# ON THE GLOBAL IDENTIFIABILITY OF CHANNEL IDENTIFICATION PROBLEMS

*Jonathan H. Manton*[1], *Walter D. Neumann*[2], *Paul T. Norbury*[2] *and Yingbo Hua*[1]

[1] Department of Electrical and Electronic Engineering
[2] Department of Mathematics and Statistics
The University of Melbourne, Parkville, Victoria 3052, Australia.
j.manton@ee.unimelb.edu.au

## ABSTRACT

This paper studies the fundamental issue of determining if a linear precoder, such as a filter bank, introduces enough redundancy to enable the receiver to identify the unknown finite impulse response channel. Prior to the work herein, identifiability had only been resolved for specially designed linear precoders for which linear algebra or z-matrix theory sufficed.

## 1. INTRODUCTION

An alternative to using a training sequence to identify a channel is to use a linear precoder to introduce algebraic redundancy into the transmitted signal. A fundamental issue is determining if a linear precoder introduces enough redundancy to enable the receiver to identify the channel. Mathematically this is equivalent to determining if a system of polynomial equations is invertible (or "observable") and is known to be a difficult problem [4]. Prior to the work herein, identifiability had only been resolved for specially designed precoders for which linear algebra or "$z$-matrix theory" sufficed [2, 3], or under extra constraints on the input, such as knowledge of its second order statistics [5].

The main results of this paper are as follows. Section 2 considers the problem of transmitting $p$ complex valued symbols through an FIR channel with $l$ unknown coefficients. The $p$ symbols are linearly mapped to $n+l$ symbols by a linear precoder prior to transmission. This results in $n$ polynomial equations in $p+l$ unknowns. Clearly, any precoder for which $n < p+l$ cannot identify the channel because there are fewer equations than unknowns. Perhaps somewhat surprisingly, almost all precoders for which $n = p+l$ cannot identify the channel either. However, almost all precoders can identify the channel if $n > p+l$. Identical results hold for zero prefix precoders. Section 3 studies the ability of filter banks to identify unknown channels from a polynomial equation perspective.

The notation and background information required for the rest of this paper is described in Sections 1.1 and 1.2.

### 1.1. Problem Formulation and Definitions

This paper studies the following channel identification problem. The $p$ complex valued source symbols $[s_1, \cdots, s_p]$ are to be sent through an unknown FIR channel of length $L$. For convenience, define $l = L-1$. Because a linear precoder can only identify a channel up to a constant scaling factor, it is assumed throughout that the leading channel coefficient is one. Thus the channel taps are $[1, h_1, \cdots, h_l]$ where the $l$ complex valued parameters $h_1, \cdots, h_l$ are to be determined by the receiver. For this to be possible, algebraic redundancy must be introduced prior to transmission. This is done by the precoder which maps the $p$ source symbols to $n+l$ encoded symbols $[x_{1-l}, x_{2-l}, \cdots, x_0, x_1, \cdots, x_n]$. The receiver observes the $n$ symbols $[y_1, \cdots, y_n]$ which are related to the encoded symbols by the convolution

$$y_i = x_i + \sum_{k=1}^{l} x_{i-k} h_k, \qquad i = 1, \cdots, n. \tag{1}$$

This can be written in matrix form as $\boldsymbol{y} = H\boldsymbol{x} = HP\boldsymbol{s}$ where $\boldsymbol{y} \in \mathbb{C}^n$ is the output vector, $\boldsymbol{x} \in \mathbb{C}^{n+l}$ the encoded vector, $\boldsymbol{s} \in \mathbb{C}^p$ the source vector, $H \in \mathbb{C}^{n \times (n+l)}$ the Toeplitz channel matrix

$$H = \begin{bmatrix} h_l & h_{l-1} & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & h_l & h_{l-1} & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & h_l & h_{l-1} & \cdots & 1 \end{bmatrix} \tag{2}$$

and $P \in \mathbb{C}^{(n+l) \times p}$ the precoder matrix. It is convenient to define the channel vector $\boldsymbol{h} = [h_1, \cdots, h_l]^T \in \mathbb{C}^l$.

Two subclasses of linear precoders are given particular attention in this paper. A **zero prefix precoder** is a precoder whose first $l$ rows are zero. Such a precoder sets the initial state of the channel to zero, that is, $x_{1-l} = \cdots = x_0 = 0$. If a zero prefix precoder is used, the channel equations can be written more compactly as $\boldsymbol{y} = HP\boldsymbol{s}$ where now $H \in \mathbb{C}^{n \times n}$ is the truncated channel matrix

$$H = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ h_1 & 1 & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ h_l & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \ddots & 0 \\ 0 & \vdots & 0 & h_l & \cdots & h_1 & 1 \end{bmatrix} \tag{3}$$

and $P \in \mathbb{C}^{n \times p}$ the truncated precoder matrix obtained by omitting the first $l$ rows of the original $P$.

The other type of linear precoder studied here is the **filter bank precoder** [3]. A filter bank precodes the infinite source sequence $\{\cdots, s_{-1}, s_0, s_1, \cdots\}$ by breaking it up into blocks of size

This work was supported by the Australian Research Council.

$r$ and linearly mapping each block to one of size $m$. Thus a filter bank can be represented by the infinite precoder matrix $P$ having the block diagonal form

$$P = \begin{bmatrix} \ddots & 0 & 0 & \ddots \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ \ddots & 0 & 0 & \ddots \end{bmatrix} \tag{4}$$

where $B \in \mathbb{C}^{m \times r}$ is the precoder matrix which maps each block of $r$ source symbols to a block of $m$ encoded symbols.

It is convenient to write $\boldsymbol{y} = HP\boldsymbol{s}$ as $\boldsymbol{y} = F(\boldsymbol{s}, \boldsymbol{h})$ where $F : \mathbb{C}^{p+l} \to \mathbb{C}^n$ is the corresponding polynomial map.

**Example 1** Let $l = 1$, $n = 6$, $p = 4$ and let $P$ be the zero prefix precoder mapping $\boldsymbol{s}$ to $[s_1, s_2, 0, s_3, s_4, 0]$. Then $F$ consists of the following equations:

$$y_1 = s_1 \qquad y_2 = s_2 + h_1 s_1 \qquad y_3 = h_1 s_2 \tag{5}$$

$$y_4 = s_3 \qquad y_5 = s_4 + h_1 s_3 \qquad y_6 = h_1 s_4 \tag{6}$$

Solving (5) for $h_1$ gives $y_1 h_1^2 - y_2 h_1 + y_3 = 0$, which in general has two solutions, not one. Solving (6) gives the extra equation $y_4 h_1^2 - y_5 h_1 + y_6 = 0$. Although this also has two solutions, in general there will only be one solution in common.

The above example illustrates a number of important points. For instance, the number of solutions depends on the actual source vector $\boldsymbol{s}$. In the extreme case $\boldsymbol{s} = [0, 0, 0, 0]$ there will be an infinite number of solutions. If $\boldsymbol{s} = [1, 2, 1, 2]$, there will be two solutions since (5) and (6) become identical. This is resolved below by stating that any system of polynomial equations has a generic number of solutions. Another important point is that there might be a finite number of solutions rather than a unique solution.

The following theorem is of paramount importance to this paper. Not only does it show that any system of polynomial equations has a generic number of solutions, it shows that the exceptional set for which there are a non-generic number of solutions is "very small". Note that for clarity, the variable $z \in \mathbb{C}^{p+l}$ is introduced to represent both $\boldsymbol{s}$ and $\boldsymbol{h}$, that is, $z = (\boldsymbol{s}, \boldsymbol{h})$.

**Theorem 2** *Let $F : \mathbb{C}^{p+l} \to \mathbb{C}^n$ be a polynomial map and define $N(z)$ to be the cardinality of the set $\{\tilde{z} : F(\tilde{z}) = F(z)\}$. There exists a cardinal $N$ and a non-zero polynomial $g : \mathbb{C}^{p+l} \to \mathbb{C}$ such that for any $z$ satisfying $g(z) \neq 0$ the equality $N = N(z)$ holds.*

The $N$ in Theorem 2 is the **generic number of solutions** of $F$; the generic number of solutions of $F(z) = z^2$ is two.

**Definition 3 (Generic)** *A property is said to hold for generic $t \in \mathbb{C}^n$ if there exists a non-zero polynomial $g$ such that the property holds for all $t$ satisfying $g(t) \neq 0$.*

The condition $g(t) \neq 0$ is very strong. For instance, the set $\{t \in \mathbb{C}^n : g(t) \neq 0\}$ is dense in $\mathbb{C}^n$ (under the usual topology). Moreover, if $t$ is chosen at random then $g(t) \neq 0$ with probability one. Here, **at random** is taken to mean that the random variable $t$ is absolutely continuous with respect to Lebesgue measure.

**Definition 4 (Invertible)** *A polynomial map $F$ is (rationally) invertible if for generic $z$ the equation $F(\tilde{z}) = F(z)$ has only one solution, namely $\tilde{z} = z$.*

A precoder which is capable of identifying an unknown channel is thus a precoder for which the corresponding system of equations $F$ is invertible. Analogously to the traditional training sequence based channel identification which requires a "persistently exciting" sequence, the term "exciting" is used here to describe precoders which enable the identification of unknown channels.

**Definition 5 (Strongly Exciting)** *A precoder $P$ is strongly exciting of order $l$ if the polynomial map $F(\boldsymbol{s}, \boldsymbol{h}) = HP\boldsymbol{s}$ is invertible, where $H$, defined in (2), represents an unknown FIR channel of length $l + 1$.*

A precoder which can identify a channel up to a finite number of possibilities is said to be weakly exciting.

**Definition 6 (Weakly Exciting)** *A precoder $P$ is weakly exciting of order $l$ if the polynomial map $F(\boldsymbol{s}, \boldsymbol{h}) = HP\boldsymbol{s}$ generically has a finite number of solutions, where $H$, defined in (2), represents an unknown FIR channel of length $l + 1$.*

Prop. 7 below provides a convenient test for determining if a precoder is weakly exciting. It is based on the **Jacobian matrix** $J$ of $\boldsymbol{y} = F(\boldsymbol{s}, \boldsymbol{h}) = HP\boldsymbol{s}$ which is defined to be the $n \times (p + l)$ matrix

$$J = \begin{bmatrix} \frac{\partial y_1}{\partial h_1} & \cdots & \frac{\partial y_1}{\partial h_l} & \frac{\partial y_1}{\partial s_1} & \cdots & \frac{\partial y_1}{\partial s_p} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial y_n}{\partial h_1} & \cdots & \frac{\partial y_n}{\partial h_l} & \frac{\partial y_n}{\partial s_1} & \cdots & \frac{\partial y_n}{\partial s_p} \end{bmatrix}. \tag{7}$$

The following proposition is a restatement of the fact that a system of polynomial equations has a finite number of solutions generically if and only if its Jacobian matrix has full column rank.

**Proposition 7** *The precoder $P$ is weakly exciting if and only if there exists a point $(\boldsymbol{s}, \boldsymbol{h})$ at which the corresponding Jacobian matrix $J$, defined in (7), has full column rank.*

Unfortunately no straightforward test exists for determining if a general system of polynomial equations is invertible. The remainder of this paper essentially studies the invertibility of polynomial maps of the form $F(\boldsymbol{s}, \boldsymbol{h}) = HP\boldsymbol{s}$. First though the behaviour of solutions of polynomial equations is described.

### 1.2. Behaviour of Solutions of Polynomial Equations

This section describes the solution set $\{\tilde{z} : F(\tilde{z}) = F(z)\}$ of a polynomial map $F : \mathbb{C}^{p+l} \to \mathbb{C}^n$ as $z$ is varied. Throughout, $\|\cdot\|$ denotes the Euclidean norm and $B(z; r)$ denotes the open ball centred at $z$ with radius $r$.

Assume $F$ generically has a finite number $N$ of solutions (see Theorem 2). Then a *non-generic point* is distinguished by the occurrence of any of the following behaviour.

A **branch point** is a point $z$ at which two or more solutions become one. The map $F(z) = z^2$ has a branch point at $z = 0$ since the two solutions $\tilde{z} = \pm z$ of $F(\tilde{z}) = F(z)$ become one as $z \to 0$. At a branch point, the Jacobian matrix is singular. (The converse need not be true.) In the above example, the Jacobian matrix is $[2z]$ which is singular at $z = 0$.

A **strongly degenerate point** is a point $z$ at which there are an infinite number of solutions. The point $z = (1, 0)$ of the invertible map $F(z) = (z_1 z_2, z_2)$ is strongly degenerate because there are an infinite number of solutions to $z_1 z_2 = 0$, $z_2 = 0$.

A **weakly degenerate point** is a point $z$ at which there are more than $N$, but still a finite number of, solutions. Weakly degenerate points only exist if $F$ is over-determined, that is, there are more equations than unknowns. The invertible map $F(z_1, z_2) = \left((z_1 + z_2)^2, (z_1 - z_2)^2, z_2\right)$ has two solutions at the point $z = (1, 0)$, namely $z_1 = \pm 1$, $z_2 = 0$.

A **solution at infinity** of the equation $F(\tilde{z}) = F(z)$ is a sequence $\{\tilde{z}_k\}_{k=1}^{\infty}$ which diverges to infinity (that is, $\|\tilde{z}_k\| \to \infty$) yet $F(\tilde{z}_k) \to F(z)$. Take $F(z_1, z_2) = (z_1 z_2, z_2(z_2 - 1))$ which generically has two solutions. The equations $z_1 z_2 = 1$, $z_2(z_2 - 1) = 0$ appear to only have one solution $z_1 = 1$, $z_2 = 1$. The other solution is hiding at infinity: $F\left(\left(k, \frac{1}{k}\right)\right) = \left(1, \frac{1}{k}\left(\frac{1}{k} - 1\right)\right) \to (1, 0)$.

A point which does not exhibit any of the above behaviour is called a generic point.

**Definition 8 (Generic Point)** *A point $z$ of a polynomial map $F$ which generically has a finite number of solutions is a generic point if there does not exist a solution $\tilde{z}$ of $F(\tilde{z}) = F(z)$ which is a branch point or a weakly or strongly degenerate point, and if there does not exist a solution at infinity of the equation $F(\tilde{z}) = F(z)$.*

The following theorem connects the two concepts of generic point and generic number of solutions.

**Theorem 9** *Let $F : \mathbb{C}^{p+l} \to \mathbb{C}^n$ be a polynomial map which generically has a finite number $N$ of solutions. Then the set $\Omega \subset \mathbb{C}^{p+l}$ of all generic points of $F$ has the following properties. It is open in $\mathbb{C}^{p+l}$. There exists a non-zero polynomial $g : \mathbb{C}^{p+l} \to \mathbb{C}$ such that $g(z) \neq 0$ implies $z \in \Omega$. Moreover, for any $z \in \Omega$ there are precisely $N$ distinct solutions $\tilde{z}$ of $F(\tilde{z}) = F(z)$.*

**Remark:** A point at which $F$ has $N$ solutions need not be a generic point due to the possible occurrence of both a weakly degenerate solution (which adds an extra solution) and a solution at infinity (which removes a solution).

**Corollary 10** *A polynomial map $F : \mathbb{C}^{p+l} \to \mathbb{C}^n$ is invertible if and only if there exists a point $z$ such that the following three criteria are satisfied. There is a unique solution $\tilde{z}$ of $F(\tilde{z}) = F(z)$. The Jacobian matrix of $F$, defined in (7), has full column rank at $z$. There does not exist a sequence $\left\{\tilde{z}_k \in \mathbb{C}^{p+l}\right\}_{k=1}^{\infty}$ for which both $\|\tilde{z}_k\| \to \infty$ and $F(\tilde{z}_k) \to F(z)$ hold.*

The following is an alternative to Corollary 10.

**Corollary 11** *A polynomial map $F : \mathbb{C}^{p+l} \to \mathbb{C}^n$ generically has $N$ solutions if and only if there exists an open set $\Omega \subset \mathbb{C}^{p+l}$ such that for every $z \in \Omega$ there are precisely $N$ solutions of $F(\tilde{z}) = F(z)$.*

The following lemma states that the set of solutions is well behaved at generic points.

**Lemma 12** *Let $F$ be a polynomial map which generically has $N$ solutions. Let $z_1$ be a generic point of $F$ and let $z_2, \cdots, z_N$ be the other $N-1$ solutions, that is, $F(z_1) = \cdots = F(z_N)$. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $z \in B(z_1; \delta)$ the set $X = \{\tilde{z} : F(\tilde{z}) = F(z)\}$ has precisely $N$ distinct points, and moreover, the intersection $X \cap B(z_i; \epsilon)$ is non-empty for each $i = 1, \cdots, N$.*

The following lemma can be used to prove that a system of polynomial equations is not invertible. It is only applicable when the number of equations is equal to the number of unknowns for otherwise the point $\boldsymbol{y}$ might be weakly degenerate.

**Lemma 13** *Let $F : \mathbb{C}^{p+l} \to \mathbb{C}^n$ denote a system of polynomial equations with $n = p + l$. If there exists a $\boldsymbol{y}$ for which $F(\boldsymbol{s}, \boldsymbol{h}) = \boldsymbol{y}$ has precisely $N$ distinct solutions then the generic number of solutions is both finite and greater than or equal to $N$.*

## 2. LINEAR PRECODERS

This section elucidates the generic behaviour of linear precoders as well as that of the subclass of zero prefix precoders. The following example illustrates the importance of distinguishing between arbitrary precoders and zero prefix precoders.

**Example 14** Let $P$ be the precoder which maps $\boldsymbol{s}$ to $[0, s_1, s_1]$. Then the output equations are $y_1 = s_1$ and $y_2 = s_1 + h_1 s_1$ which, for $y_1 \neq 0$, have a unique solution. Thus $P$ is strongly exciting of order 1. However, removing the zero prefix can destroy this property; let $P$ now map $\boldsymbol{s}$ to $[s_1, s_1, s_1]$. The output equations are $y_1 = s_1 + h_1 s_1$ and $y_2 = s_1 + h_1 s_1$ which have an infinite number of solutions. This new $P$ is not even weakly exciting.

Prop. 16 below shows that whether or not a precoder is weakly or strongly exciting is a generic property. This means that whether or not a randomly generated matrix $P \in \mathbb{C}^{(n+l)\times p}$ is strongly exciting of order $l$ with probability one depends only on $n$, $p$ and $l$. More intriguing is the fact that a table can theoretically be constructed which maps the triple $(n, p, l)$ to the number $N_{(n,p,l)}$ which specifies in advance how many solutions there will be to the equation $F(\tilde{\boldsymbol{s}}, \tilde{\boldsymbol{h}}) = F(\boldsymbol{s}, \boldsymbol{h})$ if the precoder $P \in \mathbb{C}^{(n+l)\times p}$, source $\boldsymbol{s} \in \mathbb{C}^p$ and channel $\boldsymbol{h} \in \mathbb{C}^l$ are chosen at random.

The key step in the proof is to define the polynomial map $G : \mathbb{C}^{p+l+(n+l)p} \to \mathbb{C}^{n+(n+l)p}$ to be

$$G(\boldsymbol{s}, \boldsymbol{h}, \boldsymbol{p}) = \begin{bmatrix} HP\boldsymbol{s} \\ \boldsymbol{p} \end{bmatrix} \tag{8}$$

where $P \in \mathbb{C}^{(n+l)\times p}$ denotes the matrix whose elements correspond to the elements of $\boldsymbol{p}$ (that is, $\boldsymbol{p} = \operatorname{vec} P$). As in Section 1.1, define $F(\boldsymbol{s}, \boldsymbol{h}) = HP\boldsymbol{s}$. Therefore the equation $G(\tilde{\boldsymbol{s}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{p}}) = G(\boldsymbol{s}, \boldsymbol{h}, \boldsymbol{p})$ is equivalent to the equations $F(\tilde{\boldsymbol{s}}, \tilde{\boldsymbol{h}}) = F(\boldsymbol{s}, \boldsymbol{h})$ and $\tilde{P} = P$. The interpretation is that the receiver knows both $\boldsymbol{y} = HP\boldsymbol{s}$ as well as the precoder matrix $P$. Prop. 16 is then a consequence of the following lemma.

**Lemma 15** *Define $G$ as in (8). If for generic $(\boldsymbol{s}, \boldsymbol{h}, \boldsymbol{p})$ the equation $G(\tilde{\boldsymbol{s}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{p}}) = G(\boldsymbol{s}, \boldsymbol{h}, \boldsymbol{p})$ has $N$ solutions then for generic $P$ the equation $F(\tilde{\boldsymbol{s}}, \tilde{\boldsymbol{h}}) = F(\boldsymbol{s}, \boldsymbol{h})$ has $N$ solutions for generic $(\boldsymbol{s}, \boldsymbol{h})$.*

**Proposition 16** *For fixed dimensions $n$ and $p$ and channel order $l$, let $G$ be the polynomial map defined in (8) and let $N$ be the generic number of solutions of $G$. If $N = 1$ then a generic $P$ is strongly exciting. If $N < \infty$ then a generic $P$ is weakly exciting. If $N = \infty$ then a generic $P$ is not exciting.*

**Remark:** It is clear that Prop. 16 is true for both arbitrary precoders $P \in \mathbb{C}^{(n+l)\times p}$ and zero prefix precoders $P \in \mathbb{C}^{n\times p}$.

Unfortunately Prop. 16 does not imply that if there exists a single precoder $P$ which is strongly exciting then the same is true for generic $P$. The reason is given in Corollary 10; there may be extra solutions hiding at infinity. This is now demonstrated by the following example and lemma.

**Example 17** Choose $p$ and $l$ arbitrarily but set $n = p + l$. The zero prefix precoder $P$ which maps $\boldsymbol{s}$ to $[s_1, 0, \cdots, 0, s_2, \cdots, s_p]$ where there are $l$ zeros after $s_1$ is strongly exciting of order $l$.

**Lemma 18** *For any channel order $l \geq 1$ and number of source symbols $p > 1$ there exists a zero prefix precoder $P \in \mathbb{C}^{n\times p}$ of size $n = p + l$ which is weakly but not strongly exciting of order $l$.*

It follows from Lemma 13, Prop. 16 and Lemma 18 that a randomly chosen precoder is not strongly exciting if $n = p + l$. This result is somewhat surprising because it shows that even though there are the same number of equations as unknowns, there is still not enough information to identify the channel.

**Theorem 19** *For any channel order $l \geq 1$ and number of source symbols $p > 1$, a generic precoder $P \in \mathbb{C}^{(n+l)\times p}$ of size $n = p+l$ is weakly but not strongly exciting. Moreover, the same is true for a generic zero prefix precoder $P \in \mathbb{C}^{n\times p}$.*

**Remark 1:** No precoder with $n < p+l$ is weakly exciting because $n < p + l$ implies there are fewer equations than unknowns.
**Remark 2:** If $n > p+l$ then a generic precoder is weakly exciting because adding extra equations never increases the generic number of solutions.

It might be expected that increasing $n$ or decreasing $p$ will turn a weakly exciting precoder into a strongly exciting one. The following example shows that this is not always true.

**Example 20** Set $l = 1$, $n = 4$ and $p = 3$. Then the zero prefix precoder which maps $\boldsymbol{s}$ to $[s_3, s_1, s_2, s_3]$ can be shown to be weakly exciting; it generically has three solutions. It might then be expected that the zero prefix precoder of size $n = 5$ which maps $\boldsymbol{s}$ to $[s_2, s_3, s_1, s_2, s_3]$ is strongly exciting since now $n > p + l$. However, it is still not strongly exciting; generically it too has three solutions. Note that this is an example of a *cyclic prefix* commonly used in OFDM systems [1].

Theorem 22 below restores intuition by proving that the above behaviour is non-generic; a randomly chosen precoder with $n > p + l$ is strongly exciting. The proof exploits the fact that the last element $x_n$ of the encoded vector $\boldsymbol{x} = P\boldsymbol{s}$ affects only the last element $y_n$ of the output vector $\boldsymbol{y} = H\boldsymbol{x}$. Partition the matrices $H$ and $P$ as follows (it doesn't matter if the full form (2) or the truncated form (3) is used).

$$HP\boldsymbol{s} = \begin{bmatrix} H_1 & 0 \\ \boldsymbol{u}^T & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ \boldsymbol{v}^T \end{bmatrix} \boldsymbol{s} = \begin{bmatrix} H_1 P_1 \boldsymbol{s} \\ \boldsymbol{u}^T P_1 \boldsymbol{s} + \boldsymbol{v}^T \boldsymbol{s} \end{bmatrix} \qquad (9)$$

where $\boldsymbol{u}^T = [0 \cdots 0 \; h_l \cdots h_1]$ and $\boldsymbol{v}^T$ is the last row of $P$. Partition the polynomial map $G$ defined in (8) accordingly.

$$\begin{aligned} G(\boldsymbol{s}, \boldsymbol{h}, \boldsymbol{p}) &= \begin{bmatrix} G_1(\boldsymbol{s}, \boldsymbol{h}, \boldsymbol{p}_1) \\ G_2(\boldsymbol{s}, \boldsymbol{h}, \boldsymbol{v}) \end{bmatrix}, \qquad G_1 = \begin{bmatrix} H_1 \boldsymbol{p}_1 \boldsymbol{s} \\ \boldsymbol{p}_1 \end{bmatrix}, \\ G_2 &= \begin{bmatrix} \boldsymbol{u}^T P_1 \boldsymbol{s} + \boldsymbol{v}^T \boldsymbol{s} \\ \boldsymbol{v} \end{bmatrix}. \end{aligned} \qquad (10)$$

Here, as in (8), $P$ and $P_1$ are such that $\boldsymbol{p} = \operatorname{vec} P$ and $\boldsymbol{p}_1 = \operatorname{vec} P_1$ respectively. Notice that $G_1$ is identical to $G$ in (8) if the precoder $P_1$ were used instead of $P$ in (8). Therefore $G_1$ generically has a finite number of solutions since Theorem 19 and Remark 2 following it ensure that a generic $P_1$ is weakly exciting. The following lemma is required; its proof is based on Corollary 11 and Lemma 12.

**Lemma 21** *Define $G$, $G_1$ and $G_2$ as in (10) and assume that $G_1$ generically has a finite number $N$ of solutions. Let $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{h}}, \bar{\boldsymbol{p}}_1)$ be a generic point of $G_1$, that is, the set*

$$\begin{aligned} X &= \left\{(\boldsymbol{s}, \boldsymbol{h}) : \; G(\boldsymbol{s}, \boldsymbol{h}, \bar{\boldsymbol{p}}_1) = G_1(\bar{\boldsymbol{s}}, \bar{\boldsymbol{h}}, \bar{\boldsymbol{p}}_1)\right\} \qquad (11) \\ &= \{(\boldsymbol{s}_1, \boldsymbol{h}_1), \cdots, (\boldsymbol{s}_N, \boldsymbol{h}_N)\} \end{aligned}$$

*has $N$ distinct points. If there exists a $\bar{\boldsymbol{v}}$ such that $G_2(\boldsymbol{s}_i, \boldsymbol{h}_i, \bar{\boldsymbol{v}}) \neq G_2(\boldsymbol{s}_j, \boldsymbol{h}_j, \bar{\boldsymbol{v}})$ for $i \neq j$ then $G$ is invertible.*

As in Lemma 21, let $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{h}}, \bar{\boldsymbol{p}}_1)$ be a generic point of $G_1$ and define $(\boldsymbol{s}_i, \boldsymbol{h}_i)$ as in (11). It is important to note that $\boldsymbol{s}_i \neq \boldsymbol{s}_j$ for $i \neq j$ because the channel equations are invertible if $\boldsymbol{h}$ is known (that is, $F(\boldsymbol{s}_1, \boldsymbol{h}) = F(\boldsymbol{s}_2, \boldsymbol{h})$ implies $\boldsymbol{s}_1 = \boldsymbol{s}_2$). For each $(\boldsymbol{s}_i, \boldsymbol{h}_i)$ the first element of $G_2$ takes the value $\boldsymbol{u}_i^T \bar{P}_1 \boldsymbol{s}_i + \boldsymbol{v}^T \boldsymbol{s}_i$ (where $\boldsymbol{u}_i$ depends only on $\boldsymbol{h}_i$). These can clearly be made distinct by judicious choice of $\boldsymbol{v}$. Applying Lemma 21 shows that $G$ is invertible, that is, a generic precoder $P$ is strongly exciting.

**Theorem 22** *For any channel order $l \geq 1$ and number of source symbols $p \geq 1$, if $n > p+l$ then a generic precoder $P \in \mathbb{C}^{(n+l)\times p}$ is strongly exciting. Moreover, the same is true for a generic zero prefix precoder $P \in \mathbb{C}^{n\times p}$.*

## 3. FILTER BANKS

A filter bank is an infinitely long precoder with the diagonal block structure in (4). Let $\boldsymbol{s} = \{s_i\}_{i=-\infty}^{\infty}$ be the source sequence, $\boldsymbol{y} = \{y_i\}_{i=-\infty}^{\infty}$ the output sequence and $\boldsymbol{h} = [h_1, \cdots, h_l]$ the channel vector. Then $F : \mathbb{C}^\infty \times \mathbb{C}^l \to \mathbb{C}^\infty$ is defined to be the polynomial map such that $\boldsymbol{y} = F(\boldsymbol{s}, \boldsymbol{h})$. As in Section 1.1, $F(\boldsymbol{s}, \boldsymbol{h}) = HP\boldsymbol{s}$, except here now the channel matrix $H$, precoder $P$, and source $\boldsymbol{s}$ are infinite dimensional.

The definitions of weakly and strongly exciting precoders can be modified to cope with an infinite number of polynomial equations $F$ in an infinite number of unknowns $\boldsymbol{s}$. It is easier though to side-step the issue by taking the following as definitions rather than as theorems: For any filter bank $F$ let $\tilde{\boldsymbol{y}}$ denote a finite (and non-empty) subsequence of $\boldsymbol{y}$ and let $\tilde{F}$ be the finite subset of equations such that $\tilde{\boldsymbol{y}} = \tilde{F}(\tilde{\boldsymbol{s}}, \boldsymbol{h})$ for some finite subsequence $\tilde{\boldsymbol{s}}$ of $\boldsymbol{s}$. A filter bank $F$ is strongly exciting if there exists an $\tilde{F}$ which is invertible. Similarly, $F$ is weakly exciting if there exists an $\tilde{F}$ which has a finite number of solutions for generic $(\tilde{\boldsymbol{s}}, \boldsymbol{h})$.

The results in Section 2 do not automatically hold for filter banks. The reason is because a randomly chosen precoder matrix has zero probability of having a diagonal block structure. Rather than attempt to rederive corresponding results for filter banks, this section studies what are believed to be more pertinent issues.

The following decomposition of $F$ into an infinite sequence of alternating maps plays a key role in this section. Partition the source sequence $\boldsymbol{s}$ into blocks of length $q$ and denote the $i$th block by $\boldsymbol{s}_i$. Similarly, write $\boldsymbol{y} = \{\cdots, \boldsymbol{y}_{-1}, \boldsymbol{y}_0, \boldsymbol{y}_1, \cdots\}$ where the $i$th block $\boldsymbol{y}_i$ has length $q - l$ if $i$ is odd and length $l$ if $i$ is even. Due to the block structure (4) of filter banks, it is always possible to choose $q \geq l$ such that $\boldsymbol{y} = F(\boldsymbol{s}, \boldsymbol{h})$ can be decomposed as

$$\begin{aligned} \cdots, \quad \boldsymbol{y}_1 &= F_1(\boldsymbol{s}_1, \boldsymbol{h}), \quad \boldsymbol{y}_2 = F_2(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{h}), \\ \boldsymbol{y}_3 &= F_1(\boldsymbol{s}_2, \boldsymbol{h}), \quad \boldsymbol{y}_4 = F_2(\boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{h}), \quad \cdots . \end{aligned} \tag{12}$$

If $F$ is weakly exciting then it is possible to choose $q$ large enough so that $F_1$ is weakly exciting, that is, the equation $F_1(\tilde{\boldsymbol{s}_1}, \boldsymbol{h}) = F_1(\boldsymbol{s}_1, \boldsymbol{h})$ has a finite number of solutions for generic $(\boldsymbol{s}_1, \boldsymbol{h})$.

**Definition 23 (Separating Partition)** *The pair $(F_1, F_2)$ is a separating partition of the weakly exciting filter bank $F$ if $F$ can be decomposed as in (12) and $F_1$ is weakly exciting.*

Although $F_1(\boldsymbol{s}_1, \boldsymbol{h}) = HP\boldsymbol{s}_1$ for suitable $H$ and $P$, this extra structure is not exploited below except for the implicit assumption that $\boldsymbol{y}_1 = F_1(\boldsymbol{s}_1, \boldsymbol{h})$ has a unique solution $\boldsymbol{s}_1$ if both $\boldsymbol{y}_1$ and $\boldsymbol{h}$ are known. This assumption is true if $F_1$ is weakly exciting since this implies $P$ has full rank, hence $HP$ is invertible.

Of interest are the two polynomial maps $F_{1,1}$ and $F_{1,2,1}$ defined by

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_3 \end{bmatrix} = F_{1,1}(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{h}) = \begin{bmatrix} F_1(\boldsymbol{s}_1, \boldsymbol{h}) \\ F_1(\boldsymbol{s}_2, \boldsymbol{h}) \end{bmatrix}, \tag{13}$$

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \boldsymbol{y}_3 \end{bmatrix} = F_{1,2,1}(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{h}) = \begin{bmatrix} F_1(\boldsymbol{s}_1, \boldsymbol{h}) \\ F_2(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{h}) \\ F_1(\boldsymbol{s}_2, \boldsymbol{h}) \end{bmatrix}.$$

The two maps in $F_{1,1}$ are coupled by $\boldsymbol{h}$. It will be shown below that this introduces a special structure into its solutions. Moreover, this structure occurs in $F_{1,2,1}$ despite the extra coupling introduced by $F_2$. The following definitions are required to describe this structure.

**Definition 24 (Indistinguishable)** *With respect to a filter bank $F$, two channels $\boldsymbol{h}$ and $\tilde{\boldsymbol{h}}$ are indistinguishable if for all $\boldsymbol{s}$ there exists a $\tilde{\boldsymbol{s}}$ for which $F(\boldsymbol{s}, \boldsymbol{h}) = F(\tilde{\boldsymbol{s}}, \tilde{\boldsymbol{h}})$.*

A dyslexic filter bank is one which cannot distinguish most channels. It is posed as an open problem at the end of this section whether or not weakly exciting yet dyslexic filter banks exist.

**Definition 25 (Dyslexic)** *A filter bank $F$ is dyslexic if there exists a non-zero polynomial $g$ such that for any channel vector $\boldsymbol{h}$ satisfying $g(\boldsymbol{h}) \neq 0$ there exists a channel vector $\tilde{\boldsymbol{h}}$ not equal to $\boldsymbol{h}$ yet indistinguishable from $\boldsymbol{h}$.*

Lemmas 26 and 27 describe the structures of the solutions of $F_{1,1}$ and $F_{1,2,1}$. For convenience the following sets are first defined.

$$X_{121}(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{h}) = \left\{ \tilde{\boldsymbol{h}} : F_{1,2,1}(\tilde{\boldsymbol{s}}_1, \tilde{\boldsymbol{s}}_2, \tilde{\boldsymbol{h}}) = F_{1,2,1}(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{h}) \right\} \tag{14}$$

$$X_{11}(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{h}) = \left\{ \tilde{\boldsymbol{h}} : F_{1,1}(\tilde{\boldsymbol{s}}_1, \tilde{\boldsymbol{s}}_2, \tilde{\boldsymbol{h}}) = F_{1,1}(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{h}) \right\}$$

**Lemma 26** *If $F_1$ is weakly exciting then for generic $(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{h})$ the set $X_{11}(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{h})$ depends only on $\boldsymbol{h}$. Moreover, for generic $\boldsymbol{h}$, $\tilde{\boldsymbol{h}} \in X_{11}(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{h})$ for generic $(\boldsymbol{s}_1, \boldsymbol{s}_2)$ if and only if $\boldsymbol{h}$ and $\tilde{\boldsymbol{h}}$ are indistinguishable with respect to $F_1$.*

**Lemma 27** *If $F_1$ is weakly exciting then for generic $(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{h})$ the set $X_{121}(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{h})$ depends only on $\boldsymbol{h}$. Moreover, for generic $\boldsymbol{h}$, $\tilde{\boldsymbol{h}} \in X_{121}(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{h})$ for generic $(\boldsymbol{s}_1, \boldsymbol{s}_2)$ if and only if $\boldsymbol{h}$ and $\tilde{\boldsymbol{h}}$ are indistinguishable with respect to $F_{1,2,1}$.*

**Remark:** Although Lemmas 26 and 27 appear to be virtually identical, their validity is founded on two quite different reasons. Moreover, it is possible for $F_1$ to be dyslexic but not $F_{1,2,1}$.

The following two corollaries of Lemmas 26 and 27 establish the relationship between invertibility and dyslexia.

**Corollary 28** *Let $(F_1, F_2)$ be a separating partition. Then $F_{1,1}$ is invertible if and only if $F_1$ is not dyslexic.*

**Corollary 29** *Let $(F_1, F_2)$ be a separating partition. Then $F_{1,2,1}$ is invertible if and only if $F_{1,2,1}$ is not dyslexic.*

The following theorem provides an upper bound on the number of elements of $\boldsymbol{y}$ that must be observed before an unknown channel can be identified by the receiver. Its proof is based on Lemma 27.

**Theorem 30** *Let $(F_1, F_2)$ be a separating partition for a weakly exciting filter bank $F$. Define $F_{1,2,1}$ as in (13). Then $F$ is strongly exciting if and only if $F_{1,2,1}$ is strongly exciting (invertible).*

The following is an immediate consequence of Corollary 29.

**Corollary 31** *Let $(F_1, F_2)$ be a separating partition for a weakly exciting filter bank $F$. Define $F_{1,2,1}$ as in (13). Then $F$ is strongly exciting if and only if $F_{1,2,1}$ is not dyslexic.*

**Remark:** A sufficient condition for $F_{1,2,1}$ not to be dyslexic is for $F_1$ not to be dyslexic.

As already mentioned, the above analysis did not fully exploit the structure $\boldsymbol{y} = HP\boldsymbol{s}$. This motivates the following question.

**Open Problem 1:** Does there exist a weakly exciting yet dyslexic filter bank?

A negative answer means every weakly exciting filter bank is also strongly exciting.

## 4. REFERENCES


[1] B. Le Floch, M. Alard, and C. Berrou. Coded orthogonal frequency division multiplex. *Proceedings of the IEEE*, 83(6):982–996, 1995.

[2] H. Liu and X.-G. Xia. Precoding for undersampled antenna array receiver systems. In *Proc. of the 28th Annual Asilomar Conf. on Signals, Systems, and Computers*, pages 1043–1047, November 1997.

[3] A. Scaglione, G. B. Giannakis, and S. Barbarossa. Redundant filterbank precoders and equalizers: Parts I and II. *IEEE Transactions on Signal Processing*, July 1999. To appear.

[4] E. D. Sontag. On the observability of polynomial systems, I: Finite-time problems. *SIAM J. Control and Optimization*, 17(1):139–151, 1979.

[5] M. K. Tsatsanis and G. B. Giannakis. Transmitter induced cyclostationarity for blind channel equalization. *IEEE Transactions on Signal Processing*, 45(7):1785–1794, July 1997.