VANISHING CYCLES AND MONODROMY OF COMPLEX POLYNOMIALS

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Abstract. We describe the trivial summand for monodromy around a fibre of a polynomial map $\mathbb{C}^n \to \mathbb{C}$ generalising and clarifying work of Artal Bartolo, Cassou-Noguès and Dimca, who proved similar results under strong restrictions on the homology of the general fibre and singularities of the other fibres. They also showed a polynomial map $f\colon \mathbb{C}^2 \to \mathbb{C}$ has trivial global monodromy if and only if it is "rational of simple type" in the terminology of Miyanishi and Sugie. We refine this result and correct the Miyanishi-Sugie classification of such polynomials, pointing out that there are also non-isotrivial examples.

1. Introduction

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a primitive polynomial map ("primitive" means f is not of the form $g \circ h$ with $g\colon \mathbb{C} \to \mathbb{C}$ and $h\colon \mathbb{C}^n \to \mathbb{C}$ polynomial maps and $\deg g > 1$). It is well-known that there are just finitely many points $c_1, \dots, c_m \in \mathbb{C}$ for which the fibre $f^{-1}(c_i)$ is "irregular", that is, it has different topology from the generic or "regular" fibre.

Definition 1.1. If $f^{-1}(c)$ is a fibre of $f\colon \mathbb{C}^n \to \mathbb{C}$ choose ϵ sufficiently small that all fibres $f^{-1}(c')$ with $c' \in D^2_\epsilon(c) - \{c\}$ are regular and let $N(c) := f^{-1}(D^2_\epsilon(c))$. Let $F = f^{-1}(c')$ be a regular fibre in $N(c)$. Then

$$V_q(c) := \operatorname{Ker}\big(H_q(F;\mathbb{Z}) \to H_q(N(c);\mathbb{Z})\big)$$
$$V^q(c) := \operatorname{Cok}\big(H^q(N(c);\mathbb{Z}) \to H^q(F;\mathbb{Z})\big)$$

are the groups of *vanishing q-cycles* and *vanishing q-cocycles* for $f^{-1}(c)$. They have the same rank, which we call the *number of vanishing q-cycles for $f^{-1}(c)$*.

Choose a regular value c_0 for f and paths γ_i from c_0 to c_i for $i = 1, \dots, m$ which are disjoint except at c_0. We can use these paths to refer homology or cohomology of a regular fibre near one of the irregular fibres $f^{-1}(c_i)$ to the homology or cohomology of the "reference" regular fibre $F = f^{-1}(c_0)$.

The fundamental group $\Pi = \pi_1\big(\mathbb{C} - \{c_1, \dots, c_m\}\big)$ acts on the homology $H_*(F;\mathbb{Z})$ and cohomology $H^*(F;\mathbb{Z})$. If this action is trivial we say that f has "trivial global monodromy group". This action has the following generators.

Let $h_q(c_i)\colon H_q(F) \to H_q(F)$ and $h^q(c_i)\colon H^q(F) \to H^q(F)$ be the monodromy about the fibre $f^{-1}(c_i)$ (obtained by translating the fibre F along the path γ_i until close to the fibre $f^{-1}(c_i)$, then in a small loop around that fibre, and back along γ_i). We are interested in the fixed group $H^q(F)^{h^q(c_i)} = \operatorname{Ker}\big(1 - h^q(c_i)\big)$ of this local monodromy.

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Theorem 1.2. *The maps $H^q(F;\mathbb{Z}) \to V^q(c_i)$ induce an isomorphism*

$$H^q(F;\mathbb{Z}) \cong \bigoplus_{i=1}^{m} V^q(c_i)$$

Moreover, if we denote by $K^q(c_i)$ the image of $\operatorname{Ker}\big(1-h^q(c_i)\big)$ under the natural map $H^q(F) \to V^q(c_i)$ then under this isomorphism we have:

$$\operatorname{Ker}\big(1-h^q(c_j)\big) = K^q(c_j) \oplus \bigoplus_{i\neq j} V^q(c_i),$$

so the subgroup of cohomology fixed under global monodromy is

$$H^q(F;\mathbb{Z})^{\Pi} = \bigoplus_{i=1}^{m} K^q(c_i).$$

Theorem 1.3. *Let $H_*(f^{-1}(c),\infty)$ denote $H_*(f^{-1}(c),U)$, where U is a regular neighbourhood of infinity (e.g., $U = \{z \in f^{-1}(c) : ||z|| > R\}$ for large R). Then we have a natural exact sequence:*

$$0 \to \operatorname{Cok}\big(1-h^{q-1}(c)\big) \to H_{2n-q-1}(f^{-1}(c),\infty) \to K^q(c) \to 0.$$

Under the assumptions that F has homology only in dimension $(n-1)$ and that all singularities of fibres of f are isolated, Artal Bartolo, Cassou-Nogués, and Dimca [1] proved the dimension formulae for $\operatorname{Ker}\big(1-h^{n-1}(c)\big)$ and $H^{n-1}(F;\mathbb{Z})^{\Pi}$ that follow from the above theorems. Polynomials $f(x_1,\dots,x_n) = x_1 g(x_2,\dots,x_n)$ are examples of polynomials with trivial global monodromy that do not satisfy their assumptions for $n > 2$.

We also have a homology version of these results:

Theorem 1.4. *The inclusions $V_q(c_i) \to H_q(F;\mathbb{Z})$ induce an isomorphism*

$$H_q(F;\mathbb{Z}) \cong \bigoplus_{i=1}^{m} V_q(c_i).$$

Moreover, there is a natural exact sequence

$$0 \to \operatorname{Im}\big(1-h_q(c)\big) \to V_q(c) \to H^{2n-q-1}(f^{-1}(c),\infty) \to \operatorname{Ker}\big(1-h_{q-1}(c)\big) \to 0.$$

This has the immediate corollary:

Corollary 1.5. *If r_c is the number of components of $f^{-1}(c)$ then*

$$0 \to \operatorname{Im}\big(1-h_1(q)\big) \to V_1(c) \to \mathbb{Z}^{r_c-1} \to 0,$$

In particular, the monodromy about the fibre $f^{-1}(c)$ in dimension 1 is trivial if and only if the number of components of this fibre exceeds by one the number of its vanishing 1-cycles. □

For an irreducible fibre this says this monodromy is trivial if and only if the fibre has no vanishing 1-cycles. This generalises the positive answer by Michel and Weber [5] to Dimca's question whether the local monodromy around a reduced and irreducible fibre of a polynomial $f\colon \mathbb{C}^2 \to \mathbb{C}$ is trivial if and only if the fibre is regular, since:

Theorem 1.6. *For $n = 2$ a fibre has no vanishing cycles if and only if it is regular.*

One can show that if the monodromy around an irregular fibre of a 2-variable polynomial is trivial then all of the components implied by Corollary 1.5 except one must be rational. It is not hard to find examples where the other component has any genus.

We shall prove the above theorems in Section 2. But when the fibre $f^{-1}(c)$ is reduced with isolated singularities, there is a quick proof of Corollary 1.5. Namely, let F_0 be the "non-singular core" of $f^{-1}(c)$ obtained by intersecting $f^{-1}(c)$ with a very large ball and then removing small regular neighbourhoods of its singularities. Then F_0 can be isotoped into a nearby regular fibre F and it is not hard to see (cf e.g., [9]):

Proposition 1.7. *Under the above assumption, $H_q(F,F_0)$ is isomorphic to $V_q(c)$ by an isomorphism that fits in the commutative diagram*

$$\begin{array}{ccc} H_q(F,F_0) & \xrightarrow{\cong} & V_q(c) \\ \uparrow & & \downarrow\subseteq \\ H_q(F) & \xrightarrow{1-h_q} & H_q(F) \end{array}$$

Since the number of topological components of F_0 is r_c, Corollary 1.5 follows in this case using $q=1$ and the long exact homology sequence for the pair (F,F_0).

The following consequence of the monodromy results was proved by Artal Bartolo et al. [1].

Theorem 1.8. *The polynomial $f\colon C^2 \to \mathbb{C}$ has trivial global monodromy group if and only if f is rational of simple type, in the sense of Miyanishi and Sugie* [6].

A polynomial $f\colon \mathbb{C}^2 \to \mathbb{C}$ is "rational" if its generic fibre is rational (i.e., genus zero). "Simple type" means that if we take a nonsingular compactification $Y = \mathbb{C}^2 \cup E$ of $\mathbb{C}^2$ such that f extends to a holomorphic map $\overline{f}\colon Y \to \mathbb{C}P^1$ then $\overline{f}$ is of degree 1 on each "horizontal" irreducible component of the compactification divisor E (E is a union of smooth rational curves $E_1,\dots,E_n$ with normal crossings and a component E_i is called *horizontal* if $\overline{f}|E_i$ is non-constant).

We give a simple proof of Theorem 1.8 and refinements of it in Section 3. In the final Section 4 we describe corrections to Miyanishi and Sugie's classification of rational polynomials of simple type. Details of this will appear elsewhere.

2. Proofs of the main theorems

For each irregular value c_i we construct a neighbourhood $N_i = f^{-1}(D^2_\epsilon(c_i))$ of the corresponding irregular fibre as in Definition 1.1, with ϵ chosen small enough that the disks $D^2_\epsilon(c_i)$ are disjoint. Let c_0 be a regular value outside all these disks and choose disjoint paths γ_i joining c_0 to each disk $D^2_\epsilon(c_i)$. Let $P = \bigcup_{i=1}^m \gamma_i$ and $D = \bigcup_{i=1}^m D^2_\epsilon(c_i)$ so $K = P \cup D$ is the union of these paths and disks. Then $\mathbb{C}^n$ deformation retracts onto $f^{-1}(K)$. The Mayer-Vietoris sequence for $f^{-1}(K) = f^{-1}(P) \cup f^{-1}(D)$ gives

$$0 \to H^q(F) \oplus \bigoplus_{i=1}^m H^q(N_i) \to \bigoplus_{i=1}^m H^q(F) \to 0, \quad (q \geq 0). \tag{1}$$

Since the i-th summand of the sum $\bigoplus_1^m H^q(N_i)$ maps trivially to all but the i-th summand of $\bigoplus_1^m H_q(F)$, this shows:

Proposition 2.1. *$H^q(N_i) \to H^q(F)$ is injective with cokernel (by Definition 1.1) $V^q(c_i)$.* □

Thus, factoring source and target of the middle isomorphism of (1) by the subgroup $\bigoplus H^q(N_i)$ gives the isomorphism

$$H^q(F) \xrightarrow{\cong} \bigoplus_{i=1}^{m} V^q(c_i), \tag{2}$$

of the first statement of Theorem 1.2.

The long exact sequence for the pair (N_i, F) now shows that we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^q(N_i) & \longrightarrow & H^q(F) & \longrightarrow & H^{q+1}(N_i,F) & \longrightarrow & 0 \\
 & & \Vert & & \Vert & & \downarrow\cong & & \\
0 & \longrightarrow & H^q(N_i) & \longrightarrow & H^q(F) & \longrightarrow & V^q(c_i) & \longrightarrow & 0\,.
\end{array}$$

We now claim that we can identify the long exact sequence of the triple $(N_i, \partial N_i, F)$ as follows:

$$\begin{array}{ccccccc}
H^q(\partial N_i,F) & \longrightarrow & H^{q+1}(N_i,\partial N_i) & \longrightarrow & H^{q+1}(N_i,F) & \longrightarrow & H^{q+1}(\partial N_i,F) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
H^{q-1}(F) & \longrightarrow & H_{2n-q-1}(f^{-1}(c_i),\infty) & \longrightarrow & V^q(c_i) & \longrightarrow & H^q(F)\,.
\end{array}$$

The first and fourth vertical isomorphisms are seen by thickening F within ∂N and then using excision and the Künneth formula:

$$H^q(\partial N_i, F) \cong H^q(F \times I, F \times \partial I) \cong H^{q-1}(F).$$

We have already shown the third vertical isomorphism. Thus only the second vertical isomorphism remains to be shown. Let N_i^0 be $f^{-1}(D^2_\epsilon(c_i)) \cap D^{2n}$ where D^{2n} is first chosen large enough that $f^{-1}(c_i)$ is transverse (in the sense of stratified sets) to the boundary of it and all larger disks, and ϵ is then re-chosen small enough that ∂D^{2n} is transverse to $f^{-1}(c_i')$ for all $c_i' \in D^2_\epsilon(c_i)$. Put $\partial_0 N_i^0 := \partial N_i \cap N_i^0$ and $F_i^0 := f^{-1}(c_i) \cap D^{2n}$ and $C_i := f^{-1}(c_i) - \mathrm{int}(F_i^0)$. Then the inclusion of $N_i - C_i$ in N_i is a homotopy equivalence and the inclusion of ∂N_i into $\partial N_i \cup (N_i - N_i^0 - C_i)$ is a homotopy equivalence, so we have: $H^{q+1}(N_i, \partial N_i) \cong H^{q+1}(N_i - C_i, \partial N_i \cup (N_i - N_i^0 - C_i))$. Excision then shows this is isomorphic to $H^{q+1}(N_i^0, \partial N_i^0 - \partial F_i^0)$, and this equals $H^{q+1}(N_i^0, \partial_0 N_i^0)$ by homotopy equivalence. Putting $\partial_1 N_i^0 := \partial N_i^0 - \mathrm{int}(\partial_0 N_i^0)$, Poincaré-Lefschetz duality gives $H^{q+1}(N_i^0, \partial_0 N_i^0) \cong H_{2n-q-1}(N_i^0, \partial_1 N_i^0)$. But the pair $(N_i^0, \partial_1 N_i^0)$ is homotopy equivalent to $(F_i^0, \partial F_i^0)$. By excision $H_*(F_i^0, \partial F_i^0) = H_*(f^{-1}(c_i), \infty)$. Thus the above diagram is proved.

Consider now the composition $H^q(F) \to V^q(c_i) \to H^q(F)$ where the second map is the map of the above diagram. Tracing the definitions, we see it is the composition: $H^q(F) \to H^{q+1}(\partial N_i, F) \to H^q(F)$, where the first map is boundary map for the pair. This composition is evidently $1 - h^q(c_i)$. Since $H^q(F) \to V^q(c_i)$ is surjective with kernel $\bigoplus_{j \neq i} V^q(c_j)$, it follows that $\mathrm{Ker}\big(1 - h^q(c_i)\big)$ contains $\bigoplus_{j \neq i} V^q(c_j)$. It hence has the form $K^q(c_i) \oplus \bigoplus_{j \neq i} V^q(c_j)$ in terms of the isomorphism of (2), where $K^q(c_i) = \mathrm{Ker}(V^q(c_i) \to H^q(F))$. Thus the second statement of Theorem 1.2 follows. Theorem 1.3 then follows by replacing the first term of the bottom sequence of the above diagram by its image and the last arrow by its kernel. □

The proof of the homology versions of these results is essentially the same so we omit it.

Proof of Theorem 1.6. A homology computation shows $\chi(f^{-1}(c)) = \chi(N(c))$ for any n. For instance, putting $c = c_i$ we have: $\chi(f^{-1}(c_i)) = \chi(F_i^0) = \chi(N_i^0) = \chi(N_i^0, \partial_0 N_i^0) = \chi(N_i, \partial N_i) = \chi(N_i)$. This uses homotopy equivalence for the first two equalities and the homology isomorphism of the previous proof for the fourth, and the third and fifth equalities, of the form $\chi(X) = \chi(X, Y)$, hold because the Y in both cases fibres over S^1 and therefore has $\chi(Y) = 0$ (the exact sequence of a pair shows $\chi(X, Y) = \chi(X) - \chi(Y)$).

For $n = 2$, $\tilde{H}_q(F)$ and $\tilde{H}_q(N_i)$ both vanish for $q \neq 1$, so the number of vanishing 1-cycles is $\chi(N_i) - \chi(F) = \chi(f^{-1}(c_i)) - \chi(F)$. The fact that this is positive for an irregular fibre is proved in [3]. (For a reduced fibre it was first proved by Suzuki [12], see also [8]. The case of non-reduced fibres, which was also stated by Suzuki, but without proof, is an immediate consequence of Corollary 1.5 since a non-reduced fibre of a primitive polynomial must have more than one component.) $\square$

3. Proof and discussion of Theorem 1.8

We give two proofs of Theorem 1.8. Our first proof is similar to that of [1] but avoids the use of Deligne's monodromy theorem.

Let $Y = \mathbb{C}^2 \cup E$ be as described just after Theorem 1.8. E is a union of smooth rational curves $E_1, \dots, E_n$ with normal crossings. Let δ be the number of horizontal curves. Then we have (see e.g., Kaliman [4], Corollary 2; in the rational case this is Lemma 1.6 of Miyanishi and Sugie [6] who attribute it to Saito [11]).

Lemma 3.1.

$$\delta - 1 \geq \sum_{i=1}^{m} (r_{c_i} - 1),$$

where r_a is the number of irreducible components of $f^{-1}(a)$. Moreover, equality holds if f is rational. $\square$

The following lemma, which was was first proved by Suzuki [12], is immediate from Theorem 1.4.

Lemma 3.2. *The total number of vanishing cycles for f satisfies:*

$$\sum_{i=1}^{m} \dim V_1(c_i) = 1 - \chi(F).$$

Now Corollary 1.5 implies that if the global monodromy is trivial then

$$\dim V_i = r_{c_i} - 1 \quad \text{for } i = 1, \dots, m,$$

so applying the above two lemmas gives

$$\delta - 1 \geq \sum_{i=1}^{m} (r_{c_i} - 1) = \sum_{i=1}^{m} \dim V_1(c_i) = 1 - \chi(F),$$

whence

$$\delta + \chi(F) \geq 2.$$

Let $\overline{F}$ be the generic fibre of $\overline{f}\colon Y \to \mathbb{C}P^1$. Then $\overline{F}$ intersects each horizontal curve of the compactification divisor E, so $\overline{F} - F$ consists of at least δ points. Thus

$$\chi(\overline{F}) \geq \delta + \chi(F) \geq 2.$$

It follows that these inequalities are equalities. Thus, F is a rational curve and, moreover, $\overline{F}$ intersects each horizontal curve in exactly one point, so f is of simple type.

Conversely, if f is rational of simple type, then the homology of a generic fibre has a basis consisting of small circles about all but one of its punctures. The punctures occur where the compactified fibre $\overline{F}$ intersects the horizontal curves, so the homology classes can be globally indexed by which horizontal curve they come from. It follows that the global monodromy must be trivial. □

There is also a quick proof using only Deligne's monodromy theorem [2]. Indeed, Deligne's theorem gives an epimorphism $H^1(Y) \to H^1(\overline{F})^{\Pi}$, but $H^1(Y) = 0$, so this implies the first part of the following proposition (which strengthens Theorem 1.8).

Proposition 3.3. 1. *The global monodromy on the closed fibre $\overline{F}$ is trivial if and only if f has rational generic fibres.*

2. *If we consider the subgroup $B \subset H_1(F)$ generated by small loops around the punctures of F, then the global monodromy restricted to B is trivial if and only if $\overline{f}$ is degree 1 on all horizontal curves.*

For the second part of this proposition note that if $\overline{f}$ is degree > 1 on some horizontal curve E then the homology classes represented by the punctures where $\overline{F}$ meets E get permuted non-trivially as we circle a branch point of $\overline{f}|E$. □

We can refine the last argument to obtain a stronger result. Let $p_{i1}, \dots, p_{ik_i}$ be the points where $\overline{f}^{-1}(c_i)$ meets horizontal curves and for each $j = 1, \dots, k_i$ let δ_{ij} be the degree of $\overline{f}$ on a small neighbourhood of the point p_{ij} in its horizontal curve. Thus, the generic fibre F near $f^{-1}(c_i)$ has δ_{ij} punctures near p_{ij} that are cyclically permuted by the monodromy around c_i. It follows that the restriction of $1 - h_1(c_i)$ to the subgroup B of the above proposition has image of dimension $\sum_{j=1}^{k_i}(\delta_{ij} - 1)$. Denote

$$\begin{aligned} e_{c_i} :&= \dim \operatorname{Im}\bigl(1 - h_1(c_i)\bigr) - \dim \operatorname{Im}\bigl(\bigl(1 - h_1(c_i)\bigr)|B\bigr) \\ &= \dim \operatorname{Im}\bigl(1 - h_1(c_i)\bigr) - \sum_{j=1}^{k_i}(\delta_{ij} - 1). \end{aligned}$$

This measures the "extra" part of $\operatorname{Im}\bigl(1 - h_1(c_i)\bigr)$ that does not arise from the homology at infinity.

It is clear that if $e_{c_i} = 0$ then the local monodromy $\overline{h}_1(c_i)\colon H_1(\overline{F}) \to H_1(\overline{F})$ of the closed fibre around $\overline{f}^{-1}(c_i)$ is trivial. The converse is *not* true for arbitrary maps of a surface, but the following theorem implies that it is for our local monodromy map.

Theorem 3.4. *With $\overline{V}_1(c_i) := \operatorname{Ker}(H_1(\overline{F}) \to H_1(\overline{N}_i))$, we have*

$$\operatorname{Im}\bigl(1 - \overline{h}_1(c_i)\bigr) \subset \overline{V}_1(c_i)$$

and both these groups have rank e_{c_i}. *Moreover*

$$\sum_{i=1}^{m} e_{c_i} \geq 2\,\mathrm{genus}(F).$$

Proof. The inclusion $\mathrm{Im}(1-\overline{h}_1(c_i)) \subset \overline{V}_1(c_i)$ is clear, while the fact that they have the same dimension is proved in part 2c) of section III of [1] (note that $\dim \overline{V}_1(c_i)$ is exactly the number k_{c_i} of Kaliman [4], discussed also in [1]). We have a short exact sequence

$$0 \to B \to H_1(F) \to H_1(\overline{F}) \to 0 \tag{3}$$

and taking the image of $1-h(c_i)$ applied to this sequence gives a sequence

$$0 \to \mathbb{Z}^{\sum_j (\delta_{ij}-1)} \to \mathrm{Im}\big(1-h_1(c_i)\big) \to \mathrm{Im}\big(1-\overline{h}_1(c_i)\big) \to 0. \tag{4}$$

This sequence is exact except possibly at its middle term (this holds for a homomorphic image of any short exact sequence). The cokernel of $\mathbb{Z}^{\sum_j (\delta_{ij}-1)} \to \mathrm{Im}\big(1-h_1(c_i)\big)$ has dimension, by definition, e_{c_i}. Since the sequence induces a surjection of this cokernel to $\mathrm{Im}\big(1-\overline{h}_1(c_i)\big)$ we see:

$$e_{c_i} \geq k_{c_i}. \tag{5}$$

On the other hand, Corollary 1.5 implies:

$$\dim V(c_i) = \dim \mathrm{Im}\big(1-h_1(c_i)\big) + (r_{c_i}-1) = \sum_{j=1}^{k_i} (\delta_{ij}-1) + e_{c_i} + r_{c_i} - 1.$$

Summing this over i and applying Lemma 3.2 on the left and the Riemann-Hurwitz formula on the right gives

$$1-\chi(F) = \sum (d_E - 1) + \sum_{i=1}^{m} (e_{c_i} + r_{c_i} - 1),$$

where the first sum on the right is over all horizontal curves E and d_E is the degree of $\overline{f}$ on E. Since $\sum d_E$ is the number of punctures of F this simplifies to

$$2\,\mathrm{genus}(F) = 1 - \delta + \sum_{i=1}^{m} (e_{c_i} + r_{c_i} - 1), \tag{6}$$

where δ is the number of horizontal curves. But Kaliman proves this equation in [4] with e_{c_i} replaced by k_{c_i}, so the inequalities (5) must be equalities. The final inequality of the theorem follows from (6) and Lemma 3.1. $\square$

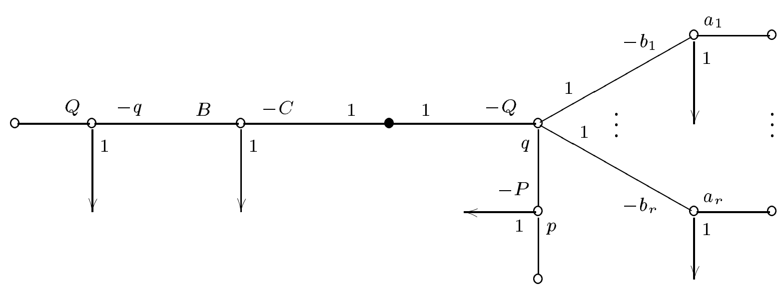
A surprising consequence of the above proof is the exactness of the kernel sequence (and hence also the image sequence (4)) of $1-h(c_i)$ applied to the short exact sequence (3). Indeed, if we replace each group A in (3) by the chain complex $0 \to A \xrightarrow{1-h} A \to 0$, then the resulting short exact sequence of chain complexes has long exact homology sequence $0 \to \mathrm{Ker}(1-h_1|B) \to \mathrm{Ker}(1-h_1) \to \mathrm{Ker}(1-\overline{h}_1) \to \mathrm{Cok}(1-h_1|B) \to \mathrm{Cok}(1-h_1) \to \mathrm{Cok}(1-\overline{h}_1) \to 0$. The equality in (5) implies that the middle map of this sequence has rank 0, and hence is the zero map since $\mathrm{Cok}(1-h_1|B)$ is free abelian.

4. Classification of rational polynomials of simple type

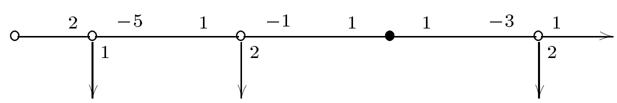
The classification in [6] mistakenly assumed isotriviality (all regular fibres of f are conformally isomorphic to each other) at one stage in the proof (page 346, lines 10–11). There are in fact also many non-isotrivial 2-variable rational polynomials of simple type, the simplest being $f(x,y) = x(1+xy)(1+axy)+xy$ of degree 5, whose regular fibres $f^{-1}(c)$ are 4-punctured $\mathbb{C}P^1$'s such that the cross-ratio of the punctures varies linearly with c.

In this section we list the non-isotrivial rational polynomials of simple type. We list their regular splice diagrams (see [7], [8]), since this gives a useful description of the topology. For each case there are several possible topologies for the irregular fibres, depending on additional parameters. We have a proof that these examples complete the classification but it is tedious and not yet written down in full detail, so the result should be considered tentative.

Let p, q, P, Q be positive integers with $Pq - pQ = 1$ and let r and $a_1, \dots, a_r$ be positive integers. Let $A = \sum_{i=1}^r a_i$, $B = AQ + P - Q$, $C = Aq + p - q$, and $b_i = qQa_i + 1$ for $i = 1, \dots, r$. Then the following is the regular splice diagram of a rational polynomial of simple type.



There is one further degree 8 example that does not fall in the above family. The splice diagram is



In all these examples the curve obtained by filling the puncture corresponding to the second arrowhead from the left has constant conformal type as we vary the regular fibre $f^{-1}(c)$, and that puncture varies linearly with $c \in \mathbb{C}$.

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