HYPERBOLIC MONOPOLES AND HOLOMORPHIC SPHERES

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ABSTRACT. We associate to an $SU(2)$ hyperbolic monopole a holomorphic sphere embedded in projective space and use this to uncover various features of the monopole.

1. INTRODUCTION

In this paper we exploit the geometry of hyperbolic space to study monopoles. We will use features of hyperbolic space that do not arise in Euclidean space, and hence expose properties of hyperbolic monopoles that have no analogues for Euclidean monopoles. The space of geodesics in $\mathbb{H}^3$ is the complex manifold

$$Z = \mathbb{P}^1 \times \mathbb{P}^1 - \bar{\Delta}$$

where the point $(w, z) \in \mathbb{P}^1 \times \mathbb{P}^1$ represents the geodesic that runs from $\hat{w} = -1/\bar{w}$, the antipodal point of w, to z considered as points on the sphere at infinity. The antidiagonal $\bar{\Delta}$ has been removed, although one aspect of this paper is that in some sense we can replace the antidiagonal, making sense of $(\hat{z}, z)$, which represents a geodesic from z to itself.

A monopole is a pair (A, Φ) consisting of a connection A with L^2 curvature F_A defined on a trivial bundle E over $\mathbb{R}^3$ with structure group $SU(2)$, and a Higgs field $\Phi : \mathbb{R}^3 \to \mathbf{su}(2)$ that solves the Bogomolny equation

$$d_A \Phi = *F_A$$

and satisfies $\lim_{r \to \infty} ||\Phi|| = m$, the mass of the monopole. The charge of the monopole is defined to be the topological degree of the map $\Phi_\infty : S^2 \to S^2$. The gauge group of maps $g : \mathbb{R}^3 \to SU(2)$ acts on the equations and we identify gauge equivalent monopoles. The metric is featured in the Hodge star, $*$. In this paper we will mainly consider the hyperbolic metric and sometimes refer to the Euclidean metric.

Hyperbolic and Euclidean monopoles have been studied using three constructions: the spectral curve, Nahm data, and the rational map [2, 6, 8, 9, 10, 16, 17]. Each construction is related, although they are different enough that a particular aspect of monopoles is often more readily seen from the perspective of one of these constructions. The rational map and the spectral curve use solutions of an ordinary differential equation ((1) in the next section) defined along geodesics, known as the *scattering equation*. The rational map arises when one restricts the scattering equation to the pencil of geodesics that contain a common point (which may be at infinity.) The spectral curve is defined to be the set of geodesics along which the scattering equation has an L^2 solution. It is a compact algebraic curve inside the

variety of geodesics. The spectral curve of a charge k monopole is a degree (k, k) curve in $\mathbb{P}^1 \times \mathbb{P}^1 - \bar{\Delta}$, or equivalently the zero set of a holomorphic section of the line bundle $\mathcal{O}(k, k)$.

In this paper, we introduce a fourth construction for hyperbolic monopoles—an embedded holomorphic sphere in projective space. Such a construction has existed previously for half-integer mass hyperbolic monopoles [6]. We define it for any real mass using completely different techniques to [6] and give various applications, some of which are known for half-integer mass monopoles.

A hyperbolic monopole has a well-defined limit at infinity given by a reducible connection A_∞ over a two-sphere. Denote by F_{A_∞} the curvature of the reducible connection.

Theorem 1. *An $SU(2)$ hyperbolic monopole (A, Φ) of charge k is determined by a degree k holomorphic embedding $q : \mathbb{P}^1 \to \mathbb{P}^k$ uniquely defined up to the action of $U(k + 1)$ on its image with the properties:*
(i) $\Sigma = \{(w, z) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid \langle q(\hat{w}), q(z) \rangle = 0\}$ is the spectral curve of (A, Φ);
(ii) $F_{A_\infty} = q^* \omega$, for ω the Kahler form on $\mathbb{P}^k$.

One consequence of the theorem is the fact that an $SU(2)$ hyperbolic monopole is determined up to gauge by its reducible connection on the sphere at infinity. This was proven in [19] by a different method, and for the half-integer mass case in [6] also using an embedded sphere in projective space.

Theorem 1 relies on the fact that $\mathbb{P}^1 \times \mathbb{P}^1 - \bar{\Delta}$, the twistor space of geodesics in $\mathbb{H}^3$, has a compactification obtained by including a totally real surface. The same situation seems to arise for spherical monopoles [20] which would lead one to predict that a monopole with one singularity on S^3 is determined by its asymptotic value near the singularity, and it is neatly described by a holomorphic sphere in projective space. The construction does not apply to Euclidean monopoles. Only the charge of a Euclidean monopole is detected from its reducible connection at infinity. The difference comes down to the asymptotic decay conditions forced on finite energy monopoles in Euclidean and hyperbolic spaces.

The *centre* of a Euclidean monopole is defined in [4]. Previously a definition of the centre has existed only for half-integer mass hyperbolic monopoles. In [23] the third author proposed a definition for a general hyperbolic monopole but could not prove that the centre is unique. Intuitively, the centre of a monopole arises from the $PSL(2, \mathbb{C})$ action on hyperbolic space. One would like to show that the $PSL(2, \mathbb{C})$ orbit of a hyperbolic monopole possesses a centred monopole unique up to the action of $SO(3) \subset PSL(2, \mathbb{C})$. The holomorphic sphere allows one to apply geometric invariant theory to obtain such a definition for the centre of the monopole.

Theorem 2. *There is a lift of the $PSL(2, \mathbb{C})$ action on the space of hyperbolic monopoles to a linear $SL(2, \mathbb{C})$ action on $\mathbb{C}^N$ whose stable points contain the space of hyperbolic monopoles. A monopole has a unique centre, and it is centred when it lies in the zero set of the moment map for $SU(2) \subset SL(2, \mathbb{C})$.*

Given a hyperbolic monopole (A, Φ) and a point $w \in S^2_\infty$, one can use the scattering equation along geodesics γ satisfying $\lim_{t \to -\infty} \gamma(t) = w$ to define a degree k rational map $f_w(z) : \mathbb{P}^1 \to \mathbb{P}^1$. Moreover, the rational map uniquely determines the monopole. Previously, it has not been understood how one might relate the

different rational maps as w is varied. The holomorphic sphere q in some sense combines all of these rational maps.

Theorem 3. *For any $w \in S_\infty^2$,*

$$f_w = \pi_w \circ q : \mathbb{P}^1 \to \mathbb{P}^k \to \mathbb{P}^1$$

where π_w is projection onto a unique line $L_w \subset \mathbb{P}^k$ that contains $q(w)$.

We have been unable to improve the theorem from an existence result to a more satisfying version that would specify L_w, and a scale (described in Section 4), in terms of q.

We prove Theorems 1, 2 and 3 in Sections 2, 3 and 4 respectively. Property (i) of Theorem 1 is not sufficient to guarantee that a curve is the spectral curve of a monopole. In general a spectral curve requires even further restrictions. Nevertheless, in Section 5 we exploit the fact that for charge two monopoles no further restrictions are necessary. In Section 6 we discuss similarities between the holomorphic sphere and previous work on massless monopoles. In the final section we prove a vanishing theorem for hyperbolic monopoles which we need to prove Theorem 1. This is of some independent interest having been conjectured in [16] and is a necessary step towards generalising that work to monopoles of non-integral mass.

2. Holomorphic sphere

Theorem 1 consists of two quite independent results. Part (i) states that the spectral curve of a hyperbolic monopole is of a specific type inside the variety of (k, k) curves in $\mathbb{P}^1 \times \mathbb{P}^1$. Part (ii) is a consequence of a more direct relationship between the spectral curve and the boundary value of the hyperbolic monopole, given in Theorem 5 in terms of the defining polynomial of the spectral curve.

The spectral curve of a hyperbolic monopole possesses a type of positivity property which can be seen explicitly in the case of charge 1 monopoles. The spectral curve of a charge 1 monopole is a real $(1, 1)$ curve corresponding to all geodesics containing a given point of $\mathbb{H}^3$. Such a $(1, 1)$ curve necessarily lies in the connected component of the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ which represents all geodesics containing $0 \in \mathbb{H}^3$. For example, if a real $(1, 1)$ curve contains the points $(0, 0)$ and (∞, ∞), it is of the form $w - az = 0$ for $a \in \mathbb{R}^*$. It is a spectral curve of a charge 1 monopole precisely when $a > 0$. The proof of the first part of Theorem 1 is a generalisation of this simple fact, using the connectivity of the moduli space and a rather deep analogue of the property $a \in \mathbb{R}^*$.

The defining "polynomial" of the spectral curve of a hyperbolic monopole is an example of a general feature used in this paper. A section of

$$
\begin{array}{c}
\mathcal{O}(k, k) \\
\downarrow \\
Z = \mathbb{P}^1 \times \mathbb{P}^1 - \bar{\Delta}
\end{array}
$$

extends to a section of

$$
\begin{array}{c}
\mathcal{O}(k, k) \\
\downarrow \\
Q = \mathbb{P}^1 \times \mathbb{P}^1
\end{array}
$$

and hence is given by a polynomial. More generally, if a holomorphic bundle over Z extends to Q then any section extends. When the bundle is trivial, this says that a holomorphic function on Z is necessarily constant, which uses the fact that

Z contains many compact holomorphic curves, in particular those $(1,1)$ curves corresponding to all geodesics containing a given point of $\mathbb{H}^3$. The more general fact can be proven in a couple of ways. In the proof of Lemma 9.1, it is shown that sections of line bundles over Z lift to homogeneous functions defined over a large enough (to contain many compact holomorphic curves) subset of $\mathbb{P}^3$. An alternative argument uses the fact that any local holomorphic function defined over a deleted neighbourhood of a totally real submanifold, in this case any open set in $\bar{\Delta}$, extends uniquely to a holomorphic function on the neighbourhood. Thus, if a holomorphic bundle over Z extends to Q, then any local holomorphic section also extends.

2.1. **Positive definite.** Along any geodesic $\gamma \subset \mathbb{H}^3$ the monopole (A, Φ) defines the scattering equation

$$(\partial_t^A - i\Phi)s = 0 \tag{1}$$

where t parametrises γ and $s(t)$ is a section of E restricted to γ. The Bogomolny equations define an integrability condition $[\partial_t^A - i\Phi, \partial_{\bar{z}}^A] = 0$ and hence local solutions satisfying $\partial_{\bar{z}}^A s = 0$ can be found. These define a holomorphic bundle $\tilde{E}$ over Z with distinguished sub-line bundles L_+, L_- given by those solutions that decay as $t \to \infty$, respectively $t \to -\infty$. The line bundles L_+ and L_- coincide over an algebraic curve $\Sigma \in \mathbb{P}^1 \times \mathbb{P}^1$ known as the *spectral curve*. Points on the spectral curve represent geodesics that possess a solution which decays both as $t \to \pm\infty$.

Corresponding to reversing the direction of a geodesic, the space of geodesics $\mathbb{P}^1 \times \mathbb{P}^1 - \bar{\Delta}$ possesses a real structure $(w, z) \mapsto (\hat{z}, \hat{w})$. The spectral curve Σ is invariant under the real structure since a solution of (1) along $\gamma(t)$ can be used to construct a solution of (1) along $\gamma(-t)$ with decay preserved.

Lemma 2.1. *The defining polynomial for the spectral curve can be chosen to satisfy*

$$\psi(\hat{z}, \hat{w}) = \overline{\psi(w, z)} \tag{2}$$

and to be positive on the anti-diagonal $w = \hat{z}$.

Proof. If we take $\psi(w, z)$ to mean a degree (k, k) polynomial in w^{-1} and z (perhaps one would prefer $P(w^{-1}, z)$ or to refer to $w^k \psi(w, z)$ as the polynomial) then one can express the reality condition quite simply. The reality condition means that $\psi(\hat{z}, \hat{w})$ and $\psi(w, z)$ have the same zero set and since $\psi(w, z)$ and the complex conjugate of $\psi(\hat{z}, \hat{w})$ both define degree (k, k) polynomials in w^{-1} and z, they are the same up to a constant $\psi(\hat{z}, \hat{w}) = c\overline{\psi(w, z)}$. The spectral curve does not intersect the anti-diagonal so ψ does not vanish there, and hence $c = \exp(2i\theta)$ for some constant θ. We can replace ψ by $\exp(-i\theta)\psi$ to get (2). Since ψ does not vanish on the anti-diagonal, it is either positive or negative there, and if the latter we can replace it by $-\psi$. $\square$

Theorem 4. *For each monopole (A, Φ) there exists a holomorphic embedding*

$$q : \mathbb{P}^1 \to \mathbb{P}^k$$

unique up to the action of $U(k+1)$ on its image satisfying

$$\langle q(\hat{w}), q(z) \rangle = \psi(w, z).$$

Proof. Let $v(z) = (1, z, z^2, \ldots, z^k)$. Then

$$\psi(w, z) = v(-1/w)^T \Psi v(z)$$

for a $(k+1) \times (k+1)$ matrix Ψ. Condition (2) is equivalent to $\Psi = \overline{\Psi}^T$.

To prove the theorem we will show that the matrix Ψ is positive definite so $\Psi = \bar{Q}^T Q$ for an invertible $(k+1) \times (k+1)$ matrix Q unique up to $Q \mapsto uQ$ for $u \in U(k+1)$. Then set $q(z) = Qv(z)$, a degree k holomorphic map.

This proves a stronger property of q than simply being an embedding—the image of q spans all of $\mathbb{P}^k$, and we call it *full*. A full map is an embedding since any singular point z would satisfy $0 = q'(z) = Qv'(z)$, and any double point would satisfy $0 = q(z_1) - \lambda q(z_2) = Q(v(z_1) - \lambda v(z_2))$, and in both cases Q would have a non-trivial kernel, contradicting the fullness of q.

The following lemma gives part of the property that the bilinear form Ψ is positive definite.

Lemma 2.2. *The matrix Ψ is non-degenerate.*

Proof. We will prove that for any (k,k) curve $\Sigma = \{(w,z) \in \mathbb{P}^1 \times \mathbb{P}^1 | \psi(w,z) = 0\}$ with coefficient matrix Ψ, the condition that Ψ be nondegenerate is equivalent to $H^0(\Sigma, \mathcal{O}(-2,k)) = 0$. The spectral curve of a mass m monopole satisfies the property $L^{2m+k}|_\Sigma \cong \mathcal{O}_\Sigma$ [17] so

$$H^0(\Sigma, \mathcal{O}(-2,k)) = H^0(\Sigma, L^{2m}(k-2,0))$$

and the latter vanishes by Theorem 7.

A section of $H^0(\Sigma, \mathcal{O}(-2,k))$ is represented by a polynomial together with the defining polynomial of Σ, expressed as $p(w), \psi(w) \in \mathbb{C}[z][w^{-1}]$ with coefficients given by sections of $\mathcal{O}(k)$, or degree k polynomials in z, such that

$$w^{-2}p(w) + \psi(w)q(w) \in \mathbb{C}[z][w] \quad \text{for some} \quad q(w) \in \mathbb{C}[w, w^{-1}]. \tag{3}$$

Put $q = \sum q_l(-1)^l w^{l-1}$. Then

$$\psi(w)q(w) = \sum_{i,j,l} \Psi_{ij} q_l (-1)^{i+l} w^{l-i-1} z^j \tag{4}$$

and the coefficient of $w^{-1}z^j$ is $\sum_i \Psi_{ij} q_i$. The degeneracy of Ψ is equivalent to the existence of a nontrivial q such that $\sum_i \Psi_{ij} q_i = 0$ for all j. But then (4) becomes (3) if we move the terms on the right hand side of (4) with negative powers of w to the left hand side, and the lemma is proven.

The difference of sections in two charts giving rise to a vector in the kernel of Ψ looks like the coboundary map in cohomology. In fact, we can express the proof of the lemma in terms of the exact sequence in cohomology given by

$$0 \to H^0(\Sigma, \mathcal{O}(-2,k)) \to H^1(Q, \mathcal{O}(-2-k,0)) \xrightarrow{\Psi} H^1(Q, \mathcal{O}(-2,k))$$

where the right-most map is multiplication by $\psi(w,z)$ and becomes the matrix $\Psi : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$. $\qquad\square$

Since a continuous family of non-degenerate Hermitian matrices has constant signature, it follows from Lemma 2.2 and the connectivity of the moduli space that we need show that only one monopole possesses a positive definite Ψ. This is true for axially symmetric monopoles by the explicit construction given in Section 7 (or we can prove it for half-integer mass monopoles using techniques from [6].) Hence Ψ is positive definite for all monopoles and the theorem is proven. $\qquad\square$

2.2. **Hermitian metrics.** A Hermitian metric on a vector space V is a linear map

$$H : \overline{V} \otimes V \to \mathbb{C} \quad \text{satisfying} \quad H(u, v) = \overline{H(\overline{v}, \overline{u})}$$

where the map $\overline{\cdot}$ gives an antilinear isomorphism from V to $\overline{V}$ and back.

A Hermitian metric on a holomorphic bundle uniquely determines a Hermitian connection on the holomorphic bundle compatible with the holomorphic structure. The reducible connection on the sphere at infinity A_∞, a $U(1)$ connection on the holomorphic line bundle $\mathcal{O}(-k)$ over S^2, can be described via a Hermitian metric on $\mathcal{O}(-k)$. In local coordinates, the Hermitian metric, h, is locally a positive valued function well-defined up to $h(z) \sim |g(z)|^2 h(z)$, for g a local holomorphic function. The h-Hermitian connection is $\partial_z \ln h \cdot dz$, or in a unitary gauge it is

$$A_\infty = -\partial_{\bar{z}} \ln \xi \cdot d\bar{z} + \partial_z \ln \xi \cdot dz \tag{5}$$

for $\xi^2 = h$, the positive square root.

Theorem 5. *Let ψ be the defining polynomial of the spectral curve of (A, Φ),*

$$\Sigma = \{(w, z)|\psi(w, z) = 0\}.$$

Then the restriction of ψ to the anti-diagonal, $\psi|_{\bar{\Delta}}$, gives rise to a Hermitian metric on the holomorphic bundle $\mathcal{O}(-k)$ over $\overline{\Delta}$ that defines the connection at infinity.

Proof. The real structure on $\mathbb{P}^1 \times \mathbb{P}^1$ given by $(w, z) \mapsto (\hat{z}, \hat{w})$ lifts to a real structure on the bundle $\mathcal{O}(k, k)$. This is reflected in Lemma 2.1 where it is proven that local trivialisations for $\mathcal{O}(k, k)$ can be chosen so that the involution on each fibre is simply complex conjugation, and the real structure fixes any section of $\mathcal{O}(k, k)$ whose zero set is preserved by the real structure.

Any section s of $\mathcal{O}(k, k)$ gives a map $s : \mathcal{O}(-k, -k) \to \mathbb{C}$. Suppose s is fixed by the real structure. Restrict s to the fixed point set of the real structure, $\overline{\Delta} \subset \mathbb{P}^1 \times \mathbb{P}^1$. We can identify $\mathcal{O}(-k, -k)|_{\overline{\Delta}} \cong \overline{\mathcal{O}(-k)} \otimes \mathcal{O}(-k)$ so s defines a Hermitian metric $s : \overline{\mathcal{O}(-k)} \otimes \mathcal{O}(-k) \to \mathbb{C}$ on the holomorphic bundle $\mathcal{O}(-k)$ over $\overline{\Delta}$.

Apply this to ψ, the defining polynomial of the spectral curve, since it is fixed under the real structure. Its restriction to the anti-diagonal defines a Hermitian metric on the holomorphic bundle $\mathcal{O}(-k)$, and hence a Hermitian connection there. It remains to show that this Hermitian connection is the $U(1)$ connection at infinity of the monopole. This is a consequence of the following three lemmas.

In order to understand the map $\psi : \mathcal{O}(-k, -k) \to \mathbb{C}$ we choose local holomorphic sections $\overline{s_+(\hat{z}, \hat{w})} \otimes s_+(w, z)$ of $\mathcal{O}(-k, -k)$ where $s_+(w, z)$ is a solution of (1) along the geodesic traveling from $\hat{w}$ to z (so $s_+(\hat{z}, \hat{w})$ is a solution of (1) along the oppositely oriented geodesic.)

Lemma 2.3. *The section ψ acts on $\mathcal{O}(-k, -k)$ by*

$$\psi(\overline{s_+(\hat{z}, \hat{w})} \otimes s_+(w, z)) = \langle s_+(\hat{z}, \hat{w}), s_+(w, z) \rangle.$$

Proof. Recall from [17] that a hyperbolic monopole defines a holomorphic bundle $\widetilde{E} \to Z$ with two extensions:

$$0 \to L^m(0, -k) \to \widetilde{E} \to L^{-m}(0, k) \to 0$$

and

$$0 \to L^{-m}(-k, 0) \to \widetilde{E} \to L^m(k, 0) \to 0$$

for $L = \mathcal{O}(1, -1)$. The sub-bundles $L^+ = L^m(0, -k)$ and $L^- = L^{-m}(-k, 0)$, defined as the space of solutions of (1) that decay as $t \to \infty$, respectively $t \to -\infty$, coincide

over the spectral curve Σ and their coincidence defines a non-vanishing section over Σ of L^{2m+k}.

The spectral curve is a (k, k) curve with defining polynomial ψ, hence

$$0 \to \mathcal{O}(-k, -k) \xrightarrow{\psi} \mathcal{O} \to \mathcal{O}_\Sigma \to 0$$

we can tensor this with $L^m(k, 0)$ to get

$$
\begin{array}{ccccccccc}
0 & \to & L^m(0, -k) & \xrightarrow{\psi} & L^m(k, 0) & \to & \mathcal{O}_\Sigma(L^m(k, 0)) & \to & 0 \\
 & & \uparrow\cong & & \uparrow & & \uparrow & & \\
0 & \to & L^m(0, -k) & \to & \widetilde{E} & \to & L^{-m}(0, k) & \to & 0
\end{array}
$$

which represents ψ as a map $\psi : L^+ \to \widetilde{E}/L^-$.

For τ the real structure on the space of geodesics, there is a natural map

$$\widetilde{E}_{\tau(w,z)} \otimes \widetilde{E}_{(w,z)} \to \mathbb{C}$$

given by $r \otimes s \mapsto \langle r(t), s(t) \rangle$ where $r \in \widetilde{E}_{\tau(w,z)}$ and $s \in \widetilde{E}_{(w,z)}$ or equivalently, $(\partial_t^A + i\Phi)r = 0$ and $(\partial_t^A - i\Phi)s = 0$. The inner product $\langle r(t), s(t) \rangle$ is independent of t, since

$$\partial_t \langle r(t), s(t) \rangle = \langle (\partial_t^A + i\Phi)r(t), s(t) \rangle + \langle r(t), (\partial_t^A - i\Phi)s(t) \rangle = 0.$$

Thus, $\psi : L^+ \to \widetilde{E}/L^-$ can be re-expressed as $\psi : (\widetilde{E}/L^-)^* \otimes L^+ \to \mathbb{C}$ and since $(\widetilde{E}/L^-)^* \cong (L^-)^\perp = \overline{\tau^* L^+}$, we have

$$\psi(\overline{s_+(\hat{z}, \hat{w})} \otimes s_+(w, z)) = \langle s_+(\hat{z}, \hat{w}), s_+(w, z) \rangle.$$

$\square$

Lemma 2.4.

$$\lim_{w \to \hat{z}} \psi(w, z) = \lim_{w \to \hat{z}} \langle s_+(\hat{z}, \hat{w}), s_+(w, z)) \rangle = \lim_{t \to \infty} \exp(2mt) \|s_+(w, z)\|^2.$$

Proof. In order to make sense of the lemma, we really need to choose local trivialisations for the bundles so that we are dealing with local functions, and so that s_+ is well-defined. We can do this as follows. Fix $\lim_{t \to \infty} \exp(mt) s_+(w, z)$ in a small neighbourhood $U \subset \mathbb{P}^1$ containing z. (One can choose any family of solutions $s_+(w_0, z)$, for fixed w_0, and use $\lim_{t \to \infty} \exp(mt) s_+(w_0, z)$.) As w moves close enough to $\hat{z}$ so that $\hat{w} \in U$, we use the same limit for $\lim_{t \to -\infty} \exp(mt) s_+(\hat{z}, \hat{w})$, where we use the parameter $-t$ for the oppositely oriented geodesic.

The proof is not yet immediate, since we have only arranged that the values at opposite ends of a geodesic in the t independent quantity $\langle s_+(\hat{z}, \hat{w}), s_+(w, z) \rangle$ are approximately the same.

Now use the fact from [21] that there exists a gauge in which

$$\partial_t^A \pm i\Phi = \partial_t \pm i \begin{pmatrix} im & 0 \\ 0 & -im \end{pmatrix} + \epsilon \cdot C \exp(-m|t|) \tag{6}$$

where C is constant and $\epsilon \to 0$ as $w \to z$. We see that we do indeed end up with the product of $\lim_{t \to -\infty} \exp(mt) s_+(\hat{z}, \hat{w})$ and $\lim_{t \to \infty} \exp(mt) s_+(w, z)$ which, by construction, tends towards $\lim_{t \to \infty} \exp(2mt) \|s_+(w, z)\|^2$. Further details can be found in [19]. $\square$

Lemma 2.5. *Let s_+ be a local holomorphic section of L_+. Then if we fix w and parametrise the sphere at infinity by z*

$$h(w, z) = \lim_{t \to \infty} \exp(2mt) \|s_+(w, z)\|^2$$

is a Hermitian metric that determines the $U(1)$ connection at infinity.

Proof. Since the $U(1)$ connection at infinity is Hermitian, it can be determined from its $(0, 1)$ part. If we fix one end of a family of geodesics (to be $\hat{w}$) and vary the other end (z), then $\partial_{\bar{z}}^A s_+ = \lambda(z) s_+$ for some $\lambda(z)$ *independent of t*. This follows from the three properties $(\partial_t^A - i\Phi)s_+ = 0$, $[\partial_{\bar{z}}^A, \partial_t^A - i\Phi] = 0$ and $\partial_{\bar{z}}^A s_+$ decays as $t \to \infty$. In particular, $\lambda(z)$ makes sense at $t = \infty$ and gives the $(0, 1)$ part of the $U(1)$ connection at infinity. The $(1, 0)$ part can be determined by the fact that the $U(1)$ connection at infinity is Hermitian with respect to $\lim_{t \to \infty} \exp(2mt)\|s_+(w, z)\|^2$. $\qquad \blacksquare$

From Lemmas 2.3, 2.4 and 2.5, we see that $\psi|_{\bar{\Delta}}$ defines a Hermitian metric that gives rise to the $U(1)$ connection at infinity.

Note that when choosing a local frame for $\mathcal{O}(-k, -k)$, if we also require

$$\partial_{\bar{z}}^A s_+(w, z) = 0 = \partial_w^A s_+(w, z)$$

and similar conditions on $s_+(\hat{z}, \hat{w})$, then local holomorphic sections for $\mathcal{O}(-k, -k)$ are simply given in terms of local holomorphic functions with respect to this frame, whereas without these extra conditions one must use the $(0, 1)$ part of a connection to detect local holomorphic sections.

It is important to understand that in order to use (5) to retrieve the connection from the Hermitian metric, one needs a local trivialisation of the holomorphic bundle in which local holomorphic sections are given by local holomorphic functions. When we choose *separable* transition functions for $\mathcal{O}(-k, -k)$, that is each transition function is given by a product of transition functions for $\mathcal{O}(-k, 0)$ and $\mathcal{O}(0, -k)$, then the holomorphic structure on the bundle $\mathcal{O}(-k)$ over the antidiagonal has local holomorphic sections given by local holomorphic functions. In particular, the choice of ψ as a *polynomial* (in the local coordinates w and z or $-1/w$ and z, etc) arises from separable transition functions. Thus, we can choose ψ to be the defining polynomial and use (5) to retrieve the connection and Theorem 5 is proven. $\qquad \blacksquare$

In the statement of Theorem 1, we express the relationship of the holomorphic map $q : \mathbb{P}^1 \to \mathbb{P}^k$ with the connection at infinity via $F_{A_\infty} = q^*\omega$, for ω the Kahler form on $\mathbb{P}^k$. The holomorphic map q pulls back a Hermitian metric, its connection and its curvature. The Hermitian metric is given by $\langle q(z), q(z) \rangle$ which, by Theorem 4 is $\psi|_{\bar{\Delta}}$. Thus, Theorem 5 can be restated as q pulls back the Hermitian metric that defines the $U(1)$ connection at infinity and in particular $F_{A_\infty} = q^*\omega$ and part (ii) of Theorem 1 is proven.

The real analyticity of ψ in a neighbourhood of $\bar{\Delta}$, and the fact that $\bar{\Delta}$ is a totally real submanifold of $\mathbb{P}^1 \times \mathbb{P}^1$ allows one to show that $\psi|_{\bar{\Delta}}$ well-defined up to multiplication by holomorphic and anti-holomorphic functions uniquely extends and hence determines ψ on $\mathbb{P}^1 \times \mathbb{P}^1$. See [19] for details. Thus

Corollary 6. *An $SU(2)$ hyperbolic monopole is determined up to gauge by its reducible connection on the sphere at infinity.*

This was proven in [19] by a slightly different method. That paper did not require Theorem 5—$\psi|_{\bar{\Delta}}$ is a Hermitian metric that defines the connection at infinity— although it did use the fact that $\psi|_{\bar{\Delta}}$ determines ψ. The proof of Corollary 6 in the half integer mass case [6] uses the discrete Nahm equations to prove that a holomorphic map $q : \mathbb{P}^1 \to \mathbb{P}^k$ uniquely determines the monopole, and a result of Calabi [7] to show that $q^*\omega$ uniquely determines q. That approach, combined with Theorem 1 can be used to give a third proof of Corollary 6, although the use of Calabi's theorem seems a bit unnecessary given the alternative local argument.

3. Centred monopoles

The holomorphic sphere $q : \mathbb{P}^1 \to \mathbb{P}^k$ associated to a monopole allows one to use geometric invariant theory to define the centre of a monopole. We can represent q by

$$q(z) = v_0 + \sqrt{k}v_1 z + \sqrt{\binom{k}{2}}v_2 z^2 + \cdots + \sqrt{\binom{k}{j}}v_j z^j + \cdots + v_k z^k$$

where each $v_j \in \mathbb{C}^{k+1}$ and the coefficients arise quite naturally as we shall see later. The $PSL(2,\mathbb{C})$ action on the domain of such maps lifts to a linear action of $SL(2,\mathbb{C})$ on $(k+1)$-tuples $(v_0, \ldots, v_k) \mapsto (w_0, \ldots, w_k)$ where,

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot q(z) &= (cz+d)^k v_0 + \cdots + (cz+d)^{k-j}(az+b)^j \sqrt{\binom{k}{j}}v_j + \ldots \\
&= w_0 + \sqrt{k}w_1 z + \cdots + \sqrt{\binom{k}{j}}w_j z^j + \cdots + w_k z^k.
\end{aligned}$$

The space of $(k+1)$-tuples is a subset of $\mathbb{C}^N$ (for $N = (k+1)^2$), so that geometric invariant applies. The norm on the space of $(k+1)$-tuples is

$$\|q\|^2 = \|v_0\|^2 + \|v_1\|^2 + \cdots + \|v_j\|^2 + \cdots + \|v_k\|^2 \tag{7}$$

which is preserved by $SU(2) \subset SL(2,\mathbb{C})$. (We are abusing notation by labeling $(v_0, \ldots, v_k) \in \mathbb{C}^N$ by q when really q is the projective class in $\mathbb{CP}^{N-1}$.)

Recall that a $(k+1)-$tuple $(v_0, \ldots, v_k)$ is *stable* under the $SL(2,\mathbb{C})$ action if and only if the map $SL(2,\mathbb{C}) \to \mathbb{C}^N$ given by $g \mapsto g \cdot (v_0, \ldots, v_k)$ is proper, so in particular the $SL(2,\mathbb{C})$ orbit is closed and we can minimise the norm $\|q\|$ in its $SL(2,\mathbb{C})$ orbit.

Lemma 3.1. *Each* $(k+1)-$*tuple* $(v_0, \ldots, v_k)$ *arising from a degree k holomorphic map q is a stable point of the $SL(2,\mathbb{C})$ action.*

Proof. By the Hilbert criterion it is enough to test the stability of a point on one-parameter subgroups of $SL(2,\mathbb{C})$. Any one parameter subgroup in $SL(2,\mathbb{C})$ is given by

$$g \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} g^{-1}.$$

Since the degree of q is k then $v_0 \neq 0$ and $v_k \neq 0$. Also, after acting by g the map q is still of degree k and hence we may assume that $v_0 \neq 0$, $v_k \neq 0$ and $g = I$. Then the action is given by

$$(v_0, \ldots, v_k) \mapsto (t^{-k}v_0, \ldots, t^{2j-k}v_j, \ldots, t^k v_k).$$

In particular, since $v_0 \neq 0$ and $v_k \neq 0$, the norm $\|q\| \to \infty$ as $t \to 0$ and $t \to \infty$ so the map is proper. $\qquad\square$

Proposition 3.2. *The moment map for the action of* $SU(2)$ *is*

$$\mu(v_0, \ldots, v_k) = (\sum_{j=0}^{k}(2j-k)\|v_j\|^2, \sum_{j=0}^{k-1}\sqrt{(j+1)(k-j)}(v_j, v_{j+1})) \in \mathbb{R} \times \mathbb{C}.$$

Proof. We wish to minimise the norm (7) on each $SL(2,\mathbb{C})$ orbit (which is closed by Lemma 3.1.) The minimum occurs on stationary points of the infinitesimal action of $\mathbf{sl}(2,\mathbb{C})$. Since $\mathbf{su}(2) \subset \mathbf{sl}(2,\mathbb{C})$ acts trivially it is enough to consider the actions of

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ e_\theta = \begin{pmatrix} 0 & \exp(i\theta) \\ 0 & 0 \end{pmatrix}$$

given by

$$e_0 \cdot (v_0, \ldots, v_k) = (-kv_0, (2-k)v_1, \ldots, (2j-k)v_j, \ldots, kv_k)$$

$$e_\theta \cdot (v_0, \ldots, v_k) = \exp(i\theta)(\sqrt{k}v_1, \ldots, \sqrt{j(k+1-j)}v_j, \ldots, \sqrt{k}v_k, 0).$$

Then

$$e_0 \cdot \|q\|^2 = 2\sum_{j=0}^{k}(2j-k)\|v_j\|^2$$

$$e_\theta \cdot \|q\|^2 = 2Re \exp(i\theta) \sum_{j=0}^{k-1}\sqrt{(j+1)(k-j)}(v_j, v_{j+1})$$

and the result follows. $\qquad\square$

Definition 1. *An* $SU(2)$ *hyperbolic monopole is* centred *at* $0 \in \mathbb{H}^3$ *if its associated holomorphic sphere* $q : \mathbb{P}^1 \to \mathbb{P}^k$ *lies in the zero set of the moment map* μ.

A consequence of the preceeding definition and the discussion of geometric invariant theory is a well-defined centre of a monopole. Each $PSL(2,\mathbb{C})$ orbit of a monopole possesses a unique $SO(3)$ orbit that lies in the zero set of the moment map μ. Hence to each element in a $PSL(2,\mathbb{C})$ orbit one can associate a unique point of $\mathbb{H}^3$ which is defined to be the centre of the monopole.

4. RATIONAL MAPS

Proof of Theorem 3. The rational map $f_w(z) : \mathbb{P}^1 \to \mathbb{P}^1$ is defined as follows. Consider all geodesics that begin at $w \in S^2_\infty$. Frame the bundle E at $w \in S^2_\infty$ and extend it to a neighbourhood. This consists of choosing vectors in each of the eigenspaces of Φ. One of these vectors extends to a unique global solution of (1) as follows. Define $s_w(z)$ to be a solution of (1) along all geodesics beginning at $w \in S^2_\infty$ and extending to S^2, satisfying
  (i) $\lim_{t\to-\infty} e^{mt}s_w(z)$ is a non-zero vector in the chosen eigenspace of Φ;
  (ii) $\lim_{z\to w} s_w(z)$ exists;
  (iii) $\partial_{\bar{z}}^A s_w(z) = 0$.
  Similarly, the other eigenspace gives rise to decaying solutions s_- that satisfy these conditions with e^{mt} in (i) replaced by e^{-mt}. This frame of solutions is unique since any other frame differs by a holomorphic gauge transformation defined over

S^2 and hence is constant, and in fact the identity since the bundle E is framed at w.

Amongst solutions of (1) along each geodesic that begins at $w \in S^2_\infty$ is a solution s_+ that decays so that $\lim_{t\to\infty} e^{mt}s_+$ is well-defined. This defines a one-dimensional subspace of the frame defined above, and hence of $\mathbb{C}^2$.

Thus, we get a map $f_w(z) : \mathbb{P}^1 \to \mathbb{P}^1$ which turns out to be holomorphic [1, 2]. The poles of $f_w(z)$ correspond to those points z_i such that the solution s_+ along the geodesic from w to z decays at both ends. Equivalently, s_+ is a multiple of s_- and has no s_w component. Thus, the poles come from points of the spectral curve, $(\hat{w}, z_i) \in \Sigma$. We have chosen a direction in the frame $\mathbb{C}^2$ to represent ∞. We choose the orthogonal direction in $\mathbb{C}^2$ to represent 0, so a zero of f_w corresponds to a solution s_+ that is a multiple of s_w and thus has no s_- component. In particular, w is a zero of f_w since in the limit $z \to w$, s_+ and s_- are orthogonal.

Choose $w \in S^2$ and let L be any line in $\mathbb{P}^k$ that contains the point $q(w)$. Let $P : \mathbb{P}^k \to L$ be projection onto the line. It is alternatively described as projection onto the plane in $\mathbb{C}^{k+1}$ defined by L using the Hermitian product on $\mathbb{C}^{k+1}$. The map $Pq(z) : \mathbb{P}^1 \to \mathbb{P}^1$ is a degree k holomorphic map. We choose the direction $q(w) \in L$ to represent the point 0, and the orthogonal direction to represent ∞. Thus $Pq(z)$ has poles given by z_i such that $Pq(z_i) \in L$ is orthogonal to $q(w) \in L$, so $\langle q(w), q(z_i) \rangle = 0$. The poles correspond to points of the spectral curve $(\hat{w}, z_i) \in \Sigma$ and coincide with the poles of the rational map.

Furthermore, since $Pq(w) = q(w)$, w is a zero of $Pq(z)$ which agrees with $f_w(w) = 0$. There are k zeros $\{w_i | i = 1, .., k\}$ ($w_1 = w$) of f_w counted with multiplicity. When the zeros are distinct, $q(w_i)$ define a k-dimensional subspace of $\mathbb{C}^{k+1}$, since q is full, and this possesses a unique orthogonal direction. Choose L_w to represent the plane spanned by this orthogonal direction and $q(w)$. Hence, the holomorphic map $Pq(z)$ has the same zeros and poles as $f_w(z)$. If a zero w_i has multiplicity $d + 1$, then $q(w_i), q'(w_i), .., q^{(d)}(w_i)$ spans a $(d + 1)$−dimensional subspace of $\mathbb{C}^{k+1}$ and the unique orthogonal direction still exists.

The rational maps $f_w(z)$ and $Pq(w)$ differ by a constant. This constant determines a scale on the line L_w. We supposed that the coefficients of the unit vector in $q(w)$ and the orthogonal unit vector in L_w determine a rational map, or in other words that the isomorphism of L_w with $\mathbb{P}^1$ respects the metric on $\mathbb{P}^k$. It may be that there is another natural scale on L_w. The question of how we might determine L_w and the scale intrinsically from q is an interesting one.

5. Charge two monopoles

For charge two hyperbolic monopoles, we can get explicit expressions for the boundary data. We will restrict to the space of centred charge 2 hyperbolic monopoles since these give rise to interesting structure. A charge two monopole is centred if after reflection in the origin, the new monopole is gauge equivalent to the original one. Since a gauge equivalent monopole produces the same equivalence class of holomorphic spheres in projective space, $q : \mathbb{P}^1 \to \mathbb{P}^2$, comes from a centred monopole when

$$q(\hat{z}) = u \cdot q(z), \quad \text{for some } u \in U(3) \tag{8}$$

(u is independent of z.) If we put $q = v_0 + v_1 z\sqrt{2} + v_2 z^2$ for $v_i \in \mathbb{C}^3$, then (8) is equivalent to $\|v_0\|^2 = \|v_2\|^2$ and $(v_0, v_1) + (v_1, v_2) = 0$ and this is the zero set of

the moment map defined in Proposition 3.2. Put

$$\mathcal{M}_2 = \{(v_0, v_1, v_2) \in \mathbb{C}^3 \otimes \mathbb{C}^3 \mid \|v_0\|^2 = \|v_2\|^2, \ (v_0, v_1) + (v_1, v_2) = 0\}/CU(3) \quad (9)$$

where $CU(3) = \mathbb{R}^+ \times U(3)$ is the conformal unitary group which acts on a triple by $(v_0, v_1, v_2) \mapsto (u^{-1}v_0, u^{-1}v_1, u^{-1}v_2)$.

If we replace the vectors v_i in (9) by vectors in $\mathbb{C}^2$ and quotient by $CU(2)$ then this gives the space of centred rational maps of degree 2 which naturally sit inside $\mathcal{M}_2$. if we replace the vectors v_i in (9) by vectors in $\mathbb{C}$ and quotient by $\mathbb{C}^*$ then we get a real structure on $\mathbb{CP}^2$ with fixed set $\mathbb{RP}^2$.

The space $\mathcal{M}_2$ is a five dimensional space that contains an open dense five-dimensional manifold $\mathcal{M}_2^0 \subset \mathcal{M}_2$ that is given by triples of independent vectors. Points of $\mathcal{M}_2^0$ precisely correspond to full maps q and these contain the space of centred charge 2 hyperbolic monopoles.

There is an $SO(3)$ action on $\mathcal{M}_2$ that preserves $\mathcal{M}_2^0$ coming from the action of $SU(2)$ on the polynomials $(1, z, z^2)$ given by

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \cdot (1, z, z^2) = ((-\bar{b}z + \bar{a})^2, (-\bar{b}z + \bar{a})(az + b), (az + b)^2).$$

It is well-defined since it commutes with the $CU(3)$ action. A convenient description of the space $\mathcal{M}_2$ is as follows.

Proposition 5.1. $\mathcal{M}_2 \cong \mathbf{su}(2) \otimes \mathbf{su}(2)/CO(3)$ *and the isomorphism respects the right $SO(3)$ actions.*

Proof. Again $CO(3) = \mathbb{R}^+ \times SO(3)$ is the conformal orthogonal group. A point of $\mathbf{su}(2) \otimes \mathbf{su}(2)/CO(3)$ is represented by a triple (r_0, r_1, r_2) for $r_i \in \mathbf{su}(2)$, and the isomorphism is given by

$$(r_0, r_1, r_2) \mapsto \left(\frac{1}{\sqrt{2}}(r_0 + r_2 i), r_1, \frac{1}{\sqrt{2}}(-r_0 + r_2 i)\right).$$

The proof requires the choice of representatives in each $CU(3)$ orbit

$$\mathcal{M}_2 \cong \{(v_0, v_1, -\bar{v}_0) \mid v_1 \in \mathbb{R}^3\}.$$

Using $CU(3)$, we may assume that $v_1 = (1, 0, 0)$ so that by (9) $v_0 = (c, \xi_0)$ and $v_2 = (-\bar{c}, \xi_2)$ for $\xi_i \in \mathbb{C}^2$ satisfying $\|xi_0\|^2 = \|xi_2\|^2$. Now use $u \in U(2)$ to realise $u\xi_2 = -\bar{u}\bar{\xi}_0$, or equivalently $u^T u \xi_2 = -\bar{\xi}_0$. We can do this since $\{u^T u | u \in U(2)\}$ acts transitively on $S^3 \subset \mathbb{C}^2$. $\qquad\blacksquare$

One would expect that $\mathcal{M}_2^0$ is a one parameter family of four-dimensional manifolds of centred charge 2 hyperbolic monopoles with given mass. In fact, it seems that half of $\mathcal{M}_2^0$ does not represent hyperbolic monopoles. Evidence for this is the fact that the point of $\mathcal{M}_2^0$ consisting of $\{v_i = e_i | i = 0, 1, 2\}$, where e_0, e_1, e_2 is an orthonormal set of basis vectors, is the unique fixed point of the $SO(3)$ action on $\mathcal{M}_2^0$. Such a point cannot correspond to a hyperbolic monopole, since no monopole is $SO(3)$ invariant. It does correspond to all charge 2 Euclidean monopoles since they each give a symmetric measure at infinity which would be pulled back by the this fixed point.

Consider the axially symmetric points in $\mathcal{M}_2^0$. These are given by orthogonal triples (v_0, v_1, v_2) and thus the spectral curve is given by

$$w^2 - 2\|v_1\|^2 wz + z^2 = 0.$$

In Section 7 we calculate the spectral curves of axially symmetric monopoles. In the charge 2 case, we find that the spectral curve is $w^2 - 2\cos(\pi/(2m+2))wz + z^2$. Thus $\|v_1\|^2 = \cos(\pi/(2m+2))$ and in particular it takes its values on the unit interval and the symmetric point is on one side of the allowed values.

In general, we expect the four dimensional spaces of monopoles with given mass to form two sided hypersurfaces in $\mathcal{M}_2^0$. We expect the symmetric point to partition $\mathcal{M}_2^0$ into two pieces, one containing hyperbolic monopoles. The piece containing hyperbolic monopoles is determined by the axially symmetric examples, and by the fact that in the limit, as the triple tends towards spanning a two-dimensional subspace, the massless monopoles emerge. One might guess that the other half of the points correspond to asymptotic values of spherical monopoles near a singular point.

To identify the mass of the monopole from the point of $\mathcal{M}_2^0$ is difficult. However, we can in a sense understand the tangential direction of changing mass as follows. Associated to a hyperbolic monopole is the rational map obtained from scattering from $0 \in \mathbb{H}^3$ [12], and when the monopole has charge 2 and is centred, it is uniquely determined by the intersection of the spectral curve with the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. We fix this rational map, and change the mass.

On $\mathbf{su}(2) \otimes \mathbf{su}(2)/CO(3)$ for $\nu = (r_0, r_1, r_2)$ the map $\nu \mapsto [\nu, \nu]$ is well-defined. Here $[\cdot, \cdot]$ is the bracket induced on $\mathbf{su}(2) \otimes \mathbf{su}(2)$ by the Lie bracket on the $\mathbf{su}(2)$. Note that this is not a Lie bracket, and in general $[\nu, \nu] \neq 0$.

Proposition 5.2. *The map $\nu \mapsto [\nu, \nu]$ is an involution with fixed point the symmetric point of $\mathcal{M}_2^0$.*

Proof. Put $\nu = e_1 \otimes r_1 + e_2 \otimes r_2 + e_3 \otimes r_3$ where e_i is an orthonormal basis of $\mathbf{su}(2)$. Then

$$[\nu, \nu] = 2e_1 \otimes [r_2, r_3] + 2e_2 \otimes [r_3, r_1] + 2e_3 \otimes [r_1, r_2]$$

since $[e_i, e_j] = \epsilon_{ijk} e_k$. The element ν is fixed by this map if $[r_i, r_j] = \lambda \epsilon_{ijk} r_k$ for a constant λ, thus $r_i \mapsto e_i$ under the action of $CO(3)$. The square of this map is given by

$$
\begin{aligned}
[[\nu, \nu], [\nu, \nu]] &= 8e_1 \otimes [[r_3, r_1], [r_1, r_2]] + 8e_2 \otimes [[r_1, r_2], [r_2, r_3]] \\
&\qquad + 8e_3 \otimes [[r_2, r_3], [r_3, r_1]] \\
&= 8\langle r_1, [r_2, r_3]\rangle \nu \\
&\equiv \nu
\end{aligned}
$$

where the last equivalence uses the fact that for monopoles r_1, r_2 and r_3 are independent and hence $\langle r_1, [r_2, r_3]\rangle r_1 \neq 0$. We have also used the identity

$$[[r_1, r_2], [r_2, r_3]] = \langle r_1, [r_2, r_3]\rangle r_1$$

which can be shown to hold in $\mathbf{su}(2)$ by linearly extending the easy identity

$$[[e_i, e_j], [e_j, e_k]] = \epsilon_{ijk} e_j.$$

$\square$

Proposition 5.3. *The expression $[\nu, \nu]$ defines a vector field on $\mathcal{M}_2^0$ and a flow along that vector field fixes the rational map of the monopole.*

Proof. The intersection of the spectral curve with the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ consists of four points, given by two pairs of antipodal points on the diagonal. These uniquely determine the the rational map obtained by scattering from $0 \in \mathbb{H}^3$.

Put $v_0 = (1/\sqrt{2})(r_0 + r_2 i)$, $v_1 = r_1$ and $v_2 = (1/\sqrt{2})(-r_0 + r_2 i)$ for real vectors r_i as in Proposition 5.1. When we translate the triple $\nu = (r_0, r_1, r_2)$ in the $[\nu, \nu]$ direction the intersection of the spectral curve with the diagonal is unchanged and hence the rational map is preserved. To see this, calculate $\langle q(\hat{z}), q(z) \rangle = 0$ in terms of the r_i to get the degree 4 polynomial

$$
\begin{aligned}
0 = & \frac{1}{2}\left[\langle r_2, r_2 \rangle - \langle r_0, r_0 \rangle + 2i\langle r_0, r_2 \rangle\right] z^4 + \frac{1}{2}\left[\langle r_2, r_2 \rangle - \langle r_0, r_0 \rangle - 2i\langle r_0, r_2 \rangle\right] \\
& + 2\left[\langle r_0, r_1 \rangle - i\langle r_2, r_1 \rangle\right] z^3 - 2\left[\langle r_0, r_1 \rangle + i\langle r_2, r_1 \rangle\right] z \\
& + \left[\langle r_0, r_0 \rangle + \langle r_2, r_2 \rangle - 2\langle r_1, r_1 \rangle\right] z^2.
\end{aligned}
$$

Now, consider the infinitesimal change given by $\nu \mapsto \nu + t[\nu, \nu]$ so

$$r_0 \mapsto r_0 + t[r_1, r_2], \ r_1 \mapsto r_1 + t[r_2, r_0], \ r_2 \mapsto r_2 + t[r_0, r_1].$$

Up to first order, this yields the change

$$\langle r_i, r_j \rangle \mapsto \langle r_i, r_j \rangle + t\delta_{ij}\epsilon_{ikl}\langle r_i, [r_k, r_l] \rangle$$

where we only sum over k and l. Thus the coefficients of the degree 4 polynomial are unchanged up to first order. (For example take the coefficient of z^4,

$$
\begin{aligned}
\langle r_0, r_0 \rangle &\mapsto \langle r_0, r_0 \rangle + 2t\langle r_0, [r_1, r_2] \rangle \\
\langle r_2, r_2 \rangle &\mapsto \langle r_2, r_2 \rangle + 2t\langle r_2, [r_0, r_1] \rangle \\
\langle r_0, r_2 \rangle &\mapsto \langle r_0, r_2 \rangle
\end{aligned}
$$

and the changes cancel.) □

For any $\xi \in \mathbf{so}(3)$, $[\xi, \nu]$ consists of trivial vectors (they point in the gauge direction) and $[\nu, \xi]$ gives the tangent space to the $SO(3)$ action. Thus we have four tangent directions in $\mathcal{M}_2^0$, three tangent to a moduli space of monopoles with fixed mass, and one "transverse" to each moduli space. It would be useful to find another mass-preserving tangent direction that would enable one to specify the fixed mass submanifold of $\mathcal{M}_2^0$.

5.1. **Mass of the monopole.**
The spectral curve Σ of a charge k monopole is a real (k, k) curve in $\mathbb{P}^1 \times \mathbb{P}^1$ with the extra condition that $\mathcal{O}(-(2m+k), 2m+k)|\Sigma \cong \mathcal{O}$, [17], where m is the mass of the monopole, $m = \lim_{r \to \infty} \|\Phi\|$. It is quite difficult to detect the mass from the spectral curve.

The case of charge 2 $SU(2)$ hyperbolic monopoles is special mainly because it is related to elliptic functions via its elliptic spectral curve, and because the spectral curve is identified with its Jacobian, the place where the Nahm data resides. Elliptic functions and isomonodromic deformations are used in [11] to find a new family of Einstein metrics and explicit expressions for them on the space of charge 2 centred monopoles, those monopoles invariant (up to gauge) under reflection in the origin.

We will describe part of the construction in [11]. For a generic choice of $(w, z) \in \Sigma$, the two lines $\{w\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{z\}$ meet Σ again, once each. Label (w, z) by P_0 and the other intersection point of the vertical line $\{w\} \times \mathbb{P}^1$ with Σ by P_1. At P_1 take a horizontal line and label the second point of Σ which it intersects by P_2. Continue this process until P_{4m+4} to get $P_0, P_1, \ldots P_{4m+4}$. Each point P_i gives a divisor on Σ, and $P_0 + P_1 \sim \mathcal{O}(0, 1)$, $P_1 + P_2 \sim \mathcal{O}(1, 0)$, $P_2 + P_3 \sim \mathcal{O}(0, 1)$ and so on,

where $\mathcal{O}(0,1)$ and $\mathcal{O}(1,0)$ mean the restriction of these line bundles to Σ. Take the alternating sum of these divisors to get $P_0+P_1-(P_1+P_2)+\cdots-(P_{4m+3}+P_{4m+4}) \sim \mathcal{O}(-(2m+2),2m+2)$. But $\mathcal{O}(-(2m+2),2m+2) \sim \mathcal{O}$ on Σ (where we assume for the moment that $m \in (1/2)*\mathbb{Z}$.) Thus $P_0 - P_{4m+4}$ is the trivial divisor and hence there is a meromorphic function on Σ with multiplicity one zero and pole given respectively by P_0 and P_{4m+4}. Non-constant meromorphic functions must have at least two zeros, thus we get a contradiction unless $P_0 = P_{4m+4}$.

Out of interest, we will mention the relation of this construction to the Poncelet polygon problem—to find n-sided polygons in the plane inscribed in one conic and circumscribed about another—described in [11]. Consider the map $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ defined by $\pi((w_0,w_1),(z_0,z_1)) = (w_0 z_0, w_0 z_1 + w_1 z_0, w_1 z_1)$, (or affinely $\pi(w,z) = (wz, w+z)$.) The preimage of any point consists of (w,z) and (z,w) thus the map is a two fold branched cover ramified on the diagonal and branched over the conic $B = (z_0^2, 2z_0 z_1, z_1^2)$. It simply relates the coefficients of a degree two polynomial to its roots. The image of any vertical or horizontal line $\{w\} \times \mathbb{P}^1$, respectively $\mathbb{P}^1 \times \{z\}$, is tangent to B.

The spectral curve of a *centred* 2 monopole is invariant under the involution that swaps the two factors $(w,z) \mapsto (z,w)$. This is because the two points represent a geodesic running from $\hat{w}$ to z, respectively a geodesic running from $\hat{z}$ to w. These geodesics are images of each other under reflection in the origin. The image of the spectral curve of a centred 2 monopole is a conic, $C = \pi(\Sigma)$. Let $(w,z) \in \Sigma$, then the images of the two lines $\{w\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{z\}$ are the two tangents of the conic B meeting C in the point $\pi(w,z)$. Hence the construction described above yields a $4m+4$-sided polygon in the plane inscribed in one conic and circumscribed about another.

When $m \in (1/4)*\mathbb{Z}$ one can conclude that $P_0 \neq P_{4m+4}$ but they lie in the same fibre of π, again giving a solution of the Poncelet problem.

This construction can be interpreted in terms of the holomorphic sphere $q : \mathbb{P}^1 \to \mathbb{P}^2$. A generic point $z_0 \in \mathbb{P}^1$ gives rise to a sequence of points $\ldots z_{-1}, z_0, z_1, z_2, \ldots$ by requiring the condition that $(q(z_i), q(z_{i\pm1})) = 0$.

Lemma 5.4. *When $m \in \mathbb{Q}$, then the sequence $\{\ldots z_{-1}, z_0, z_1, z_2, \ldots\}$ defined by $(q(z_i), q(z_{i\pm1})) = 0$ is a discrete lattice on the sphere.*

Proof. This follows from the argument described above. If $m = m_1/m_2$ then after $4m_1 + 4m_2$ steps we can conclude from $\mathcal{O}(-(2m+2), 2m+2)^{m_2} \sim \mathcal{O}$ on Σ that the sequence closes up. $\square$

The number of points in the discrete lattice is related to the sum of the numerator and denominator of the mass. It would be good to see the mass precisely from the lattice and to understand what can be done in the irrational mass case.

A lattice can be constructed in this way for any charge k monopole. At each new step, k new points on the sphere are produced. It is unlikely that this will yield a discrete lattice. The argument for this relied crucially on the property that after a finite number of steps in the construction, we are left with a question about the divisor $P_0 - P_N$ consisting of two points, and can use the fact that a meromorphic function must have at least two zeros.

Hitchin mentions that his metrics are defined via the spectral curves and have little to do with the monopole fields. The relationship between the boundary values

of the monopole fields and the spectral curve should expose a more direct link between the monopoles and the Einstein metric.

6. Massless monopoles.

By the maximum principle on the Higgs field, monopoles with zero mass are necessarily flat, and hence trivial on hyperbolic space. Still, the zero mass limit of hyperbolic monopoles, which by a rescaling corresponds to the infinite curvature limit of hyperbolic space, contains intersting features. This limit was studied in [3, 5, 13] for different reasons.

Given a rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$, one can produce a curve $C_f \subset \mathbb{P}^1 \times \mathbb{P}^1$ by

$$C_f = \{(w, z) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid f(z) = \sigma f \sigma(w)\} \qquad (10)$$

where σ is the antipodal map $\sigma(z) = -1/\bar{z}$, [3, 5]. When $f(z) = k'/(z^N - k)$, the curve C_f contains the parameters of a solution of the Yang-Baxter equation related to the Potts model. For a degree N rational map f, the curve C_f has the properties

1. C_f is a curve of bidegree (N, N) on $\mathbb{P}^1 \times \mathbb{P}^1$
2. C_f is a real curve with respect to $\tau(w, z) = (\hat{z}, \hat{w})$ and has no real points
3. $N(D^+ - D^-) \sim 0$ where D^+ and D^- are the divisors of the intersection of C_f with $\mathbb{P}^1 \times \{z_0\}$ and $\{w_0\} \times \mathbb{P}^1$.

The spectral curve Σ of a hyperbolic monopole of mass m satisfies conditions 1 and 2 and a modification of condition 3:

3*. $(N+2m)(D^+ - D^-) \sim 0$ where D^+ and D^- are the divisors of the intersection of Σ with $\mathbb{P}^1 \times \{z_0\}$ and $\{w_0\} \times \mathbb{P}^1$.

Thus, it is natural to treat the curves C_f as the zero mass limit of hyperbolic monopoles. Another way to write (10) is

$$C_f = \{(w, z) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid \langle f(w), f(z) \rangle = 0\}$$

where $\langle \cdot, \cdot \rangle$ is the Hermitian metric on $\mathbb{C}^2$, so C_f detects when the subspaces are orthogonal.

By Lemma 2.1 the holomorphic sphere satisfies $\langle q(\hat{w}), q(z) \rangle = w^{-k} \psi(w, z)$ or in other words the zero set of ψ, which defines the spectral curve of the monopole, is given by

$$\Sigma = \{(w, z) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid \langle q(w), q(z) \rangle = 0\}.$$

Thus we see that the holomorphic sphere $q : \mathbb{P}^1 \to \mathbb{P}^k$ resembles closely the holomorphic map $f : \mathbb{P}^1 \to \mathbb{P}^1$. Moreover, as the mass of the monopole tends to zero, the image of q tends toward being contained in a line in projective space, giving $f : \mathbb{P}^1 \to \mathbb{P}^1 \hookrightarrow \mathbb{P}^k$ in the limit. We interpret the image of q to be "almost" contained in a line to mean that the pull-back of the Kahler form under q is close to the pull-back of the Kahler form when the image lies in a line. When the rational map f is given by radial scattering the claim follows from [18], where it is shown that small non-integer mass hyperbolic monopoles have boundary values perturbed not too far from the pull-back of the Kahler form on $\mathbb{P}^1$ by the rational map.

The particular line in $\mathbb{P}^k$ into which the image of q tends is not significant, since q is only well-defined up to the action of $U(k+1)$ on its image. Since every line in $\mathbb{P}^k$ is equivalent up to this action of $U(k+1)$, the image of q can tend to lie inside any line.

The motivation of [3, 5] is to find solutions to the Yang-Baxter equations that use the spectral curve of a monopole and resemble the curves C_f from the Potts model. It may be that the rational map $f(z) = k'/(z^N - k)$ has an analogue q for each mass. In general it is hard to find the holomorphic maps q corresponding to monopoles, however particularly symmetric examples are more accessible such as those described in Section 5.

In [13] it was shown that the rational map associated to a hyperbolic monopole can be used to construct an explicit solution of a degenerate form of the Bogomolny equations obtained from the infinite curvature limit of hyperbolic space. This explicit solution was interpreted as an approximate monopole and the curve C_f defined in (10) naturally arises as a type of spectral curve. It was proven that the approximate monopole can flow to a unique genuine monopole under a heat flow. This viewpoint may help with the question: is there a good way to go straight from the holomorphic sphere q to the monopole field (A, Φ)? We would hope to construct from q an approximate monopole and again prove that a genuine monopole lies nearby.

7. Axially symmetric monopoles.

When the monopole is axially symmetric, the spectral curve Σ is a collection of k $(1,1)$ curves

$$\prod_i (w - a_i z) = 0.$$

The reality condition on Σ implies that for each i there is a j such that $a_i = \bar{a}_j$. The curve has mass m if $L^{k+2m}|_\Sigma$ is trivial. With respect to local trivialisations of L^{k+2m} in neighbourhoods of $(w, z) = (0, 0)$ and $(w, z) = (\infty, \infty)$ a transition function can be given by $z^{k+2m}w^{-(k+2m)}$. A non-vanishing section over Σ can be set to be the constant 1 in a neighbourhood of $(w, z) = (0, 0)$. Along the curve $w - a_i z = 0$, the transition function is $z_{k+2m}w^{-(k+2m)} = a_i^{-(k+2m)}$ thus $1 \mapsto a_i^{-(k+2m)}$ and one condition that this is a global section over Σ is that the sections over each $w - a_i z = 0$ agree at $(w, z) = (\infty, \infty)$. Thus

$$a_i^{k+2m} = a_j^{k+2m} \in \mathbb{R} \tag{11}$$

for all i and j, and since $\bar{a}_i$ is amongst the a_js, the number a_i^{k+2m} is real. When $2m$ is not an integer, the expression a_i^{k+2m} is still uniquely defined. In general, such an expression requires the choice of a branch. In our case, there is a well-defined branch of a_i^{k+2m} obtained by continuity as the mass is varied. When $m = 0$, one can still make sense of the spectral curve of a "massless" monopole (see Section 6) and in this case it is given by the equation $w^k + (-1)^k \alpha^k z^k = 0$ for some $\alpha > 0$. Since condition (11) is a discrete condition on the a_i we can again use continuity in the mass, and prove that for $m > 0$

$$a_j = \alpha \exp\left(\frac{2\pi i j}{k + 2m}\right), \ j = (1-k)/2, (3-k)/2, \dots, (k-1)/2. \tag{12}$$

Notice that, in agreement with Hitchin [9] p.188, that a_i^{k+2m} is positive when k is odd and negative when k is even. This reflects the real structure on the line bundle.

The coefficients of $\prod_j (w - a_j z)$, for a_j defined in (12) are all non-zero. This can be seen from the fact that each symmetric polynomial in the a_j's strictly increases with the mass since each $a_j(m)$ creeps along the circle towards the positive real line.

In Lemma 2.2 it is proven that the non-degeneracy of the matrix of coefficients of the defining polyomial of the spectral curve is equivalent to the vanishing of the cohomology group $H^0(\Sigma, \mathcal{O}(k, -2))$.

We have given explicit expressions for spectral curves of hyperbolic monopoles and for boundary values of hyperbolic monopoles. Here we give an explicit expression for the field over $\mathbb{H}^3$ of a charge two hyperbolic monopole. Choose coordinates (z, r) where r is the hyperbolic distance from the origin to the point, and z is a holomorphic coordinate on each sphere of constant distance from the origin. In order to give a gauge invariant expresssion it is convenient to use an associated Hermitian metric H defined over $\mathbb{H}^3$ which gives the monopole (A, Φ) in a non-unitary gauge by

$$A_{\bar{z}} = 0, \; A_z = H^{-1}\partial_z H, \; A_r = (1/2)H^{-1}\partial_r H, \; \Phi = (-i/2)H^{-1}\partial_r H.$$

The pair (A, Φ) satisfies the Bogomolny equation when H satisfies the nonlinear equation

$$\partial_r(H^{-1}\partial_r H) + \frac{(1 + |z|^2)^2}{\sinh^2(r)}\partial_{\bar{z}}(H^{-1}\partial_z H) = 0. \tag{13}$$

For an axially symmetric centred charge 2 hyperbolic monopole, H looks like

$$H = \frac{1}{D}\begin{pmatrix} a(r) + 2b(r)|z|^2 + |z|^4/a(r) & (1 - b(r)^2)^{1/2}(a(r) - 1/a(r))\bar{z}^2 \\ (1 - b(r)^2)^{1/2}(a(r) - 1/a(r))z^2 & 1/a(r) + 2b(r)|z|^2 + a(r)|z|^4 \end{pmatrix} \tag{14}$$

for $D = (1 + |z|^2)^2 - (2 - b(r)(a(r) + 1/a(r)))|z|^2$. The functions $a(r)$ and $b(r)$ satisfy a set of non-linear equations derived from putting (14) into (13). One explicit solution is given by

$$a(r) = \text{sech}(r) = b(r) \tag{15}$$

and this gives a mass $1/2$ monopole. When $a(r) = e^{-2r}$ and $b(r) \equiv 0$ we get a solution of a degenerate equation much like (13) which corresponds to a zero mass monopole.

The holomorphic sphere of the monopole arising from (15) is

$$q(z) = (\frac{1}{\sqrt{2}}(1 + z), \frac{i}{\sqrt{2}}(1 - z), z^2).$$

It would be desirable to find a one-parameter family of solutions $a_m(r)$ and $b_m(r)$ depending on the mass m, in particular to get explicit expressions for fractional mass hyperbolic monopoles.

8. The Vanishing Theorem

The proof of Theorem 1 requires $H^0(\Sigma, L^{2m}(k - 2, 0)) = 0$. In this section we will prove a more general vanishing theorem that has further applications.

Theorem 7 (Vanishing Theorem). *If $\Sigma \subset Z$ is the spectral curve of a hyperbolic monopole of mass m and charge k then*

$$H^0(\Sigma, L^s(k - 2, 0)) = 0$$

for all $1 \le s \le 2m + 1$.

Note 8.1. We have that L^{2m+k} restricted to Σ is trivial. So $H^0(\Sigma, L^s(k-2, 0)) = H^0(\Sigma, L^{s-2m-k}(k-2, 0)) = H^0(\Sigma, L^{2m+2-s}(k-2, 0))$. The last equality uses the real structure and is actually a conjugate linear isomorphism. So it suffices to prove the theorem for $1 \le s \le m+1$.

Note 8.2. The case $s = 1$ (and hence also $s = 2m + 1$) is elementary since we have $H^0(S, \mathcal{O}(k-1, -1)) = 0$ for any degree (k, k) curve $\Sigma \subset Z$.

The method of proof of the Theorem is an adaption of Hitchin [9] for the Euclidean case. In summary it is as follows.
(1) Show that the $H^0(\Sigma, L^s(k-2, 0))$ injects into $H^1(Z, L^{s-m}\widetilde{E}(-2, 0))$.
(2) Penrose transform to get a solution u of

$$(\nabla_A^* \nabla_A - 1 + \Phi_s^* \Phi_s)u = 0 \tag{16}$$

where $\Phi_s = \Phi + i(s - m - 1)$ such that $|u(x)|$ decays asympotically as $x \to 0$ like the maximum of x^s and x^{2m+2-s}.
(3) Transfer $u(x)$ to $\mathbb{R}^4 - \mathbb{R}^2$ where the operator in (20) becomes positive and we can integrate by parts to show that $u = 0$.

9. HOLOMORPHIC SECTIONS OF L

Before we begin the proof we need a result about the space of holomorphic sections of $L^s(k, 0)$ over Z. Note that $L^s(k, 0)$ extends to the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ only when s is an integer. The result we need says that if s is not an integer there are no holomorphic sections of L^s over Z and if s is an integer they are all obtained by restriction of holomorphic sections of $L^s(k, 0)$ over Q. In this latter case the Kunneth formula tells us that

$$H^0(Z, L^s(k, 0)) = H^0(Q, L^s(k, 0)) = H^0(\mathbb{P}^1, \mathcal{O}(s+k)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(-s)).$$

We have

Lemma 9.1. *For any non-negative integer* k

$$H^0(Z, L^s(k, 0)) = \begin{cases} 0 & s \ne 0, -1, \ldots, -k \\ \mathbb{C}^{(s+k)(-s)} & s = 0, -1, \ldots, -k \end{cases}$$

Proof. A local section f of $L^s(k, 0) = \mathcal{O}(k+s, -s)$ pulls back to a real analytic function $\hat{f}$ defined locally on $\mathbb{C}P^3$ satisfying $\hat{f}(\alpha u, \beta v) = \alpha^{k+s}\beta^{-s}\hat{f}(u, v)$ where u and v lie in $\mathbb{C}^2$. Equivalently $\hat{f}(\lambda^{1/2}\mu.u, \lambda^{-1/2}\mu.v) = \lambda^{k/2+s}\mu^k \hat{f}(u, v)$ and the factor μ^k shows that $\hat{f}$ can be interpreted as a section of $\mathcal{O}(k)$ that satisfies

$$\hat{f}(\lambda^{1/2}u, \lambda^{-1/2}v) = \lambda^{k/2+s}\hat{f}(u, v). \tag{17}$$

Instead of working on local open neighbourhoods of $\mathbb{C}P^3$ we can restrict to the set P defined by $P = \{[u, v] \in \mathbb{C}P^3 : \langle u, v \rangle > 0\}$ which is an open subset of the real hypersurface $\{\text{Im}\langle u, v \rangle = 0\}$. Then (17) describes a real analytic section of $\mathcal{O}(k)|_P$, holomorphic on holomorphic sub-manifolds of P, that transforms under $\lambda \in \mathbb{R}^+$. The set P misses the pull-back of the anti-diagonal and it contains the pre-images of all real $(1, 1)$ curves. One can describe P as a twistor space.

Pick any projective line $L \subset P$. Then $\hat{f}$ continues analytically to a holomorphic section of $\mathcal{O}(k)$ in an open (in $\mathbb{C}P_3$) neighbourhood W of L. We claim that $\hat{f}$ is the restriction of a polynomial of degree k. Granted that, it follows at once that

$\hat{f} = 0$ unless $s = 0, -1, \ldots, -k$ since the only possible weights for the $\mathbb{R}^+$ action on a polynomial of degree k are $k/2, k/2 - 1, k/2 - 2, \ldots, -k/2$.

It remains to show that $\hat{f}$ is a restriction of such a polynomial. The identity

$$\hat{f}(z_0, z_1, z_2, z_3) = \sum z_{j_1} \ldots z_{j_k} \partial_{j_1} \ldots \partial_{j_k} \hat{f}$$

(which follows from Euler's identity proved by repeatedly differentiating both sides of $f(\lambda z_0, \lambda z_1, \lambda z_2, \lambda z_3) = \lambda^k f(z_0, z_1, z_2, z_3)$ with respect to λ) reduces to the case $k = 0$ for $\partial_{j_1} \ldots \partial_{j_k} \hat{f} \in H^0(W, \mathcal{O})$. But this is constant as W contains lots of intersecting projective lines. $\square$

10. Proof of the vanishing theorem

10.1. **The injection.** From the short exact sequence of sheaves

$$0 \to \mathcal{O}(-k, -k) \to \mathcal{O} \to \mathcal{O}_\Sigma \to 0$$

we obtain

$$0 \to H^0(Z, L^s(k-2, 0)) \to H^0(\Sigma, L^s(k-2, 0)) \to H^1(Z, L^s(-2, -k)) \to \ldots.$$

According to Lemma 9.1, $H^0(Z, L^s(k-2, 0)) = 0$ unless $s = 0, -1, -2, \ldots 2 - k$ so for the range of s we are interested in the connecting homomorphism

$$H^0(\Sigma, L^s(k-2, 0)) \to H^1(Z, L^s(-2, -k))$$

is injective.

We also have

$$0 \to L^m(0, -k) \to \widetilde{E} \to L^{-m}(0, k) \to 0$$

and hence

$$0 \to L^s(-2, -k) \to L^{s-m}\widetilde{E}(-2, 0) \to L^{s-2m}(-2, k) \to 0$$

so that

$$H^0(Z, L^{s-2m}(-2, k)) \to H^1(Z, L^s(-2, -k)) \to H^1(Z, L^{s-m}\widetilde{E}(-2, 0)).$$

Again from Lemma 9.1 we have that $H^0(Z, L^{s-2m}(-2, k)) = 0$ unless $s - 2m = 2, 3, \ldots, -k$ or $s = 2m+2, 2m+3, \ldots, 2m+k$ which is outside the range of interest. Finally we have

$$H^0(\Sigma, L^s(k-2, 0)) \hookrightarrow H^1(Z, L^{s-m}\widetilde{E}(-2, 0))$$

for $1 \le s \le 2m + 1$.

By replacing s by $s - 2m - k$ at the outset and using the other sequence

$$0 \to L^{-m}(-k, 0) \to \widetilde{E} \to L^m(k, 0) \to 0$$

we obtain an equivalent description of the same class in $H^1(Z, L^{s-m}\widetilde{E}(-2, 0))$ which factors through the other canonical subbundle.

10.2. **The Penrose Transform for $\mathbb{H}^3$.** We describe the Penrose transform of a class

$$[f] \in H^1(Z, L^a(-2, 0))$$

and give estimates for its growth if it is compactly supported.

Identify hyperbolic three-space $\mathbb{H}^3$ with the space of positive-definite, two by two, Hermitian matices X up to scale or with the space of positive-definite, two by two, Hermitian matrices of unit determinant. We co-ordinatize these matrices by

$$X = \frac{1}{x_3} \begin{bmatrix} 1 & x_1 + ix_2 \\ x_1 - ix_2 & x_3^2 + x_1^2 + x_2^2 \end{bmatrix}$$

for $x_3 > 0$. This is the upper half-space model of hyperbolic space. Denote by M the open set of future pointing timelike vectors in $\mathbb{R}^{3,1}$ so that $M/\mathbb{R}^+ = \mathbb{H}^3$.

Theorem 8. *There is a canonical isomorphism*

$$H^1(Z, L^a(-2, 0)) \quad = \{u \in C^\infty(M) \colon \Box u = 0, u(tX) = t^{a-2}u(X)\}$$
$$= \{v \in C^\infty(\mathbb{H}^3) \colon (\triangle + a(a-2))v = 0\}$$

Moreover if f has compact support then u has a decomposition for $x \leq 0$ smooth

$$u(x_1, x_2, x_3) = x_3^{2-a}u_1(x_1, x_2, x_3) + x_3^a u_2(x_1, x_2, x_3) \tag{18}$$

where u_1 and u_2 are smooth down to $x_3 = 0$.

Remark 10.1. Notice that the expansion fits well with what is known about the boundary behaviour of eigenfunctions of the Laplacian on $\mathbb{H}^3$ [14].

Proof. Choose homogeneous coordinates $[\eta] = [\eta^0, \eta^1]$ and $[\zeta] = [\zeta^0, \zeta^1]$ for Z, so in terms of the affine coordinates, $w = \eta^1/\eta^0$ and $z = \zeta^1/\zeta^0$. By considering the framing $\langle \eta, \zeta \rangle / \langle \zeta, \zeta \rangle$ of L over Z, we see that $[f]$ can be represented by

$$f \in \Omega^{0,1}(Z, \mathcal{E}(-2, 0))$$

such that

$$\bar{\partial}\left(\left(\frac{1 + \bar{w}z}{1 + \bar{z}z} \right)^a f \right) = 0.$$

By pulling back to P, for example, we get the function

$$\psi(x) = \int_{[\zeta] \in \mathbb{P}^1} \left(\frac{\zeta^* X \zeta}{\zeta^* \zeta} \right)^a f(X\zeta, \zeta) \wedge (\zeta^0 d\zeta^1 - \zeta^1 d\zeta^0) \tag{19}$$

from the Minkowski version of the Penrose transform. Thus ψ satisfies the wave equation on M and is plainly homogeneous of degree $a - 2$ in X. One way to find the equation satisfied by ψ on restriction to $\mathbb{H}^3$ is to compute $|X|^2\Box(|X|^{2-a}\psi(X))$, for $\Box$ on functions of degree 0 is $\triangle_{\mathbb{H}^3} = -\operatorname{tr}\operatorname{Hess}$. We have

$$\partial_i(|X|^{2-a}\psi(X)) = (2-a)|X|^{-a}X_i\psi(X) + |X|^{2-a}\partial_i\psi(X).$$

and

$$\partial^i \partial_i(|X|^{2-a}\psi(X)) = (2-a)|X|^{-a}(4\psi(X) - 2\psi(X) - (2-a)\psi(X))$$
$$= a(2-a)\psi(X)|X|^{-a}.$$

So if $v(x) = |X|^{2-a}\psi(X)$ we have $\triangle v + a(a-2)v = 0$.

Now suppose for definiteness that f has support contained in the relatively compact subset V defined by the condition $|\eta^0\zeta^1 - \eta^1\zeta^0|^2 < R^2|\eta^0\bar{\zeta}^0 + \eta^1\bar{\zeta}^1|^2$ that is

$|w - z| < R|1 + w\bar{z}|$ for some $R < 0$. The set V obviously does not intersect the antidiagonal. Consider how it meets a real curve

$$C_{(x_1,x_2,x_3)} = \{\eta^1\zeta^0 + (x_1 + ix_2)\eta^1\zeta^1 - (x_1 - ix_2)\eta^0\zeta^0 - (x_1^2 + x_2^2 + x_3^2)\eta^0\zeta^0 = 0\}$$

when x_3 is small but positive. Now

$$C_{(x_1,x_2,0)} = \{\zeta^0 + (x_1 + ix_2)\zeta^1 = 0\} \cup \{\eta^1 - (x_1 - ix_2)\eta^0 = 0\}$$

and

$$V \cap C_{(0,0,0)} = V_1 \cup V_2$$

where

$$V_1 = \{\eta^1 = 0\} \times \{[\zeta]\colon |\zeta^1/\zeta^0| < R\}$$

and

$$V_2 = \{\zeta^0 = 0\} \times \{[\eta]\colon |\eta^0/\eta^1| < R\}$$

with $V_1 \cap V_2 = \emptyset$. Similarly, for sufficiently small x_3, the intersection of V with $C_{(x_1,x_2,x_3)}$ is a union of two disjoint sets V_1 and V_2 which are approximately of the form

$$V_1 = \{\eta^1 = (x_1 - ix_2)\eta^0\} \times U_1$$

and

$$V_2 = U_2 \times \{\zeta^0 + (x_1 + ix_2)\zeta^1 = 0\}$$

where U_1 and U_2 are discs. The decomposition (18) of u in the statement of the theorem corresponds to a decomposition of the integral (19) into integrals over V_1 and V_2. To see this we check the growth rates of the two contributions by taking $x_1 = x_2 = 0$. If we integrate over V_1 we obtain

$$\int_{V_1} \left(\frac{x_3^{-1}|\zeta^1|^2 + x_3|\zeta^1|^2}{|\zeta^0|^2 + |\zeta^1|^2} \right)^a f(x_3^{-1}\zeta^0, x_3\zeta^1, \zeta^0, \zeta^1)(\zeta^0 d\zeta^1 - \zeta^1 d\zeta^0)$$

$$= x_3^{2-a} \int_{V_1} \left(\frac{1 + x_3^2|z|^2}{1 + |z|^2} \right)^a f(1, x_3^2 z, 1, z) dz$$

$$= x_3^{2-a} u_1(x_3, 0, 0)$$

Similarly integrating over V_2 gives

$$\int_{V_2} \left(\frac{x_3|\eta^0|^2 + x_3^{-1}|\eta^1|^2}{x_3^2|\eta^0|^2 + x_3^{-2}|\eta^1|^2} \right)^a f(\eta^0, \eta^1, x_3\eta^0, x_3^{-1}\eta^1)(\eta^0 d\eta^1 - \eta^1 d\eta^0)$$

$$= x_3^a \int_{V_2} \left(\frac{1 + x_3^2/|w|^2}{1 + x_3^4/|w|^2} \right)^a f(1/w, 1, x_3^2/w, 1) d(1/w)$$

$$= x_3^a u_2(x_3, 0, 0).$$

The result follows as the integrals have uniformly compact supports at $x_3 \to 0$. $\quad\square$

Remark 10.2. In particular the 'boundary data' u_1 and u_2 arise as integrals over the generators of f. If we use the real structure to replace $L^a(-2, 0)$ by $L^{-a}(0, -2) = L^a(-2, 0)$ the roles of the two generators are swapped but (of course) the conclusion is the same.

10.3. **Completion of proof.** Coupling to the bundle $\widetilde{E}$ replaces the differential equation in Theorem 8 by the analogous coupled equation

$$(\nabla_A^*\nabla_A - 1 + \Phi_s^*\Phi_s)u = 0 \tag{20}$$

where $\Phi_s = \Phi + i(s - m - 1)$. The methods of Hitchin [9] can be used to obtain additional decay like x_3^m. So we have

$$|u(x_3)| \simeq \max(x_3^s, x_3^{2+2m-s}).$$

It is not clear that the operator in (20) is positive so we now transfer to $\mathbb{R}^4 - \mathbb{R}^2$. Using $\hat{\nabla}$ for the Euclidean operators we have

$$x_3^3 \hat{\nabla}^* \hat{\nabla}(x_3^{-1}u) = (\nabla^*\nabla - 1)u$$

so that if $v = x_3^{-1}u$ then we have

$$\hat{\nabla}^* \hat{\nabla}v + x_3^{-2}\Phi_s^*\Phi_s v = 0.$$

In now $1 < s < 2m + 1$ then $v = O(x_3^\epsilon)$ for $\epsilon > 0$ and this is enough to integrate by parts in $\mathbb{R}^4 - \mathbb{R}^2$ and hence $v = 0$.

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