COUNTING LATTICE POINTS IN THE MODULI SPACE OF CURVES.

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Abstract. We show how to define and count lattice points in the moduli space $\mathcal{M}_{g,n}$ of genus g curves with n labeled points. This produces a polynomial with coefficients that include the Euler characteristic of the moduli space, and tautological intersection numbers on the compactified moduli space.

1. Introduction

Let $\mathcal{M}_{g,n}$ be the moduli space of genus g curves with n labeled points. The *decorated* moduli space $\mathcal{M}_{g,n} \times \mathbb{R}^n_+$ equips the labeled points with positive numbers $(b_1, ..., b_n)$. It has a cell decomposition due to Penner, Harer, Mumford and Thurston

$$\mathcal{M}_{g,n} \times \mathbb{R}^n_+ \cong \bigcup_{\Gamma \in \mathcal{F}\mathrm{at}_{g,n}} P_\Gamma \tag{1}$$

where the indexing set $\mathcal{F}\mathrm{at}_{g,n}$ is the space of labeled fatgraphs of genus g and n boundary components. See Section 2 for definitions of a fatgraph Γ, its automorphism group $Aut\Gamma$ and the cell decomposition (1) realised as the space of labeled fatgraphs with metrics. Restricting this homeomorphism to a fixed n-tuple of positive numbers $(b_1, ..., b_n)$ yields a space homeomorphic to $\mathcal{M}_{g,n}$ decomposed into compact convex polytopes $P_\Gamma(b_1, ..., b_n)$. When the b_i are positive integers the polytope $P_\Gamma(b_1, ..., b_n)$ is an integral polytope and we define $N_\Gamma(b_1, ..., b_n)$ to be its number of positive integer points. The weighted sum of N_Γ over all labeled fatgraphs of genus g and n boundary components is the lattice count polynomial:

Definition 1. $$N_{g,n}(b_1, ..., b_n) = \sum_{\Gamma \in \mathcal{F}\mathrm{at}_{g,n}} \frac{1}{|Aut\Gamma|} N_\Gamma(b_1, ..., b_n)$$

Each integral point in the polytope $P_\Gamma(b_1, ..., b_n)$ corresponds to a Dessin d'enfants defined by Grothendieck [3] which represents a curve in $\mathcal{M}_{g,n}$ defined over $\bar{\mathbb{Q}}$. Thus the lattice count polynomial $N_{g,n}(b_1, ..., b_n)$ counts only curves defined over $\bar{\mathbb{Q}}$. This is described in Section 2 where the integral points in $P_\Gamma(b_1, ..., b_n)$ represent metrics on labeled fatgraphs with integer edge lengths, or equivalently curves equipped with a canonical meromorphic quadratic (Strebel) differential with integral residues.

Quite generally the number of integer points in a convex polytope is a piecewise defined polynomial. Nevertheless the following theorem shows that a weighted sum of the piecewise defined polynomials $N_\Gamma(b_1, ..., b_n)$ is a polynomial.

Theorem 1. *The number of lattice points $N_{g,n}(b_1, ..., b_n)$ is a degree $3g - 3 + n$ polynomial in the integers $(b_1^2, ..., b_n^2)$ depending on the parity of the b_i.*

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The dependence on the parity means that $N_{g,n}(b_1,...,b_n)$ is represented by 2^n polynomials (by symmetry at most $[\frac{n}{2}]+2$ are different.) The polynomials are symmetric under permutations of b_i of the same parity. If the number of odd b_i is odd then $N_{g,n}(b_1,...,b_n)=0$. Otherwise, the top degree homogeneous part of $N_{g,n}(b_1,...,b_n)$ is independent of the parity. Table 1 shows the simplest polynomials. The factorisations are expected from the vanishing result of Lemma 2 in Section 2.3.

Harer and Zagier [5] calculated the orbifold Euler characteristic $\chi(\mathcal{M}_{g,1})$ and Penner [10] calculated $\chi(\mathcal{M}_{g,n})$ for general n. This information is encoded in the lattice count polynomial for all even b_i.

Theorem 2. $N_{g,n}(0,...,0)=\chi(\mathcal{M}_{g,n})$.

Kontsevich [6] defined the volume polynomial

$$V_{g,n}(b_1,...,b_n)=\sum_{\Gamma\in\mathcal{F}\mathrm{at}_{g,n}}\frac{1}{|Aut\Gamma|}Vol_\Gamma(b_1,...,b_n)$$

where $Vol_\Gamma(b_1,...,b_n)$ is the volume of the convex polytope $P_\Gamma(b_1,...,b_n)$. (The Laplace transform of $V_{g,n}$ appears as I_g in [6].) He showed that the coefficients give intersection numbers of Chern classes of the tautological line bundles L_i over the compactified moduli space $\overline{\mathcal{M}}_{g,n}$. By considering finer and finer meshes it follows that the homogeneous top degree part of the lattice point count polynomial is the volume polynomial.

Theorem 3. $N_{g,n}(b_1,...,b_n)=V_{g,n}(b_1,...,b_n)+$ *lower order terms.*

Corollary 1. *For* $|\mathbf{d}|=\sum_i d_i=3g-3+n$ *and* $\mathbf{d}!=\prod d_i!$ *the coefficient* $c_{\mathbf{d}}$ *of* $b^{2\mathbf{d}}=\prod b_i^{2d_i}$ *in* $N_{g,n}(b_1,...,b_n)$ *is the intersection number*

$$c_{\mathbf{d}}=\frac{1}{2^{6g-6+2n-g}\mathbf{d}!}\int_{\overline{\mathcal{M}}_{g,n}}c_1(L_1)^{d_1}...c_1(L_n)^{d_n}.$$

Kontsevich proved that these tautological intersection numbers satisfy a recursion relation conjectured by Witten [12] that determine the intersection numbers. The lattice count polynomials satisfy a recursion relation that uniquely determine the polynomials and when restricted to the top degree terms imply Witten's recursion.

Table 1. Lattice count polynomials for even b_i

g	**n**	$N_{g,n}(b_1,...,b_n)$
0	3	1
1	1	$\frac{1}{48}\left(b_1^2-4\right)$
0	4	$\frac{1}{4}\left(b_1^2+b_2^2+b_3^2+b_4^2-4\right)$
1	2	$\frac{1}{384}\left(b_1^2+b_2^2-4\right)\left(b_1^2+b_2^2-8\right)$
2	1	$\frac{1}{2^{16}3^35}\left(b_1^2-4\right)\left(b_1^2-16\right)\left(b_1^2-36\right)\left(5b_1^2-32\right)$

Theorem 4. *The lattice count polynomials satisfy the following recursion relation which determines the polynomials uniquely from $N_{0,3}$ and $N_{1,1}$.*

$$\begin{aligned}\left(\sum_{i=1}^n b_i\right) N_{g,n}(b_1,...,b_n) &= \sum_{i\neq j}\sum_{p+q=b_i+b_j} pqN_{g,n-1}(p,b_1,..,\hat{b}_i,..,\hat{b}_j,..,b_n) \\ &+ \sum_i \sum_{p+q+r=b_i} pqr\Big[N_{g-1,n+1}(p,q,b_1,..,\hat{b}_i,..,b_n) \\ &\quad + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=\{1,..,\hat{i},..,n\}}} N_{g_1,|I|+1}(p,b_I)N_{g_2,|J|+1}(q,b_J)\Big]\end{aligned} \tag{2}$$

The proof of Theorem 4 is elementary. The recursion relation (2) is used to prove Theorem 1. It resembles Mirzakhani's recursion relation [7] between polynomials giving the Weil-Petersson volume of the moduli space. In fact the top homogeneous degree part of $N_{g,n}(b_1,...,b_n)$ coincides with the top homogeneous degree part of Mirzakhani's Weil-Petersson volume polynomial (after multiplying by an appropriate power of 2) since both of these coincide with Kontsevich's volume. Mirzakhani [8] already showed the coefficients of the Weil-Petersson volume polynomial are the intersection numbers given in Corollary 1. Do and Safnuk [2] use fatgraphs to give a simpler proof of Mirzakhani's recursion relation restricted to the top homogeneous degree part and show that it is a rescaled version of Mirzakhani's proof.

Although Table 1 shows only even b_i, the recursion relation needs the odd cases too. We will fill in the cases of odd b_i here. When $\sum b_i$ is odd, $N_{g,n}(b_1,...,b_n)\equiv 0$. The polynomial $N_{0,4}(b_1,...,b_4)$ is the same as in the table when $b_1,...,b_4$ are all odd, and when exactly two of the b_i are odd $N_{0,4}(b_1,...,b_4)=\frac{1}{4}\left(b_1^2+b_2^2+b_3^2+b_4^2-2\right)$. For genus 1 when b_1 and b_2 are odd $N_{1,2}(b_1,b_2)=\frac{1}{384}\left(b_1^2+b_2^2-2\right)\left(b_1^2+b_2^2-10\right)$.

Section 2 contains preliminaries on fatgraphs and lattice point counting. Theorems 1 and 4 are proven in Section 2.2. Section 2.3 contains a simple vanishing result for $N_{g,n}(b_1,...,b_2)$ which has powerful consequences. In Section 3 we prove Theorem 2 and treat the special case of $n=1$ labeled points.

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2. Fatgraphs

A *fatgraph* is a graph Γ with vertices of valency >2 equipped with a cyclic ordering of edges at each vertex. In Figure 1 we use the projection to define the cyclic ordering to be anticlockwise at each vertex. The two pictured fatgraphs are

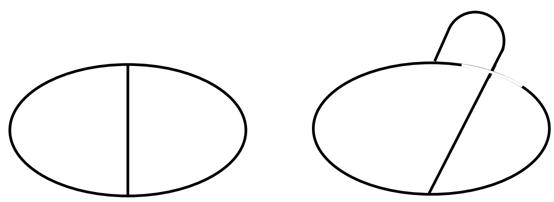

Figure 1. Fatgraphs

different, although the underlying graphs are the same. A fatgraph structure on a graph is equivalent to an embedding of a graph into a surface $\Gamma \to \Sigma$ such that $\Sigma - \Gamma$ is a union of disks. This gives a genus g and number of boundary components n to Γ. The examples in Figure 1 have genus 0 and 1 shown in Figure 2.

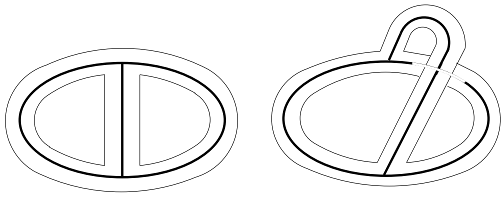

FIGURE 2. Graphs embedded in genus 0 and 1 surfaces

A *labeled* fatgraph is a fatgraph with boundary components labeled $1, ..., n$. The set of all labeled fatgraphs of genus g and n boundary components is notated by $\mathcal{F}\mathrm{at}_{g,n}$.

It is useful to describe a fatgraph in the following equivalent way [6] which makes the automorphisms transparent. Given a graph Γ with vertices of valency > 2, let X be the set of oriented edges, so each edge of Γ appears in X twice. Define the map $\tau_1 : X \to X$ that flips the orientation of each edge. A fatgraph, or ribbon, structure on Γ is a map $\tau_0 : X \to X$ that permutes cyclically the oriented edges with a common source vertex. Let X_0, X_1 and X_2 be the vertices, edges and boundary components of the fatgraph Γ. Then $X_0 = X/\tau_0$, $X_1 = X/\tau_1$ and $X_2 = X/\tau_2$ for $\tau_2 = \tau_0\tau_1$. An *automorphism* of the labeled fatgraph Γ is a permutation $\phi : X \to X$ that commutes with τ_0 and τ_1 and acts trivially on X_2. The examples in Figure 1 given any labeling have automorphism groups $\{1\}$ and $\mathbb{Z}_6$.

A metric on a labeled fatgraph Γ assigns positive numbers—lengths—to each edge of the fatgraph. If $\Gamma \in \mathcal{F}\mathrm{at}_{g,n}$ then the valency > 2 conditions on the vertices ensures that the number of edges $e(\Gamma)$ of Γ is bounded $e(\Gamma) \leq 6g - 6 + 3n$. Let P_Γ be the $6g - 6 + 3n$ cell consisting of all metrics on Γ. Construct the cell-complex

$$\mathcal{M}_{g,n}^{\text{combinatorial}} = \bigcup_{\Gamma \in \mathcal{F}\mathrm{at}_{g,n}} P_\Gamma$$

where we identify isometric metrics on fatgraphs, and when the length of an edge $l_E \to 0$ we identify this with the metric on the fatgraph with the edge E contracted. By the existence and uniqueness of meromorphic quadratic differentials with foliations having compact leaves, known as Strebel differentials, the cell complex is homeomorphic to the decorated moduli space $\mathcal{M}_{g,n}^{\text{combinatorial}} \cong \mathcal{M}_{g,n} \times \mathbb{R}_+^n$ [4].

Denote by $P_\Gamma(b_1, ..., b_n) \subset P_\Gamma$ the metrics on Γ with fixed boundary lengths $\mathbf{b} = (b_1, ..., b_n) \in \mathbb{R}_+^n$ or equivalently with specified residues of the (square root of the) associated Strebel differential. Then

$$\mathcal{M}_{g,n}^{\text{combinatorial}}(b_1, ..., b_n) = \bigcup_{\Gamma \in \mathcal{F}\mathrm{at}_{g,n}} P_\Gamma(b_1, ..., b_n) \cong \mathcal{M}_{g,n}. \tag{3}$$

2.1. **Counting lattice points in convex polytopes.** A convex polytope $P \subset \mathbb{R}^n$ can be defined as the convex hull of a finite set of vertices in $\mathbb{R}^n$. We will consider *integral* polytopes P where the vertices lie in $\mathbb{Z}^n$. Define the number of integral points in P by $N_P = \#\{P \cap \mathbb{Z}^n\}$ and $N_P(k) = \#\{kP \cap \mathbb{Z}^n\}$ where kP rescales

$\lambda_j \mapsto k\lambda_j$. Also, define $N_P^0(k)$ to be the number of integral points in the *interior* of kP.

Theorem 2.1 (Ehrhart). *If $P \subset \mathbb{R}^n$ is an n-dimensional convex polytope then*

$$N_P(k) = \mathrm{Vol}(P)k^n + ...$$

is a degree n polynomial in k with top coefficient the volume of P. Furthermore,

$$N_P^0(k) = (-1)^n N_P(-k).$$

We can define a convex polytope with positive codimension as follows. Given a linear map $A : \mathbb{R}^N \to \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$ define

$$P_A(\mathbf{b}) = \{\mathbf{x} \in \mathbb{R}_+^N | A\mathbf{x} = \mathbf{b}\}.$$

If A and b have integer entries (with respect to the standard bases) then $P_A(\mathbf{b})$ is integral and we define $N_{P_A}(\mathbf{b}) = \#\{P_A \cap \mathbb{Z}^N\}$. In this case $N_{P_A}(\mathbf{b})$ is a piecewise defined polynomial in $\mathbf{b}$ - for example, $N_{P_A}(\mathbf{b})$ may be zero for some values of $\mathbf{b}$.

The set $P_\Gamma(\mathbf{b})$ in (3) is a convex polytope defined by solutions $\mathbf{x} \in \mathbb{R}_+^{e(\Gamma)}$ of

$$A_\Gamma \mathbf{x} = \mathbf{b}$$

where A_Γ is the incidence matrix that maps the vector space generated by edges of Γ to the vector space generated by boundary components of Γ—an edge maps to the sum of its two incident boundary components. In the examples in Figure 1 the incidence matrices are

$$A_\Gamma = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad A_{\Gamma'} = \begin{pmatrix} 2 & 2 & 2 \end{pmatrix}.$$

We define

$$N_\Gamma(\mathbf{b}) = \#\{P_A \cap \mathbb{Z}_+^N\}.$$

It is natural to allow non-negative solutions although we allow only *positive* integer solutions. This is justified by the fact that if some of the x_i vanish then this will be counted using a fatgraph obtained by collapsing edges of Γ. (If the collapsing of edges of Γ does not yield a fatgraph, for example collapsing a loop, then we do not want to count such solutions.)

Since each edge is incident to exactly two (not necessarily distinct) boundary components the columns of A_Γ add to 2, or equivalently $(1, 1, ..., 1) \cdot A_\Gamma = (2, 2, ..., 2)$. Thus,

$$\sum b_i = (1, 1, ..., 1) \cdot \mathbf{b} = (1, 1, ..., 1) \cdot A_\Gamma \mathbf{x} = (2, 2, ..., 2) \cdot \mathbf{x} \in 2\mathbb{Z}$$

so $N_\Gamma(\mathbf{b}) = 0$ if $\sum b_i$ is odd. Hence the lattice count polynomial $N_{g,n}(b_1, ..., b_n)$ given in Definition 1 also vanishes when $\sum b_i$ is odd.

If we relax the condition on fatgraphs that the valency of each vertex must be > 2 then Grothendieck [3] showed that fatgraphs with all edge lengths 1 possess branched covers of $\mathbb{P}^1$ branched over 0, 1 and ∞. By a theorem of Belyi these correspond to curves defined over $\bar{\mathbb{Q}}$. When the length of each edge is a positive integer this is the same as a string of length 1 edges joined by valency 2 vertices. Thus, $N_{g,n}(b_1, ..., b_n)$ counts curves defined over $\bar{\mathbb{Q}}$ branched over of $0, 1, \infty \in \mathbb{P}^1$ with all points over 1 of ramification 2, and all points over 0 of ramification > 2.

For a convex polytope $P \subset \mathbb{R}^N$ and a polynomial ϕ on $\mathbb{R}^N$ define the following generalisation of counting lattice points.

$$N_P(\phi, k) = \sum_{\mathbf{x} \in kP \cap \mathbb{Z}^N} \phi(\mathbf{x})$$

and $N_P^0(\phi, k)$ the sum over interior integer points of kP. Later when applying the recursion relation we will need to calculate sums with a parity restriction as in Lemma 1 because the polynomials $N_{g,n}$ vanish if the sum of the arguments is odd.

Lemma 1.

$$S_m(k) = \sum_{\substack{p+q=k \\ q \text{ even}}} p^{2m+1} q, \quad R_{m,m'}(k) = \sum_{\substack{p+q+r=k \\ r \text{ even}}} p^{2m+1} q^{2m'+1} r \tag{4}$$

are odd *polynomials in k of degree $2m+3$, respectively $2m+2m'+5$, depending on the parity of k.*

Proof. The dependence on the parity means that there are two polynomials $S_m^{\text{even}}(k)$ and $S_m^{\text{odd}}(k)$ depending on whether k is even or odd. The same is true for $R_{m.m'}(k)$. Notice that

$$S_m(k) = 2N_P(\phi_1, k)$$

for $P = \{(x, y) \in \mathbb{R}_+^2 | x + 2y = 1\}$ and $\phi_1 = x^{2m+1} y$ (substitute $q = 2Q$.) Similarly,

$$R_{m,m'}(k) = 2N_{P'}(\phi_2, k)$$

for $P' = \{(x, y, z) \in \mathbb{R}_+^3 | x + y + 2z = 1\}$ and $\phi_2 = x^{2m+1} y^{2m'+1} z$.

The polytopes P and P' are *rational*, not integral. They can be expressed in terms of the integral convex polytopes of higher dimension

$$P_1 = \{x \geq 0, y \geq 0, x + 2y \leq 2\}, \quad P_2 = \{x \geq 0, y \geq 0, z \geq 0, x + y + 2z \leq 2\}.$$

For k even

$$S_m^{\text{even}}(k) = N_{P_1}(\phi_1, \frac{k}{2}) - N_{P_1}^0(\phi_1, \frac{k}{2}), \quad R_{m,m'}^{\text{even}}(k) = N_{P_2}(\phi_2, \frac{k}{2}) - N_{P_2}^0(\phi_2, \frac{k}{2}).$$

A generalisation [1] of Ehrhart's theorem states that for a dimension n integral convex polytope $P \subset \mathbb{R}^n$, $N_P(\phi, k)$ is a degree $\deg \phi + n$ polynomial in k and

$$N_P^0(\phi, k) = (-1)^{\deg \phi + n} N_P(\phi, -k).$$

For the cases at hand, $\deg \phi + n$ is even so the right hand side is $N_P(\phi, -k)$ and $S_m^{\text{even}}(k)$ and $R_{m,m'}^{\text{even}}(k)$ are *odd* polynomials in k of degree $2m+3$, respectively $2m+2m'+5$. For k odd,

$$S_m^{\text{odd}}(k) = N_{P_1}^0(\phi_1, \frac{k+1}{2}) - N_{P_1}(\phi_1, \frac{k-1}{2})$$

and $R_{m,m'}^{\text{odd}}(k)$ is the same expression with P_2 in place of P_1. Once again $S_m^{\text{odd}}(k)$ and $R_{m,m'}^{\text{odd}}(k)$ are *odd* polynomials in k of degree $2m+3$, respectively $2m+2m'+5$. $\square$

2.2. **Recursion.**

Proof of Theorem 4. The lattice count polynomial $N_{g,n}(b_1,...,b_n)$ counts labeled fatgraphs with positive integer edge lengths which we call integer fatgraphs in $P_\Gamma(b_1,...,b_n)$. We can produce an integer fatgraph in $P_\Gamma(b_1,...,b_n)$ from simpler integer fatgraphs in the three ways shown in Figures 3, 4 and 5. Choose a graph in $P_{\Gamma'}(p,b_3,...,b_n)$ and add an edge of length $q/2$ inside the boundary of length p as in Figure 3 so that $p+q=b_1+b_2$. Similarly, attach an edge and a loop of total length

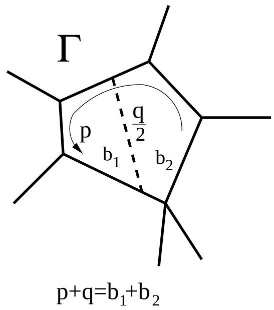

FIGURE 3. Γ is obtained from a simpler fatgraph by adding the broken line.

$q/2$ inside the boundary of length p as in Figure 4 so that $p+q=b_1+b_2$. In both

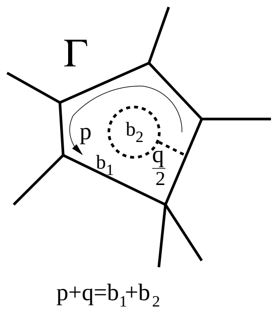

FIGURE 4. Γ is obtained by adding a line and loop of total length $q/2$.

cases for each Γ' there are p possible ways to attach the edge so this construction contributes $pN_{g,n-1}(p,b_3,...,b_n)$ to $N_{g,n}(b_1,...,b_n)$. However we have overcounted, particularly when we repeat this construction for any pair b_i and b_j, since each integer fatgraph in $P_\Gamma(b_1,...,b_n)$ can be produced in many ways like this. To deal with this, we overcount even further by taking $pqN_{g,n-1}(p,b_3,...,b_n)$, i.e. taking each constructed fatgraph q times. But now we see that for each edge that we attach of length $q/2$ we have overcounted q times. If we were to use all of the edges of Γ in this way then we would have overcounted by

$$\sum_{E\in\Gamma} l(E) = \sum_{i=1}^{n} b_i.$$

Indeed all of the edges of Γ are used, exactly once, when we include one further construction of the integer fatgraph Γ.

Choose an integer fatgraph in $P_{\Gamma'}(p, q, b_2, ..., b_n)$ for $\Gamma' \in \mathcal{F}\text{at}_{g-1,n+1}$ *or* choose two integer fatgraphs in $P_{\Gamma_1}(p, b_2, ..., b_j)$ and $P_{\Gamma_2}(q, b_{j+1}, ..., b_n)$ for $\Gamma_1 \in \mathcal{F}\text{at}_{g_1,j}$ and $\Gamma_2 \in \mathcal{F}\text{at}_{g_2,n+1-j}$ where $g_1 + g_2 = g$ and attach an edge of length $r/2$ connecting these two boundary components as in Figure 5 so that $p + q + r = b_1$.

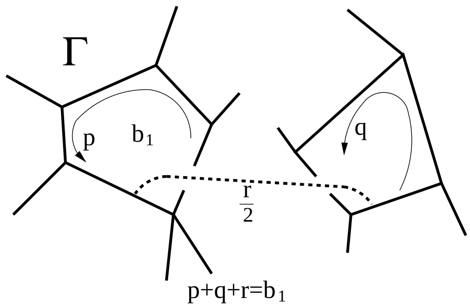

FIGURE 5. Γ is obtained from a single fatgraph or two disjoint fatgraphs by adding the broken line.

In the diagram, the two boundary components of lengths p and q are part of a fatgraph that may or may not be connected. There are pq possible ways to attach the edge so this construction contributes $pqN_{g-1,n+1}(p, q, b_2, ..., b_n)$ and $pqN_{g_1,j}(p, b_2, ..., b_j)N_{g_2,n+1-j}(q, b_{j+1}, ..., b_n)$ to $N_{g,n}(b_1, ..., b_n)$ and again we have overcounted. We overcount further by a factor of r to get $pqrN_{g-1,n+1}(p, q, b_2, ..., b_n)$ and $pqrN_{g_1,j}(p, b_2, ..., b_j)N_{g_2,n+1-j}(q, b_{j+1}, ..., b_n)$. We repeat this for each $g_1+g_2 = g$ and $I \sqcup J = \{2, ...n\}$ and then for each b_j in place of b_1.

As previewed above, each edge of Γ has been attached to construct Γ and $N_{g,n}(b_1, ..., b_n)$ has been overcounted $\sum_{i=1}^n b_i$ times yielding (2). $\square$

Remark. The idea in the proof above to overcount by the length of each edge of the graph Γ comes from the similar idea introduced by Mirzakhani [7] where she unfolds a function on Teichmüller space that sums to the analogue of b_1.

To apply the recursion we need to first calculate $N_{0,3}(b_1, b_2, b_3)$ and $N_{1,1}(b_1)$. There are seven labeled fatgraphs in $\mathcal{F}\text{at}_{0,3}$ coming from three unlabeled fatgraphs. It is easy to see that $N_{0,3}(b_1, b_2, b_3) = 1$ if $b_1 + b_2 + b_3$ is even (and 0 otherwise.) This is because for each (b_1, b_2, b_3) there is exactly one of the seven labeled fatgraphs Γ with a unique solution of $A_\Gamma \mathbf{x} = \mathbf{b}$ while the other six labeled fatgraphs yield no solutions. For example, if $b_1 > b_2 + b_3$ then only the fatgraph Γ with $A_\Gamma = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has a solution and that solution is unique.

To calculate $N_{1,1}(b_1)$, note that $A_\Gamma = [2 \quad 2 \quad 2]$ or $[2 \quad 2]$ for the 2-vertex and 1-vertex fatgraphs. Hence

$$N_{1,1}(b_1) = a_1 \binom{\frac{b_1}{2} - 1}{2} + a_2 \binom{\frac{b_1}{2} - 1}{1}$$

where a_1 is the number of trivalent fatgraphs (weighted by automorphisms) and a_2 is the number of 1-vertex fatgraphs. The genus 1 graph Γ from Figure 1 has $|Aut\Gamma| = 6$ so $a_1 = 1/6$, and a_2 uses the genus 1 figure 8 fatgraph which has automorphism group $\mathbb{Z}_4$ hence $a_2 = 1/4$. Thus

$$N_{1,1}(b_1) = \frac{1}{6}\binom{\frac{b_1}{2}-1}{2} + \frac{1}{4}\binom{\frac{b_1}{2}-1}{1} = \frac{1}{48}\left(b_1^2 - 4\right).$$

We can also calculate $N_{1,1}(b_1)$ using a version of the recursion

$$b_1 N_{1,1}(b_1) = \frac{1}{2} \sum_{\substack{p+q+p=b \\ b \text{ even}}} pq.$$

We will calculate $N_{0,4}[b_1, b_2, b_3, b_4]$ to demonstrate the recursion relation and the parity issue.

$$\left(\sum_{i=1}^{4} b_i\right) N_{0,4}(b_1, b_2, b_3, b_4) = \sum_{i \neq j} \sum_{\substack{p+q=b_i+b_j \\ q \text{ even}}} pq.$$

If all b_i are even, or all b_i are odd, then $b_i + b_j$ is always even so the sum is over p and q even. We have

$$S_0^{\text{even}}(k) = \sum_{i \neq j} \sum_{\substack{p+q=k \\ q \text{ even}}} pq = 4\binom{\frac{k}{2}+1}{3}$$

so

$$\left(\sum_{i=1}^{4} b_i\right) N_{0,4}(\mathbf{b}) = \sum_{i \neq j} 4\binom{\frac{b_i+b_j}{2}+1}{3} = \left(\sum_{i=1}^{4} b_i\right)\frac{1}{4}\left(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 4\right)$$

agreeing with Table 1. If b_1 and b_2 are odd and b_3 and b_4 are even then we need

$$S_0^{\text{odd}}(k) = \sum_{i \neq j} \sum_{\substack{p+q=k \\ q \text{ even}}} pq = \frac{1}{2}\binom{k+1}{3}$$

so

$$\begin{aligned}\left(\sum_{i=1}^{4} b_i\right) N_{0,4}(\mathbf{b}) &= \sum_{(i,j)=(1,2) \text{ or } (3,4)} 4\binom{\frac{b_i+b_j}{2}+1}{3} + \sum_{(i,j)\neq(1,2) \text{ or } (3,4)} \frac{1}{2}\binom{b_i+b_j+1}{3} \\ &= \left(\sum_{i=1}^{4} b_i\right)\frac{1}{4}\left(b_1^2 + b_2^2 + b_3^2 + b_4^2 - 2\right)\end{aligned}$$

so we see that the polynomial representatives of $N_{0,4}(\mathbf{b})$ agree up to a constant term.

Proof of Theorem 1. We can use the recursion (2) to prove that $N_{g,n}(b_1, ..., b_n)$ is a polynomial of the right degree but to prove that it is a polynomial in b_i^2 we need a different recursion formula (5). For simplicity we use (5) to prove each part of

Theorem 1.

$$b_1N_{g,n}(b_1,...,b_n) = \sum_{j>1}\frac{1}{2}\Bigg(\sum_{p+q=b_1+b_j} pqN_{g,n-1}(p,b_2,..,\hat{b}_j,..,b_n) + \sum_{p+q=b_1-b_j} pqN_{g,n-1}(p,b_2,..,\hat{b}_j,..,b_n)\Bigg) + \sum_{p+q+r=b_1} pqr\Bigg[N_{g-1,n+1}(p,q,b_2,...,b_n) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,...,n\}}} N_{g_1,|I|}(p,b_I)N_{g_2,|J|}(q,b_J)\Bigg] \tag{5}$$

This differs from the recursion formula (2) by breaking the symmetry around b_1. The sum over the term $p+q=b_1-b_j$ needs to be interpreted as follows. If $b_1-b_j>0$ it is read as written, whereas if $b_1-b_j<0$ then replace b_1-b_j by b_j-b_1 and negate the sum. (This is not the same as sending (p,q) to $(-p,-q)$.)

We will prove the recursion (5) below. Before that we will prove that given $N_{0,3}$ and $N_{1,1}$ then (5) determines polynomials $N'_{g,n}(b_1,...,b_n)$ of degree $3g-3+n$ in b_i^2. By induction, the simpler polynomials are polynomials in b_i^2 so monomials on the right hand side of the recursion are of the form

$$S_m(k) = \sum_{\substack{p+q=k\\ q \text{ even}}} p^{2m+1}q, \quad R_{m,m'}(k) = \sum_{\substack{p+q+r=k\\ r \text{ even}}} p^{2m+1}q^{2m'+1}r$$

as in (4). In Lemma 1 it is proven that $S_m(k)$ and $R_{m,m'}(k)$ are odd polynomials in k. In particular, $S_m(b_1-b_j)=-S_m(b_j-b_1)$ explaining the interpretation of the sum over $b_1-b_j<0$

The sums over $p+q+r=b_1$ yield terms which are odd in b_1 from $R_{m,m'}(b_1)$ and even in b_i for $i>1$ hence $1/b_1$ times these terms is even in all b_i^2. The sums over $p+q=b_1+b_j$ and $p+q=b_1-b_j$ have the same summands so each monomial occurs with the same coefficient. Hence the terms involving b_1 are of the form $S_m(b_1+b_j)+S_m(b_1-b_j)$ and since S_m is odd, this sum is odd in b_1 and even in b_j, and even in the all other b_i. Again $1/b_1$ times these terms is even in all b_i^2. Thus by induction the polynomials generated by the recursion relation (5) from $N_{0,3}$ and $N_{1,1}$ are polynomials in b_i^2.

We will now calculate the degree in b_i^2. By induction $\deg N_{g,n-1}=3g-3+n-1$ and by Lemma 2 $S_m(k)$ takes a term $p^{2m+1}q$ and produces a degree $2m+3$ polynomial, i.e. it increases the degree by 1. In this case $3g-3+n-1+1=3g-3+n$ as required. Similarly, by induction $\deg N_{g-1,n+1}=3g-3+n-2$ and $\deg N_{g_1,|I|}N_{g_2,|J|}=3g-3+n-2$. By Lemma 2 $R_{m,m'}(k)$ increases the degree of its summand by 2. Since $3g-3+n-2+2=3g-3+n$ the result is proven by induction starting from the degrees of $N_{0,3}$ and $N_{1,1}$.

As above, write $N'_{g,n}$ for the polynomials produced from the recursion (5). To prove the recursion (5) we use the fact that both (2) and (5) uniquely determine

$N_{g,n}$ and $N'_{g,n}$ respectively. It remains to show that (5) $\Rightarrow$ (2), hence $N_{g,n}$ and $N'_{g,n}$ necessarily coincide.

Apply (5) to each b_i to calculate $b_i N'_{g,n}(b_1,...,b_n)$ and add.

$$\begin{aligned}\left(\sum_{i=1}^n b_i\right) N'_{g,n}(b_1,...,b_n) &= \sum_{i\neq j}\sum_{p+q=b_i+b_j} pqN'_{g,n-1}(p,b_1,..,\hat{b}_i,..,\hat{b}_j,..,b_n)\\ &+\sum_i \sum_{p+q+r=b_i} pqr\Bigg[N'_{g-1,n+1}(p,q,b_1,..,\hat{b}_i,..,b_n)\\ &\qquad + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{1,..,\hat{i},..,n\}}} N'_{g_1,|I|}(p,b_I)N'_{g_2,|J|}(q,b_J)\Bigg]\\ &+\Delta\end{aligned}$$

where

$$\Delta=\sum_{i\neq j}\frac{1}{2}\left(\sum_{p+q=b_i-b_j}+\sum_{p+q=b_j-b_i}\right)pqN'_{g,n-1}(p,b_1,..,\hat{b}_i,..,\hat{b}_j,..,b_n)=0$$

since the sums contain only canceling odd terms $S_m(b_i-b_j)+S_m(b_j-b_i)=0$.

Thus $N_{g,n}$ and $N'_{g,n}$ satisfy the recursion relation (2) which uniquely determines them, hence

$$N_{g,n}=N'_{g,n}$$

so it follows that $N_{g,n}$ satisfies the recursion (5). $\square$

Remark. The top degree term of recursion (5) is a discrete version of the integration recursion for volume given by Do and Safnuk [2]. They show their recursion is a rescaled version of Mirzakhani's recursion relation [7] which give the Virasoro relations among tautological classes [8].

2.3. **Vanishing.**

Lemma 2. *If* $\sum_{i=1}^n b_i < 4g+2n$ *then* $N_{g,n}(b_1,...,b_n)=0$.

Proof. A labeled fatgraph $\Gamma\in\mathcal{F}\mathrm{at}_{g,n}$ has at least one vertex and hence at least $2g+n$ edges since $\chi(\Gamma)=1-2g-n$. Since $N_{g,n}$ counts positive integers solutions of $A_\Gamma x=b$, each $x_i\geq 1$, thus $\sum x_i\geq 2g+n$. Each edge contributes twice to the boundary of Γ so

$$\sum_{i=1}^n b_i = 2\sum_{i=1}^{e(\Gamma)} x_i \geq 4g+2n$$

and the lemma follows. $\square$

Lemma 2 can be used to get strong information about the lattice count polynomial. For example, $N_{1,1}(2)=0$ and since it is a polynomial in b_1^2 of degree 1 we get $N_{1,1}(b_1)=c(b_1^2-4)$. The genus 2 case gives $N_{2,1}(2)=0=N_{2,1}(4)=N_{2,1}(6)$ hence

$$N_{2,1}(b_1)=c_1(b_1^2-4)(b_1^2-16)(b_1^2-36)(b_1^2+c_2).$$

Although it is very difficult to calculate $N_{g,n}$ directly using fatgraphs, in the simplest cases it is calculable by extending the idea behind the vanishing Lemma 2. When $\sum_{i=1}^{n} b_i = 4g + 2n$ the argument in the proof of Lemma 2 shows that each $x_i = 1$ so $N_{g,n}$ counts 1-vertex fatgraphs.

3. Euler characteristic

Using the cell decomposition (1), the orbifold Euler characteristic of the moduli space can be calculated via a sum over labeled fatgraphs. Expressing the sum as a Feynman expansion Penner [10] calculated the following.

$$\chi(\mathcal{M}_{g,n}) = \sum_{\Gamma\in\mathcal{F}\mathrm{at}_{g,n}} \frac{(-1)^{e(\Gamma)-n}}{|Aut\Gamma|} = \begin{cases} (-1)^{n-1}(n-3)! & g=0 \\ (-1)^{n-1}\frac{(2g+n-3)!}{(2g-2)!}\zeta(1-2g) & g>0 \end{cases}$$

The exponent is the dimension of the cell since $\dim P_\Gamma = e(\Gamma) - n$.

The lattice count polynomial gives another way to calculate the Euler characteristic via $N_{g,n}(0,...,0) = \chi(\mathcal{M}_{g,n})$. We will prove this here.

Proof of Theorem 2. Define

$$R_{g,n}(z) = \sum_{\mathbf{b}\in\mathbb{Z}_+^n} N_{g,n}(b_1,...,b_n) z^{b_1+...+b_n}.$$

It has the following properties:

(1) $R_{g,n}(z)$ is a meromorphic function, holomorphic on $\bar{\mathbb{C}} - \{\pm 1\}$.
(2) $R_{g,n}(0) = 0$
(3) $R_{g,n}(\infty) = (-1)^n N_{g,n}(0,...,0)$.

Recall that $N_{g,n}(\mathbf{b})$ is represented by a collection of polynomials depending on the parity of b_i. By the symmetry of these polynomials we can set $R_{g,n}(z) = \sum_{k=0}^{n}\binom{n}{k}R_{g,n}^{(k)}(z)$ where $k =$ the number of odd b_i. The basic idea behind property (1) is that if $p(n) = \sum_{j=0}^{k} p_j n^j$ is a polynomial then

$$\sum_{n>0} p(n)z^n = \sum_{j=0}^{k} p_j \sum_{n>0} n^j z^n = \sum_{j=0}^{k} p_j \left(z\frac{d}{dz}\right)^j \frac{z}{1-z}$$

which is a meromorphic function with pole at $z = 1$ and known behaviour at $z = 0$ and $z = \infty$. If we restrict the parity of n then

$$\sum_{\substack{n>0 \\ n \text{ even}}} p(n)z^n = \sum_{j=0}^{k} p_j \left(z\frac{d}{dz}\right)^j \frac{z^2}{1-z^2}, \quad \sum_{\substack{n>0 \\ n \text{ odd}}} p(n)z^n = \sum_{j=0}^{k} p_j \left(z\frac{d}{dz}\right)^j \frac{z}{1-z^2} \tag{6}$$

which are both meromorphic functions with poles at $z = \pm 1$. Furthermore,

$$\left(z\frac{d}{dz}\right)^j \frac{z^2}{1-z^2}\bigg|_{z=\infty} = \begin{cases} -1 & j=0 \\ 0 & j>0 \end{cases}, \quad \left(z\frac{d}{dz}\right)^j \frac{z}{1-z^2}\bigg|_{z=\infty} = 0.$$

Each polynomial $N_{g,n}(b_1,...,b_n)$ is a sum of monomials of the form $\prod_1^n b_i^{2m_i}$ so $R^{(k)}_{g,n}(z)$ is a sum of finitely many series

$$R^{(k)}_{g,n}(z)=\sum_{\mathbf{m}} c_{\mathbf{m}} \sum_{\substack{\mathbf{b}\in\mathbb{Z}_+^n\\ b_i \text{ odd } i\leq k}} b_1^{2m_1}...b_n^{2m_n} z^{b_1+...+b_n}$$

which consists of terms of the form

$$\sigma^{(k)}_{\mathbf{m}}(z)=\sum_{\substack{\mathbf{b}\in\mathbb{Z}_+^n\\ b_i \text{ odd } i\leq k}} b_1^{2m_1}...b_n^{2m_n} z^{b_1+...+b_n}=\prod_{i=1}^{k}\sum_{\substack{b_i>0\\ b_i \text{ odd}}} b_i^{2m_i}z^{b_i}\cdot\prod_{i=k+1}^{n}\sum_{\substack{b_i>0\\ b_i \text{ even}}} b_i^{2m_i}z^{b_i}.$$

This is a finite product of meromorphic functions each with poles only at $z=\pm1$ by (6). Furthermore, from the evaluation at ∞ of such functions, $\sigma^{(k)}_{\mathbf{m}}(\infty)=(-1)^n$ if $\mathbf{m}=\mathbf{0}$ and $k=0$ and it vanishes otherwise. Thus, $R_{g,n}(\infty)$ contains only one non-vanishing term, $R_{g,n}(\infty)=R^{(0)}_{g,n}(\infty)=(-1)^nN_{g,n}(0,...,0)$ where we evaluate using the polynomial $N_{g,n}$ that takes in all even b_i.

We have proven (1) and (3). Property (2) follows from the strict positivity of the b_i and the convergence of the series which follows from the convergence of $1+z+z^2+...$ for $|z|<1$.

We can calculate $R_{g,n}(\infty)$ in another way. For a vector $v=(v_1,...,v_n)$ with $v_i\in\mathbb{Z}_+$ define (the semigroup homomorphism) $|v|=\sum_{i=1}^n v_i$. Recall that the incident matrix $A_\Gamma=[\alpha_1,...,\alpha_{e(\Gamma)}]$ for $\alpha_i\in\mathbb{R}^n$ of a labeled fatgraph Γ defines a convex polytope $A_\Gamma x=\mathbf{b}$ and $N_\Gamma(\mathbf{b})$ counts integral points $x\in\mathbb{Z}_+^{e(\Gamma)}$. Thus

$$\begin{aligned}R_\Gamma(z) &= \sum_{\mathbf{b}\in\mathbb{Z}_+^n} N_\Gamma(b_1,...,b_n)z^{b_1+...+b_n}=\sum_{x\in\mathbb{Z}_+^{e(\Gamma)}} z^{|A_\Gamma x|}\\ &= \sum_{x\in\mathbb{Z}_+^{e(\Gamma)}} z^{\sum_{i=1}^{e(\Gamma)}x_i|\alpha_i|}=\prod_{i=1}^{e(\Gamma)}\sum_{x_i\in\mathbb{Z}_+} z^{x_i|\alpha_i|}\\ &= \prod_{i=1}^{e(\Gamma)}\frac{z^{|\alpha_i|}}{1-z^{|\alpha_i|}}\end{aligned}$$

so $R_\Gamma(\infty)=(-1)^{e(\Gamma)}$ and

$$R_{g,n}(\infty)=\sum_{\Gamma\in\mathcal{F}\mathrm{at}_{g,n}}\frac{(-1)^{e(\Gamma)}}{|Aut\Gamma|}=(-1)^n\chi(\mathcal{M}_{g,n}).$$

Combining this with property (3) yields the theorem

$$N_{g,n}(0,...,0)=\chi(\mathcal{M}_{g,n}).$$

□

3.1. **Calculating** $N_{g,1}$**.** When $n=1$ there is a more direct proof of Theorem 2. For any $\Gamma\in\mathcal{F}\mathrm{at}_{g,1}$ the incidence matrix is $A_\Gamma=[2,2,...,2]$. The equation $Ax=b$

has $\binom{\frac{b}{2}-1}{e(\Gamma)-1}$ positive integral solutions. Hence

$$N_{g,1}(b)=c^{(g)}_{6g-3}\binom{\frac{b}{2}-1}{6g-4}+c^{(g)}_{6g-4}\binom{\frac{b}{2}-1}{6g-5}+..+c^{(g)}_{k}\binom{\frac{b}{2}-1}{k-1}+..+c^{(g)}_{2g}\binom{\frac{b}{2}-1}{2g-1}$$

where the coefficients are weighted counts of fatgraphs of genus g with $n=1$ boundary component

$$c^{(g)}_k=\sum_{\substack{\Gamma\in\mathcal{F}\mathrm{at}_{g,1}\\ e(\Gamma)=k}}\frac{1}{|Aut\Gamma|}.$$

The polynomial $\binom{\frac{b}{2}-1}{k}$ evaluates at $b=0$ to $(-1)^k$ which gives a direct proof that the Euler characteristic is given by evaluation at 0.

$$N_{g,1}(0)=\sum_{\Gamma\in\mathcal{F}\mathrm{at}_{g,1}}\frac{(-1)^{e(\Gamma)-1}}{|Aut\Gamma|}=\chi(\mathcal{M}_{g,1}).$$

When $n=1$ the weighted number of trivalent fatgraphs and 1-vertex fatgraphs are known [11].

$$c^{(g)}_{6g-3}=2\frac{1}{12^g}\frac{(6g-5)!}{g!(3g-3)!},\quad c^{(g)}_{2g}=\frac{(4g-1)!}{4^g(2g+1)!}$$

We can calculate $N_{2,1}(b)$ without using the recursion relation (except to deduce that $N_{2,1}(b)$ is a polynomial of degree 4 in b^2) by applying Lemma 2 to get $N_{2,1}(b)=0$ for $b=2,4$ and 6. This leaves two unknown coefficients which can be calculated from any two of the three pieces of known information $c^{(2)}_9$, $c^{(2)}_4$ and $N_{2,1}(0)$.

$$\begin{aligned}N_{2,1}(b)&=\frac{1}{2^{16}3^35}\left(b^2-4\right)\left(b^2-16\right)\left(b^2-36\right)\left(5b^2-32\right)\\&=\tfrac{35}{6}\tbinom{\frac{b}{2}-1}{8}+\tfrac{105}{4}\tbinom{\frac{b}{2}-1}{7}+\tfrac{93}{2}\tbinom{\frac{b}{2}-1}{6}+\tfrac{161}{4}\tbinom{\frac{b}{2}-1}{5}+\tfrac{84}{5}\tbinom{\frac{b}{2}-1}{4}+\tfrac{21}{8}\tbinom{\frac{b}{2}-1}{3}.\end{aligned}$$

The polynomial $N_{2,1}(b)$ enables us to calculate the weighted counts of fatgraphs $c^{(2)}_k$. We can similarly calculate $N_{3,1}$ and hence deduce the weighted counts of fatgraphs.

$$\begin{aligned}N_{3,1}(b)&=\tfrac{1}{2^{25}3^65^27}(b^2-4)(b^2-16)(b^2-36)(b^2-64)(b^2-100)(5b^4-188b^2+1152)\\&=\tfrac{5005}{3}\tbinom{\frac{b}{2}-1}{14}+\tfrac{25025}{2}\tbinom{\frac{b}{2}-1}{13}+41118\tbinom{\frac{b}{2}-1}{12}+\tfrac{929929}{12}\tbinom{\frac{b}{2}-1}{11}+\tfrac{183955}{2}\tbinom{\frac{b}{2}-1}{10}\\&\quad+\tfrac{283767}{4}\tbinom{\frac{b}{2}-1}{9}+\tfrac{317735}{9}\tbinom{\frac{b}{2}-1}{8}+10813\tbinom{\frac{b}{2}-1}{7}+\tfrac{25443}{14}\tbinom{\frac{b}{2}-1}{6}+\tfrac{495}{4}\tbinom{\frac{b}{2}-1}{5}.\end{aligned}$$

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