dx$ due to Mirzakhani as:

$$\frac{F^M_{2k+1}(t)}{(2k+1)!}=\sum_{i=0}^{k+1}\zeta(2i)(2^{2i+1}-4)\frac{t^{2k+2-2i}}{(2k+2-2i)!}=\sum_{i=0}^{k+1}\frac{2\pi}{\sin(2\pi x)}\frac{t^{2k+2-2i}}{(2k+2-2i)!}.$$

Using this, one can replace $D(x,y,z)$ and $R(x,y,z)$ by $D^M(x,y,z)$ and $R^M(x,y,z)$ and replace $\frac{1}{\cos(2\pi z)}$ with $\frac{2\pi}{\sin(2\pi z)}$ in the statements of Lemmas 7.8 and 7.9. The proofs of these new statements appear in the appendix of [19], using a different approach. The viewpoint here shows that the spectral curve $x=\frac{1}{2}z^2$, $y=\frac{\sin(2\pi z)}{2\pi}$ studied by Eynard and Orantin in [19] is implicit in Mirzakhani's work.

Theorem 7.7 is rather interesting since the recursion (7) arises out of (super) hyperbolic geometry, whereas topological recursion is a type of loop equation. Stanford and Witten [57] tied these together by producing a matrix model related to super JT gravity which gives rise to the spectral curve S and loop equations which coincide with topological recursion.

Theorem 7.7 and the general consequence of topological recursion (68) satisfied by any spectral curve produces another proof of the equation (8)

$$V^{\Theta}_{g,n+1}(2\pi i,L_1,...,L_n)=(2g-2+n)V^{\Theta}_{g,n}(L_1,...,L_n)$$

which was proven in Section 6.2 using pull-back properties of the cohomology classes $\Theta_{g,n}$.

Theorem 7.11. *The recursion* (7) *satisfied by*

$$V^{\Theta}_{g,n}(L_1,...,L_n)=\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\exp\left\{2\pi^2\kappa_1+\frac{1}{2}\sum_{i=1}^{n}L_i^2\psi_i\right\}$$

is equivalent to the following equality with the Brézin-Gross-Witten tau function of the KdV hierarchy

$$Z^{\Theta}(\hbar,t_0,t_1,...)=\exp\sum\frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\prod_{i=1}^{n}\psi_j^{k_i}t_{k_i}=Z^{BGW}(t_0,t_1,...).$$

Proof. Corollary 6.3 uses (7), analogous to Mirzakhani's proof of Theorem 3, to prove $Z^{\Theta}(\hbar,t_0,t_1,...)=Z^{\mathrm{BGW}}(t_0,t_1,...)$ which gives one direction of the proof.

For the converse, we use the following result from [47]. Given a regular spectral curve $S=(\Sigma,x,y,B)$ form the irregular spectral curve $S'=(\Sigma,x,dy/dx,B)$. Then for the decomposition given by (70)

$$Z^S=\hat{R}\hat{T}\hat{\Delta}Z^{\mathrm{KW}}(\hbar,\{v^{k,m+1}\})...Z^{\mathrm{KW}}(\hbar,\{v^{k,D}\})$$

we have

$$Z^{S'}=\hat{R}\hat{T}_0\hat{\Delta}Z^{\mathrm{BGW}}(\hbar,\{v^{k,m+1}\})...Z^{\mathrm{BGW}}(\hbar,\{v^{k,D}\})$$

where $T_0(z)=T(z)/z$ is the shift by 1 between the translations explained in Theorem 6.5. Moreover, if the partition function comes from a CohFT $Z^S=Z_\Omega$ then $Z^{S'}=Z_{\Omega^\Theta}$. This relation is simplified when dx has a single zero, since $R=I$ and it essentially reduces to the shift by 1 between the translations, which is clearly shown in (62) and (63).

Apply this to $S=S_{EO}$ which transforms to S' by

$$x=\frac{1}{2}z^2,\ y=\frac{\sin(2\pi z)}{2\pi}\quad\rightsquigarrow\quad x=\frac{1}{2}z^2,\ \frac{dy}{dx}=\frac{\cos(2\pi z)}{z}.$$

By Theorem 7.6,

$$Z^{S_{EO}}=\exp\left(\sum_{g,n}\frac{\hbar^{g-1}}{n!}\sum_{\vec{k}\in\mathbb{N}^n}\int_{\overline{\mathcal{M}}_{g,n}}\exp(2\pi\kappa_1)\prod_{i=1}^n\psi_i^{k_i}t_{k_i}\right)$$

hence

$$\begin{aligned}Z^{S'}&=\exp\left(\sum_{g,n}\frac{\hbar^{g-1}}{n!}\sum_{\vec{k}\in\mathbb{N}^n}\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\exp(2\pi\kappa_1)\prod_{i=1}^n\psi_i^{k_i}t_{k_i}\right)\\&=\exp\sum_{g,n}\frac{\hbar^{g-1}}{n!}V^{\Theta}_{g,n}(L_1,...,L_n)|_{\{L_i^{2k}=2^k k!t_k\}}.\end{aligned}$$

By Theorem 7.7 this implies that $V^{\Theta}_{g,n}(L_1,...,L_n)$ satisfies the recursion (7). □

Theorem 4 is a consequence of Theorem 7.7 and also a consequence of the shorter proof given in Theorem 7.11. The importance of having both proofs is that together they prove the equivalence statement of Theorem 7.11.

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