

# Morse Theory and Cohomology Operations.

## **Abstract**

There is a chain complex generated by the critical points of any Morse function on a manifold  $M$  whose homology is isomorphic to that of  $M$ . When we use more than one Morse function we can obtain operations which coincide with the natural cohomology operations on  $M$ .

## **1 Course Outline.**

In this course we will study the Morse theory of many functions over a manifold. By associating a Morse function over a fixed manifold to each edge of a given graph we can obtain an invariant of the manifold, defined by Betz [1]. The invariant is obtained by counting solutions to the graph flow equations. We will consider the moduli space of such solutions as lying in an appropriate Banach space. We will follow the treatment Schwarz gave of Morse theory for a single function [7]. His methods are based on those of Floer [3, 4]. We will study analysis over a graph, observing when the existent methods over the real line survive and when changes need to be made to these methods. There will be two main differences between the approach here and that of Schwarz (besides his being more thorough). We will consider a generic perturbation of the Morse functions rather than the metric. Also, rather than use a homotopy between Morse functions to obtain a canonical isomorphism between their homologies and homology invariants, we will use the graph invariants. In a sense this approach corresponds to a discontinuous homotopy between functions. Interspersed in the lectures will be course outlines, or updates. Hopefully these will serve as motivation, particularly for the earlier material.

## 2 Morse Functions.

Let  $M$  be a compact manifold and consider a smooth function

$$f : M \rightarrow \mathbf{R}$$

with derivative given by

$$Df : TM \rightarrow \mathbf{R} .$$

A critical point of  $f$  is a point  $p \in M$  such that  $(Df)_p = 0$ . We can define a symmetric bilinear form  $B$ , the Hessian of  $f$ , at the critical point  $p$  by

$$B(X, Y) = \tilde{X}_p(\tilde{Y}(f))$$

where  $X, Y \in T_pM$  and  $\tilde{X}, \tilde{Y}$  are respective extensions of  $X$  and  $Y$  to local vector fields. This is symmetric since

$$\tilde{X}_p(\tilde{Y}(f)) - \tilde{Y}_p(\tilde{X}(f)) = [\tilde{X}, \tilde{Y}]_p(f)$$

and the right hand side vanishes since  $p$  is a critical point. Notice that  $\tilde{X}_p(\tilde{Y}(f)) = X(\tilde{Y}(f))$  is independent of the extension  $\tilde{X}$  while  $\tilde{Y}_p(\tilde{X}(f)) = Y(\tilde{X}(f))$  is independent of  $\tilde{Y}$  so  $B(X, Y)$  depends only on the two vectors as notated. A Morse function is a function whose Hessian, defined at each critical point, is non-degenerate.

*Examples.*

- (i) The height function on the torus.
- (ii) Embed a manifold into a real vector space then project onto a fixed vector. Generically this will be Morse.

The following fundamental result whose proof can be found in [5] underlies the good behaviour of Morse functions.

**Lemma 1 (Morse)** *Let  $p \in M$  be a critical point of the Morse function  $f$ . Then there is a local parametrisation of  $M$  around  $p$  given by  $\{(x_1, \dots, x_d)\}$  where  $p = (0, \dots, 0)$ , so that  $f$  is of the form*

$$f(x_1, \dots, x_d) = \pm x_1^2 \pm \dots \pm x_d^2 \quad \square$$

### 3 The Moduli Space of Graph Flows

Let  $M$  be a closed, compact, smooth Riemannian manifold of dimension  $d$ , and let  $\Gamma$  be an oriented, finite, possibly non-compact, graph with  $m$  edges parametrized by  $[0, 1]$ ,  $(-\infty, 0]$ , and  $[0, \infty)$ . We call these edges “internal”, “incoming”, and “outgoing” respectively. Let these edges be indexed  $\{E_1, \dots, E_m\}$  such that the first  $n$  are noncompact, and the rest are internal. Among the  $n$  noncompact edges the first  $n_1$  are assumed to be incoming, the next  $n_2 = n - n_1$  are assumed to be outgoing. In this section we define the moduli space  $\mathcal{M}(\Gamma, M)$  of “graph flows”.

We begin by defining the notion of an  $M$ -structure for the graph  $\Gamma$ . The space of all such  $M$ -structures will play a significant role in our constructions.

**Definition 1** *Fix an oriented, parameterized graph  $\Gamma$  and a closed Riemannian manifold  $M$  as above. An  $M$ -structure  $\sigma$  on  $\Gamma$  consists of the following:*

1. *A real number  $\ell_i$  associated to each internal edge  $E_i$  of  $\Gamma$ . We think of  $\ell_i$  as the length of  $E_i$ , even though we allow  $\ell_i \leq 0$ .*
2. *A function  $f_i \in C^\infty(M)$  associated to each edge  $E_i$  of  $\Gamma$ . We assume the  $f_i$ 's are distinct.*

The space of all  $M$ -structures will be denoted  $\mathcal{S}(\Gamma, M)$ . Notice that there is a homeomorphism

$$\mathcal{S}(\Gamma, M) \cong \mathbf{R}^{m-n} \times F_m(C^\infty(M))$$

where  $F_m(X) \subset X^m$  is the configuration space of  $m$  distinct points in  $X$ .

For fixed choice of such a structure  $\sigma$ , we are now ready to define the moduli space  $\mathcal{M}_\sigma(\Gamma, M)$  of “ $\Gamma$ -flows in  $M$ ”.

Let  $\gamma : \Gamma \rightarrow M$  be a continuous map, smooth on the edges. For each internal edge  $E_i$  let  $\gamma_i : [0, 1] \rightarrow M$  be the restriction of  $\gamma$  to  $E_i$  composed with the parameterization of  $E_i$  by  $[0, 1]$  given as part of the data of  $\Gamma$ . For the incoming and outgoing edges we define  $\gamma_i : (-\infty, 0] \rightarrow M$  or  $\gamma_i : [0, \infty) \rightarrow M$  similarly.

**Definition 2** *The map  $\gamma$  lies in  $\mathcal{M}_\sigma(\Gamma, M)$  if and only if for each edge  $E_i$  it satisfies the differential equation*

$$d\gamma_i/dt + \ell_i \nabla f_i = 0.$$

For the noncompact edges (i.e the incoming and outgoing edges) in this equation set  $\ell_i = 1$ . Here  $\nabla f_i$  is the gradient vector field. The space  $\mathcal{M}_\sigma(\Gamma, M)$  is topologized as a subspace of  $C^0(\Gamma, M)$ .

We let  $\mathcal{M}(\Gamma, M)$  be the union of the spaces  $\mathcal{M}_\sigma(\Gamma, M)$  where the structures  $\sigma$  vary in  $\mathcal{S}(-, \mathcal{M})$ . The space  $\mathcal{M}(\Gamma, M)$  is topologized so that the natural projection map

$$\pi : \mathcal{M}(\Gamma, M) \rightarrow \mathcal{S}(-, \mathcal{M}) \tag{1}$$

is continuous.

Given  $P \subset \mathcal{S}(-, \mathcal{M})$ , let  $\mathcal{M}_P(\Gamma, M) = \pi^{-1}(P)$ . These spaces will be important in general, but we will restrict ourselves to studying  $\mathcal{M}_\sigma(\Gamma, M)$ , the moduli space associated to a single structure. We will now describe some basic properties of these moduli spaces.

Again, fix a structure  $\sigma \in \mathcal{S}(-, \mathcal{M})$ . This defines a vector of labeling functions of the edges. Let  $f = (f_1, \dots, f_n)$  be the  $n$ -tuple of functions labeling the noncompact edges. Observe that every  $\gamma \in \mathcal{M}_\sigma(\Gamma, M)$  has the property that its restriction to each noncompact edge  $\gamma_i$  is a gradient flow line, so it therefore converges to a critical point, say  $a_i$ , of the function  $f_i$ . Thus  $\gamma$  can be associated to an  $n$ -tuple  $\vec{a} = (a_1, \dots, a_n)$  where  $a_i$  is a critical point of  $f_i$ . For a fixed  $n$ -tuple  $\vec{a}$ , let

$$\mathcal{M}_\sigma(\Gamma, M; \vec{a}) \subset \mathcal{M}_\sigma(\Gamma, M)$$

be the subspace of those  $\gamma \in \mathcal{M}_\sigma(\Gamma, M)$  which converge on the  $i$ th edge to the critical point  $a_i$ .

**Theorem 2** *For a generic choice of structure  $\sigma \in \mathcal{S}(-, \mathcal{M})$ , the moduli spaces  $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$  are manifolds for every  $n$ -tuple of critical points  $\vec{a}$ . The dimension of  $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$  is given by the formula*

$$\begin{aligned} \dim(\mathcal{M}_\sigma(\Gamma, M; \vec{a})) &= \sum_{i=1}^{n_1} [\text{index}(a_i)] - \sum_{i=1}^{n_2} [\text{index}(a_{n_1+i})] - d \cdot n_1 \\ &\quad + d \cdot \dim(H_0(\Gamma, \mathbf{R})) - d \cdot \dim(H_1(\Gamma, \mathbf{R})) \end{aligned}$$

where, as above,  $n_1$  and  $n_2$  are the number of incoming and outgoing edges of  $\Gamma$  respectively. Furthermore an orientation on the manifold  $M$  induces orientations on the moduli spaces  $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$ .

To prove that a point in the solution set is a manifold point we must show that it possesses a neighbourhood of solutions homeomorphic to Euclidean space. As a first approximation we can consider only the first order differences between nearby solutions to a given solution. This amounts to linearising the non-linear ode given by the gradient flow equation. In local coordinates around a point in the interior of an edge of  $\Gamma$  the flow equation is given by

$$d\tilde{x}/dt + g(\tilde{x})\nabla f(\tilde{x}) = 0$$

where  $f$  is the Morse function associated to the edge of  $\Gamma$  and the metric is given by

$$\langle w, v \rangle = w^T g^T(\tilde{x})v, \quad w, v \in T_{\tilde{x}}M .$$

Consider the nearby path given by  $\tilde{x}(t) + \lambda v(t)$  where  $v(t)$  is a vector field along  $\tilde{x}(t)$ . The nearby path satisfies the flow equation when

$$dv(t)/dt + \nabla g(\tilde{x})v(t)\nabla f(\tilde{x}) + g(\tilde{x})\nabla\nabla f(\tilde{x})v(t) = 0$$

equivalently

$$dv(t)/dt + A(t)v(t) = 0 \tag{2}$$

By  $\nabla g(\tilde{x})v(t)$  we mean that each entry in the matrix  $g$  should be sent to its gradient vector then via its inner product with  $v$  returned to a scalar. Similarly we differentiate each term in the vector  $\nabla f$  this way. If we choose the trivialisation of the tangent bundle so that at the critical point  $g = I$  then at the critical point  $A$  coincides with the Hessian of the Morse function.

Equation (2) is the linearised flow equation. The dimension of its space of solutions depends on the limiting values of  $A(t)$  along each edge of  $\Gamma$  together with the particular choice of Morse functions. In a sense that we will describe later for a generic choice of Morse functions the dimension of the space of solutions of (2) depends only on the limiting values of  $A(t)$ . Once we have found the space of solutions to (2) we would like to understand if these integrate to solutions of the flow equation, thus showing that we have a manifold point in the moduli space of solutions. Both these issues require the appropriate setup of Banach spaces. The next few sections will be devoted to this setup.

## 4 Analysis on a graph.

Given a parametrised graph  $\Gamma$ , define

$$C^\infty(\Gamma, \mathbf{R}^n) \subset C^0(\Gamma, \mathbf{R}^n)$$

to be the subspace of continuous functions that are smooth on the edges of  $\Gamma$ . At vertices the functions should have one-sided derivatives to all orders. Define  $L^2(\Gamma, \mathbf{R}^n)$  to be the completion of  $C^\infty(\Gamma, \mathbf{R}^n)$  with respect to the norm defined by integrating the square of the function along  $\Gamma$ . Similarly, define  $W^{1,2}(\Gamma, \mathbf{R}^n)$  using the norm given by the sum of the  $L^2$  norm of the function with the  $L^2$  norm of its derivative. It does not matter that the derivative is not continuous at the vertices.

**Lemma 3 (Sobolev embedding.)**  $W^{1,2}(\Gamma, \mathbf{R}^n) \subset C^0(\Gamma, \mathbf{R}^n)$ .

**Proof.** Let  $\{\phi_i\} \subset C^\infty(\Gamma)$  be a Cauchy sequence in the  $W^{1,2}(\Gamma)$  norm. We will show that the convergence is uniform. First notice that

$$\begin{aligned} |\phi_i(s) - \phi_i(t)| &= \left| \int_t^s \dot{\phi}_i(\tau) d\tau \right| \\ &\leq \|\dot{\phi}_i\|_2 (s-t)^{\frac{1}{2}} \leq M(s-t)^{\frac{1}{2}} \end{aligned}$$

where  $M$  is a bound on the  $W^{1,2}(\Gamma)$  norm of  $\{\phi_i\}$ . This gives both a uniform bound on the constant of continuity and the maximum of the function on any compact subset of  $\Gamma$ . The latter bound follows from the former together with the uniform  $L^2(\Gamma)$  bound. Thus, the sequence  $\{\phi_i\}$  is equicontinuous on compact subsets so converges to a continuous limit.  $\square$

Denote by  $\tilde{\Gamma} \subset \Gamma$  a compact subset obtained by cutting the external edges of  $\Gamma$  off a finite distance from a vertex.

**Lemma 4 (Rellich.)** *The inclusion map*

$$W^{1,2}(\tilde{\Gamma}, \mathbf{R}^n) \hookrightarrow L^2(\tilde{\Gamma}, \mathbf{R}^n)$$

*is compact.*

**Proof.** It follows from the proof of the previous lemma that an element of  $W^{1,2}(\tilde{\Gamma})$  of norm 1 has constant of continuity less than 2,

say, and supremum norm bounded by a constant depending only on  $\tilde{\Gamma}$ . Thus, a sequence of such functions is equicontinuous and so has a uniformly convergent subsequence. Since

$$\|\phi\|_{L^2(\tilde{\Gamma})} \leq c\|\phi\|_\infty$$

where  $c$  depends only on  $\tilde{\Gamma}$ , the uniformly convergent subsequence converges in the  $L^2$  norm as well.  $\square$

## 4.1 The trajectory spaces.

Consider  $\mathbf{R}$  with the differentiable structure obtained by requiring that

$$h : \begin{cases} \overline{\mathbf{R}} & \rightarrow [-1, 1] \\ t & \mapsto t/\sqrt{1+t^2} \end{cases}$$

be a diffeomorphism. Similarly, we wish to put this differentiable structure on the external edges of any parametrised graph  $\Gamma$  so that  $\overline{\Gamma}$  is compact.

Define

$$\begin{aligned} \mathcal{P}_{\Gamma, \tilde{a}}^{1,2} &= \mathcal{P}_{\Gamma, \tilde{a}}^{1,2}(\Gamma, M) \\ &= \{\exp(s) \in C^0(\overline{\Gamma}, \mathbf{R}^n) \mid s \in W^{1,2}(\Gamma, h^*\mathcal{D}), h \in C_{\tilde{a}}^\infty(\overline{\Gamma}, M)\}. \end{aligned}$$

This is a Banach manifold modeled on  $W^{1,2}(\Gamma, \mathbf{R}^n)$ . In [7] it is shown that this Banach manifold contains all of the solutions to the flow equation that we require. It involves showing that the solutions decay rapidly enough along the non-compact edges of  $\Gamma$ .

## 4.2 Proof of Theorem 2.

Let

$$\mathcal{X} = W^{1,2}(\Gamma, \mathbf{R}^n) \quad , \quad \mathcal{Y} = L^2(\Gamma, \mathbf{R}^n) \quad ,$$

$$\mathcal{S} = \{A \in GL(n, \mathbf{R}) \mid A^T = A\} \quad ,$$

$$\mathcal{A} = \{A : \overline{\Gamma} \rightarrow \text{End}(\mathbf{R}^n) \mid A \in C^0(E_i) \text{ for } E_i \in \Gamma, A(\partial\Gamma) \in \mathcal{S}\} \quad .$$

Consider  $F : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})$  given by

$$(F_A s)(t) = \dot{s}(t) + A(t)s(t) \quad .$$

It is easy to see that  $F$  is continuous:

$$\begin{aligned} \|(F_A - F_B)(s)\|_2 &= \left( \int |A - B|s|^2 dt \right)^{\frac{1}{2}} \\ &\leq \|A - B\|_\infty \|s\|_2 \leq \|A - B\|_\infty \|s\|_{1,2} \end{aligned}$$

hence

$$\|F_A - F_B\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})} \leq \|A - B\|_\infty .$$

Before stating the next proposition we need to prove a rather standard lemma.

**Lemma 5** *Let  $\mathcal{X}$ ,  $\mathcal{Y}$  and  $\mathcal{Z}$  be Banach spaces and  $F \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$ ,  $K \in \mathcal{K}(\mathcal{X}, \mathcal{Z})$ , the space of compact operators and  $c > 0$  with*

$$\|x\|_{\mathcal{X}} \leq c(\|Fx\|_{\mathcal{Y}} + \|Kx\|_{\mathcal{Z}}), \text{ for all } x \in \mathcal{X} \quad (3)$$

*Then  $F$  is a semi-Fredholm operator.*

**Proof.** By semi-Fredholm we mean that  $F$  has finite-dimensional kernel and closed range. Notice that the image under  $K$  of any bounded sequence in the kernel has a convergent subsequence which is necessarily Cauchy. The inequality (3) then implies that the subsequence is Cauchy in  $X$ . Thus the unit ball in the kernel of  $F$  is compact, showing that the kernel is finite-dimensional. Now, consider a bounded sequence  $\{x_i\} \subset \mathcal{X}$  such that  $\{Fx_i\}$  is Cauchy in  $\mathcal{Y}$ . Choose a subsequence  $\{x_{i_j}\}$  such that  $\{Kx_{i_j}\}$  is Cauchy in  $\mathcal{Z}$ . It follows from (3) that  $\{x_{i_j}\}$  is Cauchy thus converging to  $x$ , say. Since  $F$  is continuous we have that  $\{Fx_{i_j}\}$  converges to  $Fx$ . In fact, the sequence  $\{x_i\}$  can be arranged to be bounded as follows. By the Hahn-Banach theorem there exists a closed subspace  $\mathcal{X}_0 \subset \mathcal{X}$  satisfying

$$\ker F \oplus \mathcal{X}_0 = \mathcal{X} .$$

Project  $\{x_i\}$  onto  $\{\tilde{x}_i\} \subset \mathcal{X}_0$ . This has to be bounded since otherwise a subsequence of  $\{\tilde{x}_i/\|\tilde{x}_i\|\}$  converges to  $x \in \mathcal{X}_0$  with  $\|x\| = 1$  and  $Fx = 0$  in contradiction to the construction of  $\mathcal{X}_0$ . Thus  $F$  has closed range.  $\square$

**Proposition 6** *For each  $A \in \mathcal{A}$  the map  $F_A$  is Fredholm.*

**Proof.** First we will use a result whose proof can be found in [7]. Given  $A \in \mathcal{A}$  there are constants  $T > 0$ ,  $c(T) > 0$  such that

$$\|s\|_{1,2} \leq c(T) \|F_A s\|_2 \quad \text{for all } s \in \mathcal{X}, \quad s|_{\tilde{\Gamma}} = 0$$

where  $\tilde{\Gamma} \subset \Gamma$  is the compact subset obtained by cutting  $\Gamma$  off at the parameter  $T$  on outgoing edges and  $-T$  on incoming edges. Now, given any  $A \in \mathcal{A}$ , there is a Banach space  $\mathcal{Z}$  and a  $K \in \mathcal{K}(\mathcal{X}, \mathcal{Z})$  with  $c > 0$  satisfying

$$\|x\|_{\mathcal{X}} \leq c(\|F_A\|_{\mathcal{Y}} + \|Kx\|_{\mathcal{Z}}) \quad \text{for all } x \in \mathcal{X} .$$

Let  $T(A)$  be as provided above. Then

$$\int_{\tilde{\Gamma}} |\dot{s} + As|^2 dt \geq \int_{\tilde{\Gamma}} \left( \frac{1}{2} |\dot{s}|^2 - |As|^2 \right) dt .$$

Thus using  $|A(t) \cdot s(t)| \leq \|A(t)\| \cdot |s(t)|$  and setting  $\tilde{c} = \max_{\tilde{\Gamma}} \|A(t)\|$ , we have

$$\int_{\tilde{\Gamma}} |\dot{s} + As|^2 dt \geq \frac{1}{2} \int_{\tilde{\Gamma}} |\dot{s}|^2 - \tilde{c} \int_{\tilde{\Gamma}} |s|^2 dt .$$

Hence there is a  $c > 0$  satisfying

$$\int_{\tilde{\Gamma}} (|s|^2 + |\dot{s}|^2) dt \leq c \int_{\tilde{\Gamma}} (|s|^2 + |\dot{s} + As|^2) dt .$$

Defining a cut-off function  $\beta \in C^\infty(\Gamma, [0, 1])$  with the properties

$$\beta|_{\tilde{\Gamma}} = 1, \quad \beta(t) = 0 \quad \text{for } |t| \geq T + 1,$$

$$\text{and } \dot{\beta}(t) \neq 0 \quad \text{for } |t| \in (T, T + 1)$$

we achieve

$$\begin{aligned} \|s\|_{1,2} &= \|\beta s + (1 - \beta)s\|_{1,2} \leq \|\beta s\|_{1,2} + \|(1 - \beta)s\|_{1,2} \\ &\leq c(\|\beta s\|_2 + \|F_A(\beta s)\|_2 + \|F_A((1 - \beta)s)\|_2) \end{aligned}$$

for a  $c > 0$  large enough. That is

$$\begin{aligned} \|s\|_{1,2} &\leq c(\|\beta s\|_2 + 2\|\dot{\beta}s\|_2 + \|\beta F_A s\|_2 + \|(1 - \beta)F_A s\|_2) \\ &\leq c_1(\|s\|_{L^2(\tilde{\Gamma})} + \|F_A s\|_0) . \end{aligned}$$

By Lemma 4 the operator

$$K : W^{1,2}(\Gamma) \xrightarrow{\text{rest}} W^{1,2}(\tilde{\Gamma}) \xrightarrow{\text{cpt.}} L^2(\tilde{\Gamma}) = \mathcal{Z}$$

is compact so we are in the situation of Lemma 5 and  $F_A$  is semi-Fredholm.

In order to show that  $F_A$  is Fredholm we must study its cokernel. Since  $L^2(\Gamma)$  is a Hilbert space we can identify the cokernel of  $F_A$  with the orthogonal complement of its image. Thus

$$\text{coker } F_A = \{r \in L^2(\Gamma) \mid \langle r, \dot{\phi} + A\phi \rangle = 0 \text{ for all } \phi \in C_0^\infty(\Gamma)\}.$$

We will study the local behaviour of an element of the cokernel. Let  $I \subset \Gamma$  be an open interval in the interior of an edge. Notice that  $A(t)$  is differentiable on  $I = (t_0, t_1)$ , say. If  $r$  lies in the cokernel of  $F_A$  then it satisfies

$$\langle r, \dot{\phi} + A\phi \rangle = 0 \text{ for all } \phi \in C_0^\infty(I).$$

Now  $\phi(t) = \int_{t_0}^t \dot{\phi}(\tau) d\tau$  so

$$\int_I \langle r(t), \dot{\phi}(t) \rangle dt + \int_I \langle A^T(t)r(t), \int_{t_0}^t \dot{\phi}(\tau) d\tau \rangle dt = 0$$

and by Fubini's theorem

$$\int_I \langle r(\tau), \dot{\phi}(\tau) \rangle d\tau + \int_I \int_\tau^{t_1} \langle A^T(t)r(t), \dot{\phi}(\tau) \rangle dt d\tau = 0$$

Thus

$$\int_I \langle r(\tau) - \int_{t_1}^\tau A^T(t)r(t) dt, \dot{\phi}(\tau) \rangle d\tau = 0 \text{ for all } \phi \in C_0^\infty(I).$$

Since  $\dot{\phi}$  has mean zero and the set of such functions is dense in  $L^2(I)$  we have

$$r(\tau) - \int_{t_1}^\tau A^T(t)r(t) dt = \text{constant}.$$

This integral equation supplies us with information about the behaviour of  $r$  in  $I$ . For a start it says that  $r$  is absolutely continuous with derivative equal to the integrand almost everywhere. At points

of continuity of  $A$  the derivative of  $r$  is equal to the integrand. Furthermore, regularity of  $A$  gives regularity of  $r$ . This can be seen as follows. At a point  $\tau_0$  of continuity of  $A$

$$\left| \frac{1}{2\delta} \int_{\tau_0-\delta}^{\tau_0+\delta} A^T(t)r(t)dt - A^T(\tau_0)r(\tau_0) \right| \leq \epsilon M$$

where  $\epsilon = \sup_{(\tau_0-\delta, \tau_0+\delta)} \{|A(t)|, |r(t)|\}$  tends to zero as  $\delta$  tends to zero since  $A(t)$  and  $r(t)$  are continuous at  $\tau_0$ . This shows that the derivative of  $r$  exists there and

$$\dot{r}(\tau_0) = A^T(\tau_0)r(\tau_0) \tag{4}$$

If  $A$  is differentiable in a neighbourhood of  $\tau_0$  then by (4)

$$\ddot{r}(\tau) = (\dot{A}^T(\tau) + A^T(\tau)^2)r(\tau)$$

in that neighbourhood, and so on. Notice that (4) implies

$$\|\dot{r}\|_2 \leq \|A\|_\infty \|r\|_2 < \infty .$$

Since  $r$  need not be continuous at the vertices it doesn't lie in  $W^{1,2}(\Gamma)$ .

Rather than defining  $C^\infty(\Gamma)$  to be functions smooth on the edges of  $\Gamma$  and continuous at the vertices we could have dropped the continuity condition at the edges. (Except for the Sobolev embedding theorem) the analysis goes through as before. We showed above that we can identify the cokernel of  $F_A$  with the kernel of  $F_{-A^T}$  where the domain consists of  $W^{1,2}$  functions not necessarily continuous at the vertices. The same proof now applies to show that the dimension of this kernel is finite. Hence  $F_A$  is Fredholm.  $\square$

### 4.3 Index calculation.

In order to calculate the index of  $F_A$  we will first show that the index depends only on the asymptotic values of  $A$ . This will enable us to choose a specific  $A$  where the kernel and cokernel can be studied directly.

Following Schwarz [7] define

$$\Sigma = F(\mathcal{A}) = \{F_A \in \mathcal{L}(\mathcal{X}; \mathcal{Y}) \mid A \in \mathcal{A}\} \subset \mathcal{F}(\mathcal{X}, \mathcal{Y})$$

and consider the equivalence class of operators

$$\Theta_{F_A} = \{F_B \in \Sigma \mid B(\partial\Gamma) = A(\partial\Gamma)\}, \quad A \in \mathcal{A} .$$

**Lemma 7** *Given any  $F \in \Sigma$ , the class  $\Theta_F$  is contractible within  $\Sigma$  as a subspace of  $\mathcal{F}(\mathcal{X}; \mathcal{Y})$ .*

**Proof.** [7]  $\square$

**Proposition 8** *For  $A \in \mathcal{A}$  we have*

$$\begin{aligned} \text{index } F_A &= \sum_{i=1}^{n_1} \text{index } a_i - \sum_{i=n_1+1}^n \text{index } a_i + dn_1 \\ &+ d \cdot \dim(H_0(\Gamma, \mathbf{R})) - d \cdot \dim(H_1(\Gamma, \mathbf{R})) \end{aligned}$$

where  $\text{index } a_i$  is the number of negative eigenvalues of  $A$  on the end of  $E_i$ .

**Proof.** From Lemma 7 the index map is constant on  $\Theta_{F_A}$ . Thus we may assume that  $A$  is diagonal, zero on a compact subset  $\tilde{\Gamma} \subset \Gamma$  containing all vertices and internal edges, and constant outside of a compact subset of  $\Gamma$  that contains  $\tilde{\Gamma}$ . So for  $A = \text{diag}(\lambda_1(t), \dots, \lambda_d(t))$  we have

$$\dot{s}_i = -\lambda_i(t)s(t), \quad i = 1, \dots, d.$$

We can explicitly solve this system. Since  $\lambda_i(t) = \lambda_i^j$  is constant near infinity along  $E_j \subset \Gamma$  then  $s(t) \sim e^{-\lambda_i t}$  near infinity. Thus,  $s \in W^{1,2}(\mathbf{R}^+, \mathbf{R})$  only when  $\lambda_i^j$  is negative (respectively, positive) when  $E_j$  is incoming (respectively, outgoing). Thus, if the  $i$ th eigenvalue does not satisfy this condition for a single  $j$  then the solution must vanish on  $E_j$  and hence on all of  $\Gamma$ . We see then that the dimension of the kernel is given by the number of  $\lambda_i(t)$  with  $\lambda_i^j > 0$  for  $1 \leq j \leq n_1$  (labels for incoming edges) and  $\lambda_i^j < 0$  for  $n_1 \leq j \leq n$  (labels for outgoing edges).

In the previous section we saw that an element  $r$  of the cokernel satisfies

$$\dot{r} - A^T(t)r = 0$$

by considering smooth functions with support away from the vertices. Now consider  $\phi \in C_0^\infty(\Gamma)$  whose support lies in a neighbourhood of the vertex  $v \in \Gamma$ . We have

$$0 = \int_{\Gamma} \langle r, \dot{\phi} + A\phi \rangle dt$$

$$\begin{aligned}
&= \sum_{\{i:v \in E_i\}} \epsilon_i r_i(v) \phi(v) - \int_{\Gamma} \langle \dot{r}, \phi \rangle dt + \int_{\Gamma} \langle A^T r, \phi \rangle dt \\
&= \sum_{\{i:v \in E_i\}} \epsilon_i r_i(v) \phi(v)
\end{aligned}$$

where  $r_i(v)$  is the limiting value of  $r$  along  $E_i$  and  $\epsilon_i = 1$  when  $E_i$  is incoming and  $-1$  for outgoing. Thus at each vertex  $v$  we have  $\sum_{\{i:v \in E_i\}} \epsilon_i r_i(v)$ . This means that  $r$  is free to be discontinuous up to a codimension 1 condition at each vertex.

Along each edge  $E_j$ , in considering  $-A^T$  we negate  $\lambda_i^j$  and  $r_i \in W^{1,2}(\mathbf{R}^+, \mathbf{R})$  when  $\lambda_i^j$  is positive (respectively, negative) when  $E_j$  is incoming (respectively, outgoing). It is no longer true that if  $r_i$  vanishes along one edge then it vanishes on all of  $\Gamma$ . For each  $i$  we get a contribution to the cokernel from each edge  $E_j$  that is compatible in the sense just described with  $\lambda_i^j$ . The dimension of the cokernel is given by the number of compatible external edges minus 1 for each  $i$ .

In order to calculate

$$\text{index } F_A = \dim \ker F_A - \dim \text{coker } F_A$$

we will change the sign of  $\lambda_i^j$  and observe the change in the index. On an incoming edge  $E_i$  if  $\lambda_i^j$  contributes to the kernel, so it is negative, then  $-\lambda_i^j$  cannot contribute to the cokernel and we lose 1 from the index. If  $\lambda_i^j < 0$  and does not contribute to the kernel then  $-\lambda_i^j$  contributes to the cokernel and we again lose 1 from the index. Similar arguments apply to positive  $\lambda_i^j$  and outgoing edges. Thus

$$\text{index } F_A = \sum_{i=1}^{n_1} \text{index } a_i - \sum_{1=n_1+1}^n \text{index } a_i + \text{constant} \quad (5)$$

Assume there is at least one incoming edge. If  $\lambda_i^j > 0$  for all  $i, j$  then the kernel is trivial and the cokernel gets a contribution from each incoming edge minus the codimension one condition from the vertices. In fact, each component of  $\Gamma$  imposes a codimension one condition. (If  $\Gamma$  consists of only outgoing edges then all positive eigenvalues will lead to a contribution of  $d$  to the kernel so in a sense contributing  $-d$  to the cokernel. In other words we can proceed as if there is an incoming edge.) The values that a function in the cokernel takes on the interior of the graph are completely determined by the exterior values

when there are no cycles. Each cycle contributes one dimension to the cokernel. This is because the ambiguity in extending a solution from the exterior to the interior can be seen by subtracting two different extensions, or equivalently setting the functions to be zero on the external edges. Then a lone cycle contributes a one-dimensional family of functions in the cokernel obtained by fixing the (constant) value on one internal edge and reading off the values on the other edges by traveling around the cycle. For more than one cycle we set the function to be zero on all edges except those in a particular cycle. Thus

$$\dim \operatorname{coker} F_A = dn_1 - d \cdot \dim(H_0(\Gamma, \mathbf{R})) + d \cdot \dim(H_1(\Gamma, \mathbf{R}))$$

and this supplies us with the constant in (5) so

$$\begin{aligned} \operatorname{index} F_A &= \sum_{i=1}^{n_1} \operatorname{index} a_i - \sum_{i=n_1+1}^n \operatorname{index} a_i + dn_1 \\ &+ d \cdot \dim(H_0(\Gamma, \mathbf{R})) - d \cdot \dim(H_1(\Gamma, \mathbf{R})) \end{aligned}$$

and the proposition follows.  $\square$

#### 4.4 Regularity.

Let  $\mathcal{S}^*$  be the submanifold of the Banach manifold

$$\mathcal{S}(\Gamma, M) \cong \mathbf{R}^{m-n} \times F_m(C^2(M))$$

obtained by requiring that the functions be Morse. In this section we will show that 0 is a regular value of  $F_A$  for a generic subset of the Banach manifold  $\mathcal{S}^*$ .

Now consider the map

$$\Phi : \mathcal{S}^* \times \mathcal{P}_{\Gamma, \bar{a}}^{1,2} \rightarrow \mathcal{P}_{\Gamma, \bar{a}}^2$$

given by associating to an  $M$ -structure,  $\sigma$  its flow equations. The proof that  $\Phi(b)^{-1}(0)$  is generically a manifold consists of four steps.

1. Show that 0 is a regular value of  $\Phi$  so  $Z = \Phi^{-1}(0)$  is a manifold.
2. The map

$$\pi : Z \rightarrow \mathcal{S}^*$$

is Fredholm with the same index as  $\Phi(\sigma)$ .

3. Sard-Smale then gives us a generic set  $\Sigma \subset \mathcal{S}^*$  of regular values of  $\pi$ .
4. Show that for a regular value  $\sigma \in \Sigma$  the map

$$\Phi(\sigma) : \mathcal{P}_{\Gamma, \bar{a}}^{1,2} \rightarrow \mathcal{P}_{\Gamma, \bar{a}}^2$$

has zero as a regular value.

**Lemma 9** *For all  $(\sigma, \gamma) \in Z$ , we have  $D\Phi_{(\sigma, \gamma)}$  is onto. In other words 0 is a regular value of  $\Phi$ .*

**Proof.** Put

$$D\Phi = D_1 + D_2 : \mathbf{R}^{m-n} \times C^2(M)^m \oplus W^{1,2}(\Gamma, \mathbf{R}^d) \rightarrow L^2(\Gamma, \mathbf{R}^d).$$

We have shown already that the complement of the image of  $D_2$  is finite-dimensional and that functions in the complement satisfy a differential equation. By showing that this finite-dimensional space cannot be orthogonal to the image of  $D_1$  we will have shown that  $D\Phi$  is onto. For  $\xi \in \mathbf{R}^{m-n} \times C^2(M)^m$  and  $r \in R(F_A)^\perp$  we have  $\langle D_1 \xi, r \rangle = d\xi(r) = 0$ . We have shown that  $r$  must be continuous along an edge so pick a point where it does not vanish then choose  $\xi$  so that  $\nabla \xi$  is a bump function in the neighbourhood where  $r$  does not change sign. This shows that  $r$  must vanish there so the complement is trivial and  $D\Phi$  is onto.  $\square$ .

The other three steps are given by Proposition 2.24 in [7]. This completes the proof of Theorem 2.  $\square$

**Remarks.**

(i) In the single Morse function case where there is an  $\mathbf{R}$ -action obtained by flowing along the solutions of the gradient equation put

$$\mathcal{M}(a, b) = \mathcal{M}_\sigma(\Gamma, M; a, b) / \mathbf{R} \tag{6}$$

We will call this the moduli space of gradient flows of  $f$  running from  $a$  to  $b$ .

(ii) We are interested only in *connected* graphs so  $\dim(H_0(\Gamma, \mathbf{R})) = 1$ . The more general formula, which follows by additivity, hides the fact that often a moduli space is empty.

(iii) Orientation uses the determinant line bundle over the space of Fredholm operators. This might have to wait.

## 5 Compactification.

Fix a structure  $\sigma$  satisfying the generic property. We will now construct a natural compactification of the space  $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$ . To do this we first recall the natural compactification of the space of gradient flow lines of a Morse function converging to two fixed critical points. We will refer to the space of flow-lines from critical point  $a_i$  to critical point  $b_i$  by  $\mathcal{M}_\sigma(a_i, b_i)$ . The following is a standard result in classical Morse theory.

**Proposition 10** *Let  $\overline{\mathcal{M}}(a, b)$  denote the space of “piecewise flow lines” connecting critical points  $a$  and  $b$ . That is*

$$\overline{\mathcal{M}}(a, b) = \bigcup_{a=a_0 > a_1 > \dots > a_j = b} \mathcal{M}(a, a_1) \times \dots \times \mathcal{M}(a_{j-1}, b) ,$$

where the union is taken over decreasing finite sequences of critical points. (The partial ordering is defined by  $\alpha \geq \beta$  iff  $\mathcal{M}(\alpha, \beta)$  is nonempty.) Then  $\overline{\mathcal{M}}(a, b)$  is compact and contains  $\mathcal{M}(a, b)$  as an open dense subspace.

**Proof.** Consider a sequence of unparametrised gradient flows  $\{[\gamma^j]\}$  of  $f$ , defined on  $M$ , running between the critical points  $a$  and  $b$ . The square brackets denote the fact that we are considering equivalence classes of parametrised gradient flows. Choose a point  $x_j$  on each flow. Since  $M$  is compact there is a subsequence of  $\{x_j\}$  that converges to  $x \in M$ , say. Choose parametrisations for each flow by putting  $\gamma^j(0) = x_j$ . We drop the square brackets since we are working with parametrised flows. Any point on  $\gamma(t)$ , the gradient flow of  $f$  satisfying  $\gamma(0) = x = \lim_{j \rightarrow \infty} \gamma^j(0)$  is also a limit point  $\lim_{j \rightarrow \infty} \gamma^j(t)$ . This is because if we take a path  $p : [0, 1] \rightarrow M$  satisfying  $p(0) = \gamma_i(0)$  and  $p(1) = \gamma(t_0)$  then the diffeomorphism defined by the flow of  $f$  produces a continuous map  $F : [0, 1] \times [0, t_0] \rightarrow M$  with  $F(1, t) = \gamma$  and  $F(x_i, t) = \gamma_i(t)$  for some  $\{x_i\}$ . Alternatively, the gradient vector field  $\nabla f$  is bounded and uniformly continuous since  $M$  is compact. Since  $d\gamma^j/dt = -\nabla f$  the derivatives  $\{d\gamma^j/dt\}$  are uniformly bounded so  $\{\gamma^j\}$  is an equicontinuous family and thus has a continuous limit  $\gamma(t)$ . By differentiating  $\nabla f$  we can get a uniform  $C^2$  bound on the  $\{\gamma^j(t)\}$  and thus show that  $\gamma(t)$  satisfies the gradient equation. It is not necessarily true that  $\lim_{t \rightarrow -\infty} \gamma(t) = a$  or  $\lim_{t \rightarrow \infty} \gamma(t) = b$ . Set  $c = \lim_{t \rightarrow -\infty} \gamma(t)$ , a critical point of  $f$ .

**Lemma 11** *If  $c \neq a$  then there exists a flow  $[\mu]$  that lies in the limit point set of the sequence  $\{[\gamma^j]\}$  satisfying  $\lim_{t \rightarrow \infty} \mu(t) = a$ .*

We will assume this for the moment to prove the theorem. Apply the lemma first to  $\gamma(t)$  at  $c$  then to  $\mu(t)$  at  $c_1 = \lim_{t \rightarrow -\infty} \mu(t)$  and so on. This forms a strictly increasing sequence of critical points of  $f$  since  $\mathcal{M}(c_{k+1}, c_k) = \phi$  if  $\text{index}(c_{k+1}) \leq \text{index}(c_k)$ . As  $f$  has only finitely many critical points we must have  $c_k = a$  for some  $k$ . The same argument works in traveling down towards  $b$ . Alternatively, we can canonically parametrise  $\{[\gamma^j]\}$  by  $s = f(\gamma^j(t))$ . In this case they each satisfy  $d\gamma^j(s)/ds + \nabla f/|\nabla f| = 0$ . Using a uniform  $C^1$  bound we get a continuous limit. This gets around the use of Lemma 11. We see that the limit of  $\{[\gamma^j]\}$  is an unparametrised piecewise flow.  $\square$

**Proof of Lemma 11.** Since  $c \neq a$  it cannot be a local maximum. This is because there is a neighbourhood of a local maximum which lies entirely inside the unstable manifold of that critical point thus prohibiting  $\gamma(t)$  to lie in the limit point set of the  $\{[\gamma^j]\}$ . Also, by construction,  $c$  is not a local minimum. Thus when we invoke the Morse lemma that expresses  $f$  in local coordinates around  $c$  we see that there are both negative and positive coefficients:

$$f(x_1, x_2, \dots, x_d) = x_1^2 \pm x_2^2 \pm \dots - x_d^2.$$

Assume that the first  $l$  coefficients are positive and the rest negative. We will require that the path  $\gamma(t)$  corresponds to  $x_1 = 0 = x_2 = \dots = x_{d-1}$ . Locally, solutions to the gradient equation are of the form

$$\xi(t) = (C_1 e^{-2t} + C_2 e^{\mp 2t} + \dots + C_d e^{2t}).$$

(This assumes that the metric is Euclidean. In general by taking a small enough neighbourhood we can get close enough to a Euclidean metric so that the present argument works.) If we fix  $t$  and allow the  $C_i$  to depend on a sequence  $\{s_j\}$  that converges to 0 and satisfying  $\xi^{s_j}(t) = [\gamma^j]$  then  $\lim_{j \rightarrow \infty} C_i = 0$  for  $i \neq d$  and  $\lim_{j \rightarrow \infty} C_d$  is non-zero. Now choose  $t = t(s_j)$  so that  $\lim_{j \rightarrow \infty} t(s_j) = -\infty$  and each  $C_i e^{-2t}$  converges, with at least one of these expressions tending to a non-zero limit. By passing to a subsequence we can be assured of doing this since  $(C_1 e^{-2t} + C_2 e^{-2t} + \dots + C_l e^{-2t})$  lies in the compact manifold  $\mathbf{RP}^{l-1}$  and thus this converges projectively. Clearly we can choose

$t = t(s_j)$  to decrease at the right rate to guarantee convergence in  $\mathbf{R}^l$ , moreover in a small enough disk so that the Morse lemma still applies.

Thus there is a point in the stable manifold of  $c$  that lies in the limit point set of  $\{[\gamma^j]\}$ . The gradient flow that passes through this point we will call  $\mu$ .  $\square$

There is a similar compactification for the moduli spaces of  $\Gamma$ -flows. Namely, let

$$\overline{\mathcal{M}}_\sigma(\Gamma, M; \vec{a}) = \bigcup_{\vec{b}} \mathcal{M}_\sigma(\Gamma, M; \vec{b}) \times \overline{\mathcal{M}}_\sigma(b_1, a_1) \times \cdots \times \overline{\mathcal{M}}_\sigma(a_n, b_n).$$

Whether we use  $\overline{\mathcal{M}}_\sigma(b_i, a_i)$  or  $\overline{\mathcal{M}}_\sigma(a_i, b_i)$  in the above union depends on whether the  $i$ th edge is incoming or outgoing. The space  $\overline{\mathcal{M}}_\sigma(\Gamma, M; \vec{a})$  consists of  $\Gamma$ -flows that are allowed to be piecewise flows on the non-compact edges. We refer to these as “piecewise  $\Gamma$ -flows”. There is an obvious way to topologize  $\overline{\mathcal{M}}_\sigma(\Gamma, M; \vec{a})$ .

**Theorem 12** *The space  $\overline{\mathcal{M}}_\sigma(\Gamma, M; \vec{a})$  is compact and contains the space  $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$  as an open dense subspace.*

**Proof.** As above the non-compact edges converge to piece-wise flows. The internal edges remain as true flows since by looking at the limit of a single point on an edge we can reconstruct the limiting edge by flowing according to the prescribed equation. Since the parameter runs from 0 to 1 no critical point will be met. (Unless the entire path is a single critical point.) Again by continuity of the limiting map of the graph, the limit is a map of the graph.  $\square$

We would like to be more precise about this compactification. Further than describing the limit point of each sequence we can describe all sequences that converge to a given limit point. Equivalently we will describe the subset of the moduli space given by a deleted neighbourhood of the points added, or the “ends” of the moduli space.

For example, consider  $S^4$ ,  $\mathbf{CP}^2$  and  $\overline{B^4}$ —different compactifications of  $\mathbf{R}^4$  obtained by respectively adding a point,  $S^2$  and  $S^3$ . A neighbourhood of the point at infinity in  $S^4$  is given by a ball which intersects  $\mathbf{R}^4$  in the complement of a ball. A neighbourhood of the sphere at infinity in  $\mathbf{CP}^2$  is given by a non-trivial complex line bundle

over  $S^2$ . Given a point  $z$  in  $S^2$ , nearby points in  $\mathbf{R}^4 = \mathbf{C}^2$  are pairs of complex numbers  $(w_1, w_2)$  lying in the complement of a ball in  $\mathbf{R}^4$  satisfying  $w_1/w_2 = z$ . A neighbourhood of the boundary of  $\overline{B^4}$  is the product of  $S^3$  with an interval. A point in the complement of a ball in  $\mathbf{R}^4$  represents a vector and it is close to that point in  $S^3$  which describes the direction of the vector. These three examples give a notion of a “larger” compactification. There are sequences in  $\mathbf{R}^4$  that converge to the same point in  $\mathbf{CP}^2$  but different points in  $\overline{B^4}$  whereas the converse cannot be true. In this sense  $\overline{B^4}$  is a larger compactification than  $\mathbf{CP}^2$  and both of these are larger than  $S^4$ . We will see that the compactification of the moduli space of graph flows is quite large indeed. This is reflected by a gluing map from broken flows to real flows leading to a uniqueness property of the ends of the moduli space.

To describe the ends of the moduli space  $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$  we will set up the following notation. For  $n$ -tuples of critical points  $\vec{a}$  and  $\vec{b}$  associated to the structure  $\sigma$ , consider the oriented spaces of flow lines

$$\begin{aligned} \mathcal{M}_i &= \mathcal{M}_{f_i}(b_i, a_i) \text{ for incoming } E_i \\ \mathcal{M}_i &= \mathcal{M}_{f_i}(a_i, b_i) \text{ for outgoing } E_i. \end{aligned}$$

**Theorem 13** *There exist “gluing” maps*

$$\Phi_{\vec{a}, \vec{b}} : \overline{\mathcal{M}}_\sigma(\Gamma, M; \vec{a}) \times \prod_{a_i \neq b_i} \overline{\mathcal{M}}_i \times [0, 1) \rightarrow \overline{\mathcal{M}}_\sigma(\Gamma, M; \vec{b}),$$

*that are orientation preserving homeomorphisms onto disjoint images. Moreover the complement of the images,*

$$\overline{\mathcal{M}}_\sigma(\Gamma, M; \vec{b}) - \bigcup_{\vec{a}} \Phi_{\vec{a}, \vec{b}}$$

*is compact.*

**Proof.** We will begin with this result for a single Morse function on a graph with no vertices. In this case the broken flow consists of a single parametrised flow  $\gamma \in \mathcal{M}(\Gamma, M; a^0, a^1)$  together with a collection of unparametrised flows  $[\gamma^i] \in \mathcal{M}(a^i, a^{i+1})$  where  $-k \leq i \leq l$ . Our strategy will be to use these flows to construct an approximate flow between  $a^{-k}$  and  $a^l$  and show that there is a true flow nearby.

For a path  $\gamma$  running between  $\alpha, \beta \in M$  define

$$\begin{aligned} \mathcal{E}(\gamma) &= \frac{1}{2} \int_{-\infty}^{\infty} (|d\gamma/dt(s)|^2 + |\nabla_{\gamma(s)}(f)|^2) ds \\ &= f(\alpha) - f(\beta) + \frac{1}{2} \int_{-\infty}^{\infty} (|d\gamma/dt(s) + \nabla_{\gamma(s)}(f)|^2) ds \end{aligned}$$

where the first expression shows that  $\mathcal{E}$  is non-negative and the second expression shows that  $\mathcal{E}$  is minimised by the gradient flow. For  $\mathcal{E}$  to make sense we must restrict the paths to satisfy  $\int_{-\infty}^{\infty} |d\gamma/dt(s)|^2 ds < \infty$ .

A broken flow yields a path with small energy—an approximate flow. The implicit function theorem shows that there is a unique true flow nearby. Details can be found in [7]. The same argument goes through for the external edges of graph flows.  $\square$

We will be concerned with the moduli spaces of dimension zero and one,  $\mathcal{M}_\sigma^0(\Gamma, M; \vec{a})$  and  $\mathcal{M}_\sigma^1(\Gamma, M; \vec{a})$ . These theorems tell us that  $\mathcal{M}_\sigma^0(\Gamma, M; \vec{a}) = \overline{\mathcal{M}}_\sigma^0(\Gamma, M; \vec{a})$  is a finite set of points with signs (orientation). Moreover if an end of one of these isolated  $\Gamma$ -flows glues to an isolated flow line, then the pair forms one end of a compact interval of  $\Gamma$ -flows. The other end of this interval is modeled by another such pair.

## 6 Chain complexes.

A chain complex is a sequence of abelian groups

$$C_0 \xrightarrow{\partial_0} C_1 \xrightarrow{\partial_1} \dots \xrightarrow{\partial_{n-2}} C_{n-1} \xrightarrow{\partial_{n-1}} C_n$$

satisfying  $\partial_{k+1} \circ \partial_k = 0$ . We wish to consider the the chain complex generated by the critical points of a Morse function on a compact manifold.

By considering the moduli space of solutions to the gradient flow equation converging to critical points of consecutive degrees we can define a boundary operator on the space of critical points graded by their degree. The boundary operator is defined by counting points

in the moduli space. We can show that the moduli space is a zero-dimensional compact manifold which ensures that the notion of counting is well-defined. What is startling is that this operator has square zero and thus defines a complex. The *uniqueness* in the gluing construction lies behind the fact that  $\partial^2 = 0$ .

Let  $a$  and  $b$  be critical points of  $f$  of index  $k+1$  and  $k$  respectively. We have shown that  $\mathcal{M}(a, b)$  is a zero-dimensional oriented compact manifold. Thus it makes sense to count the points, with sign, in  $\mathcal{M}(a, b)$ . Put  $n(a, b) = \#\mathcal{M}(a, b)$  and define the linear operator

$$\partial a = \sum n(a, b)b$$

where the sum is over all critical points  $b$  of index  $k$ .

**Lemma 14**

$$\partial^2 = 0 .$$

**Proof.** By linearity

$$\partial \partial a = \sum n(a, b) \partial b = \sum n(a, b) n(b, c) c$$

where the sum is over all critical points  $b$  of index  $k$  and  $c$  of index  $k-1$ . We will show that for fixed  $c$  the sum  $\sum n(a, b) n(b, c) c$  over all intermediate critical points  $b$  vanishes. By Theorem 13 the compactified one-dimensional moduli space  $\overline{\mathcal{M}(a, c)}$  is a manifold with boundary. That is the boundary points, which are piecewise flows, each correspond to a unique edge. Since one-dimensional compact manifolds can only be a finite collection of closed intervals this means that the ends come in pairs. Thus the contributions to  $\partial^2(a)$  come in pairs. This immediately gives the vanishing of each component modulo two.  $\square$

## 7 Cohomology Operations.

Fix a generic structure  $\sigma \in \mathcal{S}(-, \mathcal{M})$  as above. Given the Morse-Smale function  $f_i$ , let  $C_*(M, f_i)$  be the associated Morse-Smale chain complex generated by the critical points, and let  $C^*(M, f_i)$  be the dual cochain complex.

We define a class  $q(\Gamma, M)$  to be an element of the complex

$$\bigotimes_{1 \leq i \leq n_1} C^*(M, f_i) \quad \bigotimes_{n_1+1 \leq i \leq n} C_*(M, f_i)$$

in the following manner. Consider those  $n$ -tuples of critical points  $\vec{a}$  such that  $\dim(\mathcal{M}_\sigma(\Gamma, M; \vec{a})) = 0$ . These spaces contain a finite number of oriented points which can be counted with sign (if  $M$  is oriented—otherwise this is well defined mod 2, and we take coefficients to be  $\mathbf{Z}_2$ ).

**Definition 3**

$$q(\Gamma, M) = \sum \# \mathcal{M}_\sigma(\Gamma, M; \vec{a}) [\vec{a}] \in \bigotimes_{1 \leq i \leq n_1} C^*(M, f_i) \quad \bigotimes_{n_1+1 \leq i \leq n} C_*(M, f_i).$$

Using the gluing theorem above and the definition of the boundary and coboundary operators in the Morse-Smale complex, we will show the following.

**Lemma 15**

$$dq = 0.$$

**Proof.** Extend the boundary operator to products as a derivation.

$$\partial : \bigotimes_{1 \leq i \leq k} C^*(M, f_i) \rightarrow \bigotimes_{1 \leq i \leq k} C^*(M, f_i)$$

$$(a_1, \dots, a_n) \mapsto \Sigma_i(a_1, \dots, \partial_i(a_i), \dots, a_k)$$

where  $\partial_i$  is defined using  $f_i$ . Then if we think of  $q$  as a map

$$q : \bigotimes_{1 \leq i \leq n_1} C_*(M, f_i) \rightarrow \bigotimes_{n_1+1 \leq i \leq n} C_*(M, f_i),$$

the requirement that  $dq = 0$  is equivalent to the requirement that  $q$  and  $\partial$  commute:  $\partial q = q\partial$ . Choose  $\vec{a} = (\vec{b}, \vec{c})$  so that  $\dim \mathcal{M}_\sigma(\Gamma, M; \vec{a}) = 1$ . We have divided  $\vec{a}$  into critical points  $\vec{b}$  corresponding to incoming flows and  $\vec{c}$  corresponding to outgoing flows. Notice that for  $\partial \vec{b} = \Sigma \vec{b}^j$   $\dim \mathcal{M}_\sigma(\Gamma, M; (\partial \vec{b}^j, \vec{c})) = 0$  so  $q(\partial \vec{b}) \in \bigotimes_{n_1+1 \leq i \leq n} C_*(M, f_i)$  makes sense and we can compare it with  $\partial q(\vec{b}) \in \bigotimes_{n_1+1 \leq i \leq n} C_*(M, f_i)$ . The one-dimensional manifold  $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$  is compact with boundary so it is a finite collection of closed intervals. Each boundary point of an interval corresponds to a piecewise graph flow with only one external edge not a true gradient flow. This is the key fact behind the proof. If more than one external edge were to break then the true graph flow inside this piecewise graph flow would lie in a moduli space of negative

dimension, thus contradicting its existence. These boundary piecewise graph flows are paired by the interval they bound. There are three types of components of the one-dimensional manifold  $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$  and thus three types of pairings of piecewise flows. The first type of component consists of an interval whose two ends correspond to incoming piecewise gradient flows. These two piecewise graph flows contribute  $1 - 1 = 0$  to  $q(\partial\vec{a})$ . The second type of component consists of an interval whose two ends correspond to an incoming and outgoing piecewise gradient flow, respectively. These two piecewise graph flows contribute, respectively, 1 to  $q(\partial\vec{a})$  and 1 to  $\partial q(\vec{a})$ . The third type of component consists of an interval whose two ends correspond to outgoing piecewise gradient flows. These two piecewise graph flows contribute  $1 - 1 = 0$  to  $\partial q(\vec{a})$ . Thus  $q(\partial\vec{a}) = \partial q(\vec{a})$  and the lemma is proven.  $\square$

We shall therefore view  $q(\Gamma, M)$  as an element of the associated homology,

$$q(\Gamma, M) \in H^*(M)^{\otimes n_1} \otimes H_*(M)^{\otimes n_2}.$$

We shall now describe four basic examples of these invariants.

Example 1.  $\Gamma =$

In this case  $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$  has dimension zero if and only if  $\vec{a} = (a)$  is a maximum. Thus  $q(\Gamma, M) \in H_d(M)$ , and it can easily be seen to be the fundamental class. (Coefficients should be taken in  $\mathbf{Z}_2$  if  $M$  is not orientable).

Example 2.  $\Gamma =$

In this case  $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$  has dimension  $ind(a_1) + ind(a_2) - d$ , where  $\vec{a} = (a_1, a_2)$ . Thus  $q(\Gamma, M) \in \oplus_q H^q(M) \otimes H^{d-q}(M)$ , which defines an element in  $\oplus_q Hom(H^q(M), H_{d-q}(M))$ . This is the Poincaré duality isomorphism, given by taking the cap product with the fundamental class.

Example 3.  $\Gamma =$

In this case  $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$  has dimension  $ind(a_1) - ind(a_2) - ind(a_3)$ , where  $\vec{a} = (a_1, a_2, a_3)$ . Thus

$$q(\Gamma, M) \in \bigoplus_{r \leq k} H^k(M) \otimes H_r(M) \otimes H_{k-r}(M)$$

and defines an element in  $\bigoplus_{r \leq k} Hom(H^r(M) \otimes H^{k-r}(M), H^k(M))$ . This is the cup product operation.

Example 4.  $\Gamma =$

In this case  $\mathcal{M}_\sigma(\Gamma, M; \vec{a})$  has dimension  $ind(a) - d$ , where  $\vec{a} = (a)$ . Thus  $q(\Gamma, M) \in H^d(M)$ . It is easily seen to be the Euler class (or Stiefel - Whitney class  $w_d$  if  $M$  is not orientable).

We end this section by discussing some basic structure properties of the invariants  $q(\Gamma, M)$ . In particular the following results say that the four examples above can be used to compute the invariant for any graph and these invariants are independent of the choice of metric and  $M$ -structure.

**Proposition 16** *If  $\Gamma_1$  and  $\Gamma_2$  are homotopy equivalent via a homotopy that preserves orientations on their end, then  $q(\Gamma_1, M) = q(\Gamma_2, M)$ .*

**Proof.** By varying the length of a single edge of  $\Gamma$  and showing the invariance of  $q$  under this homotopy the general result follows. Also notice that solutions to a graph flow equation with one edge of zero length are the same as solutions to the graph flow equation for the graph with that edge contracted to a point.

Denote by  $\{\sigma^s\}$  the path of  $M$ -structures where a single internal edge takes on the variable length  $l_s$ . Define

$$\tilde{\mathcal{M}}_{\sigma^s}(\Gamma, M; \vec{a}) = \{\tilde{\gamma} : [0, 1] \times \Gamma \rightarrow M \mid \tilde{\gamma}(s, \cdot) \in \mathcal{M}_{\sigma^s}(\Gamma, M; \vec{a})\} .$$

By setting up the appropriate Banach spaces we can show that for a generic choice of  $M$ -structure the *path* described above is regular. We find that  $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_{\sigma^s}(\Gamma, M; \vec{a})$  is a smooth oriented manifold with boundary. It has dimension one greater than the dimension of the boundary components  $\mathcal{M}_0 = \mathcal{M}_{\sigma^0}(\Gamma, M; \vec{a})$  and  $\mathcal{M}_1 = \mathcal{M}_{\sigma^1}(\Gamma, M; \vec{a})$ .

In the case that  $\tilde{\mathcal{M}}$  is a compact 1-manifold, it forms an oriented cobordism between  $\mathcal{M}_0$  and  $\mathcal{M}_1$ . This gives an isomorphism, on the level of chains, between the two invariants  $q_0$  and  $q_1$  corresponding, respectively, to the zero dimensional moduli spaces  $\mathcal{M}_0$  and  $\mathcal{M}_1$ . This is because a compact 1-manifold must have two endpoints. Either a point in  $\mathcal{M}_0$  is matched with a point in  $\mathcal{M}_1$  or two points in  $\mathcal{M}_0$ , respectively  $\mathcal{M}_1$ , are matched with opposite orientations and thus cancel.

In general  $\tilde{\mathcal{M}}$  will not be compact. In this case the equivalence of the invariants occurs only at the level of homology. We will construct a chain homotopy equivalence

$$\Phi: \bigotimes_{1 \leq i \leq n_1} C^*(M, f_i) \quad \bigotimes_{n_1+1 \leq i \leq n} C_*(M, f_i) \rightarrow \bigotimes_{1 \leq i \leq n_1} C^*(M, f_i) \quad \bigotimes_{n_1+1 \leq i \leq n} C_*(M, f_i)$$

where  $q_0 - q_1 = \partial \circ \Phi - \Phi \circ \partial$ .

The compactness properties of the space of solutions to the time-dependent graph flow equations mimic those of the time-independent flows which we have already studied. We saw that by including piecewise graph flows the moduli space became compact. In the case at hand again we find that the boundary of the one-dimensional moduli space consists of the zero-dimensional moduli spaces as well as piecewise graph flows. If we fix an  $s \in [0, 1]$  then the formal dimension of  $\mathcal{M}_s$  is zero. This, we would expect, should prohibit a sequence of graph flows (and thus a boundary point of the 1-manifold) from degenerating to a piecewise flow since the piecewise flow will consist of a graph flow in a formally negative dimensional space. However, since it is the path of  $M$ -structures that is regular so there might be an  $s$  for which the  $M$ -structure is not regular, at such an  $s$  a formal negative dimension does not obstruct the existence of a solution. Thus compactness fails precisely at the  $M$ -structures that are not regular. (Note that the Morse functions can be chosen to be regular since we are changing only a length. Thus any broken graph flows can only

contain true gradient flows.) Define

$$\Phi: \bigotimes_{1 \leq i \leq n_1} C^*(M, f_i) \bigotimes_{n_1+1 \leq i \leq n} C_*(M, f_i) \rightarrow \bigotimes_{1 \leq i \leq n_1} C^*(M, f_i) \bigotimes_{n_1+1 \leq i \leq n} C_*(M, f_i)$$

by summing over all moduli spaces of formal dimension negative one at the  $M$ -structures that are not regular. So  $\langle \Phi(a_1, \dots, a_{n_1}), (a_{n_1+1}, \dots, a_n) \rangle$  equals the number of solutions of the graph flow equations when

$$\begin{aligned} \dim(\mathcal{M}_{\sigma^s}(\Gamma, M; \vec{a})) &= \sum_{i=1}^{n_1} [\text{index}(a_i)] - \sum_{i=1}^{n_2} [\text{index}(a_{n_1+i})] - d \cdot n_1 \\ &\quad + d \cdot \dim(H_0(\Gamma, \mathbf{R})) - d \cdot \dim(H_1(\Gamma, \mathbf{R})) \\ &= -1 \end{aligned}$$

and we sum over all  $\sigma^s$  that are not regular. Strictly we need to prove that these moduli spaces consist of a finite number of points. Really we need only be concerned with  $\Phi$  defined on those chains that represent an end of the one-dimensional moduli space defined above. Using this perspective we can also introduce an orientation, or sign, in the definition of  $\Phi$ . Then we see that  $q_0$  and  $q_1$  fail to be the same when these new endpoints are introduced. We have  $q_0 - q_1 = \partial \circ \Phi - \Phi \circ \partial$ . This requires a gluing theorem to show that each end is used only once. A codimension argument should show that the moduli spaces of formal dimension less than negative one are never non-empty for a good choice of path.  $\square$

Now let  $\Gamma_1$  and  $\Gamma_2$  be oriented graphs. Let  $\Gamma_{1,2}^{i\#j}$  be the oriented graph obtained by gluing incoming edge  $i$  of  $\Gamma_1$  to outgoing edge  $j$  of  $\Gamma_2$ .

**Proposition 17**

$$q(\Gamma_{1,2}^{i\#j}, M) = q(\Gamma_1, M) \diamond^{i,j} q(\Gamma_2, M),$$

where  $\diamond^{i,j}$  denotes tensorial contraction of cohomology in the  $i$ th coordinate with homology in the  $j$ th coordinate.

**Proof.** We mean to combine the respective  $M$ -structures in the obvious way—assume that the Morse functions associated to the glued

edges are the same. The length we assign to this edge is arbitrary. By Proposition 16 this does not affect the invariant. The reason that we have not included the  $M$ -structures in the statement of the proposition is that we will see in Theorem 21 (which uses the conclusion of this proposition) that the invariant associated to the graph is independent of the choice of  $M$ -structure.

We first notice that for large enough  $l$ , the length of the internal edge of  $\Gamma_{1,2}^{i\#j}$  obtained by the gluing, there is a one-to-one correspondence between solutions of the graph flow equation for  $\Gamma^{i\#j}$  and pairs of solutions for the graphs  $\Gamma_1$  and  $\Gamma_2$  with common critical point at the  $i$ th and  $j$ th edges, respectively. But this exactly describes the desired contraction. By the homotopy invariance proven above we can replace a large  $l$  with any  $l$ .  $\square$

**Corollary 18** *Changing the orientation of a non-compact edge induces the Poincare duality isomorphism on the relevant tensor coordinate of the invariant  $q(\Gamma, M)$ .*

**Proof.** Let  $\Gamma$  be a given graph with outgoing edge  $E$ . Glue the graph of Example 2 to  $\Gamma$  at  $E$  to get  $\Gamma'$ . By Proposition 17  $q_{\Gamma'}$  is the composition of  $q_{\Gamma}$  with the Poincare duality isomorphism. Contract the internal glued edge to a point. By Proposition 16 this does not change the invariant.  $\square$

**Proposition 19** *The graph  $\Gamma$  consisting of an incoming edge a vertex, an outgoing edge and no other edges defines a canonical isomorphism between Morse homologies for different Morse functions.*

**Proof.** Given two Morse functions  $f$  and  $g$ , we will show that the map

$$q_{\Gamma} : H_*^f \rightarrow H_*^g$$

is an isomorphism by constructing an inverse. Use the same graph to get

$$q_{\Gamma} : H_*^g \rightarrow H_*^f .$$

By Proposition 17 the composition

$$q_{\Gamma} \circ q_{\Gamma} : H_*^f \rightarrow H_*^f$$

coincides with the map

$$q_{\Gamma'} : H_*^f \rightarrow H_*^f$$

where  $\Gamma'$  is the graph obtained by gluing the outgoing edge of one copy of  $\Gamma$  with the incoming edge of another copy. By Proposition 16 we can homotope the length of the internal edge of  $\Gamma'$  to zero and remove it so we are left with  $\Gamma$  once more but now the same Morse function  $f$  is associated to each edge. Clearly this map is the identity.  $\square$

**Corollary 20** *The Morse homology  $H_*^f(M)$  of a generic Morse function  $f : M \rightarrow \mathbf{R}$  is isomorphic to the simplicial homology of  $M$ ,  $H_*(M)$ .*

**Proof.** By Proposition 19 we need only confirm  $H_*^f(M) \cong H_*(M)$  for a single generic Morse function  $f$ . Triangulate  $M$  and choose  $f$  to have a critical point of index( $k$ ) in the centre of each  $k$ -simplex and no other critical points. Make the choice also so that there is exactly one gradient flow line from each centre of a simplex to the centre of each face of the simplex. It is easy to see that such an  $f$  exists and that it satisfies the Morse-Smale condition of transversality. The Morse complex based on critical points and boundary operator supplied by gradient flows coincides precisely with the cell complex based on simplices and the boundary operator. Thus the homologies coincide.  $\square$

The following says that  $q(\Gamma, M)$  is indeed an invariant of  $M$ .

**Theorem 21** *The homology class  $q(\Gamma, M)$  does not depend on the choice of structure  $\sigma \in \mathcal{S}(-, \mathcal{M})$ .*

**Proof.** If  $f$  is a Morse function associated to the external edge  $E$  of  $\Gamma$  then we can replace it by the Morse function  $g$  by first gluing the graph of Proposition 19 to  $\Gamma$  at  $E$ . We then contract the internal glued edge to a point. By Propositions 16, 17 and 19 the new invariant is related to the old by the isomorphism of Proposition 19. One by one we can change the Morse functions associated to external edges in this way to pass from one  $M$ -structure to another. For the internal edges we simply use the homotopy invariance result to contract these to points and thus demonstrating independence.  $\square$

**Proposition 22** *The homology class  $q(\Gamma, M)$  does not depend on the choice of metric  $g$ .*

**Proof.** Since the space of metrics is connected we can find curves connecting any two generic metrics that induce chain homotopy equivalences that preserve the  $q(\Gamma, M)$ 's.

For a path of metrics  $\{g^s\}$ , define

$$\tilde{\mathcal{M}}_\sigma(\Gamma, M, g_s; \vec{a}) = \{\tilde{\gamma} : [0, 1] \times \Gamma \rightarrow M \mid \tilde{\gamma}(s, \cdot) \in \mathcal{M}_\sigma(\Gamma, M, g_s; \vec{a})\}$$

where the latter moduli space is taken with respect to the fixed metric  $g_s$ .

As with the argument in Proposition 16 we wish to sum over all the bad metrics to get a chain homotopy equivalence. There is a further complication here in that the boundary operator changes as the metric changes. Thus first we must show that a path of metrics induces an isomorphism between the different homologies obtained from the different boundary operators. In the case of varying  $M$ -structure the Morse functions associated to external edges remained Morse-Smale. Since this is not the case here we have two types of “irregular” graph flows to consider. These are broken flows where either the gradient flow or the graph flow lies in a formally negative dimensional space. In Proposition 16 the gradient flow had to be true. There is much analysis needed for this argument.  $\square$

## 8 Graphs and Symmetry.

The cohomology operations defined by considering solutions to many gradient equations with the same initial data can be thought of as arising from a graph where each edge of a graph is a gradient flow of a Morse function. We can exploit any symmetries of the graph to obtain higher order cohomology operations.

*Examples.*

- (i) Stiefel-Whitney classes.
- (ii) Steenrod squares.

## 9 Current Research.

More generally than using the critical points of a Morse function to describe homology and its algebraic structures we can add other objects in the set of critical points to obtain further structure. On a symplectic manifold we can include periodic orbits of a family of symplectomorphisms with critical points of Hamiltonians and obtain the

“quantum” cohomology of the symplectic manifold. This is isomorphic to the usual homology as a vector space though not as an algebra.

## 10 Physics

In [8] Witten treats the space of differential forms on a Riemannian manifold as a model for supersymmetric quantum mechanics. In this section we will follow his treatment of Morse theory in this context and generalise it to the case of a multiplet of Morse functions.

Acting on forms we have the operators

$$Q_1 = d + d^*, \quad Q_2 = i(d - d^*), \quad H = dd^* + d^*d$$

so  $H$  is the Laplacian. These satisfy the supersymmetry relations.

$$Q_1^2 = Q_2^2 = H, \quad Q_1Q_2 + Q_2Q_1 = 0 \quad (7)$$

We interpret  $p$ -forms as being bosonic or fermionic depending on whether  $p$  is even or odd so the  $Q_i$  are supersymmetry operators which swap bosons and fermions.

Let  $\mathbf{f} = (f_1, \dots, f_n)$  be an  $n$ -tuple of Morse functions over the Riemannian manifold  $M$  and let  $\mathbf{t} = (t_1, \dots, t_n)$  be a vector in  $\mathbf{R}^n$ . Put

$$d_{\mathbf{t}} = e^{-\mathbf{f} \cdot \mathbf{t}} de^{\mathbf{f} \cdot \mathbf{t}}, \quad d_{\mathbf{t}}^* = e^{-\mathbf{f} \cdot \mathbf{t}} d^* e^{\mathbf{f} \cdot \mathbf{t}}$$

and consider the operators

$$Q_{1\mathbf{t}} = d_{\mathbf{t}} + d_{\mathbf{t}}^*, \quad Q_{2\mathbf{t}} = i(d_{\mathbf{t}} - d_{\mathbf{t}}^*), \quad H_{\mathbf{t}} = d_{\mathbf{t}}d_{\mathbf{t}}^* + d_{\mathbf{t}}^*d_{\mathbf{t}}.$$

These still satisfy the algebra (7) for any  $\mathbf{t}$ .

Witten showed that for each critical point of each Morse function  $f_i$  there is a corresponding differential form. This is the differential form whose energy remains finite as  $t_i$  is sent to infinity. Using this correspondence we will show the relationship between the algebra induced by the graph moduli spaces and the algebra of differential forms.

The gradient flow lines which define the boundary operator for the Morse complex have two physical interpretations. One is that they minimise the action of the Lagrangian and the other is that they contribute most to certain  $s$ -matrix elements. This latter interpretation, the WKB approximation, should generalise to the case of graphs and multiplets of Morse functions.

## A Finite-dimensional methods.

In this appendix we will describe the more intuitive finite-dimensional approach to proving the results. First we will start with a description of the stable and unstable manifolds a Morse function associates to its critical points.

Define the stable manifold of a critical point  $a$  of  $f$  to be the set of points  $\mathcal{W}_a^s \subset M$  which lie on a flow line of  $f$  that converges to  $a$ . Similarly, the unstable manifold of  $a$  is the set of points  $\mathcal{W}_a^u \subset M$  that lie on a flow line of  $f$  that originates at  $a$  or converges to  $a$  for negative time.

**Proposition 23** *The stable and unstable manifolds of a critical point are diffeomorphic to disks.*

$$\mathcal{W}_a^s \cong D^{d-\text{index}(a)}, \quad \mathcal{W}_a^u \cong D^{\text{index}(a)} .$$

**Proof.** We will consider only the unstable manifold since by negating the Morse function we can include the stable manifold. As in Section 4.1 define the Banach manifold

$$\mathcal{P}_a^{1,2} = \{ \exp(s) \in C^0([0, \infty], \mathbf{R}^n) \mid s \in W^{1,2}([0, \infty), h^* \mathcal{D}), \\ h \in C^\infty([0, \infty], M), h(\infty) = a \} .$$

Then as before we find that  $F_A$  is a Fredholm operator with index given by the index of the critical point  $a$ . The boundary condition on elements of the cokernel now ensures that  $F_A$  is surjective. It follows that  $\mathcal{W}_a^u$  is a manifold of dimension  $\text{index } a$ . In particular,  $a \in \mathcal{W}_a^u$  is a manifold point. using the  $\mathbf{R}$ -action on  $\mathcal{W}_a^u$  given by reparametrising, we see that  $\mathcal{W}_a^u \cong D^{\text{index}(a)}$ .  $\square$

Before describing the alternative approach to the proof of Theorem 2 we shall give a short description of the Morse theory of a single function  $f$ . In this case the graph  $\Gamma$  has no vertices and one edge which is both incoming and outgoing. For the critical points  $a$  and  $b$  of  $f$  the theorem states that  $\mathcal{M}_\sigma(\Gamma, M; a, b)$  is a manifold and its dimension is given by  $\text{index}(a) - \text{index}(b)$ .

A flow of  $f$  is determined uniquely by a point on the flow so the space  $\mathcal{M}_\sigma(\Gamma, M; a, b)$  is the subset of  $M$  given by the intersection of the unstable manifold of  $a$ ,  $\mathcal{W}_a^u$ , with the stable manifold of  $b$ ,  $\mathcal{W}_b^s$ . When

these two submanifolds are transverse their intersection is a manifold of dimension  $\text{index}(a) - \text{index}(b)$ . This transversality condition known as the ‘‘Morse-Smale transversality property’’ is the generic condition that we will require in the theorem.

The tangent space at a critical point  $a$  possesses a subspace  $V_a$  corresponding to the unstable manifold of that point. To orient the moduli spaces we first choose orientations, at each critical point, for each of these vector spaces. Using  $f$  we can compare these orientations. We choose the orientation on the moduli space  $\mathcal{M}(a, b)$  which combines with the orientation of  $V_b$  to give the orientation of  $V_a$ . Our use of these moduli spaces will be independent of the choices of orientation at each critical point.

**Proof of Theorem 2.** The argument we give is intuitive though not open to generalisation. It uses the fact that we can identify the moduli space as a submanifold of  $M$ .

First consider a graph without internal edges. Since the graph is connected we may assume that there is exactly one vertex. The case of no vertices is the standard Morse theory described above. By putting a Riemannian metric on  $M$ , the vertex uniquely determines the graph since for each Morse function there is a unique gradient flow containing the vertex. The vertex is characterised by the fact that it lies inside the intersection of the stable and unstable manifolds of the critical points. Since the intersection of these manifolds is transverse then it is a manifold with the stated dimension.

The gradient flow of a Morse function produces diffeomorphisms on the manifold. If we introduce an internal edge to a graph then rather than considering the intersection of stable and unstable manifolds we consider the intersection of diffeomorphic copies of stable and unstable manifolds.

We require that the diffeomorphic copies be transverse so the intersection, and thus the moduli space of graphs, is a manifold. The dimension remains the same unless the new edge introduces a cycle. In this case the moduli space is cut down by dimension  $d$ . We can see this as follows. Now

$$\mathcal{M}_\sigma(\Gamma, M; \vec{a}) = \mathcal{W}_{a_1}^s \times \mathcal{W}_{a_2}^s \times \dots \times \mathcal{W}_{a_k}^u \times \dots \times \mathcal{F}(\mathcal{W}_{a_m}^s) \times \dots \cap \Delta(M) \subset M^{n+h_1}$$

where  $\mathcal{F}$  is a diffeomorphism corresponding to an edge of the graph,  $\Delta(M)$  is the diagonal in  $M^{n+h_1}$  and  $h_1 = \dim(H_1(\Gamma, \mathbf{R}))$ . The prod-

uct in this expression has dimension

$$\sum_{i=1}^{n_1} [\text{index}(a_i)] - \sum_{i=1}^{n_2} [\text{index}(a_{n_1+i})] + d \cdot n_2$$

and since  $\Delta(M)$  has codimension  $d + h_1 - 1$  and the intersection is transverse the theorem follows. The key new feature introduced by a cycle is that the product involves graphs of diffeomorphisms which have relatively small dimension, i.e.  $\dim U \times \mathcal{F}(U) = 2 \dim U$  whereas  $\dim \Gamma_{\mathcal{F}} = \dim U$  where  $\Gamma_{\mathcal{F}}$  denotes the graph of  $\mathcal{F}|_U$ . The orientation is induced by the orientation on  $M^{n+h_1}$ .  $\square$

**Remark.** In this proof, the generic condition on the structure  $\sigma$  is that the labeling functions  $f_i$  are Morse, and satisfy the Morse-Smale transversality properties—the stable and unstable manifolds of the critical points all intersect transversally. Also we require that stable and unstable manifolds for different Morse functions intersect transversally and that diffeomorphisms induced by gradient flows yield transversal intersections.

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