A q-analogue of Mirzakhani's recursion for Weil–Petersson volumes

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Abstract. We define q-analogues of Mirzakhani's recursion for Weil–Petersson volumes and the Stanford–Witten recursion for super Weil–Petersson volumes. Okuyama recently introduced a q-deformation of the Gaussian Hermitian matrix model which produces quasi-polynomials that recover the Weil–Petersson volumes via a rescaled $q \to 1$ limit. The q-deformations of the Weil–Petersson volumes produced here agree with the top degree terms of Okuyama's quasi-polynomials and suggest a variation of Okuyama's methods to the super setting.

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1 Introduction

The Weil–Petersson volume of the moduli space $\mathcal{M}_{g,n}$ of genus g curves with n marked points is defined from the Kähler form ω^{WP} of the Weil–Petersson metric by

$$V_{g,n}^{\mathrm{WP}} = \int_{\mathcal{M}_{g,n}} \exp \omega^{\mathrm{WP}}.$$

The Weil–Petersson form ω^{WP} can be defined purely in terms of the complete hyperbolic structure associated to a genus g curve with n marked points. Wolpert proved that the Fenchel–Nielsen coordinates on the Teichmüller space $\mathcal{T}_{g,n}$ are Darboux coordinates for ω^{WP} [14]. More generally, one can use Fenchel–Nielsen coordinates on the Teichmüller space $\mathcal{T}_{g,n}(L_1, \ldots, L_n)$ of hyperbolic surfaces with geodesic boundary components of lengths $(L_1, \ldots, L_n) \in \mathbb{R}^n_{\geqslant 0}$. Wolpert's theorem then enables one to define a family of deformations $\omega^{\mathrm{WP}}(L_1, \ldots, L_n)$ of the Weil–Petersson form $\omega^{\mathrm{WP}} = \omega^{\mathrm{WP}}(0, \ldots, 0)$ with associated volumes

$$V_{g,n}^{\mathrm{WP}}(L_1, \ldots, L_n) = \int_{\mathcal{M}_{g,n}} \exp \omega^{\mathrm{WP}}(L_1, \ldots, L_n).$$

Mirzakhani produced relations – see equation (18) in Section 2 – between the volumes $V_{g,n}^{\mathrm{WP}}(L_1, \ldots, L_n)$ that determine the volumes recursively from $V_{0,3}^{\mathrm{WP}}(L_1, L_2, L_3)$ and $V_{1,1}^{\mathrm{WP}}(L_1)$, [9]. She used this recursion to prove that the volumes are polynomial in the lengths and moreover that $V_{g,n}^{\mathrm{WP}}(L_1, \ldots, L_n) \in \mathbb{Q}[\pi^2][L_1^2, \ldots, L_n^2]$.

The main purpose of this paper is to introduce a natural q-deformation of Mirzakhani's recursion that produces a family of polynomials $V_{g,n}(L_1, \ldots, L_n) \in \mathbb{Q}[[q]][L_1^2, \ldots, L_n^2]$, whose coefficients are q-series. The recursion requires integration using the kernels $D_q(x, y, z)$ and $R_q(x, y, z)$, which we now introduce. These are defined from the function

$$H_q(x, y) = \frac{1}{2} \sum_{m=1}^{\infty} (-1)^{m-1} q^{m^2/2} (q^{m/2} + q^{-m/2}) \Big(e^{\frac{1}{2}(x+y)(q^{m/2} - q^{-m/2})} + e^{\frac{1}{2}(x-y)(q^{m/2} - q^{-m/2})} \Big), \qquad (1)$$

via the formulas

$$\frac{\partial}{\partial x} D_q(x,y,z) = H_q(y+z,x) \qquad \text{and} \qquad \frac{\partial}{\partial x} R_q(x,y,z) = \frac{1}{2}\big(H_q(z,x+y) + H_q(z,x-y)\big),$$

together with the initial conditions $D_q(0,y,z) = R_q(0,y,z) = 0$.

Define the polynomials $V_{g,n}(L_1,\dots,L_n) \in \mathbb{Q}[[q]][L_1^2,\dots,L_n^2]$ from the base cases

$$\begin{aligned} V_{0,1}(L) &:= 0 \\ V_{0,2}(L_1,L_2) &:= 0 \\ V_{0,3}(L_1,L_2,L_3) &:= 1 \\ V_{1,1}(L) &:= \frac{1}{2L}\int_0^\infty x D_q(L,x,x)\,\mathrm{d}x, \end{aligned} \tag{2}$$

and the recursion

$$\begin{aligned} L_1 V_{g,n}(L_1, \boldsymbol{L}_K) &= \sum_{j=2}^{n} \int_0^\infty x R_q(L_1,L_j,x)\, V_{g,n-1}(x, \boldsymbol{L}_{K\setminus\{j\}})\,\mathrm{d}x \\ &+ \frac{1}{2}\int_0^\infty\int_0^\infty xy D_q(L_1,x,y)\Big[V_{g-1,n+1}(x,y,\boldsymbol{L}_K) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J = K}} V_{g_1,|I|+1}(x,\boldsymbol{L}_I)\, V_{g_2,|J|+1}(y,\boldsymbol{L}_J)\Big]\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{3}$$

Here, we use the notation $K = \{2,3,\dots,n\}$ and write $\boldsymbol{L}_I = (L_{i_1}, L_{i_2}, \dots, L_{i_m})$ for $I = \{i_1, i_2, \dots, i_m\}$.

Calculating the integral of equation (2) and applying the recursion of equation (3) yields the formulas

$$\begin{aligned} V_{1,1}(L) &= \frac{1}{48}L^2 + \frac{1}{2}\zeta_q(2) \\ V_{0,4}(L_1,L_2,L_3,L_4) &= \frac{1}{2}\sum_{i=1}^{4} L_i^2 + 12\zeta_q(2) \\ V_{1,2}(L_1,L_2) &= \frac{1}{192}(L_1^2+L_2^2)^2 + \frac{1}{2}\zeta_q(2)(L_1^2+L_2^2) + 5\zeta_q(4) + 7\zeta_q(2)^2. \end{aligned}$$

These use the q-zeta function evaluated at the even integers, which we define via the formula

$$\zeta_q(2k) = \sum_{m=1}^{\infty} \frac{q^{mk}}{(1-q^m)^{2k}}. \tag{4}$$

Observe that this is a q-series that converges for $|q| < 1$ and satisfies

$$\lim_{q\to 1} (1-q)^{2k}\, \zeta_q(2k) = \zeta(2k).$$

Theorem 1. *The recursion of equation* (3) *defines symmetric polynomials* $V_{g,n}(L_1,\dots,L_n) \in \mathbb{Q}[[q]][L_1^2,\dots,L_n^2]$ *which tend to the Weil–Petersson volumes in the following rescaled* $q \to 1$ *limit.*

$$\lim_{q\to 1} (1-q)^{6g-6+2n}\, V_{g,n}\Big(\frac{L_1}{1-q},\dots,\frac{L_n}{1-q}\Big) = V_{g,n}^{\mathrm{WP}}(L_1,\dots,L_n)$$

Theorem 1, which is the first main result of the paper, comprises two parts. The first shows that the recursion produces symmetric polynomials with q-series coefficients. The second makes sense of the $q \to 1$ limit using the fact that the coefficients are convergent for $|q| < 1$ and then calculates the limit via a comparison of the recursion of equation (3) with Mirzakhani's recursion for Weil–Petersson volumes. Theorem 1 justifies considering $V_{g,n}(L_1,\dots,L_n)$ as a q-deformation of the Weil–Petersson volume $V_{g,n}^{\mathrm{WP}}(L_1,\dots,L_n)$.

The main content of the first part of Theorem 1 is nicely demonstrated by the calculation of the integral given by equation (2) that defines $V_{1,1}(L)$. Similar integrals arise more generally in the calculation of

$V_{g,n}(L_1,\dots,L_n)$ using the recursion of equation (3). A priori, we observe that $V_{1,1}(L) \in \mathbb{Q}[[q,q^{-1}]][[L^2]]$ that is, the integral could contain terms with arbitrarily large powers of L and negative powers of q.

$$\begin{aligned}
V_{1,1}(L) &= \frac{1}{2L}\int_0^\infty x D_q(L,x,x)\,\mathrm{d}x \\
&= \sum_{m=1}^{\infty}(-1)^{m-1}q^{\binom{m}{2}}(1+q^m)\frac{\sinh\frac{1}{2}L(q^{m/2}-q^{-m/2})}{L(q^{m/2}-q^{-m/2})}\int_0^\infty x e^{\frac{1}{2}x(q^{m/2}-q^{-m/2})}dx \\
&= \sum_{m=1}^{\infty}(-1)^{m-1}q^{\binom{m}{2}}(1+q^m)\frac{\sinh\frac{1}{2}L(q^{m/2}-q^{-m/2})}{L(q^{m/2}-q^{-m/2})}\frac{4}{(q^{m/2}-q^{-m/2})^2} \\
&= \sum_{k=0}^{\infty}\frac{L^{2k}}{2^{2k+1}(2k+1)!}\sum_{m=1}^{\infty}(-1)^{m-1}q^{\binom{m}{2}}(1+q^m)(q^{m/2}-q^{-m/2})^{2k-2} \\
&= \frac{1}{48}L^2+\frac{1}{2}\zeta_q(2)
\end{aligned}$$

The second equality uses the series for D_q and the dominated convergence theorem to interchange the sum and the integral—see the proof of Proposition 4—and the fourth equality uses absolute convergence to interchange the order of summation. The final equality requires non-trivial q-series identities – see equation (8) in Section 2 – to produce the coefficients $\frac{1}{48}$ and $\frac{1}{2}\zeta_q(2)$, and also to prove that the coefficients of L^{2k} vanish for all $k \geqslant 2$. Observe that this vanishing requires the cancellation of terms that include negative powers of q.

The appearance of $\zeta_q(2)$ in $V_{1,1}(L)$ is a special case of a more general phenomenon. As a consequence of Proposition 4 below, the coefficients of $V_{g,n}(L_1,\dots,L_n)$ are polynomials in values of the q-zeta function.

$$V_{g,n}(L_1,\dots,L_n) \in \mathbb{Q}[\zeta_q(2),\zeta_q(4),\dots,\zeta_q(6g-6+2n)][L_1^2,\dots,L_n^2]$$

The second main result of the paper is an analogue of Theorem 1 in the context of super Weil–Petersson volumes. The notion of super Weil–Petersson volumes was defined by Stanford and Witten in their work on JT gravity and its generalisations [13]. They produced a super analogue of Mirzakhani's recursion for Weil–Petersson volumes which was later proved by the second author [11].

We introduce a natural q-deformation of the Stanford–Witten recursion that produces a family of polynomials $\widehat{V}_{g,n}^{(m)}(L_1,\dots,L_n) \in \mathbb{Q}[[q]][L_1^2,\dots,L_n^2]$ for integers $g \geqslant 0$, $n \geqslant 1$ and $m \geqslant 0$, following previous work of the second author [10]. In fact, we prove that for m odd, we have $\widehat{V}_{g,n}^{(m)}(L_1,\dots,L_n) = 0$, while for m even, we have

$$\widehat{V}_{g,n}^{(m)}(L_1,\dots,L_n) \in \mathbb{Q}[\zeta_q^{\mathrm{odd}}(2),\zeta_q^{\mathrm{odd}}(4),\dots,\zeta_q^{\mathrm{odd}}(2g-2+m)][L_1^2,\dots,L_n^2].$$

Here, we use the odd q-zeta function evaluated at the even integers, which we define via the following sum over the odd positive integers.

$$\zeta_q^{\mathrm{odd}}(2k) = \sum_{m\ \mathrm{odd}}\frac{q^{mk}}{(1-q^m)^{2k}}$$

This is a q-series that converges for $|q|<1$.

It is convenient to assemble the polynomials $\widehat{V}_{g,n}^{(m)}(L_1,\dots,L_n)$ into series via the equation

$$\widehat{V}_{g,n}(L_1,\dots,L_n;s) := \sum_{m=0}^{\infty}\frac{s^m}{m!}\widehat{V}_{g,n}^{(m)}(L_1,\dots,L_n), \tag{5}$$

where we will often omit the variable s in the notation as it is implicit.

Our q-deformation of the Stanford–Witten recursion requires integration using the kernels $\widehat{D}_q(x,y,z)$ and $\widehat{R}_q(x,y,z)$, which we now introduce. These are defined from the sum over the odd positive integers

$$\begin{aligned}
\widehat{H}_q(x,y) = \frac{1}{8\psi(q)^2}\sum_{m\ \mathrm{odd}}(-1)^{(m-1)/2}q^{(m^2-1)/4}(q^{m/2}+q^{-m/2}) \\
\left(e^{\frac{1}{4}(x-y)(q^{m/2}-q^{-m/2})}-e^{\frac{1}{4}(x+y)(q^{m/2}-q^{-m/2})}\right),
\end{aligned} \tag{6}$$

via the formulas

$$\widehat{D}_q(x,y,z) = \widehat{H}_q(y+z,x) \qquad \text{and} \qquad \widehat{R}_q(x,y,z) = \frac{1}{2}\big(\widehat{H}_q(z,x+y) + \widehat{H}_q(z,x-y)\big).$$

The definition of $\widehat{H}_q(x,y)$ requires the following Ramanujan theta function, which converges for $|q|<1$.

$$\psi(q) = \prod_{m=1}^{\infty} \frac{1-q^{2m}}{1-q^{2m-1}} = \frac{(1-q^2)(1-q^4)(1-q^6)\cdots}{(1-q)(1-q^3)(1-q^5)\cdots} = 1+q+q^3+q^6+q^{10}+\cdots$$

Define $\widehat{V}_{g,n}(L_1,\ldots,L_n;s) \in \mathbb{Q}[[q]][L_1^2,\ldots,L_n^2][[s^2]]$ from the initial conditions $\widehat{V}_{0,1}^{(0)}(L)=0$, $\widehat{V}_{0,1}^{(1)}(L)=0$, $\widehat{V}_{0,1}^{(2)}(L)=1$ and $\widehat{V}_{1,1}^{(0)}(L_1)=\frac{1}{8}$, and the same recursion as equation (3) although with the new kernels $\widehat{D}_q(x,y,z)$ and $\widehat{R}_q(x,y,z)$.

$$\begin{aligned} L_1\widehat{V}_{g,n}(L_1,\boldsymbol{L}_k;s) &= \sum_{j=2}^{n}\int_0^\infty x\widehat{R}_q(L_1,L_j,x)\,\widehat{V}_{g,n-1}(x,L_2,\boldsymbol{L}_{K\setminus\{j\}};s)\,\mathrm{d}x \\ &+ \frac{1}{2}\int_0^\infty\int_0^\infty xy\widehat{D}_q(L_1,x,y)\Big[\widehat{V}_{g-1,n+1}(x,y,L_K;s) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=K}} \widehat{V}_{g_1,|I|+1}(x,\boldsymbol{L}_I;s)\,\widehat{V}_{g_2,|J|+1}(y,\boldsymbol{L}_J;s)\Big]\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{7}$$

For example, the recursion produces all terms of the disk series, the first of which are as follows.

$$\widehat{V}_{0,1}(L;s) = \frac{s^2}{2!} + \left(\frac{1}{2}L^2 + 48\zeta_q^{\mathrm{odd}}(2)\right)\frac{s^4}{4!} + \left(\frac{3}{8}L^4 + 240\zeta_q^{\mathrm{odd}}(2)L^2 + 5760\zeta_q^{\mathrm{odd}}(4) + 17280\zeta_q^{\mathrm{odd}}(2)^2\right)\frac{s^6}{6!} + \cdots$$

Theorem 2. *The recursion of equation (7) defines symmetric polynomials $\widehat{V}_{g,n}^{(m)}(L_1,\ldots,L_n) \in \mathbb{Q}[[q]][L_1^2,\ldots,L_n^2]$. The series defined by equation (5) tend to corresponding series for super Weil–Petersson volumes in the following rescaled $q\to 1$ limit.*

$$\lim_{q\to 1}(1-q)^{2g-2}\,\widehat{V}_{g,n}\Big(\frac{L_1}{1-q},\ldots,\frac{L_n}{1-q};(1-q)s\Big) = \widehat{V}_{g,n}^{\mathrm{WP}}(L_1,\ldots,L_n;s)$$

The polynomials $V_{g,n}(L_1,\ldots,L_n)$ introduced here arise naturally in recent work of Okuyama on the double-scaled SYK model, in which he defined quasi-polynomials $N_{g,n}^q(b_1,\ldots,b_n)$ that he interpreted as discrete analogues of the Weil–Petersson volumes [12]. Okuyama constructed $N_{g,n}^q(b_1,\ldots,b_n)$ from a certain Hermitian matrix model and its associated loop equations. It is proven in [6] that $V_{g,n}(b_1,\ldots,b_n)$ defined via (3) agrees with the top degree terms of $N_{g,n}^q(b_1,\ldots,b_n)$ with respect to the grading $\deg b_i = 1$ and $\deg \zeta_q(2k) = 2k$. Together with Theorem 1, this proves Okuyama's conjectural relationship of the double-scaled SYK model with Weil–Petersson volumes.

The geometry underlying the constructions in this paper provides an interesting topic that deserves further study. The functions $D_q(x,y,z)$ and $R_q(x,y,z)$ are related to the functions $D(x,y,z)$ and $R(x,y,z)$ defined by Mirzakhani via a rescaled $q\to 1$ limit – see equation (19). Mirzakhani produced the kernels $D(x,y,z)$ and $R(x,y,z)$ from the geometry of a hyperbolic pair of pants with geodesic boundary components of lengths x,y,z [9]. They arise as natural lengths along these geodesic boundaries and appear in Mirzakhani's generalisation of McShane's identity [8]. Similarly, the rescaled $q\to 1$ limits of the functions $\widehat{D}_q(x,y,z)$ and $\widehat{R}_q(x,y,z)$ are the functions $\widehat{D}(x,y,z)$ and $\widehat{R}(x,y,z)$ appearing in the recursion for super Weil–Petersson volumes – see equation (23). Stanford and Witten produced geometric constructions for $\widehat{D}(x,y,z)$ and $\widehat{R}(x,y,z)$ using supergeometry [13]. Geometric interpretations of the q-analogues $D_q(x,y,z)$ and $R_q(x,y,z)$, as well as their super counterparts $\widehat{D}_q(x,y,z)$ and $\widehat{R}_q(x,y,z)$, would be extremely interesting.

In another direction, one might hope to interpret the polynomials we introduce here as "q-volumes" defined by integration of q-de Rham cohomology classes, following Aomoto [4]. This may be related to the recently defined notion of the Habiro ring [7], which raises the question of whether the constructions in this paper exhibit interesting behaviour as q approaches roots of unity.

Section 2 contains the proof of Theorem 1, which is divided into Propositions 3, 4 and 7. In Section 3, we define the Weil–Petersson super volumes before giving the proof of Theorem 2, which is again divided into more basic pieces – namely, Propositions 10, 11 and 13.

2 A q-deformation of Weil–Petersson volumes

In this section, we prove the existence and properties of the integrals in the recursion of equation (3). The function $H_q(x,y)$ is treated as a formal power series in y, hence $D_q(x,y,z)$ and $R_q(x,y,z)$ are formal power series in x, respectively x and y. We prove convergence of integration of $H_q(x,y)$ with respect to x, which gives convergence of integration of $D_q(x,y,z)$, respectively $R_q(x,y,z)$, with respect to y and z, respectively z. The same holds for $\widehat{H}_q(x,y)$, $\widehat{D}_q(x,y,z)$ and $\widehat{R}_q(x,y,z)$.

We begin with a sequence of q-series identities.

Proposition 3. *For any $k \in \mathbb{Z}$,*

$$\sum_{m=1}^{\infty} (-1)^{m-1} q^{m(m-1)/2} (1+q^m) \frac{q^{mk}}{(1-q^m)^{2k}} = s_k\big(\zeta_q(2), \zeta_q(4), \dots\big), \tag{8}$$

where s_k is defined by

$$\exp\left(\sum_{m=1}^{\infty} \frac{p_m}{m}\, t^m\right) = \sum_{k=-\infty}^{\infty} s_k(p_1, p_2, \dots)\, t^k. \tag{9}$$

For example, we have $s_k(p_1,p_2,\dots) = 0$ for $k<0$ and

$$s_0(p_1,p_2,\dots) = 1,\ s_1(p_1,p_2,\dots) = p_1, \quad s_2(p_1,p_2,\dots) = \frac{1}{2}(p_2+p_1^2), \quad s_3(p_1,p_2,\dots) = \frac{1}{6}(2p_3+3p_1p_2+p_1^3).$$

Proof. The case $k=0$ is equivalent to the following telescoping sum, which converges for $|q|<1$.

$$\sum_{m=1}^{\infty} (-1)^{m-1} q^{m(m-1)/2} (1+q^m) = (1+q) - (q+q^3) + (q^3+q^6) - (q^6+q^{10}) + \cdots = 1 \tag{10}$$

For the case $k>0$, we begin with the following infinite partial fraction decomposition, which appears in the work of Andrews [3, Equation (2.1)].

$$\prod_{m=1}^{\infty} \frac{(1-q^m)^2}{(1-zq^m)(1-z^{-1}q^{m-1})} = \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)/2}}{1-z^{-1}q^m} - z \sum_{m=1}^{\infty} \frac{(-1)^m q^{m(m+1)/2}}{1-zq^m}$$

Multiply both sides by $1-z^{-1}$ to obtain

$$\begin{aligned}
\prod_{m=1}^{\infty} \frac{(1-q^m)^2}{(1-zq^m)(1-z^{-1}q^m)} &= 1 + (1-z^{-1}) \sum_{m=1}^{\infty} (-1)^m q^{m(m+1)/2} \left(\frac{1}{1-z^{-1}q^m} - \frac{z}{1-zq^m}\right) \\
&= 1 + \sum_{m=1}^{\infty} (-1)^m q^{m(m+1)/2} \frac{(1-z^{-1})(1-z)(1+q^m)}{(1-z^{-1}q^m)(1-zq^m)} \\
&= 1 + \sum_{m=1}^{\infty} (-1)^m q^{m(m-1)/2} (1+q^m) \left(1 - \frac{(1-q^m)^2}{(1-z^{-1}q^m)(1-zq^m)}\right) \\
&= \sum_{m=1}^{\infty} (-1)^{m-1} q^{m(m-1)/2} (1+q^m) \frac{(1-q^m)^2}{(1-z^{-1}q^m)(1-zq^m)},
\end{aligned}$$

where the final equality uses equation (10). Observe that this expression converges for $1 \leqslant |z| < |q|^{-1}$. Now change coordinates to $t = i(z^{1/2} - z^{-1/2})$ so that $t^2 = 2 - z - z^{-1}$ to obtain the identity

$$\prod_{m=1}^{\infty} \frac{(1-q^m)^2}{(1-q^m)^2 + t^2 q^m} = \sum_{m=1}^{\infty} (-1)^{m-1} q^{m(m-1)/2} (1+q^m) \frac{(1-q^m)^2}{(1-q^m)^2 + t^2 q^m}. \tag{11}$$

This expression now converges for t away from the poles and in particular for all $|t| < 1-q$.

Equating the coefficients of t^{2k} in the Taylor expansions at $t=0$ of both sides of equation (11) produces the desired identity, as we will now see. For the left side of equation (11), consider the Taylor series at $t=0$ of

its logarithm.

$$\log \prod_{m=1}^{\infty} \frac{(1-q^m)^2}{(1-q^m)^2+t^2q^m} = -\sum_{m=1}^{\infty} \log\left(1+\frac{t^2q^m}{(1-q^m)^2}\right)$$
$$= \sum_{k=1}^{\infty}(-1)^k \frac{t^{2k}}{k} \sum_{m=1}^{\infty} \frac{q^{mk}}{(1-q^m)^{2k}} = \sum_{k=1}^{\infty}(-1)^k \frac{t^{2k}}{k}\zeta_q(2k)$$

Now take the exponential of both sides and use equation (9) to obtain

$$\prod_{m=1}^{\infty} \frac{(1-q^m)^2}{(1-q^m)^2+t^2q^m} = \sum_{k=0}^{\infty}(-1)^k\, s_k(\zeta_q(2), \zeta_q(4), \ldots)\, t^{2k}. \tag{12}$$

For the right side of equation (11), the Taylor series at $t = 0$ can be calculated termwise, as the factor $q^{m(m-1)/2}$ guarantees the convergence of the partial sums. The result is given by

$$\sum_{k=0}^{\infty}(-1)^k \sum_{m=1}^{\infty}(-1)^{m-1} q^{m(m-1)/2}(1+q^m)\frac{q^{mk}}{(1-q^m)^{2k}}\, t^{2k}. \tag{13}$$

Equating the t^{2k} coefficients of the Taylor series in equations (12) and (13) yields the desired result for $k > 0$.

For the case $k < 0$ put $\ell = -k$, denote by $F(t)$ the function given by the right hand side of equation (11) and integrate $t^{2\ell-1}F(t)$ over large circles. Using

$$\frac{1}{2\pi i}\oint_{|t|=R} \frac{t^{2\ell-1}}{1+at^2}\,\mathrm{d}t = \begin{cases} (-1)^{\ell-1}a^{-\ell}, & |a|^{-1} < R^2, \\ 0, & |a|^{-1} > R^2, \end{cases} \tag{14}$$

leads to

$$\frac{(-1)^{\ell-1}}{2\pi i}\oint_{|t|=R} t^{2\ell-1}\,\mathrm{d}t \sum_{m=1}^{\infty}(-1)^{m-1}q^{\binom{m}{2}}(1+q^m)\frac{(1-q^m)^2}{(1-q^m)^2+t^2q^m} = \sum_{m=1}^{N}(-1)^{m-1}q^{\binom{m}{2}}(1+q^m)\frac{(1-q^m)^{2\ell}}{q^{m\ell}} \tag{15}$$

Here, N is determined by $q^{-m/2} - q^{m/2} < R \Leftrightarrow m \leqslant N$ and is necessarily finite. The dominated convergence theorem allows us to interchange the integral and the sum.

The poles of the integrand of the left hand side of equation (15) occur at $\pm i a_m$ for

$$a_m := q^{-m/2} - q^{m/2},$$

which is a geometrically increasing sequence, due to $a_m/a_{m+1} < q^{1/2}$. This allows us to choose the contour in equation (15) so that its distance from the poles is bounded below, leading to an upper bound on the integrand as follows. Define

$$R_M^2 = a_M a_{M+1}$$

so that $a_m < R_M \Leftrightarrow m \leqslant M$. For $|t| = R_M$ and $m \leqslant M$,

$$|a_m^2 + t^2| \geqslant \left||t|^2 - a_m^2\right| = R_M^2 - a_m^2 > R_M^2 - a_M^2 > (1-q^{1/2})R_M^2.$$

From

$$q^{-m/2} > a_m \geqslant (1-q)q^{-m/2}$$

we have

$$\prod_{m=1}^{M} a_m^2 < q^{-M(M+1)/2},$$

and

$$R_M^{2M} = (a_M a_{M+1})^M > (1-q)^{2M} q^{-(2M+1)M/2}.$$

Thus

$$\prod_{m=1}^{M}\left|\frac{a_m^2}{a_m^2+t^2}\right| < \frac{\prod_{m=1}^{M} a_m^2}{(1-q^{1/2})^M R_M^{2M}} < \frac{q^{(2M+1)M/2}q^{-M(M+1)/2}}{(1-q^{1/2})^M(1-q)^{2M}} = C_q^M q^{M^2/2}$$

for $C_q = (1-q^{1/2})^{-1}(1-q)^{-2}$.

For the remaining factors, $m > M$, we have $R_M^2/a_m^2 < q^{m-M-1/2}$, hence on $|t| = R_M$,

$$\prod_{m>M}\left|\frac{a_m^2}{a_m^2+t^2}\right| \leqslant \prod_{m>M}\frac{1}{1-R_M^2/a_m^2} < \prod_{j=1}^{\infty}\frac{1}{1-q^{j-1/2}} = \frac{1}{(q^{1/2};q)_\infty} < \infty$$

for $(q^{1/2};q)_\infty \neq 0$ the q-Pochhammer symbol. Now $R_M < q^{-(M+1)/2}$, so

$$\left|\frac{1}{2\pi}\oint_{|t|=R_M} t^{2\ell-1}\,dt\prod_{m=1}^{\infty}\frac{a_m^2}{a_m^2+t^2}\right| < \frac{1}{(q^{1/2};q)_\infty}q^{-\ell(M+1)}C_q^M q^{M^2/2} \underset{M\to\infty}{\longrightarrow} 0$$

since $q^{M^2/2}$ decays faster than any exponential in M. Hence, for all $\ell > 0$,

$$\lim_{N\to\infty}\sum_{m=1}^{N}(-1)^{m-1}q^{m(m-1)/2}(1+q^m)\frac{(1-q^m)^{2\ell}}{q^{m\ell}} = 0,$$

which corresponds precisely to the desired result in the case $k = -\ell < 0$. ∎

From the function $H_q(x,y)$ defined in equation (1), we define

$$F_{2k+1}(y) = \int_0^\infty x^{2k+1}H_q(x,y)\,\mathrm{d}x.$$

Here, $H_q(x,y)$ is to be understood as a series in y^2 – more precisely,

$$H_q(x,y) = \cosh\Big(y\frac{\mathrm{d}}{\mathrm{d}x}\Big)h_q(x) = \sum_{n=0}^{\infty}\frac{y^{2n}}{(2n)!}\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}h_q(x),$$

where

$$h_q(x) = \sum_{m=1}^{\infty}(-1)^{m-1}q^{m(m-1)/2}(1+q^m)\,e^{\frac{1}{2}x(q^{m/2}-q^{-m/2})}.$$

Integrability of $H_q(x,y)$ in x is guaranteed by the following result.

Proposition 4. *For each non-negative integer k, we have*

$$\frac{F_{2k+1}(y)}{(2k+1)!} = \int_0^\infty \frac{x^{2k+1}}{(2k+1)!}\,H_q(x,y)\,\mathrm{d}x = \sum_{n=0}^{k+1}b_n\frac{y^{2k+2-2n}}{(2k+2-2n)!},$$

where $b_0, b_1, b_2, \ldots \in \mathbb{Q}[\zeta_q(2), \zeta_q(4), \ldots]$ are defined by

$$\sum_{n=0}^{\infty}b_n z^{2n} = \exp\left(\sum_{j=1}^{\infty}\frac{\zeta_q(2j)}{j}(4z^2)^j\right) = 1 + 4\zeta_q(2)z^2 + 8(\zeta_q(2)^2+\zeta_q(4))z^4 + \cdots.$$

Proof. Integrability of $x^{2k+1}H_q(x,y)$ is a consequence of integrability of $x^{2k+1}\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}h_q(x)$ for all $n \geqslant 0$, together with vanishing of the integrals for $n > k+1$.

We first prove integrability of $x^{2k+1}\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}h_q(x)$ in the $n = 0$ case. Consider the following inequality, which is satisfied for $0 < q < 1$ and any positive integer m.

$$q^{-m/2} - q^{m/2} = q^{-m/2}(1+q+q^2+\cdots+q^{m-1})(1-q) > (q^{-m/2}+m-1)(1-q) > m(1-q) \tag{16}$$

Here, the first inequality uses $q^{-m/2+k} + q^{m/2-k} > 2$ for $k = 1, 2, \ldots, m-1$. A consequence of equation (16) is the bound

$$|h_q(x)| < \sum_{m=1}^{\infty}q^{m(m-1)/2}(1+q^m)\,e^{\frac{1}{2}x(q^{m/2}-q^{-m/2})} < 2\sum_{m=1}^{\infty}e^{-\frac{1}{2}xm(1-q)} = \frac{2}{e^{\frac{1}{2}(1-q)x}-1}.$$

We know that $\frac{2x^{2k+1}}{e^{\frac{1}{2}(1-q)x}-1}$ is integrable for each $k \geqslant 0$. Furthermore, equation (16) allows us to deduce pointwise convergence of the following partial sums, via upper bounds on the tail of the series.

$$\lim_{N\to\infty} \sum_{m=1}^{N} (-1)^{m-1} q^{m(m-1)/2}(1+q^m)\, e^{\frac{x}{2(1-q)}(q^{m/2}-q^{-m/2})} \\ = \sum_{m=1}^{\infty} (-1)^{m-1} q^{m(m-1)/2}(1+q^m)\, e^{\frac{x}{2(1-q)}(q^{m/2}-q^{-m/2})}$$

By the dominated convergence theorem, $x^{2k+1}h_q(x)$ is Lebesgue integrable in x, and we can interchange the sum and integral to obtain the following for $0 < q < 1$.

$$\begin{aligned}\int_0^\infty \frac{x^{2k+1}}{(2k+1)!}\, h_q(x)\,\mathrm{d}x &= \sum_{m=1}^{\infty} (-1)^{m-1} q^{m(m-1)/2}(1+q^m) \int_0^\infty \frac{x^{2k+1}}{(2k+1)!}\, e^{\frac{1}{2}x(q^{m/2}-q^{-m/2})}\,\mathrm{d}x \\ &= \sum_{m=1}^{\infty} (-1)^{m-1} q^{m(m-1)/2}(1+q^m) \frac{2^{2k+2}}{(q^{m/2}-q^{-m/2})^{2k+2}} \\ &= 2^{2k+2}\, s_{k+1}\left(\zeta_q(2), \zeta_q(4), \ldots\right)\end{aligned}$$

The last equality uses the q-series identity of equation (8), where s_k is the polynomial defined by equation (9), which leads to the definition of b_n in the statement of the proposition.

When $n > 0$, the proof of integrability of $x^{2k+1}\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}h_q(x)$ is similar to the $n = 0$ case. Partition $h_q(x) = h_q^{(2n)}(x) +$ tail into the sum of the first $2n$ terms and the tail. Integrability of the first $2n$ terms of $x^{2k+1}\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}h_q(x)$ again uses the inequality given in equation (16):

$$\begin{aligned}\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}} h_q^{(2n)}(x) &= \sum_{m=1}^{2n} q^{m(m-1)/2}(1+q^m) 2^{-2n} \left(q^{m/2}-q^{-m/2}\right)^{2n} e^{\frac{x}{2}(q^{m/2}-q^{-m/2})} \\ &< 2^{1-2n}(1-q)^{2n} \sum_{m=1}^{2n} m^{2n} e^{-\frac{1}{2}xm(1-q)}\end{aligned}$$

The tail is dominated by an integrable function as follows.

$$\sum_{m=2n+1}^{\infty} q^{m(m-1)/2}(1+q^m)(q^{m/2}-q^{-m/2})^{2n} e^{\frac{x}{2}\left(q^{m/2}-q^{-m/2}\right)} < 2\sum_{m=1}^{\infty} e^{-\frac{1}{2}xm(1-q)} = \frac{2}{e^{\frac{1}{2}(1-q)x}-1}$$

The condition $m > 2n$ ensures that $m(m-1)/2 - mn \geqslant 0$ so that $q^{m(m-1)/2}(1+q^m)\left(q^{m/2}-q^{-m/2}\right)^{2n} < 2$. Hence, $\left|x^{2k+1}\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}h_q(x)\right|$ is bounded above by an integrable function. The same comparison on the tail proves pointwise convergence for the series. So as in the $n = 0$ case, by the dominated convergence theorem, $x^{2k+1}\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}h_q(x)$ is Lebesgue integrable in x and we can interchange the sum and integral. The integral can be obtained from the $n = 0$ case via integration by parts or it can be calculated as follows.

$$\begin{aligned}&\int_0^\infty \frac{x^{2k+1}}{(2k+1)!}\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}h_q(x)\,\mathrm{d}x \\ &= \sum_{m=1}^{\infty} (-1)^{m-1} q^{m(m-1)/2}(1+q^m)\tfrac{1}{2^{2n}}(q^{m/2}-q^{-m/2})^{2n} \int_0^\infty \frac{x^{2k+1}}{(2k+1)!} e^{\frac{x}{2}(q^{m/2}-q^{-m/2})}\,\mathrm{d}x \\ &= \sum_{m=1}^{\infty} (-1)^{m-1} q^{m(m-1)/2}(1+q^m) \frac{2^{2k+2-2n}}{(q^{m/2}-q^{-m/2})^{2k+2-2n}} \\ &= 2^{2k+2-2n}\, s_{k+1-n}\left(\zeta_q(2), \zeta_q(4), \ldots\right).\end{aligned}$$

When $n > k+1$, the q-series identity of equation (8) implies the vanishing

$$\sum_{m=1}^{\infty} (-1)^{m-1} q^{m(m-1)/2}(1+q^m)(q^{m/2}-q^{-m/2})^{2n-2k-2} = 0.$$

Putting this together, we obtain

$$\int_0^\infty \frac{x^{2k+1}}{(2k+1)!}\,H(x,y)\,\mathrm{d}x = \sum_{i=0}^{k+1} s_i(\zeta_q(2),\zeta_q(4),\ldots)\,\frac{(2y)^{2k+2-2i}}{(2k+2-2i)!}.$$

Equivalently, $F_{2k+1}(y)$ is a polynomial in y given by

$$\frac{F_{2k+1}(y)}{(2k+1)!} = \sum_{n=0}^{k+1} b_n \frac{y^{2k+2-2n}}{(2k+2-2n)!}$$

where the factor of $2^{2k+2-2n}$ is absorbed by the definition of b_n. ∎

Proposition 5. *The quantity $V_{g,n}(L_1,\ldots,L_n)$ produced by the recursion of equation (3) is a symmetric polynomial in $L_1^2,\ldots,L_n^2$.*

Proof. The kernel $D_q(x,y,z)$ in the recursion of equation (3) can be written explicitly as

$$D_q(x,y,z) = \sum_{m=1}^{\infty} (-1)^{m-1}\frac{q^{m(m-1)/2}(1+q^m)}{q^{m/2}-q^{-m/2}}\Big(e^{\frac{x+y+z}{2}[m]_q} - e^{\frac{-x+y+z}{2}[m]_q}\Big),$$

where we interpret it as a series in x and introduce the shorthand $[m]_q = q^{m/2}-q^{-m/2}$. Similarly, the kernel $R_q(x,y,z)$ can be written explicitly as

$$R_q(x,y,z) = \frac{1}{2}\sum_{m=1}^{\infty} \frac{(-1)^{m-1}q^{m(m-1)/2}(1+q^m)}{q^{m/2}-q^{-m/2}}\Big(e^{\frac{x+y+z}{2}[m]_q} + e^{\frac{x-y+z}{2}[m]_q} - e^{\frac{-x+y+z}{2}[m]_q} - e^{\frac{-x-y+z}{2}[m]_q}\Big),$$

where we interpret it as a series in x and y.

Integrability of $z^{2k+1}R_q(x,y,z)$ in z is equivalent to integrability of $z^{2k+1}\frac{\partial}{\partial x}R_q(x,y,z)$ since $R_q(x,y,z)$ is a formal power series in x and y and $R_q(0,y,z)=0$, i.e. it is the *coefficients* of the series in x and y that we require to be integrable in z, and their integrability is unaffected by $\frac{\partial}{\partial x}$. Integrability of $z^{2k+1}\frac{\partial}{\partial x}R_q(x,y,z)$ is an immediate consequence of integrability of $x^{2k+1}H_q(x,y)$. Furthermore,

$$\int_0^\infty z^{2k+1}\frac{\partial}{\partial x}R_q(x,y,z)\,\mathrm{d}z = \frac{1}{2}(F_{2k+1}(x+y)+F_{2k+1}(x-y)),$$

from which we can immediately retrieve $\int_0^\infty z^{2k+1}R_q(x,y,z)\,\mathrm{d}z$ since the polynomial in x stores coefficients and the linear operator $\frac{\partial}{\partial x}$ does not actually differentiate under the integral sign.

Integrability of $y^{2i+1}z^{2j+1}D_q(x,y,z)$ in y and z is equivalent to integrability of $y^{2i+1}z^{2j+1}\frac{\partial}{\partial x}D_q(x,y,z)$, again since $D_q(0,y,z)=0$ and it is the coefficients of the series in x that we require to be integrable. The derivative $\frac{\partial}{\partial x}$ cancels the $q^{m/2}-q^{-m/2}$ denominator, leaving the same comparison integrable function as used for $H_q(x,y)$ to apply the dominated convergence theorem and deduce integrability. Furthermore, Fubini's theorem allows one to use the change of coordinates $y=\frac{1}{2}(u+v)$, $z=\frac{1}{2}(u-v)$ to prove

$$\begin{aligned}\int_0^\infty\int_0^\infty y^{2j+1}z^{2j+1}\frac{\partial}{\partial x}D_q(x,y,z)\,\mathrm{d}y\,\mathrm{d}z &= \int_0^\infty\int_0^\infty y^{2i+1}z^{2j+1}H(y+z,x)\,\mathrm{d}y\,\mathrm{d}z\\ &= \frac{(2i+1)!\,(2j+1)!}{(2i+2j+3)!}\,F_{2i+2j+3}(x).\end{aligned}$$

Again, we can immediately retrieve $\int_0^\infty\int_0^\infty y^{2j+1}z^{2k+1}D_q(x,y,z)\,\mathrm{d}y\,\mathrm{d}z$ since $\frac{\partial}{\partial x}$ does not differentiate under the integral sign.

We have proven that the recursion of equation (3) is integrable and defines a polynomial $L_1V_{g,n}(L_1,\ldots,L_n)$, by the inductive assumption that all $V_{g',n'}$ generated earlier by the recursion are polynomial. Since we have $D_q(0,y,z) = R_q(0,y,z) = 0$, the polynomial is divisible by L_1, proving that $V_{g,n}(L_1,\ldots,L_n)$ is also polynomial. Moreover, $V_{g,n}(L_1,\ldots,L_n)$ is polynomial in L_1^2, since the recursion (3) can be used to

produce the polynomial $(\partial/\partial L_1)(L_1 V_{g,n}(L_1,\dots,L_n))$ as a linear combination of $F_1(L_1), F_3(L_1),\dots$, which are inherently polynomial in L_1^2. By induction, $V_{g,n}(L_1,\dots,L_n)$ is polynomial in L_j^2 for $j>1$.

To prove that $V_{g,n}(L_1,\dots,L_n)$ is symmetric, we use the following fact: for any polynomial $P(L_1,\dots,L_n)$, if $P(L_1,L_2,\dots,L_n)+P(L_2,L_1,\dots,L_n)=(L_1+L_2)\,Q(L_1,\dots,L_n)$ for a polynomial Q, then Q is symmetric in L_1 and L_2. Furthermore, it is enough to prove that $P(L_1,-L_1,L_3,\dots,L_n)+P(-L_1,L_1,L_3,\dots,L_n)=0$.

Apply the recursion of equation (3) with each of L_1 and L_2 as the distinguished argument, and add to obtain the following polynomial symmetric in L_1 and L_2.

$$L_1 V_{g,n}(L_1,L_2,\dots,L_n)+L_2 V_{g,n}(L_2,L_1,\dots,L_n)=\int_0^\infty (\,\cdot\,)\,\mathrm{d}x+\int_0^\infty\int_0^\infty (\,\cdot\,)\,\mathrm{d}x\,\mathrm{d}y \tag{17}$$

Now set $L_2=-L_1$ in the right side of equation (17). For $j\neq 1,2$ in the first term on the right side of equation (3), the integrands cancel

$$R_q(L_1,L_j,x)V_{g,n-1}(x,-L_1,\boldsymbol{L}_{K\setminus\{2,j\}})+R_q(-L_1,L_j,x)V_{g,n-1}(x,L_1,\boldsymbol{L}_{K\setminus\{2,j\}})=0,$$

since $V_{g,n-1}(x,-L_1,\dots)=V_{g,n-1}(x,L_1,\dots)$ and $R_q(-L_1,L_j,x)=-R_q(L_1,L_j,x)$.

For $j=1$ or 2, the integrands also cancel

$$R_q(L_1,L_j,x)V_{g,n-1}(x,L_3,\dots,L_n)+R_q(-L_1,L_j,x)V_{g,n-1}(x,L_3,\dots,L_n)=0,$$

since $R_q(-L_1,L_j,x)=-R_q(L_1,L_j,x)$.

Similarly, in the remaining terms on the right side of equation (3), the integrands cancel under $L_2=-L_1$, since

$$D_q(L_1,x,y)P(L_2^2)+D_q(L_2,x,y)P(L_1^2)\overset{L_2=-L_1}{\longrightarrow}\big(D_q(L_1,x,y)+D_q(-L_1,x,y)\big)P(L_1^2)=0,$$

where $P(L_2^2)$ is a polynomial obtained from simpler volumes and $D_q(-x,y,z)=-D_q(x,y,z)$. Thus, the expression in equation (17) vanishes and we conclude that $V_{g,n}(L_1,L_2,\dots,L_n)$ is symmetric in L_1 and L_2. By induction, $V_{g,n}(L_1,L_2,\dots,L_n)$ is symmetric in $L_2,\dots,L_n$ and hence, it is symmetric in $L_1,L_2,\dots,L_n$, as claimed. ∎

To prove that the rescaled $q\to 1$ limit of $V_{g,n}(L_1,\dots,L_n)$ gives the Weil–Petersson volumes, we show that the recursion of equation (3) tends to the following recursion of Mirzakhani for Weil–Petersson volumes.

Theorem 6 (Mirzakhani [9]). *The Weil–Petersson volumes satisfy the following recursion for* $2g-2+n\geqslant 2$.

$$\begin{aligned}L_1 V_{g,n}^{\mathrm{WP}}(L_1,\boldsymbol{L}_K)&=\sum_{j=2}^{n}\int_0^\infty xR(L_1,L_j,x)\,V_{g,n-1}^{\mathrm{WP}}(x,\boldsymbol{L}_{K\setminus\{j\}})\,\mathrm{d}x\\&+\frac{1}{2}\int_0^\infty\int_0^\infty xyD(L_1,x,y)\Big[V_{g-1,n+1}^{\mathrm{WP}}(x,y,\boldsymbol{L}_K)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=K}} V_{g_1,|I|+1}^{\mathrm{WP}}(x,\boldsymbol{L}_I)\,V_{g_2,|J|+1}^{\mathrm{WP}}(y,\boldsymbol{L}_J)\Big]\,\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{18}$$

Here, we use the notation $K=\{2,3,\dots,n\}$ *and write* $\boldsymbol{L}_I=(L_{i_1},L_{i_2},\dots,L_{i_m})$ *for* $I=\{i_1,i_2,\dots,i_m\}$.

The kernels in equation (18) are defined by

$$\frac{\partial}{\partial x}D(x,y,z)=H(x,y+z)\qquad\text{and}\qquad\frac{\partial}{\partial x}R(x,y,z)=\frac{1}{2}\big(H(z,x+y)+H(z,x-y)\big)$$

and the initial conditions $D(0,y,z)=R(0,y,z)=0$, from the function

$$H(x,y)=\frac{1}{1+e^{\frac{x+y}{2}}}+\frac{1}{1+e^{\frac{x-y}{2}}}.$$

More explicitly, we have

$$D(x,y,z)=2\log\left(\frac{e^{\frac{x}{2}}+e^{\frac{y+z}{2}}}{e^{-\frac{x}{2}}+e^{\frac{y+z}{2}}}\right)\qquad\text{and}\qquad R(x,y,z)=x-\log\left(\frac{\cosh\frac{y}{2}+\cosh\frac{x+z}{2}}{\cosh\frac{y}{2}+\cosh\frac{x-z}{2}}\right).$$

The following limits relate Mirzakhani's kernels to the q-kernels introduced in Section 1.

$$\begin{aligned}\lim_{q\to 1} H_q\Big(\frac{x}{1-q},\frac{y}{1-q}\Big) &= H(x,y)\\ \lim_{q\to 1}(1-q)\,D_q\Big(\frac{x}{1-q},\frac{y}{1-q},\frac{z}{1-q}\Big) &= D(x,y,z)\\ \lim_{q\to 1}(1-q)\,R_q\Big(\frac{x}{1-q},\frac{y}{1-q},\frac{z}{1-q}\Big) &= R(x,y,z)\end{aligned} \tag{19}$$

The recursion of equation (18) uniquely determines $V_{g,n}^{\mathrm{WP}}(L_1,\dots,L_n)$ from the base cases

$$\begin{aligned}V_{0,1}^{\mathrm{WP}}(L) &:= 0\\ V_{0,2}^{\mathrm{WP}}(L_1,L_2) &:= 0\\ V_{0,3}^{\mathrm{WP}}(L_1,L_2,L_3) &:= 1\\ V_{1,1}^{\mathrm{WP}}(L) &:== \frac{1}{2L}\int_0^\infty xD(L,x,x)\,\mathrm{d}x.\end{aligned} \tag{20}$$

Calculating the integral of equation (20) and applying the recursion of equation (18) yields the formulas

$$\begin{aligned}V_{1,1}^{\mathrm{WP}}(L) &= \frac{1}{48}L^2+\frac{\pi^2}{12}\\ V_{0,4}^{\mathrm{WP}}(L_1,L_2,L_3,L_4) &= \frac{1}{2}\sum_{i=1}^4 L_i^2+2\pi^2\\ V_{1,2}^{\mathrm{WP}}(L_1,L_2) &= \frac{1}{192}(L_1^2+L_2^2)^2+\frac{\pi^2}{12}(L_1^2+L_2^2)+\frac{\pi^4}{4}.\end{aligned}$$

Proposition 7. *The polynomials $V_{g,n}(L_1,\dots,L_n)\in\mathbb{Q}[[q]][L_1^2,\dots,L_n^2]$ tend to the Weil–Petersson volumes in the following rescaled $q\to 1$ limit.*

$$\lim_{q\to 1}(1-q)^{6g-6+2n}\,V_{g,n}\Big(\frac{L_1}{1-q},\dots,\frac{L_n}{1-q}\Big)=V_{g,n}^{\mathrm{WP}}(L_1,\dots,L_n)$$

Proof. The limit clearly holds in the case $(g,n)=(0,3)$ since $V_{0,3}(L_1,L_2,L_3)=1$ and $V_{0,3}^{\mathrm{WP}}(L_1,L_2,L_3)=1$. In the case $(g,n)=(1,1)$, we can calculate explicitly as follows.

$$\lim_{q\to 1}(1-q)^2\,V_{1,1}\Big(\frac{L_1}{1-q}\Big)=\lim_{q\to 1}(1-q)^2\left(\frac{1}{48}\Big(\frac{L_1}{1-q}\Big)^2+\frac{1}{2}\zeta_q(2)\right)=\frac{1}{48}L_1^2+\frac{1}{2}\zeta(2)=V_{1,1}^{\mathrm{WP}}(L_1)$$

Using the definition $F_{2k+1}^M(y)=\int_0^\infty x^{2k+1}H(x,y)\,\mathrm{d}x$, Mirzakhani [9] proved that

$$\frac{F_{2k+1}^M(y)}{(2k+1)!}=\sum_{n=0}^{k+1}b_n^M\frac{y^{2k+2-2n}}{(2k+2-2n)!},\qquad\text{where}\quad\sum_{n=0}^\infty b_n^M z^{2n}=\frac{2\pi z}{\sin(2\pi z)}.$$

By Proposition 4, we have

$$\frac{F_{2k+1}(y)}{(2k+1)!}=\sum_{n=0}^{k+1}b_n\frac{y^{2k+2-2n}}{(2k+2-2n)!},\qquad\text{where}\quad\sum_{n=0}^\infty b_n z^{2n}=\exp\left(\sum_{j=1}^\infty\frac{\zeta_q(2j)}{j}(4z^2)^j\right).$$

The Weierstrass product

$$\frac{\sin(2\pi z)}{2\pi z}=\prod_{n=1}^\infty\left(1-\frac{4z^2}{n^2}\right)$$

together with the expansion of $\log(1-t)$ gives the Taylor expansion

$$\log\left(\frac{\sin(2\pi z)}{2\pi z}\right)=-\sum_{j=1}^\infty\frac{\zeta(2j)}{j}(4z^2)^j\qquad\Rightarrow\qquad\frac{2\pi z}{\sin(2\pi z)}=\exp\left(\sum_{j=1}^\infty\frac{\zeta(2j)(4z^2)^j}{j}\right).$$

This proves that

$$\lim_{q\to 1}(1-q)^{2k+2}\,F_{2k+1}\Big(\frac{y}{1-q}\Big) = F^M_{2k+1}(y).$$

The polynomials $F^M_{2k+1}(y)$ uniquely determine the Weil–Petersson volumes $V^{\mathrm{WP}}_{g,n}(L_1,\dots,L_n)$ from the recursion of equation (18), as shown by Mirzakhani [9]. Similarly, the polynomials $F_{2k+1}(y)$ uniquely determine the polynomials $V_{g,n}(L_1,\dots,L_n)$ from the recursion of equation (3), as shown here in the proof of Proposition 5. We have shown that the rescaled limits of the initial polynomials $V_{0,3}$ and $V_{1,1}$ produce the Weil–Petersson volumes, and that the rescaled limit of the recursion of equation (3) reproduces Mirzakhani's recursion. By induction, this proves the proposition. ■

The q-deformations that we introduce here may possess structure analogous to that of their limits, namely the Weil–Petersson volumes. For example, in previous work of the authors, we prove various properties of Weil–Petersson volumes, such as the fact that $V^{\mathrm{WP}}_{g,1}(2\pi i) = 0$ [5]. It would be interesting to see if this has consequences for the polynomials $V_{g,1}(L)$ defined here.

3 A q-deformation of super Weil–Petersson volumes

In this section, we define a q-deformation of the super Weil–Petersson volumes originally defined by Stanford and Witten [13]. We begin with a brief description of the supergeometry underlying these volumes. Roughly, a supermanifold consists of an ordinary manifold together with additional anticommuting, or *odd*, directions. The ordinary coordinates are called *even* or *bosonic*, while the odd coordinates are also called *fermionic*. Functions in the odd coordinates take values in an exterior algebra and are therefore nilpotent.

More precisely, a supermanifold is a locally ringed space $\widehat{M} = (M, \mathcal{O}_{\widehat{M}})$ such that locally

$$\mathcal{O}_{\widehat{M}}(U) \cong \mathcal{O}_M(U) \otimes \Lambda^*(V)$$

for some vector space V. The quotient of $\mathcal{O}_{\widehat{M}}$ by its nilpotent ideal recovers $\mathcal{O}_M$, so that M is the *reduced space* of $\widehat{M}$. In this sense, $\widehat{M}$ may be viewed as an infinitesimal thickening of M by odd directions.

A basic example arises from a vector bundle $E \to M$ over a smooth manifold M. The sheaf of smooth sections of $\Lambda^* E^\vee$ defines a smooth supermanifold

$$\widehat{M} = (M, \Gamma(\Lambda^* E^\vee))$$

whose odd directions are modelled on the fibres of E. Integration in these odd directions is given by Berezin integration. If (M,ω) is symplectic, integration over the odd directions reduces the super volume to an ordinary integral over the reduced space,

$$\mathrm{Vol}(\widehat{M}) = \int_M e(E^\vee)\exp(\omega),$$

where $e(E^\vee)$ denotes the Euler class of $E^\vee$, or an Euler form when M is non-compact.

The moduli space of genus g super Riemann surfaces with n marked points $\widehat{\mathcal{M}}_{g,n}$ arises in this way—the odd directions are defined smoothly by the sheaf of smooth sections of $\Lambda^* E^\vee_{g,n}$, where $E_{g,n}$ is a natural vector bundle defined over the moduli space $\mathcal{M}^{\mathrm{spin}}_{g,n}$ of spin curves with n marked points:

$$\mathcal{M}^{\mathrm{spin}}_{g,n} = \left\{ (\mathcal{C}, \theta, p_1, \dots, p_n, \phi) \;\middle|\; \phi : \theta^2 \xrightarrow{\cong} \omega^{\log}_{\mathcal{C}} \right\},$$

where $\mathcal{C}$ is an orbifold curve with isotropy subgroup $\mathbb{Z}_2$ at the marked points $\{p_1,\dots,p_n\}$, known as a twisted curve [1], and θ is a line bundle over $\mathcal{C}$.

A spin structure on an orbifold curve with marked points $(\mathcal{C}, p_1, \dots, p_n)$ defines a line bundle $\theta \to \mathcal{C}$ that is a square root of the log-canonical bundle – that is, $\theta^2 \cong \omega^{\log}_{\mathcal{C}}$. Both θ and $\omega^{\log}_{\mathcal{C}}$ are line bundles, or orbifold line bundles, defined over $\mathcal{C}$. An orbifold line bundle has a well-defined degree which may be a half-integer since the points with non-trivial isotropy can contribute one half to the degree. In particular, we have

$$\deg \omega^{\log}_{\mathcal{C}} = 2g - 2 + n \qquad \text{and} \qquad \deg \theta = g - 1 + \frac{1}{2}n.$$

Since $\deg \theta^\vee = 1 - g - \frac{1}{2}n < 0$, the bundle $\theta^\vee$ possesses no holomorphic sections and $h^0(\mathcal{C}, \theta^\vee) = 0$. The index $h^0(\mathcal{C}, \theta^\vee) - h^1(\mathcal{C}, \theta^\vee)$ is constant over any family, so $h^0(\mathcal{C}, \theta^\vee) = 0$ implies that $H^1(\mathcal{C}, \theta^\vee)$ is a vector space of constant dimension, which defines a vector bundle $E_{g,n}$ over any family of orbifold curves. Denote by $\mathcal{E}$ the universal spin structure defined over the universal curve $\mathcal{U}^{\text{spin}}_{g,n} \xrightarrow{\pi} \mathcal{M}^{\text{spin}}_{g,n}$ and define the vector bundle $E_{g,n}$ with fibre $H^1(\mathcal{C}, \theta^\vee)$ as follows.

Definition 8. Define the bundle $E_{g,n} := -R\pi_* \mathcal{E}^\vee \to \mathcal{M}^{\text{spin}}_{g,n}$.

There are two types of behaviour of the spin structure at a marked point. The spin structure either extends over the point, in which case it is called *Neveu–Schwarz*, or it does not extend over the point, in which case it is called *Ramond*. This decomposes the moduli space of spin curves into components

$$\mathcal{M}^{\text{spin}}_{g,n} = \bigsqcup_{\sigma \in \{0,1\}^n} \mathcal{M}^{\text{spin}}_{g,\sigma},$$

where we write $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{0,1\}^n$ and set $\sigma_j = 1$ if p_j is Neveu–Schwarz and $\sigma_j = 0$ if p_j is Ramond. The vector bundle $E_{g,n}$ restricts to a bundle $E_{g,\sigma} \to \mathcal{M}^{\text{spin}}_{g,\sigma} \subset \mathcal{M}^{\text{spin}}_{g,n}$ of rank $2g - 2 + \frac{1}{2}(n + |\sigma|)$, where we write $|\sigma| = \sum \sigma_j$. There is a natural Euler form $e(E_{g,\sigma}) = \text{Pf}(F_A)$, defined in previous work of the second author [10]. Here, F_A is the curvature of the Chern connection A on $E_{g,\sigma}$, defined from a natural Hermitian metric on $E_{g,\sigma}$ that comes from the super Weil–Petersson metric. The Euler form has degree $\deg e(E^\vee_{g,\sigma}) = 4g - 4 + n + |\sigma|$.

The super Weil–Petersson volume on each component reduces to the following integral over $\mathcal{M}^{\text{spin}}_{g,\sigma}$.

$$\widehat{V}^{\text{WP}}_{g,\sigma} := 2^{g-1+\frac{1}{2}(n+|\sigma|)} \int_{\mathcal{M}^{\text{spin}}_{g,\sigma}} e(E^\vee_{g,\sigma}) \exp \omega^{\text{WP}} \tag{21}$$

Let $\sigma = (1^n, 0^m)$ so that $\mathcal{M}^{\text{spin}}_{g,\sigma}$ parametrises spin curves with n Neveu–Schwarz marked points and m Ramond marked points. Consider $\omega^{\text{WP}}(L_1, \ldots, L_n, 0^m)$, which deforms the Weil–Petersson symplectic form only at the Neveu–Schwarz marked points. Then define

$$\widehat{V}^{\text{WP},(m)}_{g,n}(L_1, \ldots, L_n) := 2^{g-1+n+\frac{1}{2}m} \int_{\mathcal{M}^{\text{spin}}_{g,(1^n,0^m)}} e(E^\vee_{g,(1^n,0^m)}) \exp \omega^{\text{WP}}(L_1, \ldots, L_n), \tag{22}$$

which turns out to be a polynomial in $L_1, \ldots, L_n$. Analogous to equation (5), define the series

$$\widehat{V}^{\text{WP}}_{g,n}(L_1, \ldots, L_n; s) := \sum_{m=0}^{\infty} \frac{s^m}{m!} \widehat{V}^{\text{WP},(m)}_{g,n}(L_1, \ldots, L_n),$$

Furthermore, define the functions

$$\begin{aligned}
\widehat{H}(x,y) &:= \lim_{q\to 1} \widehat{H}_q\Big(\frac{x}{1-q}, \frac{y}{1-q}\Big) = \frac{1}{4\pi}\left(\frac{1}{\cosh\frac{x-y}{4}} - \frac{1}{\cosh\frac{x+y}{4}}\right) \\
\widehat{D}(x,y,z) &:= \lim_{q\to 1} \widehat{D}_q\Big(\frac{x}{1-q}, \frac{y}{1-q}, \frac{z}{1-q}\Big) = \widehat{H}(x, y+z) \\
\widehat{R}(x,y,z) &:= \lim_{q\to 1} \widehat{R}_q\Big(\frac{x}{1-q}, \frac{y}{1-q}, \frac{z}{1-q}\Big) = \frac{1}{2}\big(\widehat{H}(x+y,z) + \widehat{H}(x-y,z)\big).
\end{aligned} \tag{23}$$

The super Weil–Petersson volumes are uniquely determined from the initial conditions $\widehat{V}^{\text{WP},(0)}_{0,1}(L) = 0$, $\widehat{V}^{\text{WP},(1)}_{0,1}(L) = 0$, $\widehat{V}^{\text{WP},(2)}_{0,1}(L) = 1$ and $\widehat{V}^{\text{WP},(0)}_{1,1}(L_1) = \frac{1}{8}$, and the following recursion due to Stanford and Witten [13].

Theorem 9 (Alexandrov and Norbury [2]). *The super Weil–Petersson volumes satisfy the following recursion for* $g \geqslant 0$ *and* $n > 0$.

$$\begin{aligned}
L_1 \widehat{V}^{\text{WP}}_{g,n}(L_1, \boldsymbol{L}_K; s) &= \sum_{j=2}^{n} \int_0^\infty x \widehat{R}(L_1, L_j, x)\, \widehat{V}^{\text{WP}}_{g,n-1}(x, \boldsymbol{L}_{K\setminus\{j\}}; s)\, \mathrm{d}x \\
&+ \frac{1}{2}\int_0^\infty \int_0^\infty xy \widehat{D}(L_1, x, y) \Big[\widehat{V}^{\text{WP}}_{g-1,n+1}(x, y, \boldsymbol{L}_K; s) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J = K}} \widehat{V}^{\text{WP}}_{g_1,|I|+1}(x, \boldsymbol{L}_I; s)\, \widehat{V}^{\text{WP}}_{g_2,|J|+1}(y, \boldsymbol{L}_J; s)\Big]\, \mathrm{d}x\, \mathrm{d}y.
\end{aligned} \tag{24}$$

The $s = 0$ case of equation (24) was proven heuristically using supergeometry by Stanford and Witten in [13], and proven using algebraic geometry by the second author [11].

We now provide the proof of Theorem 2, which is similar to that of Theorem 1, but requires a new collection of q-series identities.

Proposition 10. *For any $k \in \mathbb{Z}$,*

$$\sum_{m \text{ odd}} (-1)^{(m-1)/2} q^{(m^2-1)/4}(1+q^m)\frac{q^{mk}}{(1-q^m)^{2k+1}} = \psi(q)^2 \cdot s_k\big(\zeta_q^{\text{odd}}(2), \zeta_q^{\text{odd}}(4), \dots\big), \tag{25}$$

where the sum is over the odd positive integers and s_k is defined as in Proposition 3 by

$$\exp\left(\sum_{m=1}^{\infty} \frac{p_m}{m}\, t^m\right) = \sum_{k=-\infty}^{\infty} s_k(p_1, p_2, \dots)\, t^k.$$

Proof. For $|q| < 1$, the following series and product converge to holomorphic functions uniformly for t, in compact sets disjoint from the poles.

$$\sum_{m \text{ odd}} (-1)^{(m-1)/2} q^{(m^2-1)/4} \frac{1+q^m}{1-q^m} \frac{(1-q^m)^2}{(1-q^m)^2 + t^2 q^m} = \psi(q)^2 \prod_{m \text{ odd}} \frac{(1-q^m)^2}{(1-q^m)^2 + t^2 q^m} \tag{26}$$

To obtain equation (26), put $t^2 = z$ and calculate the residue at each pole in the product as follows.

$$\begin{aligned}\operatorname*{Res}_{z=-(q^{-N/2}-q^{N/2})^2} \psi(q)^2 \prod_{m \text{ odd}} \frac{(1-q^m)^2}{(1-q^m)^2 + z q^m} &= \psi(q)^2 \frac{(1-q^N)^2}{q^N} \prod_{\text{odd } m \neq N} \frac{(1-q^m)^2}{(1-q^{m-N})(1-q^{m+N})} \\ &= (-1)^{(N-1)/2} q^{(N^2-1)/4-N}(1-q^{2N}),\end{aligned}$$

which uses $\big((1-q^m)^2 + z q^m\big)|_{z=-(q^{-N/2}-q^{N/2})^2} = (1-q^{m-N})(1-q^{m+N})$.

The coefficient of t^{2k} in the Taylor expansion about $t = 0$ of equation (26) produces the kth identity of equation (25) as follows. The coefficient of t^{2k} in the Taylor expansion about $t = 0$ of the left side of equation (26) is the sum over m of the Taylor coefficient of each summand, due to the factor $q^{(m^2-1)/4}$, which guarantees convergence of the partial sums. It is given by

$$(-1)^k \sum_{m \text{ odd}} (-1)^{(m-1)/2} q^{(m^2-1)/4}(1+q^m)\frac{q^{mk}}{(1-q^m)^{2k+1}}.$$

Take the Taylor expansion of the logarithm of the right side of equation (26) divided by $\psi(q)^2$.

$$\begin{aligned}\log \prod_{m \text{ odd}} \frac{(1-q^m)^2}{(1-q^m)^2 + t^2 q^m} &= -\sum_{m \text{ odd}} \log\left(1 + \frac{t^2 q^m}{(1-q^m)^2}\right) = \sum_{k=1}^{\infty} (-1)^k \frac{t^{2k}}{k} \sum_{m \text{ odd}} \frac{q^{mk}}{(1-q^m)^{2k}} \\ &= \sum_{k=1}^{\infty} (-1)^k \frac{t^{2k}}{k} \zeta_q^{\text{odd}}(2k)\end{aligned}$$

Now exponentiate both sides and use equation (9) to obtain

$$\prod_{m \text{ odd}} \frac{(1-q^m)^2}{(1-q^m)^2 + t^2 q^m} = \sum_{k=0}^{\infty} (-1)^k s_k(\zeta_q^{\text{odd}}(2), \zeta_q^{\text{odd}}(4), \dots, \zeta_q^{\text{odd}}(2k))\, t^{2k}.$$

Putting this together, we have equation (25) for $k \geqslant 0$.

To prove equation (25) for $k < 0$, denote by $G(t)$ the function given by equation (26), integrate $t^{2k-1} G(t)$ over large circles, and use equation (14). Writing $f_m(q) = (-1)^{(m-1)/2} q^{(m^2-1)/4}(1+q^m)$, we have

$$\frac{1}{2\pi i} \oint_{|t|=R} t^{2k-1}\, \mathrm{d}t \sum_{m \text{ odd}} f_m(q) \frac{(1-q^m)}{(1-q^m)^2 + t^2 q^m} = \sum_{\text{odd } m \leqslant N} f_m(q) \frac{(1-q^m)^{2k-1}}{q^{mk}}.$$

Again, uniform estimates show that the left side converges to 0 as $N \to \infty$. So for all $k > 0$, we have

$$\lim_{N\to\infty} \sum_{\text{odd } m\leqslant N} (-1)^{(m-1)/2} q^{(m^2-1)/4}(1+q^m)\frac{q^{mk}}{(1-q^m)^{2k+1}} = 0,$$

as required. ■

Note that although the odd part of the classical zeta function is simply a scalar multiple of the Riemann zeta function,

$$(1-2^{-2k})\zeta(2k) = \zeta^{\text{odd}}(2k) = \sum_{m \text{ odd}} \frac{1}{m^{2k}} = \lim_{q\to 1}(1-q)^{2k}\,\zeta_q^{\text{odd}}(2k)$$

there is no analogous relation between $\zeta_q^{\text{odd}}(2k)$ and $\zeta_q(2k)$, and Proposition 10 is not a consequence of Proposition 3.

For $\widehat{H}_q(x,y)$ defined in equation (6), define

$$\widehat{F}_{2k+1}(y) = \int_0^\infty x^{2k+1}\widehat{H}_q(x,y)\,\mathrm{d}x.$$

The function $\widehat{H}_q(x,y)$ is to be understood as the series in y^2 given by

$$\widehat{H}_q(x,y) = \frac{1}{4\psi(q)^2}\sinh\left(y\frac{\mathrm{d}}{\mathrm{d}x}\right)\widehat{h}_q(x) = \frac{1}{4\psi(q)^2}\sum_{n=0}^\infty \frac{y^{2n+1}}{(2n+1)!}\frac{\mathrm{d}^{2n+1}}{\mathrm{d}x^{2n+1}}\widehat{h}_q(x),$$

where

$$\widehat{h}_q(x) = \sum_{m \text{ odd}} (-1)^{(m+1)/2} q^{(m^2-2m-1)/4}(1+q^m)e^{\frac{x}{4}(q^{m/2}-q^{-m/2})}.$$

Proposition 11. *For each non-negative integer k, we have*

$$\frac{\widehat{F}_{2k+1}(y)}{(2k+1)!} = \int_0^\infty \frac{x^{2k+1}}{(2k+1)!}\widehat{H}_q(x,y)\,\mathrm{d}x = \sum_{n=0}^k \widehat{b}_n \frac{y^{2k+1-2n}}{(2k+1-2n)!},$$

where $\widehat{b}_0, \widehat{b}_1, \widehat{b}_2, \ldots \in \mathbb{Q}[\zeta_q^{\text{odd}}(2), \zeta_q^{\text{odd}}(4), \ldots]$ are defined by

$$\sum_{n=0}^\infty \widehat{b}_n z^{2n} = \exp\left(\sum_{m \text{ odd}} \frac{\zeta_q^{\text{odd}}(2m)}{m}(4z^2)^m\right) = 1 + 4\zeta_q^{\text{odd}}(2)z^2 + 8(\zeta_q^{\text{odd}}(2)^2 + \zeta_q^{\text{odd}}(4))z^4 + \cdots.$$

Proof. The proof of integrability of $x^{2k+1}\widehat{H}_q(x,y)$ follows the argument of the proof of Proposition 4. It is a consequence of integrability of $\widehat{h}_q(x)$, which uses the same estimates as the proof of Proposition 4, together with the inequality $\frac{1}{4\psi(q)^2} < 1$.

We begin by calculating the integral of each term of $4\psi(q)^2\widehat{H}_q(x,y)$.

$$\begin{aligned}
&\int_0^\infty \frac{x^{2k+1}}{(2k+1)!}\frac{1}{(2n+1)!}\frac{d^{2n+1}}{dx^{2n+1}}\widehat{h}_q(x)\,\mathrm{d}x \\
&\quad= \sum_{m \text{ odd}} (-1)^{(m-1)/2} q^{(m^2-1)/4}(1+q^m)\frac{(1-q^m)^{2n+1}}{q^{m(n+1)}4^{2n+1}}\int_0^\infty \frac{x^{2k+1}}{(2k+1)!}e^{\frac{x}{4}(q^{m/2}-q^{-m/2})}\,\mathrm{d}x \\
&\quad= \sum_{m \text{ odd}} (-1)^{(m-1)/2} q^{(m^2-1)/4}(1+q^m)\frac{4^{2k-2n+1}q^{m(k-n)}}{(1-q^m)^{2(k-n)+1}} \\
&\quad= 4^{2k-2n+1}\psi(q)^2 \cdot s_{k-n}(\zeta_q^{\text{odd}}(2), \zeta_q^{\text{odd}}(4), \ldots)
\end{aligned}$$

where the last equality uses the q-series identity in equation (25). In particular the integral vanishes when $n > k$ so only finitely many terms contribute to each integral of $x^{2k+1}\widehat{H}_q(x,y)$.

Hence

$$\int_0^\infty \frac{x^{2k+1}}{(2k+1)!}\widehat{H}_q(x,y)\,\mathrm{d}x = \sum_{n=0}^{k} s_n\big(\zeta_q^{\mathrm{odd}}(2),\zeta_q^{\mathrm{odd}}(4),\dots\big)\frac{(4y)^{2k-2n+1}}{(2k-2n+1)!}.$$

Since $\widehat{b}_n = 4^n s_n(\zeta_q^{\mathrm{odd}}(2),\zeta_q^{\mathrm{odd}}(4),\dots)$ then $\widehat{F}_{2k+1}(y)$ is a polynomial in y given by

$$\frac{\widehat{F}_{2k+1}(y)}{(2k+1)!} = \sum_{n=0}^{k+1}\widehat{b}_n\frac{y^{2k-2n+1}}{(2k-2n+1)!}.$$

■

Proposition 12. *The recursion of equation (7) produces series in s^2 with coefficients given by symmetric polynomials in $L_1^2,\dots,L_n^2$.*

Proof. The proof is similar to the proof of Proposition 5. Integrability of the functions $z^{2k+1}\widehat{R}_q(x,y,z)$ and $y^{2i+1}z^{2j+1}\widehat{D}_q(x,y,z)$ is an immediate consequence of integrability of $x^{2k+1}\widehat{H}_q(x,y)$, as in the proof of Proposition 5.

Furthermore, the integrals of $\widehat{D}_q(x,y,z)$ and $\widehat{R}_q(x,y,z)$ multiplied by the polynomials appearing in equation (7) are uniquely determined by $\widehat{F}_{2k+1}(y) = \int_0^\infty x^{2k+1}\widehat{H}_q(x,y)\,\mathrm{d}x$. These are polynomial due to the q-series identities of equation (25), which now replace equation (8).

The kernels $\widehat{R}_q(x,y,z)$ and $\widehat{D}_q(x,y,z)$ share two properties with the kernels $R_q(x,y,z)$ and $D_q(x,y,z)$ that allow us to adapt the proof of Proposition 5 to show that $\widehat{V}_{g,n}(L_1,\dots,L_n)$ is a symmetric polynomial. First, the vanishing $\widehat{D}_q(0,y,z) = \widehat{R}_q(0,y,z) = 0$ holds. This proves divisibility of the right side of equation (7) by L_1 and hence, gives polynomiality of $\widehat{V}_{g,n}(L_1,\dots,L_n)$. Second, $\widehat{R}_q(-L_1,L_j,x) = -\widehat{R}_q(L_1,L_j,x)$ and $\widehat{D}_q(-x,y,z) = -\widehat{D}_q(x,y,z)$ hold, which can then be used to prove that the polynomial $\widehat{V}_{g,n}(L_1,\dots,L_n)$ is symmetric. ■

Proposition 13. *The series $\widehat{V}_{g,n}(L_1,\dots,L_n)\in\mathbb{Q}[[q]][L_1^2,\dots,L_n^2][[s^2]]$ tend to the super Weil–Petersson volumes in the following rescaled $q\to 1$ limit.*

$$\lim_{q\to 1}(1-q)^{2g-2}\,\widehat{V}_{g,n}\Big(\frac{L_1}{1-q},\dots,\frac{L_n}{1-q};(1-q)s\Big) = \widehat{V}_{g,n}^{\mathrm{WP}}(L_1,\dots,L_n;s).$$

Proof. The proof is similar to that of Proposition 7, with the departure being the use of the following Weierstrass product

$$\cos(2\pi z) = \prod_{m\ \mathrm{odd}}\Big(1-\frac{16z^2}{m^2}\Big) \quad\Rightarrow\quad \log\cos(2\pi z) = -\sum_{k=1}^{\infty}\frac{\zeta^{\mathrm{odd}}(2k)}{k}(4z)^{2k}$$
$$\Rightarrow\quad \frac{1}{\cos(2\pi z)} = \exp\Big(\sum_{k=1}^{\infty}\frac{\zeta^{\mathrm{odd}}(2k)}{k}(4z)^{2k}\Big)$$

The second author [11, Lemma 5.2] introduced the function

$$G_{2k+1}(y) := \int_0^\infty x^{2k+1}\widehat{H}(x,y)\,\mathrm{d}x$$

for $\widehat{H}(x,y) = \frac{1}{4\pi}\big(\mathrm{sech}\frac{1}{4}(x-y) - \mathrm{sech}\frac{1}{4}(x+y)\big)$ and showed that it satisfies

$$\frac{G_{2k+1}(y)}{(2k+1)!} = \sum_{n=0}^{k} c_n\frac{y^{2k+1-2n}}{(2k+1-2n)!}, \qquad \text{where} \qquad \frac{1}{\cos(2\pi z)} = \sum_{n=0}^{\infty}c_n z^{2n}.$$

Since

$$\lim_{q\to 1}(1-q)^{2k}\,\zeta_q^{\mathrm{odd}}(2k) = \zeta^{\mathrm{odd}}(2k) \qquad \text{and} \qquad \lim_{q\to 1}(1-q)\,\psi(q)^2 = \frac{\pi}{2},$$

we have

$$\lim_{q\to 1}(1-q)^{2k+2}\,\widehat{F}_{2k+1}\Big(\frac{y}{1-q}\Big) = G_{2k+1}(y).$$

The polynomials $G_{2k+1}(y)$ uniquely determine the super Weil–Petersson volumes $\widehat{V}_{g,n}^{\mathrm{WP}}(L_1, \ldots, L_n; s)$, as shown by Alexandrov and the second author [2]. Similarly, the polynomials $\widehat{F}_{2k+1}(y)$ uniquely determine $\widehat{V}_{g,n}(L_1, \ldots, L_n)$ from the recursion of equation (7), as outlined above in the proof of Proposition 12. One can check that the rescaled limits of the initial data match as expected and the desired result then follows by induction. ■

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