NONTRIVIAL RATIONAL POLYNOMIALS IN TWO VARIABLES HAVE REDUCIBLE FIBRES

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We shall call a polynomial map $f\colon \mathbb{C}^2 \to \mathbb{C}$ a "coordinate" if there is a g such that $(f,g)\colon \mathbb{C}^2 \to \mathbb{C}^2$ is a polynomial automorphism. Equivalently, by Abhyankar-Moh and Suzuki, f has one and therefore all fibres isomorphic to $\mathbb{C}$. Following [7] we call a polynomial $f\colon \mathbb{C}^2 \to \mathbb{C}$ "rational" if the general fibres of f (and hence all fibres of f) are rational curves. The following theorem, which says that a rational polynomial map with irreducible fibres cannot be part of a counterexample to the 2-dimensional Jacobian Conjecture, has appeared in the literature several times. It appears with an algebraic proof in Razar [12]. It is Theorem 2.5 of Heitmann [4] (as corrected in the Corrigendum), and Lê and Weber, who give a geometric proof in [6], also cite the reference Friedland [3], which we have not seen.

Theorem 1. *If $f\colon \mathbb{C}^2 \to \mathbb{C}$ is a rational polynomial map with irreducible fibres and is not a coordinate then f has no jacobian partner (i.e., no polynomial g such that the jacobian of (f,g) is a non-zero constant).*

In this note we prove the above theorem is empty:

Theorem 2. *There is no f satisfying the assumptions of the above theorem. That is, a rational f which is not a coordinate has a reducible fibre.*

Proof. Suppose f is rational. As in [7], [6], etc., we consider a nonsingular compactification $Y = \mathbb{C}^2 \cup E$ of $\mathbb{C}^2$ such that f extends to a holomorphic map $\overline{f}\colon Y \to \mathbb{P}^1$. Then E is a union of smooth rational curves $E_1, \dots, E_n$ with normal crossings. An E_i is called *horizontal* if $\overline{f}|E_i$ is non-constant. Let δ be the number of horizontal curves. Then we have

Lemma 3.
$$\delta - 1 = \sum_{a\in\mathbb{C}} (r_a - 1),$$
where r_a is the number of irreducible components of $f^{-1}(a)$.

This is Lemma 4 of Lê-Weber [6] who attribute it to Kaliman [5], corollary 2. This lemma also appears in [7] where it is attributed to Saito [10]. The proof is simple arithmetic from the topological observation that on the one hand the euler characteristic of Y is $n+2$ and on the other hand it is $4 + \sum_{a\in\mathbb{P}^1}(\overline{r}_a - 1)$, where $\overline{r}_a$ is the number of components of $\overline{f}^{-1}(a), a \in \mathbb{P}^1$.

By this lemma, if f has irreducible fibres then there is just one horizontal curve. The theorem then follows from Lemma 1.7 of [7]. It also follows from the following proposition and its proof, which implies that the generic fibres of f have just one point at infinity and are thus isomorphic to $\mathbb{C}$. ☐

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Proposition 4. *If $f: \mathbb{C}^2 \to \mathbb{C}$ is a polynomial map and $\overline{f}: Y \to \mathbb{P}^1$ is an extension as above, and if d is the greatest common divisor of the degrees of $\overline{f}$ on the horizontal curves of Y then the general fibre of f has d components (so $f = h \circ f_1$ for some polynomials $f_1: \mathbb{C}^2 \to \mathbb{C}$ and $h: \mathbb{C} \to \mathbb{C}$ with* $\mathrm{degree}(h) = d$*).*

Proof. Let $E_1, \dots, E_\delta$ be the horizontal curves and $d_1, \dots, d_\delta$ be the degrees of $\overline{f}$ on these. Note that the points at infinity of a general fibre $f^{-1}(a)$ are the points where $\overline{f}^{-1}(a)$ meet the horizontal curves E_i, so there are d_i such points on E_i for $i = 1, \dots, \delta$. The relationship between plumbing diagram and splice diagram (cf. [9, 2] says that the splice diagram Γ for a regular link at infinity for f (cf. [8]) has δ nodes with arrows at them, and the number of arrows at these nodes are $d_1, \dots, d_\delta$ respectively. Let Γ_0 be the same splice diagram but with $d_1/d, \dots, d_\delta/d$ arrows at these nodes. Then a minimal Seifert surface S for the link represented by Γ will consist of d parallel copies of a minimal Seifert surface for the link represented by Γ_0, so this S has d components. But the general fibre of f is such a minimal Seifert surface ([8], Theorem 1), completing the proof. ∎

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