SIMPLE GEODESICS AND MARKOFF QUADS

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Abstract. We study the action of the mapping class group of the thrice-punctured projective plane on its $PGL(2,\mathbb{C})$ character variety in terms of an algorithm for generating the simple length spectra of hyperbolic 3-cusped projective planes. We apply this algorithm to prove a growth rate bound for the simple closed geodesic lengths, a sharp upper-bound for the length of the shortest geodesics and establish a McShane identity for quasi-Fuchsian representations of the corresponding fundamental group.

Introduction

Closed geodesics on hyperbolic surfaces have extremely rich properties, arising in geometry, topology and number theory. Given a Riemannian surface, its *length spectrum* is the multiset of lengths of closed geodesics on the surface and its *simple length spectrum* is the multiset of lengths of simple (i.e. non-self-intersecting) closed geodesics.

The number of closed geodesics of length less than L on an oriented hyperbolic surface grows exponentially. This can be proven using the Selberg trace formula, which regards the length spectrum in terms of the spectrum of the Laplace-Beltrami operator [19]. Whereas, the growth rate for the simple closed geodesic spectrum is polynomial in L, and was first established using combinatorial arguments [10, 17]. This was later refined by Mirzakhani [13] using the ergodicity of the mapping class group action on the space of geodesic measured laminations, and her calculations of moduli spaces volumes [14] via objects intimately related to the simple length spectrum: *McShane identities* — a sum over functions of lengths of simple closed geodesics on a punctured hyperbolic surface that is independent of the hyperbolic structure on the surface. The first such identity was obtained by McShane [8] on the 1-cusped torus and generalisations to other hyperbolic surfaces and quasi-Fuchsian representations have steadily followed [1, 5, 7, 8, 9, 14, 15, 20].

Given a surface S, denote a quasi-Fuchsian representation of $\pi_1(S)$ by X, which in the Fuchsian case may be identified with a hyperbolic surface homeomorphic to S, and more generally with a hyperbolic 3-manifold homeomorphic to $(0,1)\times S$. Any closed curve $\gamma\subset S$ defines a function $\ell_\gamma(\cdot)$ on quasi-Fuchsian representations, given by the complex length of the unique closed geodesic in X homotopy equivalent to $\gamma\subset S\subseteq X$. The complex length of a geodesic has real and imaginary parts respectively given by its geometric length and the angle of twisting of

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the normal bundle around the closed geodesic.

Consider a 4-tuple $(\alpha, \beta, \gamma, \delta)$ of one-sided simple loops on a 3-cusped projective plane that pairwise intersect once. The figure shows two such 4-tuples $(\alpha, \beta, \gamma, \delta), (\alpha, \beta, \gamma, \delta')$. The crossed circle is a cross-cap: an embedded Möbius

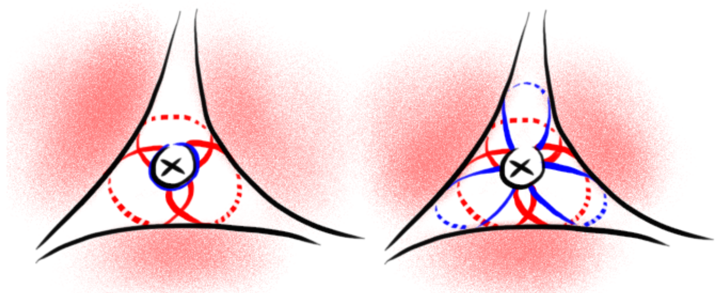

FIGURE 1. Flipping the blue geodesic.

strip. The geodesics δ and δ' are the only one-sided simple geodesics that intersect each of the geodesics α, β, γ exactly once. And we call the process of replacing δ with δ', or vice versa, a *flip*. A trace identity shows that the 4-tuple

$$(a, b, c, d) = (2\sinh \tfrac{1}{2}\ell_\alpha(X), 2\sinh \tfrac{1}{2}\ell_\beta(X), 2\sinh \tfrac{1}{2}\ell_\gamma(X), 2\sinh \tfrac{1}{2}\ell_\delta(X))$$

satisfies the equation

$$(a + b + c + d)^2 = abcd. \tag{1}$$

We call general (possibly non-geometric) solutions of (1) *Markoff quads*. From any solution (a, b, c, d) of (1), a new Markoff quad may be obtained via the following transformation:

$$(a, b, c, d) \mapsto (a, b, c, abc - 2a - 2b - 2c - d) \tag{2}$$

where $d' = abc - 2a - 2b - 2c - d$ corresponds to the flipped geodesic δ' described above. By symmetry, we may also flip any of the other three geodesics to obtain new Markoff quads with corresponding expressions for the flipped coordinate. Successively flipping geodesics produces all one-sided simple loops, and hence gives an algorithm for generating the complex lengths of all one-sided simple closed curves on S for any given quasi-Fuchsian representation X of $\pi_1(S)$.

Markoff quads are points on the $GL(2, \mathbb{C})$ relative character variety of the thrice-punctured projective plane which is the hypersurface defined by (1). The transformation (2), combined with the following permutations

$$(a, b, c, d) \mapsto (b, a, d, c), (c, d, a, b), (d, c, b, a) \tag{3}$$

generate the pure mapping class group of the thrice-punctured projective plane, and specify its action on the corresponding relative character variety. *Relative* here means that peripheral elements are parabolic, and *pure* means that we restrict to elements of the mapping class group that fix punctures.

The set of positive solutions $(a, b, c, d) \in \mathbb{R}_+^4$ of (1) is preserved under flips (2) and may be real-analytically identified with the standard simplex:

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}_+^4 \mid x_1 + x_2 + x_3 + x_4 = 1\}$$

via the involution of $\mathbb{R}_+^4$ given by

$$(x_1, x_2, x_3, x_4) = \left(\sqrt{\frac{a}{bcd}}, \sqrt{\frac{b}{acd}}, \sqrt{\frac{c}{abd}}, \sqrt{\frac{d}{abc}}\right).$$

Thus the set of positive Markoff quads is a mapping class group invariant contractible set. In fact, it is the Teichmüller component of the real character variety: a model for the Teichmüller space of the thrice-punctured projective plane. See section 5.1.

The algorithm that generates the simple length spectrum allows us to establish the maximum of the systole function over the moduli space of hyperbolic projective planes with three cusps. Recall that the *systole* of a Riemannian surface is the length of its shortest closed geodesic, which is necessarily simple.

Theorem 1. *The maximum of the systole over the moduli space of all hyperbolic 3-cusped projective planes is* $2\operatorname{arcsinh}(2)$.

This maximum is uniquely realised by the symmetric hyperbolic surface which is doubly covered by a hyperbolic surface conformally equivalent to a sphere minus the 6 vertices of the symmetric octahedron it circumscribes. Its simple length spectrum is generated from the integral Markoff quad $(4, 4, 4, 4)$.

The next main result is a McShane identity for quasi-Fuchsian representations of the fundamental group of the thrice-punctured projective plane. It gives a collection of summands depending on the quasi-Fuchsian structure X whose sum is independent of this structure.

Theorem 2. *Let X be a quasi-Fuchsian thrice-punctured projective plane. Then*

$$\sum_{\gamma} \frac{1}{1 + \exp \frac{1}{2}\ell_\gamma(X)} = \frac{1}{2},$$

where the sum is over free homotopy classes of essential, non-peripheral two-sided simple closed curves γ on S.

Note that by regarding hyperbolic surfaces as Fuchsian representations, Theorem 2 generalises the corresponding result in [15].

Finally, for any hyperbolic surface X and $L > 0$ define

$$\mathcal{S}_X(L) = \mathrm{Card}\{\gamma \text{ is a 1-sided simple closed geodesic on } X \mid \ell_\gamma(X) < L\}.$$

We prove the following estimates for $\mathcal{S}_X(L)$ when X is any hyperbolic projective plane with three cusps.

Theorem 3. *There exist constants $C(X)$ and $C'(X)$ such that*

$$C(X) \cdot L^{1.8} < \mathcal{S}_X(L) < C'(X) \cdot L^3.$$

In particular, we explore the relationship between the asymptotic growth rates of the simple length spectrum for orientable and non-orientable surfaces.

Our approach to analysing orbits of the mapping class group and in particular to proving Theorem 2 follows that of Bowditch [2], who studied the simple length spectra of hyperbolic 1-cusped tori using *Markoff triples* — solutions $(x, y, z) \in \mathbb{C}^3$ to the equation:

$$x^2 + y^2 + z^2 = xyz. \tag{4}$$

The relation of solutions of (4) to the simple length spectra of hyperbolic 1-cusped tori uses trace relations and was first studied in [4]. The hypersurface (4) is the relative character variety of the punctured torus. One can determine the simple length spectrum of a hyperbolic 1-cusped torus via sequences of the following transformations on solutions of (4):

$$(x, y, z) \mapsto (x, y, xy - z) \text{ and } (x, y, z) \mapsto (y, z, x), \tag{5}$$

which generate the (pure) mapping class group, and describe its action on the corresponding relative character variety.

This paper is loosely structured in alignment with Bowditch's proof:

Section 1: Use trace relations to derive an algorithm for generating the complex lengths of all simple closed curves on a 3-cusped projective plane, and store these data in a curve complex.

Section 2: Use geodesics lengths on X to define discrete dynamics on this cell complex. Facts established about these dynamics then translate back to results about the lengths of geodesics on 3-cusped projective planes.

Section 3: Using the dynamics described in the previous section, we generate a sequence of finite sums totaling 1. The summands in this sequence of sums eventually decrease in magnitude. In the limit, we derive Theorem 2.

Section 4: The Fibonacci bounds on the growth of geodesic lengths obtained in the previous section are invoked to obtain bounds for the asymptotic growth rate of the simple length spectrum.

Section 5: Show that flips generate a finite index normal subgroup of the mapping class group of a thrice punctured projective plane and use this to derive a cellularisation of the moduli space.

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1. Markoff quads

A modified Fricke trace identity [11] shows that the 4-tuple

$$(a, b, c, d) = (2\sinh \tfrac{1}{2}\ell_\alpha(X), 2\sinh \tfrac{1}{2}\ell_\beta(X), 2\sinh \tfrac{1}{2}\ell_\gamma(X), 2\sinh \tfrac{1}{2}\ell_\delta(X))$$

satisfies equation (1):

$$(a+b+c+d)^2 = abcd.$$

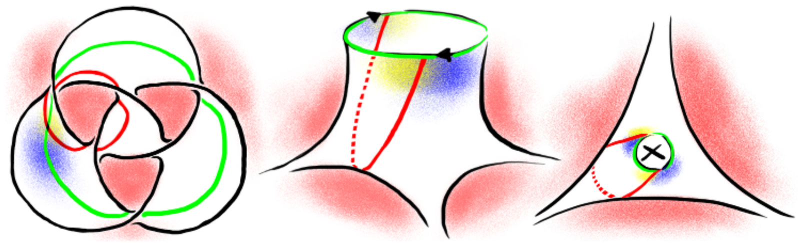

FIGURE 2. Diagrams of the thrice-punctured projective plane.

Recall from the introduction that sequences of flips generate the traces of all one-sided simple closed curves on S for any given quasi-Fuchsian representation X of $\pi_1(S)$. We begin by considering the trace identities needed for this algorithm, before detailing how to store the combinatorics of Markoff quads (and hence the simple length spectrum) for a 3-cusped projective plane in its curve complex.

1.1. **Trace identities.** Since $\pi_1(S)$ is a surface group, a representation of $\pi_1(S) = F_3$ into $PGL(2,\mathbb{C})$ admits a lift to $GL(2,\mathbb{C})$. In particular, since $PGL(2,\mathbb{C}) = PSL(2,\mathbb{C})$, any representation can be lifted to $SL(2,\mathbb{C})$; although we choose to consider trace identities in $GL(2,\mathbb{C})$. We state Fricke's relation [11] here for $GL(2,\mathbb{C})$ — in fact for $PGL(2,\mathbb{C})$ since the identity is homogeneous. Given $A_1, A_2, A_3 \in GL(2,\mathbb{C})$, set $A_0 = A_1A_2A_3$. Then:

$$\begin{aligned} 4\det A_0 = &(\mathrm{tr}\ A_0)^2 + \mathrm{tr}\ A_1 \cdot \mathrm{tr}\ A_2 \cdot \mathrm{tr}\ A_3 \cdot \mathrm{tr}\ A_0 + \mathrm{tr}\ A_1A_2 \cdot \mathrm{tr}\ A_2A_3 \cdot \mathrm{tr}\ A_3A_1 \\ &+ \frac{1}{2}\sum_{\mathrm{sym}} \left\{ \begin{array}{l} (\mathrm{tr}\ A_i)^2 \cdot \det A_jA_k - \det A_i \cdot \mathrm{tr}\ A_j \cdot \mathrm{tr}\ A_k \cdot \mathrm{tr}\ A_jA_k \\ + \det A_i \cdot (\mathrm{tr}\ A_jA_k)^2 - \mathrm{tr}\ A_0 \cdot \mathrm{tr}\ A_i \cdot \mathrm{tr}\ A_jA_k \end{array} \right\}. \end{aligned} \tag{6}$$

The sum is over all $\{i,j,k\} = \{1,2,3\}$ and the factor of $\frac{1}{2}$ in (6) is there because indices such as $(i,j,k) = (1,2,3)$ and $(1,3,2)$ give the same terms. The proof of (6) uses the fact that it extends to a relation on $M(2,\mathbb{C})$ which is quadratic in each entry of A_i. By restricting A_1, A_2, A_3 to $SL(2,\mathbb{C})$ then (6) defines the $SL(2,\mathbb{C})$ character variety of the free group F_3 which is a hypersurface inside

$$\mathbb{C}^7 = \{(\mathrm{tr}\ A_1, \mathrm{tr}\ A_2, \mathrm{tr}\ A_3, \mathrm{tr}\ A_1A_2, \mathrm{tr}\ A_2A_3, \mathrm{tr}\ A_3A_1, \mathrm{tr}\ A_1A_2A_3)\}, \tag{7}$$

We consider *type-preserving* quasi-Fuchsian representations of $\pi_1(S) = F_3$ (i.e. where peripheral elements are parabolic). The peripheral elements of the thrice-punctured projective plane correspond to A_1A_2, A_2A_3 and A_3A_1, hence

$$\mathrm{tr}\ A_1A_2 = \mathrm{tr}\ A_2A_3 = \mathrm{tr}\ A_3A_1 = 2. \tag{8}$$

In [12], they consider a different *relative character variety* of representations of F_3 into $SL(2,\mathbb{C})$ which arises from the four-punctured sphere — A_i must be parabolic for $i = 0,1,2,3$. In our case, generators are represented by one-sided curves and we choose

$$\det A_1 = \det A_2 = \det A_3 = -1. \tag{9}$$

Since $\det A_i = -1$ then A_i and A_i^{-1} are not conjugate (as they would be in $SL(2,\mathbb{C})$), we need to specify orientations on the simple closed curves representing their conjugacy classes. In figure 3, we see that there is a choice of orientation

FIGURE 3. Representative curves for A_1, A_2, A_3, $A_1A_2A_3$ and $A_1A_3A_2$.

for simple close curves representing A_1, A_2 and A_3 (anticlockwise) so that the curves representing $A_1A_2A_3$ and $A_1A_3A_2$ are simple. These are the two choices of $A_1^{\pm1}A_2^{\pm1}A_3^{\pm1}$ (up to conjugation and inversion) which are simple, and we choose

$$A_4 = A_0^{-1} = (A_1A_2A_3)^{-1} \text{ and } A_4' = (A_1A_3A_2)^{-1}. \tag{10}$$

Set $a = \text{tr } A_1$, $b = \text{tr } A_2$, $c = \text{tr } A_3$, $d = \text{tr } A_4$ and $d' = \text{tr } A_4'$, then (6) reorganises to yield (1):

$$(a+b+c+d)^2 = abcd \text{ and } (a+b+c+d')^2 = abcd'$$

which means that (a,b,c,d) and (a,b,c,d') are Markoff quads. In addition, since d and d' are the roots of the polynomial

$$p(x) = x^2 + (2a+2b+2c-abc)x + (a+b+c)^2 = (x-d)(x-d'),$$

the following identities must hold:

$$d + d' + 2a + 2b + 2c = abc \text{ and } dd' = (a+b+c)^2. \tag{11}$$

The choice $\det A_k = -1$, $k = 1,2,3$ may seem a little unnatural in $GL(2,\mathbb{C})$ since one can simply replace A_k by $iA_k \in SL(2,\mathbb{C})$. We choose $\det A_k = -1$ since it is natural when restricting to Fuchsian representations: Fuchsian representations have real (a,b,c,d) and when lifting from $PGL(2,\mathbb{R})$ to $GL(2,\mathbb{R})$ one-sided curves necessarily have negative determinant.

A pair of one-sided simple closed curves γ_i, γ_j intersecting exactly once lives inside a punctured Möbius strip as shown in Figure 4. They uniquely induce a two-sided simple closed curve as a boundary component (with the other boundary component peripheral in S).
The trace identity in $GL(2,\mathbb{C})$

$$\text{tr } A_iA_j + \det A_j \cdot \text{tr } A_iA_j^{-1} = \text{tr } A_i \cdot \text{tr } A_j \tag{12}$$

relates the complex lengths of peripheral curves α and β of a punctured Möbius strip to the complex lengths of γ_i and γ_j:

$$\cosh\left(\tfrac{1}{2}\ell_\alpha\right) + \cosh\left(\tfrac{1}{2}\ell_\beta\right) = 2\sinh\left(\tfrac{1}{2}\ell_{\gamma_i}\right)\sinh\left(\tfrac{1}{2}\ell_{\gamma_j}\right). \tag{13}$$

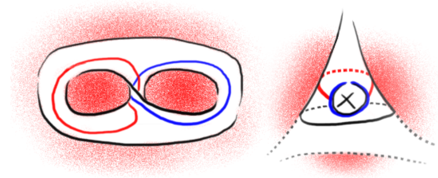

FIGURE 4. Punctured Möbius strip

Since any 2-sided geodesic on a 3-cusped projective plane necessarily bounds a pair of pants, equation (13) allows one to obtain the length of any 2-sided geodesic from the length spectrum of simple one-sided geodesics. Hence, the algorithm which generates the length spectrum of all the one-sided simple closed geodesics on a 3-cusped projective plane can be used to generate the length spectrum of all the two-sided simple closed geodesics.

1.2. **The Curve Complex.** Consider the geometric realisation of the abstract simplicial complex Ω^* with its n-simplices given by subsets of $n+1$ distinct isotopy classes of one-sided simple closed curves in S that pairwise intersect once. Identifications of simplices as the faces of higher dimensional simplicies is given by inclusion. This is a pure simplicial 3-complex, and its 1-skeleton has been previously described by Scharlemann [18] as being the 1-skeleton of the cell complex formed from a tetrahedron by repeated stellar subdivision of the faces, but not the edges.

The curve complex Ω that we're concerned with is the dual of Ω^*. The decision to take the dual accords with Bowditch's conventions in [2, 3]. We now describe and assign notation for the cells of Ω.

The vertices, or 0-cells of Ω are:

$$\Omega^0 := \left\{ \{\alpha,\beta,\gamma,\delta\} \;\middle|\; \begin{array}{l} \alpha,\beta,\gamma,\delta \text{ are isotopy classes of one-sided simple closed curves} \\ \text{that pairwise geometrically intersect once} \end{array} \right\}$$

The edges, or 1-cells of Ω are:

$$\Omega^1 := \left\{ \{\alpha,\beta,\gamma\} \;\middle|\; \begin{array}{l} \alpha,\beta,\gamma \text{ are isotopy classes of one-sided simple closed curves} \\ \text{that pairwise geometrically intersect once} \end{array} \right\}$$

Observe that each edge may be interpreted as a *flip* from one 0-cell to another. Hence, the 1-skeleton of Ω is a 4-regular tree (i.e. each vertex has degree 4). Further, the connectedness of this cell-complex described in [18, Theorem 3.1] means that flips generate all possible 0-cells, and hence all one-sided simple closed geodesics.

The faces, or 2-cells of Ω are:

$$\Omega^2 := \left\{ \{\alpha,\beta\} \;\middle|\; \begin{array}{l} \alpha,\beta \text{ are isotopy classes of one-sided simple closed curves} \\ \text{that intersect geometrically once} \end{array} \right\}$$

It follows from the observation in the previous subsection regarding punctured Möbius strips embedded in S that each face represents a unique isotopy class of essential, non-peripheral two-sided simple closed curves on S.

The 3-cells Ω^3 of Ω are:

$$\Omega^3 := \{\, \{\alpha\} \mid \alpha \text{ is an isotopy class of one-sided simple closed curves}\}$$

Later, we will sometimes denote 3-cells by capital letters, and use:

$$\vec{\Omega}^1 = \{\vec{e} = \{\alpha, \beta, \gamma; \delta' \to \delta\} \mid \{\alpha, \beta, \gamma\} \in \Omega^1\}.$$

to denote the collection of *oriented edges* of Ω. In particular, $\{\alpha, \beta, \gamma; \delta' \to \delta\}$ points from $\{\delta'\}$ to $\{\delta\}$. Figure 5 illustrates the local geometry of an oriented edge.

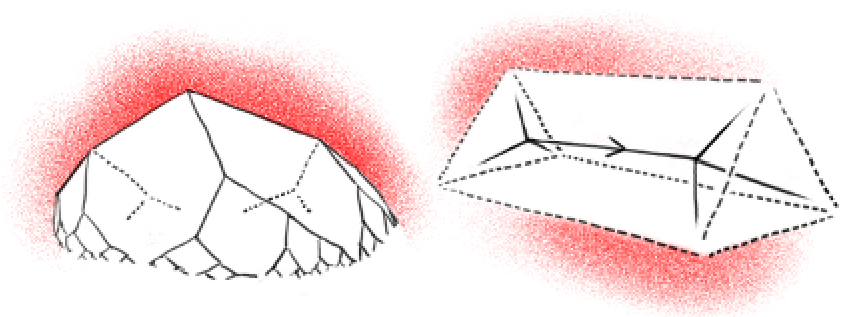

FIGURE 5. A 3-cell (left) and an oriented edge (right).

1.3. **Markoff Maps.** Given a representation $\rho : F_3 \to GL(2, \mathbb{C})$ satisfying the trace condition (8) and the determinant condition (9), we will use a Greek letter for any simple closed curve and the corresponding Latin letter for the trace of the image of the conjugacy class it defines. We decorate Ω with trace data by assigning to every 3-cell $\{\alpha\} \in \Omega^3$ its corresponding trace: $\operatorname{tr} \rho(\alpha) = a$. Thus defining a function:

$$\phi : \Omega^3 \to \mathbb{C} \text{ by } \phi(\alpha) = \operatorname{tr} \rho(\alpha),$$

where $[\alpha]$ is the conjugacy class defined by the one-sided simple closed curve α. Lower-dimensional simplices in Ω may then by interpreted as mnemonics for encoding the following relations:

Vertex relation: for $\{\alpha, \beta, \gamma, \delta\} \in \Omega^0$, (1) is equivalent to:

$$\frac{d}{a+b+c+d} = \frac{a+b+c+d}{abc}. \tag{14}$$

where $a = \operatorname{tr} \rho[\alpha]$, $b = \operatorname{tr} \rho[\beta]$, $c = \operatorname{tr} \rho[\gamma]$ and $d = \operatorname{tr} \rho[\delta]$. Note that the set of values of any four 3-cells which meet at a vertex corresponds to a Markoff quad.

Edge relation: an edge $e = (\alpha, \beta, \gamma) \in \Omega^1$ lies in the intersection of the two 0-cells $(\alpha, \beta, \gamma, \delta)$ and $(\alpha, \beta, \gamma, \delta')$, and (11) yields:

$$\frac{a+b+c+d}{abc} + \frac{a+b+c+d'}{abc} = 1, \tag{15}$$

where $d' = \operatorname{tr} \rho[\delta']$ and the others are as previously defined. Since each edge joins two vertices, the edge relation therefore tells us how to flip from one Markoff quad to another.

Face relation: given $\{\alpha, \beta\} \in \Omega^2$ and ϵ the unique non-peripheral two-sided simple closed isotopy class disjoint from α and β, from (12) we have:

$$ab = e + 2 \tag{16}$$

where $a = \text{tr}\ \rho[\alpha]$, $b = \text{tr}\ \rho[\beta]$ and $e = \text{tr}\ \rho[\epsilon]$.

These three relations allow us to generate the simple trace spectrum for ρ from a starting Markoff quad: the vertex and edge relations generate the simple trace spectrum for all the one-sided curves and the face relation then produces the simple trace spectrum for all of the two-sided curves.

Thus, we're led to consider general maps $\phi : \Omega^3 \to \mathbb{C}$ satisfying the edge and vertex relations. We call such functions *Markoff maps*, and let Φ denote the collection of all Markoff maps. In keeping with our notation for representations, we use Greek and Latin letters respectively for 3-cells (1-sided curves) and their image under some $\phi \in \Phi$. Note that we restrict ourselves to nowhere-zero Markoff maps.

In order to establish facts about short geodesics on 3-cusped projective planes, we will want to consider subsets of the simple trace spectrum below specified values.

For $k \geqslant 0$ and $\phi \in \Phi$, define the set $\Omega^3_\phi(k) \subseteq \Omega^3$ by:

$$\Omega^3_\phi(k) := \left\{\{\alpha\} \in \Omega^3 \mid |\phi(\{\alpha\})| = |a| \leqslant k\right\}. \tag{17}$$

This set allows us to keep track of one-sided simple curves with trace less than k, and we similarly define $\Omega^2_\phi(k) \subset \Omega^2$ for two-sided simple curves. Every two-sided simple curve corresponds to a unique 2-cell $\{\alpha, \beta\}$ — the shared face of the two 3-cells $\{\alpha\}$ and $\{\beta\}$. We define:

$$\Omega^2_\phi(k) := \left\{\{\alpha, \beta\} \in \Omega^2 \mid |\phi(\{\alpha\})\phi(\{\beta\})| = |ab| \leqslant k\right\}. \tag{18}$$

Note that in using $|ab|$ instead of $|ab - 2|$ for the conditions imposed, the set $\Omega^2_\phi(k)$ doesn't quite correspond to the set of two-sided simple curves with trace less than k. Although we will find this definition more suited to our analysis. In addition, we later focus on the following collection of Markoff maps $\Phi_{BQ} \subset \Phi$:

$$\Phi_{BQ} := \left\{ \phi \in \Phi \;\middle|\; \begin{array}{l} \Omega^2_\phi(k) \text{ is finite for any } k, \\ \text{and for any } \{\alpha, \beta\} \in \Omega^2_\phi(4),\ ab \notin [0, 4] \end{array} \right\}. \tag{19}$$

We show in section 3.2 that these are sufficiently conditions to guarantee the existence of a McShane identity for a given Markoff map. These conditions are similar to Bowditch's BQ-condition, which is a conjectural trace-based characterisation of quasi-Fuchsian representations. The following result shows that our condition is also necessary for quasi-Fuchsian representations of the thrice-punctured projective plane.

Lemma 4. *Markoff maps obtained from quasi-Fuchsian representations lie in Φ_{BQ}.*

Proof. Given a Markoff map ϕ arising from a quasi-Fuchsian representation ρ, consider the multiset of complex numbers obtained from evaluating ϕ on $\Omega^3_\phi(m)$.

Since this multiset is a subset of the simple trace spectrum of ρ, which is obtained (up to sign) from taking $2\sinh(\frac{1}{2}\cdot)$ of the simple length spectrum, the discreteness of the simple length spectrum ensures that $\Omega^3_\phi(m)$ is finite. Any 2-cell in $\Omega^2_\phi(m)$ is the intersection of precisely one pair of 3-cells, hence the cardinality of $\Omega^2_\phi(m)$ is bounded by the square of the cardinality of $\Omega^3_\phi(m)$ and is finite.

Next, if $ab \in [0,4]$ for some $\{a,b\} \in \Omega^2$, then there is a 2-sided non-peripheral homotopy class ϵ whose trace is $ab-2 \in [-2,2]$, thus contradicting the fact that quasi-Fuchsian representations have neither parabolics nor elliptics. $\square$

2. Analysis on the Curve Complex

Given a Markoff map ϕ, for every edge $e = \{\alpha,\beta,\gamma\}$ fix an oriented edge

$$\vec{e} = \{\alpha,\beta,\gamma;\delta' \to \delta\} \text{ to satisfy } |d'| \geqslant |d|,$$

where as usual $(a,b,c,d) = (\phi(\alpha),\phi(\beta),\phi(\gamma),\phi(\delta))$. For most edges, this choice is canonical, and for edges with equality in $|d'| = |d|$, an arbitrary orientation is chosen. This produces an orientation on Ω^1, where edges may be thought of as pointing from 3-cells corresponding to longer geodesics to 3-cells corresponding to shorter geodesics. Thus, analysis of the dynamics (in terms of the directions) of these edges informs us about the behavior of geodesic length growth for ϕ.

The following lemma gives alternative algebraic characterisations of this trace comparison.

Lemma 5. *For a Markoff quad $(a,b,c,d) \in \mathbb{C}^4$, the following conditions are equivalent:*

$$\mathrm{Re}\left(\tfrac{a+b+c+d'}{abc}\right) \geqslant \mathrm{Re}\left(\tfrac{a+b+c+d}{abc}\right) \quad \Leftrightarrow \quad \mathrm{Re}\left(\tfrac{d'}{a+b+c+d'}\right) \geqslant \tfrac{1}{2} \quad \Leftrightarrow \quad |d'| \geqslant |d|.$$

Proof. The edge relation (15) proves the first equivalence. Furthermore (15) also proves that $\mathrm{Im}\left(\tfrac{a+b+c+d'}{abc}\right) = -\mathrm{Im}\left(\tfrac{a+b+c+d}{abc}\right)$ hence the first inequality is equivalent to

$$\left|\frac{a+b+c+d'}{abc}\right| \geqslant \left|\frac{a+b+c+d}{abc}\right|$$

which is equivalent to

$$\left|\frac{(a+b+c+d')^2}{abc}\right| \geqslant \left|\frac{(a+b+c+d)^2}{abc}\right|.$$

By the vertex relation (14), this is precisely $|d'| \geqslant |d|$. $\square$

2.1. Local analysis.

Definition 1. *Call a vertex with all outward pointed oriented edges a* source*, a vertex with all inwardly pointed oriented edges a* sink *and a vertex with precisely one outwardly pointed oriented edge a* funnel*. The remaining two types of vertices are called* saddles.

Lemma 6. *There are no sources.*

Proof. Given such a vertex with adjoining 3-cells A, B, C, D, the vertex relation (14) gives:

$$\begin{aligned} 1 =& \mathrm{Re}\left(\frac{a+b+c+d}{abc}+\frac{a+b+c+d}{abd}+\frac{a+b+c+d}{acd}+\frac{a+b+c+d}{bcd}\right) \\ \geqslant& \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=2 \end{aligned}$$

which is a contradiction. □

Remark 1. In the Fuchsian case, Markoff maps take on real positive values and the proof of Lemma 6 shows that any vertex has at most one outgoing edge. This means that Fuchsian Markoff maps can have at most one sink (given some choice of orientation). We later show in theorem 11 that a sink always exists, and this may be geometrically interpreted as saying that there is a unique (up to permutation) Markoff quad (a, b, c, d) for any 3-cusped projective plane where the flips of a, b, c of d are (non-strictly) longer. However, the Markoff quad $(a, b, c, d) = \frac{i}{\sqrt{2}}(1, 1, 1, -2)$ is an example of a vertex with three outgoing edges.

Lemma 7. *At a sink vertex, a Markoff quad contains an element of magnitude less than or equal to 4.*

Proof. Let the real parts of

$$\frac{a+b+c+d}{abc}, \frac{a+b+c+d}{abd}, \frac{a+b+c+d}{acd}, \frac{a+b+c+d}{bcd},$$

respectively be $s \leqslant r \leqslant q \leqslant p$.

Then the sink part tells us that $p \leqslant \frac{1}{2}$ and the size ordering and the fact that $s + r + q + p = 1$ tells us that $p \geqslant \frac{1}{4}$. And we also know that the next largest number $q \geqslant \frac{1}{3}(1 - p)$. Therefore,

$$pq \geqslant \frac{1}{3}p(1-p) \geqslant \frac{1}{3} \times \frac{1}{4} \times \frac{3}{4} = \frac{1}{16}.$$

Then, we see that:

$$\frac{1}{|cd|} = |\frac{a+b+c+d}{bcd}| \times |\frac{a+b+c+d}{acd}| \geqslant pq \geqslant \frac{1}{16}.$$

Therefore, $|cd| \leqslant 16$ and the lesser of the magnitudes of these two traces must be less than or equal to 4. □

Lemma 8. *Given a saddle vertex* $\{\alpha, \beta, \gamma, \delta\}$ *with two outgoing oriented edges*

$$\{\alpha, \beta, \gamma; \delta' \to \delta\} \text{ and } \{\alpha, \beta, \delta : \gamma' \to \gamma\},$$

then the 2-cell $\{\alpha, \beta\}$ *lies in* $\Omega^2_\phi(4)$ *and at least one of* $\{\alpha\}, \{\beta\}$ *lies in* $\Omega^3_\phi(2)$.

Proof. The outwards pointing condition tells us that:

$$\left|\frac{c}{a+b+c+d}\right| \geqslant \frac{1}{2} \text{ and } \left|\frac{d}{a+b+c+d}\right| \geqslant \frac{1}{2}.$$

Multiplying these terms together, we have:

$$\left|\frac{cd}{(a+b+c+d)^2}\right| = \frac{1}{|ab|} \geqslant \frac{1}{4} \Rightarrow |ab| \leqslant 4 \Rightarrow \min\{|a|, |b|\} \leqslant 2.$$

□

2.2. **Global analysis.**

Lemma 9. *For $k \geqslant 2$, the cell complex comprised of all the 3-cells in $\Omega^3_\phi(k)$ is connected.*

Proof. Assume that $\Omega^3_\phi(k)$ isn't connected and consider a shortest path of oriented edges between two distinct connected components:

$$\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{p-1}, \vec{e}_p.$$

Note that by assumption, any 3-cell X that contains one of these edges must satisfy $|\phi(X)| > k$.

If $p = 1$, then $\vec{e}_1 = \{\alpha, \beta, \gamma; \delta \to \delta'\}$ such that $|d|, |d'| \leqslant k$. Then the edge relation (15) gives $abc = (a+b+c+d) + (a+b+c+d')$ hence:

$$\begin{aligned}\sqrt{k^3} < \sqrt{|abc|} \leqslant & \frac{|a+b+c+d| + |a+b+c+d'|}{\sqrt{|abc|}} \\ = & \sqrt{|d|} + \sqrt{|d'|} \leqslant 2\sqrt{k} \\ \Rightarrow & \, k^3 \leqslant 4k \Rightarrow k \leqslant 2,\end{aligned}$$

where the first equality uses the vertex relation (14). This contradicts the assumption.

On the other hand, if $p \geqslant 2$, then $\vec{e}_1$ must point away from $\vec{e}_2$ and $\vec{e}_p$ must point away from $\vec{e}_{p-1}$. But this means that at least one of the interior vertices of the path $\{\vec{e}_n\}_{n=1,\dots,p}$ must have two arrows pointing away from it, and hence by Lemma 8 one of the adjacent 3-cells X of this vertex must satisfy $|\phi(X)| \leqslant 2$, thus contradicting the assumption. □

Lemma 10. *Given an infinite ray of oriented edges $\{\vec{e}_n\}_{n\in\mathbb{N}}$ such that each $\vec{e}_n$ is directed towards $\vec{e}_{n+1}$, then this ray either:*

1. *eventually spirals along the boundary of some 2-cell $\{\xi, \eta\} \in \Omega^2_\phi(4)$, or*
2. *eventually enters and remains on the boundary of some 3-cell $\{\xi\} \in \Omega^3_\phi(2)$, or*
3. *there are infinitely many 3-cells in $\Omega^3_\phi(2)$.*

Proof. We begin by four-coloring Ω^3 with the colors $\alpha, \beta, \gamma, \delta$. In particular, we label the 3-cells meeting $\{\vec{e}_n\}$ by $\{\alpha_i\}, \{\beta_j\}, \{\gamma_k\}, \{\delta_l\}$ where the letter type is determined by the color of the cell and the subscripts grow according to how early we encounter this 3-cell as we traverse along $\{\vec{e}_n\}$.

At each vertex along $\{\vec{e}_n\}$, we encounter six 2-cells of different color-types:

$$\{\alpha\beta, \alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta, \gamma\delta\}.$$

Now consider all the 2-cells that we encounter as we go along $\{\vec{e}_n\}$. Since $\{\vec{e}_n\}$ does not repeat its edges, if we ever meet only finitely many 2-cells of a certain color-type, then $\{\vec{e}_n\}$ eventually just stays on the last 2-cell we meet of that cell-type. This also means that we can't meet only finitely many 2-cells of two different color-types because we must then stay on two distinct 2-cells - impossible because the intersection of any two 2-cells is either empty or consists of a single edge.

Assume that we encounter only finitely many 2-cells of the (wlog) $\alpha\beta$ color-type and that the 2-cell that we stay on is (with a little notation abuse) $\{\alpha, \beta\}$. This means that the vertices of $\{\vec{e}_n\}$ eventually take the following form:

$$\dots, \{\alpha, \beta, \gamma_i, \delta_j\}, \{\alpha, \beta, \gamma_{i+1}, \delta_j\}, \{\alpha, \beta, \gamma_{i+1}, \delta_{j+1}\}, \{\alpha, \beta, \gamma_{i+2}, \delta_{j+1}\}, \dots$$

and the sequences $\{|c_i|\}, \{|d_j|\}$ are (monotonically) non-increasing due to the directions of the oriented edges. Alternatively, we phrase this as the statements that:

$$\mathrm{Re}\left(\frac{c_i}{a+b+c_i+d_j}\right) \geqslant \frac{1}{2} \text{ and } \mathrm{Re}\left(\frac{d_j}{a+b+c_{i+1}+d_j}\right) \geqslant \frac{1}{2},$$

noting that the latter statement implies that:

$$2\sqrt{|d_j|} \geqslant \frac{|a+b+c_{i+1}+d_j|}{\sqrt{|d_j|}} = \sqrt{|abc_{i+1}|}.$$

Now, if the sequence $\{|c_i|\}$ is bounded below by 2, it must converge. Thus for any $\epsilon > 0$, by choosing i to be sufficiently large, $\sqrt{|c_{i+1}|} \leqslant \sqrt{|c_i|} \leqslant \sqrt{|c_{i+1}|} + \epsilon$. Then the edge relation (15) for $\{\alpha, \beta, \delta_j; \gamma_i \to \gamma_{i+1}\}$ tells us that:

$$\begin{aligned}\sqrt{|abd_j|} &\leqslant \frac{|a+b+c_i+d_j|}{\sqrt{|abd_j|}} + \frac{|a+b+c_{i+1}+d_j|}{\sqrt{|abd_j|}} \\ &= \sqrt{|c_i|} + \sqrt{|c_{i+1}|} \leqslant 2\sqrt{|c_{i+1}|} + \epsilon.\end{aligned}$$

Combining this with the inequality above, we see that:

$$|ab| \leqslant \frac{4\sqrt{|c_{i+1}|} + 2\epsilon}{\sqrt{|c_{i+1}|}} \leqslant 4 + \sqrt{2}\epsilon.$$

Therefore, $|ab| \leqslant 4$.

We have now covered the case where we meet only finitely many 2-cells of one of the color-types. The alternative is that we meet infinitely many 2-cells of all six color-types, and we produce from this four sequences of 3-cells:

$$\{\{\alpha_i\}\}, \{\{\beta_j\}\}, \{\{\gamma_k\}\}, \{\{\delta_l\}\}$$

Now, the second case arises when one of these sequences is finite — that is, we stick to the surface of some 3-cell. Assume wlog that this is for the color α, and by truncating our ray (and abusing notation), we may take $a_j = a = \phi(\{\alpha\})$ for all j. Moreover, unless we're in case 3, we may further truncate our ray so that the non-increasing sequence $\{|b_j|\}, \{|c_k|\}, \{|d_l|\}$ remains bounded above 2. Then the same analysis tells us that:

$$|ab_i|, |ac_j|, |ac_k| \to 4, \tag{20}$$

and we can see from this that $|a| \leqslant 2$.

Finally, in the case that we meet infinitely many 3-cells of every color-type, assume that the monotonically non-increasing sequences $\{|a_i|\}, \{|b_j|\}, \{|c_k|\}, \{|d_l|\}$ are bounded below by 2 and hence converge. The same analysis as in case one tells us that

$$|a_i b_j|, |a_i c_k|, |a_i d_l|, |b_j c_k|, |b_j d_l|, |c_k d_l| \to 4,$$

and since these numbers are the bound below by 2, we see that:

$$|a_i|, |b_j|, |c_k|, |d_l| \to 2.$$

Now, for the oriented edge $\{\alpha, \beta, \gamma; \delta \to \delta'\}$ sufficiently far along $\{\vec{e}_n\}$ so that $|a|, |b|, |c|, |d|, |d'|$ are each close to 2, the edge relation (15)

$$\frac{a+b+c+d}{abc} + \frac{a+b+c+d'}{abc} = 1$$

tells us that:

$$\frac{a+b+c+d}{abc}, \frac{a+b+c+d'}{abc} \approx \frac{1}{2}$$

By symmetry, this also holds for:

$$\frac{a+b+c+d}{abc}, \frac{a+b+c+d}{abd}, \frac{a+b+c+d}{acd}, \frac{a+b+c+d}{bcd} \approx \frac{1}{2}.$$

By mutliplying pairs of these terms and invoking the vertex relation (14), we obtain that:

$$ab, ac, ad, bc, bd, cd \approx 4,$$

and hence either a, b, c, d are approximately all 2 or all -2. But the vertex relation (14) then tells us that

$$64 \approx (a+b+c+d)^2 \approx abcd \approx 16,$$

giving us the desired contradiction for our assumption that these sequences could be bounded below by 2.

In particular, this shows us that we must touch some 3-cell in $\Omega^3_\phi(2)$, and the subsequent infinitely many 3-cells of the same color as X must all be in $\Omega^3_\phi(2)$. □

Theorem 11. *The set of 3-cells $\Omega^3_\phi(4)$ is non-empty. Further, if $\Omega^3_\phi(2) = \varnothing$, then there is a unique sink.*

Proof. If $\Omega^3_\phi(2)$ is non-empty then we're done. But if it is empty, then lemma 10 tells us that following oriented edges according to their directions must eventually result in a sink. If there are multiple sinks, they obviously cannot be distance 1 from each other. And one of the interior vertices of any path joining two sinks must have two arrows coming out of it and hence by lemma 8, $\Omega^3_\phi(2) \neq \varnothing$. □

We obtain our first main theorem as a corollary:

Theorem 1. *The maximum of the systole length function over the moduli space of all hyperbolic thrice punctured projective planes is* $2\,\mathrm{arcsinh}(2)$.

Proof. By Theorem 11, $\Omega^3_\phi(4)$ is non-empty: on any hyperbolic surface X there exists a one-sided simple geodesic γ with $\mathrm{tr}\,A = 2\sinh\frac{1}{2}\ell_\gamma(X) \leqslant 4$ and hence $\ell_\gamma(X) \leqslant 2\,\mathrm{arcsinh}(2)$. Thus, the maximum of the systole length function over the moduli space of all hyperbolic thrice punctured projective planes is less than or equal to $2\,\mathrm{arcsinh}(2)$.

To prove equality, first assume that the Markoff quad $(4, 4, 4, 4)$ arises from a Fuchsian representation. Then, any new Markoff quad generated from $(4, 4, 4, 4)$

must be integral and with each entry a positive multiple of 4. Hence, the corresponding Markoff map has 4 as its minimum. This in turn means that the shortest one-sided geodesic has length 2arcsinh(2). On the other hand, the shortest two-sided geodesic has trace $14 = 4\times 4 - 2$, is of length $2\operatorname{arccosh}(7) > 2\operatorname{arcsinh}(2)$ and hence does not give the shortest geodesic.

It remains to prove the existence of a 3-cusped projective plane Y corresponding to the Markoff quad $(4,4,4,4)$. A $\mathrm{PGL}_2(\mathbb{Z})$-representation for Y is given by:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 5 & 2 \\ -2 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}, A_4 = \begin{bmatrix} 5 & -2 \\ 2 & -1 \end{bmatrix}.$$

We can recognise Y geometrically due to its large symmetry group. Consider the spherical symmetric octahedron: the octahedron on the round two-sphere S^2 with great circle edges and full A_4 symmetry. Label the 6 vertices of this octahedron to get a symmetric element Σ in the moduli space $\mathcal{M}_{0,6}$, and note that the 6 labeled points are invariant under the antipodal map. There exists a unique hyperbolic cusped surface X with conformal structure Σ. And by the uniqueness of X, the vertex-fixing antipodal maps on S^2 uniformise to isometric $\mathbb{Z}_2$-actions on X. The 4 greater circles on S^2 which lie in the plane orthogonal to the vector between the centers of any two opposing faces uniformise to simple closed geodesics $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in X$. By symmetry, each γ_i is invariant under antipodal $\mathbb{Z}_2$-actions and descends to a geodesic $\overline{\gamma}_i \in Y = X/\mathbb{Z}_2$, where $Y = X/\mathbb{Z}_2$ is the desired 3-cusped projective plane. By symmetry, the four geodesics $\overline{\gamma}_i$ in Y have the same length, and hence their traces give rise to a Markoff quad (ℓ,ℓ,ℓ,ℓ) which is necessarily $(4,4,4,4)$. □

3. Divide and Conquer

Starting with a 4-tuple of simple closed one-sided curves $\{\alpha, \beta, \gamma_0, \gamma_1\}$, consider the sequence of 4-tuples produced by repeatedly flipping γ_{2i} to get γ_{2i+2} and γ_{2i+1} to get γ_{2i+3}.

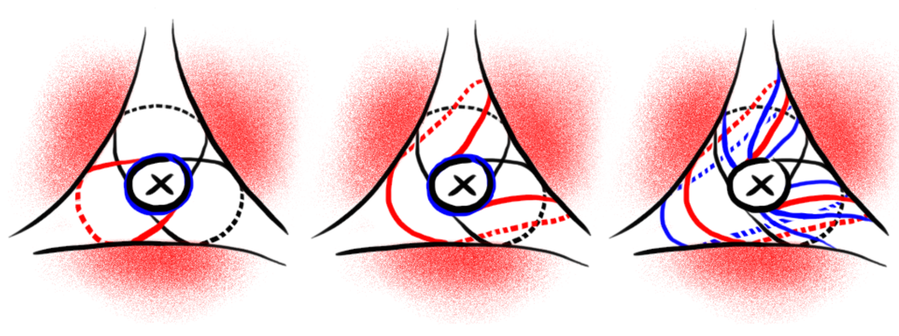

Figure 6. Flipping γ_{2i} followed by γ_{2i+1}.

In this manner, we obtain a sequence of Markoff quads:

$$(a, b, c_0, c_1), (a, b, c_2, c_1), (a, b, c_2, c_3), (a, b, c_4, c_3), (a, b, c_4, c_5), \ldots$$

By comparing the vertex relation (14) at $\{\alpha, \beta, \gamma_k, \gamma_{k+1}\}$ and the edge relation (15) at $\{\alpha, \beta, \gamma_k\}$, we obtain that:

$$\frac{c_{k-1}}{a+b+c_{k-1}+c_k} = \frac{c_k}{a+b+c_k+c_{k+1}} + \frac{a}{a+b+c_k+c_{k+1}} + \frac{b}{a+b+c_k+c_{k+1}}.$$

In this decomposition for $\frac{c_{k-1}}{a+b+c_{k-1}+c_k}$, there is another summand of the same form. Thus, starting with the the vertex relation (14), we may iteratively decompose the $\frac{c_{k-1}}{a+b+c_{k-1}+c_k}$ term to obtain:

$$\begin{aligned}
1 &= \frac{c_0}{a+b+c_0+c_1} + \tfrac{c_1}{a+b+c_0+c_1} + \tfrac{a}{a+b+c_0+c_1} + \tfrac{b}{a+b+c_0+c_1} \\
&= \frac{c_1}{a+b+c_1+c_2} + \tfrac{a}{a+b+c_0+c_1} + \tfrac{b}{a+b+c_0+c_1} + \tfrac{a}{a+b+c_1+c_2} + \tfrac{b}{a+b+c_1+c_2} + \tfrac{c_1}{a+b+c_0+c_1} \\
&\quad \vdots \\
&= \frac{c_n}{a+b+c_n+c_{n+1}} + \sum_{i=0}^{n} \left(\tfrac{a}{a+b+c_i+c_{i+1}} + \tfrac{b}{a+b+c_i+c_{i+1}} \right) + \tfrac{c_1}{a+b+c_0+c_1}.
\end{aligned}$$

Since the edge relation (15) for $\{\alpha, \beta, \gamma_n\}$ is a second order difference equation, we may explicitly calculate

$$\lim_{n\to\infty} \frac{c_n}{a+b+c_n+c_{n+1}} = \frac{1}{1+\lambda}, \text{ where } \lambda = \frac{ab-2+\sqrt{ab(ab-4)}}{2}.$$

Since we may apply this splitting algorithm to every summand that arises, we might intuitively expect that this gives a series that sums to one, whose summands each take the form $\frac{1}{1+\lambda}$ for some λ — this series is the McShane identity.

In the Fuchsian case, these summands are real numbers and correspond to the lengths of intervals in the length 1 horocycle around one of the cusps on X. The complement of all of these intervals is, by construction, the union of a Cantor set and a countable set. Some analysis is needed to show that this Cantor set is of measure 0 for us to conclude that the series sums to 1. In [2, 3], Bowditch shows that one may bound the measure of this Cantor set by smaller and smaller tails of the series. This is of course contingent upon the convergence of this series of $\frac{1}{1+\lambda}$, which can be done for BQ-Markoff maps by understanding the growth rate of its simple geodesic spectrum and showing that it grows sufficiently quickly.

Specifically, for any Markoff quad, solve for d to get

$$d = \frac{abc}{4}\left(1 \pm \sqrt{1-4(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc})}\right)^2.$$

For $|a|$, $|b|$, $|c|$ large, choose d to be the larger of the two solutions, hence

$$\log|d| \approx \log|a| + \log|b| + \log|c|.$$

In particular, $|d|$ is greater than $|a|$, $|b|$ and $|c|$ and this shows the growth of Markoff maps—if we continue and flip from a to a', then $\log|a| < \log|a'| \approx \log|b| + \log|c| + \log|d|$. This gives rise to the notion of *Fibonacci growth* for Markoff quads—in keeping with Bowditch's Fibonacci growth for Markoff triples [3].

3.1. **Fibonacci Growth.** Given an edge $e \in \Omega^1$, define the *Fibonacci function* $F_e : \Omega^3 \to \mathbb{R}$ by:

(i) $F_e(\alpha) = 1$ if $e = \{\alpha, \beta, \gamma\}$.
(ii) For $\{\alpha, \beta, \gamma, \delta \to \delta'\} \in \vec{\Omega}^1$ oriented so that it points away from e (or is either of the two possible oriented edges for e itself)

$$F_e(\{\delta\}) = F_e(\{\alpha\}) + F_e(\{\beta\}) + F_e(\{\gamma\}).$$

Hence $F_e : \Omega^3 \to \mathbb{R}$ takes the value of 1 for the three 3-cells in Ω^3 that contain e and subsequently define values for the rest of the tree by assigning to every hitherto unassigned 3-cell meeting three assigned 3-cells at some vertex the sum of the values of those already assigned 3-cells.

Definition 2. *Given a function* $f : \Omega^3 \to [0, \infty)$ *and* $\Omega' \subset \Omega^3$*, we say that* f *has:*

- *a* lower Fibonacci bound *on* Ω' *if there's some positive* κ *such that:*
$$\frac{1}{\kappa} F_e(X) \leqslant f(X) \textit{ for cofinitely many } X \in \Omega';$$
- *an* upper Fibonacci bound *on* Ω' *if there's some positive* κ *such that:*
$$f(X) \leqslant \kappa F_e(X) \textit{ for all } X \in \Omega';$$
- Fibonacci growth *on* Ω' *if there's some positive* κ *such that:*
$$\frac{1}{\kappa} F_e(X) \leqslant f(X) \leqslant \kappa F_e(X) \textit{ for cofinitely many } X \in \Omega';$$
or in other words: it has both lower and upper Fibonacci bound. We also opt to omit "on Ω'" *whenever* $\Omega' = \Omega^3$.

We assumed the choice of an edge e for these definitions, and now show that the existence of a κ satisfying these conditions is independent of this choice.

Lemma 12. *Given some edge* e *that is the intersection of the three* 3*-cells* X_1, X_2, X_3 *and a function* $f : \Omega^3 \to [0, \infty)$ *satisfying:*

$$f(D) \leqslant f(A) + f(B) + f(C) + 2c, \ 0 \leqslant c,$$

where A, B, C, D *meet at the same vertex and* D *is strictly farther from* e *than* A, B, C*. Then:*

$$f(X) \leqslant (M + c)F_e(X) - c, \textit{ for all } X \in \Omega^3,$$

where $M = \max\{f(X_1), f(X_2), f(X_3)\}$.

Proof. We prove this by induction on the distance of a region from e. The base case is due to:

$$f(X_i) \leqslant (\max\{f(X_1), f(X_2), f(X_3)\} + c) - c.$$

The induction step is similarly established:

$$f(D) \leqslant (M + c)(F_e(A) + F_e(B) + F_e(C)) - 3c + 2c = (M + c)F_e(D) - c.$$

□

Note that by essentially the same proof, we obtain the following result:

Lemma 13. *Given some edge* e *that is the intersection of the three* 3*-cells* X_1, X_2, X_3 *and a function* $f : \Omega^3 \to [0, \infty)$ *satisfying:*

$$f(D) \geqslant f(A) + f(B) + f(C) - 2c,\ 0 \leqslant c < m := \min\{f(X_1), f(X_2), f(X_3)\},$$

where A, B, C, D *meet at the same vertex and* D *is strictly farther from* e *than* A, B, C*. Then:*

$$f(X) \geqslant (m - c)F_e(X) + c, \text{ for all } X \in \Omega^3.$$

However, this is insufficient for our purposes. We shall require:

Lemma 14. *Given some oriented edge* $\vec{e}$ *that is the intersection of the three* 3*-cells* X_1, X_2, X_3 *and a function* $f : \Omega^3 \to [0, \infty)$ *satisfying:*

$$f(D) \geqslant f(A) + f(B) + f(C) - 2c,\ 0 \leqslant c < \mu := \min\{f(X_i) + f(X_j)\}_{i \neq j},$$

where A, B, C, D *meet at the same vertex and* D *is strictly farther from* e *than* A, B, C*. Then:*

$$f(X) \geqslant (\mu - 2c)F_e(X) + c, \text{ for all } X \in \Omega^3_-(\vec{e}) - \Omega^3_0(\vec{e}).$$

Proof. We first use induction to show that any two adjacent 3-cells in $\Omega^3_-(\vec{e})$ satisfy:

$$f(X) + f(Y) \geqslant (\mu - 2c)(F_e(X) + F_e(Y)) + 2c.$$

The base case where X and Y are both in $\Omega_0(e)$ follows from the definition of μ. We proceed by induction on the total distance of X and Y from e. Assume that Y is farther than X from e. The tree structure of $\Omega^3_-(\vec{e})$ means that there is a unique closest vertex between the edge e and the face $X \cap Y$. Denote the two other 3-cells at this vertex by W and Z, we then have:

$$\begin{aligned} f(X) + f(Y) \geqslant& f(X) + f(W) + f(X) + f(Z) - 2c \\ \geqslant& (\mu - 2c)(F_e(X) + F_e(W) + F_e(X) + F_e(Z)) + 4c - 2c \\ =& (\mu - 2c)(F_e(X) + F_e(Y)) + 2c, \end{aligned}$$

completing the induction.

Now consider a 3-cell $D \in \Omega^3_-(\vec{e}) - \Omega^3_0(e)$, and denote by A, B, C the three other 3-cells meeting D at the closest vertex between e and D. Then we have:

$$\begin{aligned} f(D) \geqslant& \frac{1}{2}(f(A) + f(B) + f(C) + f(D) - 2c) \\ \geqslant& \frac{1}{2}(\mu - 2c)(F_e(A) + F_e(B) + F_e(C) + F_e(D)) + 2c - c \\ =& (\mu - 2c)F_e(D) + c. \end{aligned}$$

□

Since for any edge e', the function $F_{e'}$ satisfies the criteria for these last two lemmas, we see that there is some $\kappa > 0$ such that:

$$\frac{1}{\kappa}F_e(X) \leqslant F_{e'}(X) \leqslant \kappa F_e(X), \text{ for all } X \in \Omega^3.$$

Which shows that our Definition 2 is indeed independent of the choice of the edge e.

Lemma 15. *If a function* $f : \Omega^3 \to \mathbb{R}^+$ *has a lower Fibonacci bound, then for any* $\sigma > 3$*, the following sum converges:*

$$\sum_{X \in \Omega} f(X)^{-\sigma} < \infty.$$

Proof. It suffices for us to show that this sum converges for $f = F_e$. We do this by bounding the growth of the level sets of F_e. We will prove that:

$$\operatorname{Card}\left\{ X \in \Omega^3 \mid F_e(X) = n \right\} < 2J_2(n) \tag{21}$$

where J_k is the Jordan totient function. For the remainder of this proof, we shall think of F_e not just as a function on the 3-cells Ω^3, but also as a function on the edges Ω^1: assigning to each edge $\{\alpha, \beta, \gamma\}$ the unordered 3-tuple

$$\{F_e(\{\alpha\}), F_e(\{\beta\}), F_e(\{\gamma\})\}.$$

When $n > 1$, there is a $1:3$ correspondence between

$$\{X \in \Omega^3 \mid F_e(X) = n\} \text{ and } \left\{\{\alpha, \beta, \gamma\} \in \Omega^1 \mid \max F_e(\{\alpha, \beta, \gamma\}) = n\right\}$$

defined by assigning to $X \in \Omega^3$ the three edges closest to e that lie on X. We can show by induction that the preimage of any unordered triple $\{l, m, n\}$ in the image of F_e has cardinality:

- 1, if $\{l, m, n\} = \{1, 1, 1\}$,
- 6, if $\{l, m, n\} = \{1, 1, n\}$ and
- 12, if $\{l, m, n\}$ are all distinct integers.

Thus, the relation given by assigning to a 3-cell X the unordered 3-tuples of the edges on X closest to e is at most $4:1$. Now, any triple $\{l, m, n\}$ that's in the image of F_e must be relatively prime. Otherwise, a common factor would inductively propagate back to e and contradict the starting value of $\{1, 1, 1\}$. Thus, for $n > 1$,

$$\operatorname{Card}\left\{X \in \Omega^3 \mid F_e(X) = n\right\} < 4\operatorname{Card}\{\{l, m, n\} \mid l, m < n \text{ and } \gcd(l, m, n) = 1\}$$

The right hand side is precisely $2J_2(n)$, and so (21) holds and we obtain:

$$\sum_{X \in \Omega} F_e(X)^{-\sigma} < \sum_{n \geqslant 1} 2J_2(n)n^{-\sigma} = \frac{2\zeta(\sigma-2)}{\zeta(\sigma)}.$$

As desired, the sum converges for $\sigma > 3$. □

These results enable us to conclude that: if the function

$$\log^+ |\phi| : \Omega^3 \to [0, \infty)$$

satisfies the following inequality at every vertex $\{a, b, c, d\} \in \Omega^0$:

$$\log^+ |d| \leqslant \log^+ |a| + \log^+ |b| + \log^+ |c| + 2\log\left(\frac{1+\sqrt{13}}{2}\right), \tag{22}$$

where $\log^+(x) := \max\{0, \log(x)\}$, then:

Claim: $\log^+ |\phi|$ has an upper Fibonacci bound on Ω^3.

Proof. By the preceding comment, we only need to show that (22) holds. To begin with, we see that when $|d| \leqslant 1$, the desired identity is trivially satisfied. We therefore confine ourselves to when $|d| > 1$, that is: when $\log|d| = \log^+|d|$. We now assume without loss of generality that $|a| \leqslant |b| \leqslant |c|$ and case-bash the desired result.

(1) If $1 \leqslant |a|, |b|, |c|$, then:

$$\begin{aligned}\log^+|d| = \log|d| = \log|abc| &+ 2\log\left|\frac{1}{2}\left(1 \pm \sqrt{1 - 4(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc})}\right)\right| \\ &\leqslant \log|abc| + 2\log\left|\frac{1}{2}\left(1 + \sqrt{1 + 4(\frac{1}{|ab|} + \frac{1}{|ac|} + \frac{1}{|bc|})}\right)\right| \\ &\leqslant \log|a| + \log|b| + \log|c| + 2\log\left(\frac{1+\sqrt{13}}{2}\right)\end{aligned}$$

(2) If $|a| < 1 \leqslant |b|, |c|$, then:

$$\begin{aligned}\log^+|d| = \log|bc| &+ 2\log\left|\frac{1}{2}\left(\sqrt{|a|} \pm \sqrt{1 - 4(\frac{|a|}{ab} + \frac{|a|}{ac} + \frac{|a|}{bc})}\right)\right| \\ &\leqslant \log|bc| + 2\log\left|\frac{1}{2}\left(\sqrt{|a|} + \sqrt{1 + 4(\frac{1}{|b|} + \frac{1}{|c|} + \frac{|a|}{|bc|})}\right)\right| \\ &\leqslant \log|b| + \log|c| + 2\log\left(\frac{1+\sqrt{13}}{2}\right)\end{aligned}$$

(3) And similarly, if $|a|, |b| < 1 \leqslant |c|$, then:

$$\begin{aligned}\log^+|d| = \log|c| &+ 2\log\left|\frac{1}{2}\left(\sqrt{|ab|} \pm \sqrt{1 - 4(\frac{|ab|}{ab} + \frac{|ab|}{ac} + \frac{|ab|}{bc})}\right)\right| \\ &\leqslant \log|c| + 2\log\left|\frac{1}{2}\left(\sqrt{|ab|} + \sqrt{1 + 4(1 + \frac{|b|}{|c|} + \frac{|a|}{|c|})}\right)\right| \\ &\leqslant \log|c| + 2\log\left(\frac{1+\sqrt{13}}{2}\right)\end{aligned}$$

(4) And finally, if $|a|, |b|, |c| < 1$, then:

$$\begin{aligned}\log^+|d| =& 2\log\left|\frac{1}{2}\left(\sqrt{|abc|} \pm \sqrt{1 - 4(\frac{|abc|}{ab} + \frac{|abc|}{ac} + \frac{|abc|}{bc})}\right)\right| \\ \leqslant& 2\log\left(\frac{\sqrt{|ab|} + \sqrt{1 + 4(|a| + |b| + |c|)}}{2}\right) \leqslant 2\log\left(\frac{1+\sqrt{13}}{2}\right)\end{aligned}$$

□

We now aim to show that $\log|\phi|$ has a lower Fibonacci bound, and introduce the following notation: given an oriented edge $\vec{e}$ on Ω, the removal of the edge e from the tree in Ω results in two connected components. We denote the collection of

3-cells containing edges from the tree on the *head-side* of $\vec{e}$ by

$$\Omega^3_+(\vec{e}); \text{ and } \Omega^3_-(\vec{e})$$

for the the collection of 3-cells containing edges from the tree on the *tail-side* of $\vec{e}$. We also use the notation $\Omega^3_0(e) = \Omega^3_+(\vec{e}) \cap \Omega^3_-(\vec{e})$ to refer to the three edges containing e.

Lemma 16. *Given an oriented edge $\vec{e} \in \vec{\Omega}^1$ such that $\Omega^3_0(e) \cap \Omega^3_\phi(2) = \varnothing$, then $\Omega^3_\phi(2)$ lies on the head-side of e, that is:*

$$\Omega^3_\phi(2) \subseteq \Omega^3_+(\vec{e}).$$

Furthermore, all oriented edges in $\Omega^3_-(\vec{e})$ must point toward e.

Proof. Due to the connectedness of $\Omega^3_\phi(2)$, it must lie in either $\Omega^3_+(\vec{e})$ or $\Omega^3_-(\vec{e})$. If it lies on the tail side of $\vec{e}$ then consider a shortest path containing $\vec{e}$ and touching $\Omega^3_\phi(2)$. No 3-cell in $\Omega^3_\phi(2)$ may be in direct contact with $\vec{e}$ as this would force the region on the other end of $\vec{e}$ be in $\Omega^3_\phi(2)$ — yielding a contradiction.

Hence, we have a path of length at least 2 with outwardly oriented edges at the two end of this path. Thus resulting in at least one internal vertex with two outward pointing edges and hence an adjacent result in $\Omega^3_\phi(2)$. This then contradicts the shortest assumption we placed on our path.

We have shown that $\Omega^3_\phi(2)$ is on the head-side of $\vec{e}$ and by lemma 8, every vertex on the tail-side of $\vec{e}$ must have three incoming edges and one outgoing edge. Then lemma 10 forces all of these edges to point toward e. □

Lemma 17. *Given the hypotheses of the above result, define:*

$$\mu := \min\{\log^+|\phi(X_i)| + \log^+|\phi(X_j)| \mid X_k \in \Omega^3_0(\vec{e})\} > 2\log(2),$$

then for every tail-side 3-cell $X \in \Omega^3_-(\vec{e}) - \Omega^3_0(e)$, we have:

$$\log^+|\phi(X)| \geqslant (\mu - 2\log(2))F_e(X) + \log(2),$$

and hence $\log^+|\phi|$ has a lower Fibonacci bound over $\Omega^3_-(\vec{e})$.

Proof. Let $\{\alpha\},\{\beta\},\{\gamma\},\{\delta\} \in \Omega^3_-(\vec{e})$ be the adjacent 3-cells to an arbitrarily chosen tail-side vertex, such that $\{d\}$ is farthest from e. Then we know from every edge being naturally directed towards e that:

$$\sqrt{\frac{|d|}{|abc|}} \geqslant \frac{1}{2} \Rightarrow \log|d| \geqslant \log|a| + \log|b| + \log|c| - 2\log(2).$$

Since this is satisfied for every tail-side vertex, lemma 14 then gives the desired conclusion. □

Given a 2-cell $\{\alpha,\beta\} \in \Omega^2$, label the bi-infinite path bounding $\{\alpha,\beta\}$ by $\{e_n\}_{n\in\mathbb{Z}}$. Each edge e_n is the intersection of three distinct 3-cells, two of which are $\{\alpha\},\{\beta\}$ and the last we'll denote by $\{\gamma_n\}$. The edge relation (15) then tells us that:

$$c_{n+1} + (2 - ab)c_n + c_{n-1} + 2(a+b) = 0.$$

If $ab \neq 0, 4$, we may solve for this difference equation:

$$c_n = A\lambda^n + B\lambda^{-1} - \frac{2(a+b)}{4-ab}, \text{ where } \lambda^{\pm 1} = \frac{1}{2}(ab - 2 \pm \sqrt{ab(ab-4)})$$

and the constants A, B satisfy that:

$$(a+b)^2 + ab(a+b)(A+B) = ab(ab-4)AB.$$

In particular, since $ab = (\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}})^2$, then $|\lambda| = 1$ if and only $ab \in [0, 4]$.

There remain two cases: $ab = 0$ and $ab = 4$. For the former, we assume wlog that $b = 0$, then by the vertex relation (14), $c_n + c_{n_1} + a = 0$ and the sequence is either constant or oscillates between two values. For $ab = 4$, the vertex relation says that $(a + b + c_n + c_{n-1})^2 = 4c_n c_{n-1}$, and a little algebra suffices to show:

Lemma 18. *Given the above hypotheses, then:*

(1) *If* $ab \notin [0, 4]$*, then* $|c_n|$ *grows exponentially as* $n \to \pm\infty$*.*
(2) *If* $ab \in [0, 4)$*, then* $|c_n|$ *remains bounded.*
(3) *If* $ab = 4$*, then we may choose the square roots of* $\{c_i\}$ *and* $(a + b)$ *such that* $\sqrt{c_n} = z + in\sqrt{a+b}$*, for some* $z \in \mathbb{C}$*.*

Theorem 19. *If* $\phi \in \Phi_{BQ}$*, then* $\log^+ |\phi|$ *has Fibonacci growth.*

Proof. If $\Omega^2_\phi(4) = \varnothing$, then there is a unique sink in Ω^0. Otherwise, a path between two sinks would contain some vertex with at least two outward pointing oriented edges and hence be adjacent to a 2-cell in $\Omega^2_\phi(4)$ by lemma 8. Then apply lemma 17 to the four oriented edges pointing into this unique sink to obtain the desired Fibonacci lower bound.

Otherwise,

$$\Omega^2_\phi(4) = \{\{\alpha_1, \beta_1\}, \{\alpha_2, \beta_2\}, \ldots, \{\alpha_l, \beta_l\}\}$$

is finite but non-empty. Then let T denote the smallest tree in Ω containing the boundaries of all the 2-cells in $\Omega^2_\phi(4)$. We claim that T must contain every sink and saddle. Firstly, it's clear from lemma 8 that every saddle lies on the boundary of some 2-cell in $\Omega^2_\phi(4)$ and hence in T.

Now take an arbitrary sink ν. Since $\Omega^2_\phi(4)$ is non-empty, $\Omega^3_\phi(2)$ must also be non-empty. Consider the shortest path between $\Omega^3_\phi(2)$ and ν, if the length of this path is 2 or more, then we reach a contradiction because there must be an internal vertex adjacent to a 2-cell in $\Omega^2_\phi(4)$ and hence a 3-cell in $\Omega^3_\phi(2)$. And if the length of this path is 1, then we contradict the connectedness of $\Omega^3(2)$. Hence ν lies on the boundary of some 3-cell $A \in \Omega^3_\phi(2)$.

Now, thanks to the connectedness of $\Omega^3_\phi(2)$, the boundary of A must contain some 2-cell in $\Omega^2_\phi(4)$. Thus the shortest path from ν to $\Omega^2_\phi(4)$ lies on the boundary of A. Note that the 3-cell at the tail of the chosen edge on this path closest to $\Omega^2_\phi(4)$ must point towards $\Omega^2_\phi(4)$ or else produce a closer 2-cell in $\Omega^2_\phi(4)$. Hence, by similar arguments as used in the previous paragraph, ν must lie on the boundary of a 2-cell in $\Omega^2_\phi(4)$.

We now show that all but finitely many vertices are funnels. Observe that all but finitely many edges in T lie on the boundary of some 2-cell in $\Omega^2_\phi(4)$. Then lemma 18 tells us that since $|\phi|$ grows exponentially as we traverse the boundary of a 2-cell, there can only be finitely many sinks. Further observe that by lemma 10, every oriented edge outside of T must point into T. This means that along the boundary of any of the 2-cells in $\Omega^2_\phi(4)$, there must (in all but finitely many cases) be a sink in between two saddles. Hence, we see that the number of saddles is also finite.

We now show that a Fibonacci lower bound holds over the set:

$$\Omega^3_0(T) := \{\text{ 3-cells touching } T\}.$$

We know that all but finitely many of the 3-cells in $\Omega^3_0(T)$ spiral around some 2-cell in $\Omega^2_\phi(4)$. And lemma 18 tells us that $\log^+|\phi|$ over each of these spirals grows linearly, and hence for the spiral $\Omega^3_0(\{\alpha_i,\beta_i\})-\{\{\alpha_i\},\{\beta_i\}\}$ around $\{\alpha_i,\beta_i\}\in\Omega^2_\phi(4)$, we have:

$$\log^+|\phi(X)| \geqslant \kappa_i F_e(X) + \mu_i,$$

where κ_i is a function of $|ab|$ and the minimum of F_e on this spiral around $\{\alpha_i,\beta_i\}$, and μ_i may be negative. Since there are finitely many such spirals, only finitely many 3-cells in $\Omega^3_0(T)$ not on a spiral and the minimum of $\log^+|\phi|$ is greater than 0, we see that:

$$\log^+|\phi(X)| \geqslant \kappa F_e(X), \text{ for all } X\in\Omega^3_0(T).$$

Finally, label all the oriented edges touching but not contained in T by $\{\vec{\epsilon}_i\}$ (in order of increasing distance from e if you so wish), and for each $\vec{\epsilon}_i$, label the the three 3-cells in $\Omega^3_0(\epsilon_i)$ by:

$$\Omega^3_0(\epsilon_i) = \{X_i, Y_i, Z_i\}, \text{ such that } \log^+|\phi(X_i)| \leqslant \log^+|\phi(Y_i)| \leqslant \log^+|\phi(Z_i)|.$$

Then lemma 17 tells us that for any 3-cell $X\in\Omega^3_-(\vec{\epsilon}_i)$,

$$\begin{aligned}\log^+|\phi(X)| &\geqslant (\log|\phi(X_i)\phi(Y_i)| - \log(4))F_{\epsilon_i}(X), \text{ and hence}\\ &\geqslant \frac{\log|\phi(X_i)\phi(Y_i)|-\log(4)}{\max\{F_e(X_i),F_e(Y_i),F_e(Z_i)\}}F_e(X).\end{aligned}$$

Therefore, if we can show that

$$\inf_i\left\{\frac{\log|\phi(X_i)\phi(Y_i)|-\log(4)}{\max\{F_e(X_i),F_e(Y_i),F_e(Z_i)\}}\right\} > 0$$

then we'll have shown that $\log^+|\phi|$ has a lower Fibonacci bound over all of Ω^3. And to see that this holds, first notice that by going out sufficiently far from e, we may effectively ignore the $\log(4)$ term. Then, because X_i and Y_i are in $\Omega^3_0(T)$, we see that:

$$\frac{\log|\phi(X_i)\phi(Y_i)|}{\max\{F_e(X_i),F_e(Y_i),F_e(Z_i)\}} \geqslant \frac{\kappa(F_e(X_i)+F_e(Y_i))}{\max\{F_e(X_i),F_e(Y_i),F_e(Z_i)\}} \geqslant \frac{\kappa}{2},$$

thus yielding the desired Fibonacci lower bound. We complete this proof by invoking lemma 12 for the upper Fibonacci bound. □

3.2. **Proof of the McShane Identity.** Define the function $\Psi : \vec{\Omega}^1 \to [0,1]$ by:

$$\Psi(\vec{e}) = \Psi(\{\alpha,\beta,\gamma;\delta' \to \delta\}) := \frac{d}{a+b+c+d} = \frac{a+b+c+d}{abc}.$$

Then, the edge relation (15) becomes:

$$\Psi(\vec{e}) + \Psi(\overleftarrow{e}) = 1,$$

and the vertex relation (14) is the following relation on four incoming oriented edges $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$:

$$\Psi(\vec{e}_1) + \Psi(\vec{e}_2) + \Psi(\vec{e}_3) + \Psi(\vec{e}_4) = 1.$$

These two properties in turn tell us that for a funnel with oriented edges $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and outgoing edge $\vec{e}_4$:

$$\Psi(\vec{e}_4) = \Psi(\vec{e}_1) + \Psi(\vec{e}_2) + \Psi(\vec{e}_3),$$

and so we may iteratively expand either the edge or the vertex relation (14) into a statements about a collection of many terms of the form $\Psi(\vec{e})$ totaling 1. For a tree T in the 1-skeleton of Ω, if we use the notation $C(T)$ to denote

$$C(T) := \{\vec{e} \in \vec{\Omega}^1 \mid \vec{e} \text{ points into, but is not contained in } T\},$$

then we have:

Lemma 20. *For any finite subtree T in the 1-skeleton of Ω,*

$$\sum_{\vec{e} \in C(T)} \Psi(\vec{e}) = 1.$$

Next, define the function $h : \mathbb{C} - [0,4] \to \mathbb{C}$,

$$h(x) = \frac{1}{2}(1 - \sqrt{1 - 4/x}) = \frac{2}{x(1 + \sqrt{1 - 4/x})}.$$

For an edge $e = \{\alpha,\beta,\gamma\}$, we define

$$h(e) = h(\{\alpha,\beta,\gamma\}) := h(\frac{abc}{a+b+c}) = h\left((\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc})^{-1}\right).$$

A little algebraic manipulation shows that:

$$h(e) = \Psi(\vec{e}) \text{ if and only if } \mathrm{Re}(\Psi(\vec{e})) \leqslant \mathrm{Re}(\Psi(\overleftarrow{e})).$$

In other words, Ψ of a chosen edge $\vec{e}$ is equal to $h(e)$. In fact, the main point of Theorem 19 is to prove the following result:

Lemma 21. *The following infinite series taken over Ω^2 converges absolutely for all $s > 0$,*

$$\sum_{\{\xi,\eta\} \in \Omega^2} |xy|^{-s} < \infty.$$

Proof. We see from Theorem 19 that the following series converges (absolutely):

$$\sum_{\{\xi\} \in \Omega^3} |\log|x||^{-3} < \infty.$$

Hence, the following series converges:

$$\sum_{\{\xi\} \in \Omega^3} |x|^{-\frac{s}{2}} < \infty.$$

Squaring this series, we obtain an absolutely convergent series that's strictly greater than our desired quantity. □

Before we state and prove the main theorem of this section, we introduce one piece of notation. Given a subset E consisting of edges in the 1-skeleton of Ω, we define:

$$\Omega^2(E) := \left\{\{\xi, \eta\} \in \Omega^2 \mid \{\xi, \eta\} \text{ contains an edge in } E\right\}.$$

Theorem 22. *If $\phi \in \Phi_{BQ}$, then*

$$\sum_{\{\alpha,\beta\}\in\Omega^2} h(ab) = \frac{1}{2}. \tag{23}$$

Proof. We first note that $h(x)$ is roughly order $O(|x|^{-1})$, and so lemma 21 tells us that:

$$\sum_{\{\alpha,\beta\}\in\Omega^2} h(ab) < \infty.$$

Next, we prove an inequality of the following form:

$$|h(\{\alpha, \beta, \gamma\}) - h(ab)| \leqslant \kappa |h(ac) + h(bc)|,$$

where $\kappa > 0$ is independent of a, b and c. We begin by noting that outside of a finite set of edges $\{\alpha, \beta, \gamma\}$, either $|a| \gg 0, |b| \gg 0$ or $|c| \gg 0$. If $|a|$ or $|b| \gg 0$, then:

$$\frac{|h(\{\alpha, \beta, \gamma\}) - h(ab)|}{|h(ac) + h(bc)|} \approx 1,$$

and if $|c| \gg 0$, then:

$$\frac{|h(\{\alpha, \beta, \gamma\}) - h(ab)|}{|h(ac) + h(bc)|} \approx \frac{1}{2}\left|1 + \sqrt{1 - \frac{4}{ab}}\right|^{-1} < \frac{1}{2}.$$

Therefore, we know that a κ satisfying our demands exists.

In the proof of Theorem 19, we construct a finite tree T outside of which every vertex is a funnel. Now, if we take $B_n(T)$ to be the distance n neighbourhood of T in the 1-skeleton of Ω, then lemma 20 tells us that:

$$1 = \sum_{\vec{e}\in C(B_n(T))} \Psi(\vec{e}) = \sum_{\vec{e}\in C(B_n(T))} h(e).$$

Given $\vec{e} = \{\alpha, \beta, \gamma; \delta' \to \delta\} \in C(B_n(T))$, and suppose that $\vec{e}$ joins directly onto the oriented edge $\{\alpha, \beta, \delta; \gamma \to \gamma'\} \in C(B_{n-1}(T))$, then of the three 2-cells $\{\alpha, \beta\}$, $\{\alpha, \gamma\}$, $\{\beta, \gamma\}$ containing e, we know that $\{\alpha, \beta\} \in \Omega^2(B_{n-1}(T))$ and $\{\alpha, \gamma\}$, $\{\beta, \gamma\} \in \Omega^2_n(B_n(T)) - \Omega^2(B_{n-1}(T))$. Hence, summing over all of $C(B_n(T))$, we obtain the following inequality:

$$\begin{aligned}\left|1 - 2\sum_{\{\alpha,\beta\}\in\Omega^2(B_{n-1}(T))} h(ab)\right| &= \left|\sum_{C(B_n(T))} h(e) - 2\sum_{\{\alpha,\beta\}\in\Omega^2(B_{n-1}(T))} h(ab)\right| \\ &\leqslant 2\kappa \left|\sum_{\{\gamma,\delta\}\in\Omega^2(B_n(T)) - \Omega^2(B_{n-1}(T))} h(cd)\right|,\end{aligned}$$

noting that we'd made use of the fact that any 2-cell meets either two or no edges in $C(B_n(T))$. Then, by taking $n \to \infty$ and observing that the second term tends to 0, we obtain that:

$$1 = 2 \sum_{\{\alpha,\beta\} \in \Omega^2} h(ab).$$

$\square$

Our second main theorem follows as a corollary:

Theorem 2. *Let X be a quasi-Fuchsian thrice-punctured projective plane. Then*

$$\sum_{\gamma} \frac{1}{1 + \exp \frac{1}{2}\ell_\gamma(X)} = \frac{1}{2},$$

where the sum is over free homotopy classes of essential, non-peripheral two-sided simple closed curves γ on S.

Proof. Given a simple closed two-sided geodesics γ, there is a unique pair of once-intersecting simple closed one-sided geodesics α, β that do not intersect γ (and vice versa). Firstly, this bijection affords us the desired change in the summation indices. Secondly, by invoking the face relation (16):

$$ab = e + 2 = 2\cosh(\tfrac{1}{2}\ell_\gamma) + 2,$$

$h(ab)$ yields the desired summand. $\square$

Remark 2. It is not yet clear to us whether Theorem 2 and Theorem 22 are equivalent: if every BQ-Markoff map arises from a quasi-Fuchsian representation, then the two theorems are equivalent (and we'd have an algebraic characterisation for whether a representation is quasi-Fuchsian). If not, then Theorem 22 is strictly stronger. It should be noted that it is still an open question whether Bowditch's original BQ-conditions characterise the quasi-Fuchsian punctured torus representations [3].

4. Asymptotic growth of the simple length spectrum

In this section we state results for the growth of the number of simple closed geodesics of length less than L on any hyperbolic 3-cusped projective plane. We will treat this more generally and consider any $\phi \in \Phi_{BQ}$. Since our estimates are not sharp, we do not find coefficients.

The growth as $L \to \infty$ of $\mathcal{S}_X(L)$, the number of simple closed geodesic on X of length less than L, has been well studied for orientable surfaces. Mirzakhani showed [13] that for an orientable hyperbolic surface $X \in \mathcal{M}(S) = \mathcal{M}_{g,n}$

$$\mathcal{S}_X(L) \sim C(X) \cdot L^{\dim \mathcal{M}(S)} = C(X) \cdot L^{6g-6+2n} \tag{24}$$

where $C(X)$ is a proper continuous function on $\mathcal{M}(S)$. In fact, Mirzakhani proves a refinement of this statement: for the number $\mathcal{S}_X(L,\gamma)$ of simple closed geodesics in the mapping class group orbit of γ with length less than L,

$$\mathcal{S}_X(L,\gamma) \sim C_\gamma(X) \cdot L^{\dim \mathcal{M}(S)}.$$

This generalised the corresponding result for the once punctured torus proven by McShane and Rivin [10], and by Zagier [22] for the modular torus using integer

Markoff triples.

A punctured Klein bottle K has a unique two-sided simple closed curve α, and a family α_i, $i \in \mathbb{Z}$ of one-sided simple closed curves. Put $A = 2\cosh\frac{1}{2}\ell_\alpha(X)$ and $a_i = \sinh\frac{1}{2}\ell_{\alpha_i}(X)$ for a hyperbolic 1-cusped Klein bottle X. A trace identity yields:

$$a_i^2 + a_{i+1}^2 - a_i a_{i+1} A = -1. \tag{25}$$

hence:

$$\lambda_\pm := \lim_{i\to\pm\infty} \frac{a_i}{a_{i+1}} \text{ satisfy } \lambda_\pm^2 - A\lambda_\pm + 1 = 0, \text{ and } \lambda_\pm = \exp(\pm\tfrac{1}{2}\ell_\alpha).$$

Thus, for $k \gg 0$, the sequence of traces for $\{\alpha_i\}$ is eventually approximated by:

$$\ldots, a_{\pm k}, \exp(\tfrac{1}{2}\ell_\alpha)a_{\pm k}, \exp(\tfrac{2}{2}\ell_\alpha)a_{\pm k}, \exp(\tfrac{3}{2}\ell_\alpha)a_{\pm k}, \ldots$$

And since $2\text{arcsinh}(\frac{1}{2}\cdot)$ is roughly the same as $2\log(\cdot)$ for large numbers, the tails of the lengths for $\{\alpha_i\}$ resemble:

$$\ldots, \log(a_{\pm k}), \ell_\alpha + \log(a_{\pm k}), 2\ell_\alpha + \log(a_{\pm k}), 3\ell_\alpha + \log(a_{\pm k}), \ldots$$

Since $\mathcal{S}_X$ grows linearly in ℓ_α,

$$\mathcal{S}_X(L) \sim C(X)\cdot L = C(X)\cdot L^{\dim \mathcal{M}(K)}$$

and (24) also holds for the punctured Klein bottle K. Thus, the natural question arises:

Question: does (24) *still hold for non-orientable surfaces?*

An obvious first strategy is to try to generalise Mirzakhani's proof:

4.1. **Integration over the moduli space.** The exponent of the polynomial in (24) can be explained by Mirzakhani's method [13, 14] for integrating a special class of functions $F : \mathcal{M}_{g,n}(L_1, ..., L_n) \to \mathbb{R}$ over $\mathcal{M}_{g,n}(L_1, ..., L_n)$ equipped with the Weil-Petersson volume form. Her method applies to any F of the form

$$F = \sum_{\gamma = h\cdot\gamma_0} f(\ell_\gamma) \tag{26}$$

where f is an arbitrary function and the sum is over the orbit of a geodesic under the mapping class group. The integral of F unfolds to an integral over a moduli space $\widehat{\mathcal{M}}_{g,n}(\mathbf{L})$ of pairs (Σ, γ) consisting of a hyperbolic surface Σ and a geodesic $\gamma \subset \Sigma$.

$$\begin{array}{c} \mathcal{T}\text{eich}_{g,n}(\mathbf{L}) \\ \downarrow \\ \widehat{\mathcal{M}}_{g,n}(\mathbf{L}) \\ \downarrow \\ \mathcal{M}_{g,n}(\mathbf{L}). \end{array}$$

The unfolded integral

$$\int_{\mathcal{M}_{g,n}(\mathbf{L})} F\cdot d\text{vol} = \int_{\widehat{\mathcal{M}}_{g,n}(\mathbf{L})} f(\ell_\gamma)\cdot d\text{vol}$$

can be expressed in terms of an integral over the simpler moduli space obtained by cutting Σ along the geodesic γ.

Mirzakhani applied this to the two functions $F = \mathcal{S}_X(L)$ and $F =$ constant. The characteristic function $f = \chi_{[0,L]}$ in (26) gives rise to $F = \mathcal{S}_X(L)$, or $\mathcal{S}_X(L, \gamma)$. The exponent $\dim \mathcal{M}(S)$ in (24) comes from the integral of $\chi_{[0,L]}$ over a simpler moduli space $\mathcal{M}_{g',n'}(L)$ whose volume has polynomial L dependence of degree $\dim \mathcal{M}(S) - 1$. McShane's identity expresses the constant function $F = 1$ as a sum of functions f of lengths over orbits of the mapping class group and hence it is of the right form for Mirzakhani's method. In this case,

$$V_{g,n}(\mathbf{L}) = \int_{\mathcal{M}_{g,n}(\mathbf{L})} F \cdot \mathrm{dvol} = \int_{\widehat{\mathcal{M}}_{g,n}(\mathbf{L})} f \cdot \mathrm{dvol}$$

expresses the volume $V_{g,n}(\mathbf{L}) = \mathrm{volume}(\mathcal{M}_{g,n})$ recursively in terms of simpler volumes of moduli spaces obtained by removing a pair of pants from Σ in topologically different ways.

When the surface is orientable, the volume of $\mathcal{M}_{g,n}$ is finite and the bounded functions $F = \mathcal{S}_X(L)$ and $F = 1$ are integrable. However, when S is nonorientable, the volume [15] of $\mathcal{M}(S)$ is infinite and these two functions are not integrable. Any attempt to generalise Mirzakhani's work for non-orientable surfaces must therefore take into account these technical difficulties.

We now explore this question for the thrice-punctured projective plane.

4.2. **Thrice-punctured projective plane.**

Theorem 3. *There exist constants* $C(X)$ *and* $C'(X)$ *such that*

$$C(X) \cdot L^{1.8} < \mathcal{S}_X(L) < C'(X) \cdot L^3.$$

Proof. We prove growth estimates for any $\phi \in \Phi_{BQ}$, which includes the hyperbolic case. Define:

$$\mathcal{S}_\phi(L) = \mathrm{Card}\left\{\alpha \in \Omega^3 \mid |\phi(\alpha)| < 2\sinh(L/2)\right\}$$

and an analogous function for the Fibonacci function F_e:

$$\mathcal{S}_{F_e}(L) = \mathrm{Card}\left\{\alpha \in \Omega^3 \mid F_e(\alpha) < L\right\}.$$

When ϕ arises as $\mathrm{tr}\,\rho$ from a quasi-Fuchsian surface X defined by the representation $\rho : \pi_1(S) \to GL(2, \mathbb{C})$, then $\mathcal{S}_\phi(L) = \mathcal{S}_X(L, \text{1-sided})$ is the number of simple closed one-sided geodesics on X of length less than L.

By Theorem 19, for any $\phi \in \Phi_{BQ}$, $\log^+|\phi|$ has Fibonacci growth on Ω^3:

$$\frac{1}{\kappa} F_e(X) \leqslant \log^+|\phi(X)| \leqslant \kappa F_e(X) \text{ for cofinitely many } X \in \Omega^3,$$

and we have the following comparison:

Lemma 23. *For sufficiently large* L,

$$C_1 \cdot L^{e_1} < \mathcal{S}_{F_e}(L) < C_2 \cdot L^{e_2} \quad \Longleftrightarrow \quad C_1' \cdot L^{e_1} < \mathcal{S}_\phi(L) < C_2' \cdot L^{e_2}.$$

Proof. If $\mathcal{S}_{F_e}(L) < C_2 \cdot L^{e_2}$ then there is a set $U \subset \Omega^3$ of size $C_2 \cdot L^{e_2}$ outside which $F_e(X) > L$. Hence $|\phi(X)| > e^{L/\kappa}$ outside U, or $|\phi(X)| > 2\sinh L'/2$ for $L' = 2L/\kappa$. Hence

$$\mathcal{S}_\phi(L') < C_2 \cdot L^{e_2} = C_2 \cdot \left(\frac{\kappa L'}{2}\right)^{e_2} = C_2' \cdot (L')^{e_2}.$$

Similarly, if $\mathcal{S}_{F_e}(L) > C_1 \cdot L^{e_1}$ then there is a set $U \subset \Omega^3$ of size $C_1 \cdot L^{e_1}$ on which $F_e(X) < L$. Hence $|\phi(X)| < e^{\kappa L}$ on U, or $|\phi(X)| < 2\sinh L'/2$ for $L' = 2(\kappa+\epsilon)L$ where $\epsilon > 0$ takes care of the fact that $2\sinh L'/2$ is slightly less than $e^{L'/2}$. Hence

$$\mathcal{S}_\phi(L') > C_1 \cdot L^{e_1} = C_1 \cdot \left(\frac{L'}{2(\kappa+\epsilon)}\right)^{e_1} = C_1' \cdot (L')^{e_1}.$$

The proof of the converse is the same by symmetry. □

To complete the proof of the theorem, it is sufficient to study the growth of $\mathcal{S}_{F_e}(L)$. An upper bound for $\mathcal{S}_{F_e}(L)$ is

$$\mathcal{S}_{F_e}(L) < C \cdot L^3$$

for some constant C. This uses the fact that $\mathcal{S}_{F_e}(L) = \sum_{n<L} \mathrm{Card}\left(F_e^{-1}(n)\right)$ and the 4 : 1 correspondence described in Lemma 15 between occurrence of

$$X \in \Omega^3 \text{ such that } F_e(X) = n$$

and triples of values $\{l, m, n\}$ (where $n > l + m$) of F_e on an edge. The number of possibilities for $\{l, m\}$ is less than $\frac{1}{2}n^2$ (or we can use J_2, the Jordan totient function, as in (21)) and $\sum_{k=1}^n k^2$ behaves like $C \cdot n^3$.

A lower bound for $\mathcal{S}_{F_e}(L)$ is achieved as follows: a Fibonacci quad (a, b, c, n), where $a + b + c = n$, can be represented by the triple (a, b, c). After k steps generating the values of F_e, there are $2 \cdot 3^k$ terms. The maximum value M_k of F_e after k steps satisfies the Tribonacci recursion $M_k = M_{k-1} + M_{k-2} + M_{k-3}$ and hence $M_k \sim C \cdot \lambda^k$ where $\lambda \approx 1.84$ is the real root of $\lambda^3 = \lambda^2 + \lambda + 1$. Hence, $\mathcal{S}_{F_e}(C \cdot \lambda^k) \geqslant 2 \cdot 3^k$ i.e. $\mathcal{S}_{F_e}(L) \geqslant C_1 L^\mu$ where $\mu = \log_\lambda 3 \approx 1.803$. Thus:

$$C_1 L^{1.8} < \mathcal{S}_X(L) < C_2 L^3.$$

□

Remark 3. Theorem 3 shows that the growth of $\mathcal{S}_X(L)$ is at most polynomial, but does not confirm that the thrice-punctured projective plane satisfies (24). The estimates given here for $\mathcal{S}_{F_e}(L)$ are rather weak, and the generating function for all triples of values of F_e on Ω^1 is given by

$$G(x, y, z) = \sum_{(a,b,c)} x^a y^b z^c$$

where the sum is over all triples $(a, b, c) = (F_e(X), F_e(Y), F_e(Z))$ for $(X, Y, Z) \in \Omega^1$. Then G satisfies the following functional equation:

$$G(x, y, z) = xyz + G(x, xy, xz) + G(xy, y, yz) + G(xz, yz, z).$$

One might get better estimates for $\mathcal{S}_{F_e}(L)$ from this functional equation, although numerical estimates suggest that (24) fails for the thrice-punctured projective plane. Thus, any generalisation of Mirzakhani's work should take a different form to (24).

5. The mapping class group action on the character variety.

In this final section, we make explicit the relationship between the flips (2) and the mapping class group $\Gamma(S)$ of the thrice-punctured projective plane S. We also relate Penner coordinates to Markoff quads and use this to prove that the set of positive Markoff quads is the Teichmüller space of the thrice punctured projective plane.

5.1. **Teichmüller space.** The Teichmüller space $\mathcal{T}(S)$ of a surface S encodes all the ways of assigning a complete finite-area hyperbolic metric to S, up to isotopy. Concretely, it may be expressed as:

$$\mathcal{T}(S) := \{\, (X, f) \mid f : S \to X \text{ is a homeomorphism } \}/\sim$$

where $(X_1, f_1) \sim (X_2, f_2)$ if and only

$$f_2 \circ f_1^{-1} : X_1 \to X_2$$

is isotopy equivalent to a hyperbolic isometry. We denote these equivalence classes, or *marked surfaces*, by $[X, f]$.

An *ideal triangulation* of S is, up to isotopy, a triangulation of S with vertices at the punctures of S. Given a marked surface $[X, f]$, the image $f(\sigma)$ of an arc σ on S pulls tight to a unique homotopy equivalent geodesic arc on X. Thus, any ideal triangulation on S is represented by an (geodesic) ideal triangulation on X — a maximal collection of simple bi-infinite geodesic arcs with both ends up cusps. For our purposes, we restrict to ideal triangulations $\triangle$ on thrice-punctured projective planes S representable by paths with *distinct end points*.

Horocycles of length 1 around a cusp are always simple on a complete hyperbolic surface. Thus, given an ordered ideal triangulation $(\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3)$ on X, we obtain lengths $(s_1, s_2, s_3, t_1, t_2, t_3)$ of these infinite geodesic arcs truncated at the three length 1 horocycles bounding cusps $1, 2, 3$. The λ-lengths for X with respect to this ordered ideal triangulation is then given by:

$$(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3) = (\exp \tfrac{1}{2}s_1, \exp \tfrac{1}{2}s_2, \exp \tfrac{1}{2}s_3, \exp \tfrac{1}{2}t_1, \exp \tfrac{1}{2}t_2, \exp \tfrac{1}{2}t_3).$$

In [16], Penner shows that these λ-lengths form global coordinates on the Teichmüller space of any punctured surface. This is also true for the Teichmüller space of punctured non-orientable surfaces.

The following lemma is a topological correspondence which will be promoted to a geometric correspondence below.

Lemma 24. *There is a natural bijection between Ω^0 and*

$$\{ \textit{ the collection of ideal triangulations of } S \textit{ with distinct end points } \}$$

given by sending $\{\alpha, \beta, \gamma, \delta\} \in \Omega^0$ to the unique (up to isotopy) ideal triangulation where each arc intersects precisely two of the geodesics in $\{\alpha, \beta, \gamma, \delta\}$.

Proof. Any essential two-sided simple closed curve is a boundary component of a thickening of a unique pair of once-intersecting one-sided simple closed curves (figure 4), and a boundary component of a thickening of a unique arc joining distinct punctures. By alternatingly thinking of a two-sided curve as boundary components of these two thickenings, we see that pairs of intersection points between two 2-sided simple closed curves correspond to single intersection points between two one-sided simple closed curves and between two arcs joining distinct punctures (where the punctures count as single intersection points). Hence the six arcs obtained in this way are disjoint outside the punctures iff. $\{\alpha, \beta, \gamma, \delta\} \in \Omega^0$ pairwise intersect exactly once. □

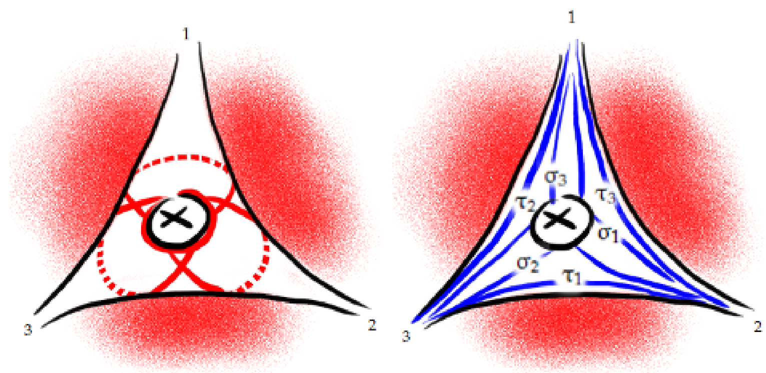

FIGURE 7. A 4-tuple of curves corresponding to a triangulation.

Lemma 25. *The λ-lengths for an ideal triangulation $\triangle$ of S identifies the Teichmüller space $\mathcal{T}(S)$ as:*

$$\left\{ (\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3) \in \mathbb{R}^6_+ \;\middle|\; \begin{array}{l} \mu_1\mu_2\mu_3 + \mu_1\lambda_2\lambda_3 + \lambda_1\mu_2\lambda_3 + \lambda_1\lambda_2\mu_3 = \lambda_1\lambda_2\mu_1\mu_2 \\ \mu_1\mu_2\mu_3 + \mu_1\lambda_2\lambda_3 + \lambda_1\mu_2\lambda_3 + \lambda_1\lambda_2\mu_3 = \lambda_1\lambda_3\mu_1\mu_3 \\ \mu_1\mu_2\mu_3 + \mu_1\lambda_2\lambda_3 + \lambda_1\mu_2\lambda_3 + \lambda_1\lambda_2\mu_3 = \lambda_2\lambda_3\mu_2\mu_3 \end{array} \right\}.$$

These λ-lengths of an ideal triangulation may be expressed in terms of the Markoff quad of the associated quadruple of one-sided geodesics in Ω^0 corresponding to the used to the define these λ-lengths.

$$(a, b, c, d) = \left(\frac{\lambda_2\lambda_3}{\lambda_1}, \frac{\lambda_1\lambda_3}{\lambda_2}, \frac{\lambda_1\lambda_2}{\lambda_3}, \frac{\mu_1\mu_2}{\lambda_3} = \frac{\mu_1\mu_3}{\lambda_2} = \frac{\mu_2\mu_3}{\lambda_1} \right), \tag{27}$$
$$(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3) = (\sqrt{bc}, \sqrt{ac}, \sqrt{ab}, \sqrt{ad}, \sqrt{bd}, \sqrt{cd}\,).$$

Thus, we may also use positive Markoff quads to globally parametrise the Teichmüller space:

Proposition 26. *Given an ordered 4-tuple $(\alpha, \beta, \gamma, \delta)$ intersecting a fixed triangulation on S as per figure 7, then the map*

$$\mathcal{T}(S) \to \{(a, b, c, d) \in \mathbb{R}^4_+ \mid (a + b + c + d)^2 = abcd\}$$
$$[X, f] \mapsto (2\sinh \tfrac{1}{2}\ell_\alpha(X), 2\sinh \tfrac{1}{2}\ell_\beta(X), 2\sinh \tfrac{1}{2}\ell_\gamma(X), 2\sinh \tfrac{1}{2}\ell_\delta(X))$$

is a real-analytic diffeomorphism, where $\ell_\alpha(X)$ denotes the length of the geodesic representative of $f_*(\alpha)$ on X. We call this global coordinate patch the trace coordinates *for* $\mathcal{T}(S)$.

Proof. With a little hyperbolic trigonometry and successive applications of the ideal Ptolemy relation [6], we can show that (27) explicitly gives the desired diffeomorphism between the trace coordinates and the λ-coordinates for $\mathcal{T}(S)$. The fact that this map is real-analytic is then a simple consequence of the real-analyticity of the λ-lengths. $\square$

Corollary 27. *The set of positive Markoff quads is the Teichmüller component of the real character variety.*

Proof. Proposition 26 proves that the set of positive Markoff quads is real-analytically diffeomorphic to Teichmüller space. It remains to show that it is a connected component of the real character variety. Suppose that one of the coordinates vanishes, say $d = 0$. Then by (1), $a + b + c = 0$. But if this point lies in the limit of a path in the set of positive Markoff quads then each of a, b and c must tend to 0 along the path. In particular, at some point on the path $abcd < 256$. But this contradicts (1) since

$$(a + b + c + d)^2 \geqslant 16\sqrt{abcd} > abcd$$

where the first inequality is the arithmetic mean-geometric mean inequality. $\square$

5.2. **The mapping class group.** Having identified the Teichmüller space with the space of real Markoff quads we now know that every positive real Markoff map arises from a Fuchsian representation. We now use this to help interpret flips as elements of the mapping class group.

We construct an explicit homeomorphism $f_4 : S \to S$ that takes $(\alpha, \beta, \gamma, \delta)$ to $(\alpha, \beta, \gamma, \delta')$. Consider the hexagonal fundamental domain of S obtained by cutting along σ_1, σ_2 and σ_3. From figure 8, we see that a rotation by π of this fun-

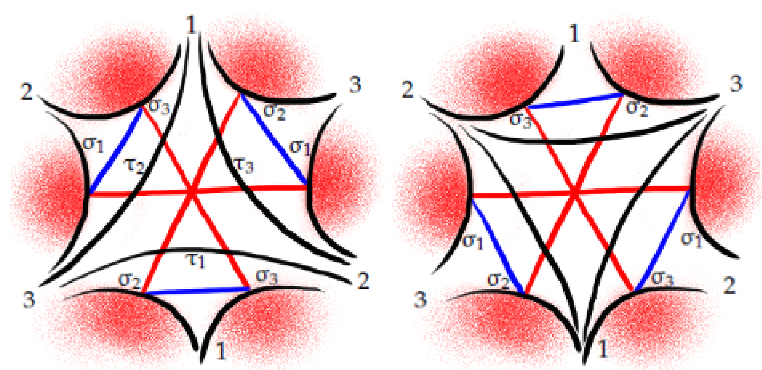

FIGURE 8. The map f_4 fixing α, β, γ, but switching δ and δ'.

damental domain fixes the labeling of the punctures and fixes each of α, β and γ whilst taking α to α'. The action of the mapping class $[f_4] \in \Gamma(S)$ therefore takes the Markoff quad (a, b, c, d) corresponding to a marked surface $[X, f]$ to

$$[f_4](a, b, c, d) = (a, b, c, abc - 2a - 2b - 2c - d),$$

that is: $[f_4]$ corresponds to a flip in the fourth entry. By symmetry, there are four flips $[f_1], [f_2], [f_3], [f_4] \in \Gamma(S)$ which flip the corresponding entries of (a, b, c, d). Let $F \leqslant \Gamma(S)$ denote the subgroup generated by these four flips.

Lemma 28. $F \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, *where each* $\mathbb{Z}_2$ *is generated by one of the* $[f_i]$.

Proof. First observe that each $[f_i]$ is indeed order 2. To see that there are no other relations, consider the action of a reduced string of flips on the 1-skeleton of the curve complex: since the 1-skeleton is a 4-regular tree, performing each flip in a sequence of flips necessarily takes us farther from the origin. □

We now consider a different subgroup in $\Gamma(S)$: the stabiliser of $\{\alpha, \beta, \gamma, \delta\}$. Due to lemma 24, this subgroup must also stabilise $\triangle$ — the triangulation corresponding to $\{\alpha, \beta, \gamma, \delta\}$.

Lemma 29. $\mathrm{Stab}(\triangle) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. There are four triangles T_1, T_2, T_3, T_4 induced by the triangulation $\triangle$ on S, and any element of $\mathrm{Stab}(\triangle)$ must take T_1 to one these four triangles. Since there is a unique way to map T_1 to any of these four triangles so as to preserve puncture-labeling, knowing the image of T_1 determines the entire mapping class. By symmetry, these mapping classes must have the same order, hence $\mathrm{Stab}(\triangle)$ is the Klein four-group. □

This stabiliser subgroup is given by:

$$\left\{ \begin{array}{rr} & [\varphi_1] : (a, b, c, d) \mapsto (b, a, d, c), \\ [\mathrm{id}], & [\varphi_2] : (a, b, c, d) \mapsto (c, d, a, b), \\ & [\varphi_3] : (a, b, c, d) \mapsto (d, c, b, a), \end{array} \right\}$$

when thought of as acting on the trace coordinates for $\mathcal{T}(S)$.

Lemma 30. *The subgroups* F *and* $\mathrm{Stab}(\triangle)$ *generate the whole mapping class group* $\Gamma(S)$.

Proof. Given an arbitrary element $[h] \in \Gamma(S)$, the action of $[h]$ on $(a, b, c, d) = [X, f] \in \mathcal{T}(S)$ produces another Markoff quad $(\bar{a}, \bar{b}, \bar{c}, \bar{d})$ corresponding to the lengths of $(h_*(\alpha), h_*(\beta), h_*(\gamma), h_*(\delta))$. Since the four flips $[f_i]$ generate all Markoff quads associated to the Fuchsian representation for X, there is an element $[g] \in F$ such that $[g] \circ [h] = [g \circ h]$ simply permutes a, b, c, d. By choosing X to be a surface where there are only four simple one-sided geodesics with traces $\{a, b, c, d\}$ (e.g.: the $(4, 4, 4, 4)$ surface), we see that $[g \circ h] \in \mathrm{Stab}(\triangle)$. □

Lemma 31. F *is a normal subgroup of* $\Gamma(S)$.

Proof. Note that it suffices to show that $\mathrm{Stab}(\triangle)$ preserves $\{[f_1], [f_2], [f_3], [f_4]\}$. We perform this check for $[f_1]$, the rest follow by symmetry:

$$[\varphi_1]^{-1} \circ [f_1] \circ [\varphi_1] = [f_2],\ [\varphi_2]^{-1} \circ [f_1] \circ [\varphi_2] = [f_3], \text{ and } [\varphi_3]^{-1} \circ [f_1] \circ [\varphi_3] = [f_4].$$

□

Since F and Stab generate $\Gamma(S)$ and their intersection is the trivial group, we obtain the following result:

Theorem 32. $\Gamma(S) = F \rtimes \mathrm{Stab}(\triangle) \cong (\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2)$. *In particular:*

$$\Gamma(S) \cong \left\langle \begin{array}{r} f_1, f_2, f_3, f_4, \\ g, h \end{array} \middle| \begin{array}{l} f_1^2 = f_2^2 = f_3^2 = f_4^2 = g^2 = h^2 = 1, gh = hg \\ g^{-1} f_1 g = f_2, h^{-1} f_1 h = f_3, g^{-1} f_3 g = f_4 \end{array} \right\rangle$$
$$\cong \left\langle f, g, h \mid f^2 = g^2 = h^2 = 1, gh = hg \right\rangle.$$

5.3. **The moduli space.** Recall that the moduli space $\mathcal{M}(S)$ of hyperbolic structures on S is given by $\mathcal{T}(S)/\Gamma(S)$. Since F is a normal subgroup of $\Gamma(S)$, the space $\mathcal{T}(S)/F$ must be a finite cover of $\mathcal{M}(S)$. To better see what $\mathcal{T}(S)/F$ looks like, we first define another global coordinate chart for $\mathcal{T}(S)$.

Lemma 33. *The Teichmüller space $\mathcal{T}(S)$ may be real-analytically identified with the following (open) 3-simplex:*

$$\{(H_a, H_b, H_c, H_d) \in \mathbb{R}_+^4 \mid H_a + H_b + H_c + H_d = 1\},$$

we call this the horocyclic coordinate *for $\mathcal{T}(S)$.*

Proof. The explicit diffeomorphisms between the horocyclic coordinates and the trace coordinates is given as follows:

$$H_a = \sqrt{\frac{a}{bcd}} = \frac{a}{a+b+c+d}, H_b = \sqrt{\frac{b}{acd}} = \frac{b}{a+b+c+d}$$
$$H_c = \sqrt{\frac{c}{abd}} = \frac{c}{a+b+c+d}, H_d = \sqrt{\frac{d}{abc}} = \frac{d}{a+b+c+d},$$

and the inverse map is given by:

$$a = \sqrt{\frac{H_a}{H_b H_c H_d}}, b = \sqrt{\frac{H_b}{H_a H_c H_d}}, c = \sqrt{\frac{H_c}{H_a H_b H_d}}, d = \sqrt{\frac{H_d}{H_a H_b H_c}}.$$

□

Remark 4. The horocyclic coordinates are so named because they correspond to the lengths of horocyclic segments on the length 1 horocycles at the cusps of a marked surface $[X, f]$. Coupled with the labeling in figure 7, figure 9 illustrates this correspondence.

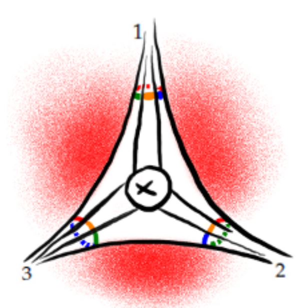

FIGURE 9. Horocyclic segments (each) of length H_a, H_b, H_c, H_d.

In these horocyclic coordinates, the flips generating F act as follows:

$$[f_1] : (H_a, H_b, H_c, H_d) \mapsto (1 - H_a, H_b \frac{H_a}{1 - H_a}, H_c \frac{H_a}{1 - H_a}, H_d \frac{H_a}{1 - H_a}),$$
$$[f_2] : (H_a, H_b, H_c, H_d) \mapsto (H_a \frac{H_b}{1 - H_b}, 1 - H_b, H_c \frac{H_b}{1 - H_b}, H_d \frac{H_b}{1 - H_b}),$$
$$[f_3] : (H_a, H_b, H_c, H_d) \mapsto (H_a \frac{H_c}{1 - H_c}, H_b \frac{H_c}{1 - H_c}, 1 - H_c, H_d \frac{H_c}{1 - H_c}),$$
$$[f_4] : (H_a, H_b, H_c, H_d) \mapsto (H_a \frac{H_d}{1 - H_d}, H_b \frac{H_d}{1 - H_d}, H_c \frac{H_d}{1 - H_d}, 1 - H_d).$$

From this, we see that the fixed points of $[f_1], [f_2], [f_3], [f_4]$ are respectively given by imposing the following conditions on the horocyclic coordinates:

$$H_a = \tfrac{1}{2}, H_b = \tfrac{1}{2}, H_c = \tfrac{1}{2}, H_d = \tfrac{1}{2}.$$

The region in $\mathcal{T}(S)$ enclosed by these four planes is therefore a fundamental domain for $\mathcal{T}(S)/F$. In this case, this fundamental domain is an octahedron. Since $[f_1]$ acts by swapping the two regions separated by $H_a = \frac{1}{2}$, the image of these fixed points in $\mathcal{T}(S)/F$ are order 2 (reflection) orbifold points. Similar comments hold for each of the $[f_i]$. Thus, $\mathcal{T}(S)/F$ is an open octahedron with four triangles of order 2 orbifold points glued onto a collection of four non-adjacent sides.

Finally, by noting that $\mathrm{Stab}(\triangle)$ acts on the horocyclic coordinates by:

$$[\varphi_1] : (H_a, H_b, H_c, H_d) \mapsto (H_b, H_a, H_d, H_c),$$
$$[\varphi_2] : (H_a, H_b, H_c, H_d) \mapsto (H_c, H_d, H_a, H_b),$$
$$[\varphi_3] : (H_a, H_b, H_c, H_d) \mapsto (H_d, H_c, H_b, H_a).$$

We obtain the result:

Theorem 34. *The moduli space $\mathcal{M}(S)$ of a thrice-puncture projective plane is homeomorphic to an open 3-ball with an open hemisphere of order 2 orbifold points glued on, and a line of orbifold points running straight through the center of this 3-ball — joining two antipodal points of this orbifold hemisphere. The orbifold points on this line are of order 2, except for the very center point of the 3-ball, which is order 4.*

Remark 5. The interior 3-ball of $\mathcal{M}(S)$ may be geometrically interpreted as the set of 3-cusped projective planes which have a unique unordered 4-tuple of geodesics whose flips are strictly longer. This was hinted at in remark 1.

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