CHARACTERISTIC CLASSES

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ABSTRACT. These are the notes of a short course given at Melbourne University in December 1999.

INTRODUCTION

There are many approaches to characteristic classes and perhaps the cleanest approach is the most sophisticated. It can be found in the book *Characteristic classes* of Milnor and Stasheff. Rather than attempting to head straight for the approach used in that book, we will gain entry via the more accessible alternative approaches. One consequence of this will be that we will continually recall the definition of characteristic classes as we gain further appreciation for the terms in the definition.

Definition 1. *A characteristic class associates invariants—numbers or cohomology classes—to the tangent bundle—and more general vector bundles.*

Course summary.

(i) Euler's formula for a warm-up.
(ii) Intersections, transversality, local vector fields.
(iii) Stiefel-Whitney classes, line bundles and families of vector fields.
(iv) Orientation, complex bundles and Chern classes.
(v) Grassmannians and cohomology.
(vi) Pontryagin classes.
(vii) Differential forms, connections and curvature.
(viii) Characteristic numbers and special vector fields.

One might think of this course as training to be able to read the book of Milnor and Stasheff. That book is best read via its fantastic exercises. You can attempt the first exercise in each chapter, moving through the chapters, then return to the start to attempt the second exercise of each chapter. There should be no problem moving forwards and backwards through the book like that, particularly after having done this short course.

Further justification for not directly defining characteristic classes in their most natural context comes from recent developments in mathematics, such as four-manifold moduli space invariants and quantum cohomology, where a particular view of characteristic classes can sometimes generalise to situations not covered by the basic theory of characteristic classes.

What you are and are not expected to know. Pre-requisites for this course are knowledge of manifolds and the tangent space. Vector bundles, homology and cohomology will appear throughout the course in various guises, although no previous knowledge of these objects is assumed.

Some topics in these notes will have to be discarded as time restricts. There are topics that are not needed in the sequel so this will be easy to do. The eighth lecture requires enough digestion that we will probably spread parts of it through earlier lectures.

1. Vector fields and Euler's formula

Euler's formula for polyhedrons is $F - E + V = 2$ where F, E, V are the numbers of faces, edges and vertices, respectively, of the polyhedron. How might we prove this?

A cube gives $6 - 12 + 8 = 2$. If we add a diagonal to one of the faces of the cube we get $7 - 13 + 8 = 2$. More generally, we can add an edge joining two existing vertices and get $7 - 13 + 8 = 2$ or add a vertex to an existing edge and get $6 - 13 + 9 = 2$. We see that the sum $F - E + V$ is unchanged under such moves. That can be used to refine any polyhedron so that all of its faces are triangles. Any two polyhedrons have a common refinement so the calculation for the cube is enough to prove the result.

Here's another approach that suits our purposes. We may assume that we are working with a triangulated sphere (by refining the polyhedron and using the fact that a polyhedron is homeomorphic to the sphere.) Consider the vector field on the sphere given by a flow from the north to the south poles. Choose the north and south poles to lie in the interior of two faces of the triangulation. Refine the triangulation so that it is made up of small enough triangles that the vector field intersects the triangulation in such a way as to point each vertex and edge towards a unique face. In that way, besides the triangles that contain the north and south poles, there are two types of triangles: one in which one vertex and two edges point towards the face and the other in which one edge points towards the face. This enables us to associate the two edges with the face and vertex, respectively the face and an edge. Thus we have $F - E + V = 1 - 2 + 1 = 0$ respectively

$F - E + V = 1 - 1 + 0 = 0$. This fails at the north pole where no edge or vertex points towards the face so we get $F - E + V = 1 - 0 + 0 = 1$ and at the south pole where all edges and vertices point towards the face so we get $F - E + V = 1 - 3 + 3 = 1$. each face, edge or vertex appears once in this association, so we get $F - E + V = 2$ and the 2 seems to come from the two zeros of the vector field.

Let's turn this around and use Euler's formula to say something about a general vector field. Take any vector field and choose a triangulation that satisfies the property that each zero of the vector field lies in a unique interior of some face and each edge and vertex points to a unique face. Again we find that $F - E + V = 0$ on most triangles and at each zero we get $F - E + V = \pm 1$. Thus, the signed sum of zeros of any vector field is $F - E + V = 2$.

These arguments work over any surface.

Definition 2. *The Euler class assigns to the tangent bundle a number.*

The Euler class is an example of a characteristic class, the tangent bundle a vector bundle, and the number a cohomology class, so a *characteristic class assigns to a vector bundle a cohomology class.*

We will finish with a quick reminder of the Gauss-Bonnet theorem. For any surface Σ we have $\int_\Sigma dA$ =Area(Σ). More interestingly, $\int_\Sigma K dA = 2\pi\chi(\Sigma)$ where K is the curvature of a an embedding $\Sigma \hookrightarrow \mathbb{R}^N$. Later we will see that KdA is a differential 2-form and the embedding essentially encodes a metric on Σ.

The proof of this: $K = \lim_{\Delta\to 0} \frac{\alpha+\beta+\gamma}{\Delta}$ where Δ is the area of a small triangle with angles $\alpha, \beta +, gamma$.

Then $\sum_i K\Delta_i = \sum_i \alpha_i + \beta_i + \gamma_i = 2\pi V - \pi F$ and since the faces are triangles $2E = 3F$ so $2\pi V - \pi F$ =$2\pi(V + F - E) = 2\pi\chi(\Sigma)$. The integral arises in the limit.

2. Intersections and transversality

Definition 3. *Two subspaces of a vector space $V_1, V_2 \subset W$ intersect transversally if $V_1 + V_2 = W$.*

Two submanifolds of a manifold $\Sigma_1, \Sigma_2 \subset X$ intersect transversally at $p \in \Sigma_1 \cap \Sigma_2$ if $T_p\Sigma_1 + T_p\Sigma_2 = T_pX$. Curves in a surface give easy examples of this.

Facts: (Theorems)

(i) If Σ_1^l and Σ_2^m intersect transversally in M^n then $\Sigma_1 \cap \Sigma_2$ is a submanifold of dimension $l + m - n$.

(ii) If $[0,1]\times\Sigma_1$ and Σ_2 intersect transversally (but perhaps $\{t\}\times\Sigma_1$ and Σ_2 don't intersect transversally for some $t\in[0,1]$ then $[0,1]\times\Sigma_1\cap\Sigma_2$ is a submanifold with boundary.

(iii) In particular, if $\dim\Sigma_1\cap\Sigma_2=0$ then $[0,1]\times\Sigma_1\cap\Sigma_2$ is a 1-manifold with boundary so the number of ends of the intersection is even. Thus, $\Sigma_1\cap\Sigma_2$ is well-defined mod 2 if we allow Σ_1 to be deformed.

The third fact is important when two submanifolds don't intersect transversally so we deform them to do so.

One way to think of a tangent space is via an embedding of a manifold M into $\mathbb{R}^N$. For example, $S^2\hookrightarrow\mathbb{R}^3$ and

$$TS^2=\{(x_1,x_2,x_3),(v_1,v_2,v_3)\mid |x|=1,x\cdot v=0\}.$$

This is a 4-dimensional manifold.

Often it is better to think of a vector field locally. An n-dimesional manifold M can be given in terms of charts $\cup U_i=M$ where U_i is diffeomorphic to a subset of $\mathbb{R}^n$. The tangent bundle is trivial on $\mathbb{R}^n$, i.e. $T\mathbb{R}^n=\mathbb{R}^n\times\mathbb{R}^n$ and so with respect to the chart U_i we can think of a vector field as $((x_1,...,x_n),(v_1(x_1,...,x_n),...,v_n(x_1,...,x_n)))$ or $v_1(x_1,...,x_n)\partial_{x_1}+...+v_n(x_1,...,x_n)\partial_{x_n}$.

The zeros of a vector field $v:M\to TM$ are given by the intersection of $v(M)\subset TM$ with $M\subset TM$ where the latter embedding comes from the zero section. Since $\dim TM=2n$ then if we we can deform the vector field so that it intersects the zero section transversally, then we get a finite set of points which gives a well-defined number mod 2. Furthermore, any vector field deforms to any other vector field, so we get a number that *only depends on the tangent bundle.* This is a characteristic class.

Vector fields on the two-sphere give a good example of this. Later we will define orientation which will allow us to count the zeros with sign to get an integer. Further examples of intersections: $\mathbb{RP}^1\subset\mathbb{RP}^2$ has non-trivial self-intersection whilst $S^1\subset\Sigma^2$ has trivial self-intersection when Σ^2 is an orientable surface. Perhaps consider the self-intersection of $\mathbb{CP}^1\subset\mathbb{CP}^2$.

3. Stiefel-Whitney classes

Along with tangent bundles, line bundles are vector bundles over a manifold with interesting characteristic classes. A line bundle can be described as a continuous family of 1-dimensional subspaces of $\mathbb{R}^N$ parametrised by M. In other words, $L\subset M\times\mathbb{R}^N$ and the subspaces are $L_x\subset\mathbb{R}^N$ for $x\in M$. (More gennerally, a k-dimensional vector bundle consists of a continuous family of k-dimensional subspaces $V_x\subset\mathbb{R}^N$.) Strictly, a line bundle is over M^n is an $(N+1)$-dimensional manifold

L equipped with a map $\pi : L \to M$ such that that π is locally trivial, i.e. there exists a cover $\{U_i\}$ of M such that $\pi^{-1}(U_i) = U_i \times \mathbb{R}$.

Alternatively, take a finite cover $M = \cup U_i$ of balls U_i and take a set of functions $\lambda_{ij} : U_i \cap U_j \to \mathbb{R}^*$ satisfying $\lambda_{ij}\lambda_{jk}\lambda_{ki} = 1$ and $\lambda_{ij} = \lambda_{ji}^{-1}$. Then these define a line bundle by mapping $U_i \times \mathbb{R} \equiv U_j \times \mathbb{R}$ on the overlap by identifying $(x, v) \sim (x, \lambda_{ij} v)$ for $x \in U_i \cap U_j$.

A section of a line bundle $s : M \to L$ is the analogue of a vector field. We can deform a section to intersect the zero section transversally and get a characteristic class that is in some sense independent of the section we choose. In what sense? Associated to each embedded circle in M, a section gives a number mod 2 (count zeros.) This number depends only on the bundle. In this case the characteristic class assigns 1 or 0 to each embedde circle. this is an example of a Stiefel-Whitney class of the line bundle.

Let $\mathbb{RP}^N$ be the manifold given by i-dimensional subspaces in $\mathbb{R}^{N+1}$. It possesses a natural line bundle $L \to \mathbb{RP}^N$ given by

$$L = \{(x, v) \in \mathbb{RP}^N \times \mathbb{R}^{N+1} \mid v \in x\}$$

called the canonical bundle.

Stiefel-Whitney classes. For any n-manifold M, we define the $n-k$th Stiefel Whitney class $w_{n-k}(TM)$ as the set of points where a generic set of k vector fields on M is linearly dependent. This defines a k-dimensional submanifold of M and we think of it as a function from $n-k$-dimensional submanifolds of M to $\mathbb{Z}_2$. More generally, if $E \to M$ is an r-dimensional bundle then $w_{r-k}(E)$ is obtained from the linear dependence set of k sections of E. Later we will give a definition which makes it clear that we get an $n-k$-dimensional submanifolds of M.

For the moment, here are some examples. Consider the canonical line bundle $L \to \mathbb{RP}^2$ defined by $L = \{(x, v) \in \mathbb{RP}^2 \times \mathbb{R}^3 \mid v \in x\}$. Let's calculate $w_1(L)$. The projective coordinates $[x_0 : x_1 : x_2]$ actually give sections of L. Take one such section x_0. It vanishes on $[0 : x_1 : x_2]$ which is $\mathbb{RP}^1 \subset \mathbb{RP}^2$ and it sends 1-dimensional submanifolds of $\mathbb{RP}^2$ to 0 or 1 depending on whether they are deformable to a point or to the natural $\mathbb{RP}^1 \subset \mathbb{RP}^2$.

Consider TS^2. We already know that $w_2(TS^2) \equiv 0$ since it counts the zeros of a vector field. For $w_1(TS^2)$ choose tow vector fields, one coming from a flow from the north pole to the south pole and the other from a flow from the east pole to the west pole. These are linearly dependent precisely along the great circle containing the four poles. This circle intersects any 1-dimensional submanifold in S^2 an even number of times so $w_1(TS^2)$ is trivial.

Here as an alternative definition of Syiefel-Whitney classes. Given an r-dimensional bundle $E \to M$, consider $E \otimes L \to M \times \mathbb{RP}^N$ where $L \to \mathbb{RP}^\infty$ is the universal bundle. Take a generic section s of $E \otimes L$ and restrict it to $M \times \mathbb{RP}^k$. The zero set $Z(s) \subset M \times \mathbb{RP}^k$ has dimension $n+k-r$ where $n = \dim M$ and E is an r-bundle. Project $Z(s)$ onto M to get an $(n+k-r)$-dimensional set or a cohomology class of dimension $r-k$. This is the Stiefel-Whitney class $w_{r-k}(E)$.

This coincides with the previous definition since if $k+1$ vector fields $\{v_0, v_1, ..., v_k\}$ are linearly dependent then there exists a tuple $\{y_0, ..., y_k\}$ such that $y_0 v_0 + ... + y_k v_k = 0$. Since the y_i are only projectively defined, they actually give a section of the canonical line bundle over $\mathbb{RP}^N$ so $y_0 v_0 + ... + y_k v_k \in \Gamma(E \otimes L)$. This view allows us to see that the zero set arises as a transversal intersection so generically we can arrange this and the mod 2 class we get is independent of any deformations. Note that if the projection of the zero set is not a submanifold we can perturb it to be that way.

4. Orientation, complex bundles and Chern classes

So far we have counted intersections mod 2 because that is all that is well-defined when we deform the intersecting manifolds. In the first lecture over surfaces we managed to count points with sign, so each point of intersection contributes +1 or -1, and get an integer that is well-defined as the intersecting manifolds are deformed. We expect to be able to do this in general. An *orientation* of a manifold allows us to do just that. It enables us to tell when a point should be counted with +1 or -1.

An orientation of a vector space V is an equivalence class of bases of V. Two bases are equivalent if the linear map that relates the two has positive determinant. An orientation of V is a choice of $+1$ for one equivalence class and -1 for the other. If $V = V_1 \oplus V_2$ then an orientation on two of these vector spaces induces an orientation on the third.

An orientation of a manifold is a (continuous) choice of orientation on each tangent space. Local orientations always exist, but global orientations might not.

If X^k and Y^{n-k} are two oriented submanifolds of an oriented manifold M^n then if they intersect transversally, the points of intersection can be counted with sign. At a point $p \in X \cap Y$ simply compare the orientation on T_pM with the induced orientation on $T_pX \oplus T_pY$. Examples: curve on a surface and $\overline{\mathbb{CP}}^2$.

We have already defined a general vector bundle to be $E \overset{\pi}{\to} M$ with the property that π is locally trivial: there exists a cover $\{U_i\}$ of M such that $\pi^{-1}(U_i) = U_i \times \mathbb{R}^k$. Alternatively, we can define E as a continuous family of k-dimensional subspaces $E_x \subset \mathbb{R}^N$ so $E \subset M \times \mathbb{R}^N$. If we replace the real vector space $\mathbb{R}^k$ by $\mathbb{C}^k$ (and $\mathbb{R}^N$ by $\mathbb{C}^N$) then we have a complex vector bundle. Examples are complex line bundles and a tangent bundle equipped with an almost complex structure.

A vector bundle is oriented if it is oriented as a manifold. For example, if M is an oriented manifold then its tangent bundle can be oriented. Any vector field v, after perhaps being deformed, gives rise to an isolated set of zeros that can be counted with sign.

Exercise. Show that this agrees with the method of giving a sign used in the first section.

Any complex vector space has a natural orientation given by $\{v_i, Jv_i\}$ for n vectors $v_i \in \mathbb{C}^n$. Thus, any complex bundle over an oriented manifold has a natural orientation.

We can now define Chern classes. The Chern classes of a complex bundle are defined analogously to Stiefel-Whitney classes where we now consider $E \otimes_{\mathbb{C}} L$ over $M \times \mathbb{CP}^\infty$. The canonical bundle L over $\mathbb{CP}^\infty$ is $\{(z, v) \in \mathbb{CP}^\infty \times \mathbb{C} \mid v \in z\}$. The tensor product is over $\mathbb{C}$ so we need a complex structure on the bundle E.

If E is a complex r-dimensional bundle we define $c_k(E)$ to be the projection onto M of the zero set of a generic section of $E \otimes L$ restricted to $\mathbb{CP}^{r-k}$. Equivalently, take $r - k + 1$ sections of E and $c_k(E)$ is the set in M where these sections are dependent. This characteristic class is a function on oriented $2k$-dimensional submanifolds of the oriented manifold M to the integers.

We will calculate the Chern classes of $T\mathbb{CP}^2$. Immediately $c_0(T\mathbb{CP}^2) = 1$, meaning that it maps any point in $\mathbb{CP}^2$ to 1 since any 5 vector fields are dependent at every point of $\mathbb{CP}^2$. We know $c_2(T\mathbb{CP}^2) = 3$ since it counts the number of zeros of a vector field and hence is the same as the Euler characteristic.

For $c_1(T\mathbb{CP}^2)$ we take two vector fields as follows. There is an action of $SU(3)$ on $\mathbb{CP}^2$ and the isotropy subgroup of each point is isomorphic to $U(2)$. The infinitesimal action of $su(3)$ on $\mathbb{CP}^2$ gives an identification of the four dimensional tangent space of a point with $su(3)/I_x$ for the isotropy subalgebra $I_x \cong u(2)$. Thus, an element of $su(3)$ defines a vector field on $\mathbb{CP}^2$. For $\xi \in su(3)$ the vector field at $[x_0 : x_1 : x_2]$ is given by $\xi \cdot x \in \mathbb{C}^3/x$. Consider the two vector fields arising from $diag(i, -i, 0), diag(0, i, -i) \in su(3)$. They are linearly dependent when $(x_0, -x_1, 0), (0, x_1, -x_2), (x_0, x_1, x_2)$ are linearly dependent. The determinant of these three vectors is $-3x_0x_1x_2$. This vanishes on the union of the three lines $x_0 = 0 = x_1 = x_2$. As a map from oriented 2-dimensional submanifolds to the integers it sends the standrad $\mathbb{CP}^1$ to 3. Each intersection is counted positively because the complex structure gives canonical orientations which agree.

5. Grassmannians and cohomology

Given $E \stackrel{\pi}{\to} Y$ let $f : X \to Y$. Then there is a bundle $f^*E \to X$. Its total space is given by $\{(e, x) \in E \times X \mid \pi(e) = f(x)\}$. Simply need to check that the composition map is locally a product. In terms of the construction $\bigcup U_i = Y$ and $\lambda_{ij} : U_i \cap U_j \to \mathbb{R}^*(\mathbb{C}^*)$, simply use $f^{-1}(U_i)$ and λ_{ij} defined on $f^{-1}(U_i) \cap f^{-1}(U_j)$.

The characteristic classes are natural meaning that they are preserved under any map $f : X \to Y$. On $\Sigma^k \subset X$ define $w_k(f^*E) \cdot \Sigma^k = w_k(E) \cdot f(\Sigma^k)$. Here we see the contravariant nature of characteristic classes. Easy examples are a map to a point, projection of a product, embeddings and the Hopf map.

There is a "classifying space" Y with bundle $\gamma \to Y$ such that for each $\mathbb{R}^n$ bundle $E \to X$ there exists a map $f : X \to Y$ such that $f^*\gamma = E$. The Stiefel-Whitney classes of γ are all known so this gives a way to calculate the Stiefel-Whitney classes of E.

The classifying space is given by the Grassmannian, $Gr(k, N)$, of k planes in $\mathbb{R}^N$ for $N >> k$ space. The canonical bundle $\gamma^n \to Y$ is given by $\{(z, v) \in Gr(k, N) \times R^N \mid v \in z\}$. Given an $\mathbb{R}^k$ bundle E over X we construct the map $f : X \to Gr(k, N)$ as follows. Embed the bundle E in $X \times R^N$. Then we have a map $f : X \to Gr(k, N)$ given by $x \mapsto E_x$. The pull-back of the canonical bundle clearly gives E.

If instead we choose $M > N$ then $Gr(k, N) \hookrightarrow Gr(k, M)$ and the canonical bundle pulls back to the canonical bundle so the pull-back of the Stiefel-Whitney classes is unaffected.

Associated to each manifold is a ring called the cohomology ring. It is a contravariant functor from manifolds to rings. Characteristic classes naturally live in this ring. The cohomology of the Grassmannian is described completely by the Stiefel-Whitney classes of the canonical bundle. Thus, instead of defining the cohomology ring, one can think of the characteristic classes of the canonical bundle. The ring structure, or cup product, comes from intersections and is most easily seen in these two examples.

We have calculated $w_1(T\mathbb{RP}^2)$ and $c_1(T\mathbb{CP}^2)$. In both cases we can produce a characteristic number from the cup product.

The analogous story works for Chern classes where the Grassmannian is now given by $Gr_{\mathbb{C}}(k, N)$ of complex subspaces $\mathbb{C}^k \subset \mathbb{C}^N$.

Theorem 1. *The cohomology ring $H^*(Gr(n, \infty); \mathbb{Z}_2)$ is a polynomial algebra over $\mathbb{Z}_2$ freely generated by the Stiefel-Whitney classes*

$$w_1(\gamma^n), ..., w_n(\gamma^n).$$

The proof requires showing that the Stiefel-Whitney classes are independent by constructing a bundle over any space with that independence, and then showing that the cohomology algebra they generate is already big enough to give all cohomology classes.

Theorem 2. *The cohomology ring $H^*(Gr_{\mathbb{C}}(n, \infty); \mathbb{Z})$ is a polynomial algebra over $\mathbb{Z}$ freely generated by the Chern classes $c_1(\gamma^n), ..., c_n(\gamma^n)$.*

6. Pontryagin classes and applications

Given a real vector bundle E, the Pontryagin classes are defined by almost as Chern classes via $L \to \mathbb{CP}^\infty$ and tensoring $E \otimes_{\mathbb{R}} L$. The significant difference is that the tensor product is over $\mathbb{R}$, not $\mathbb{C}$, and in particular the dimension of the budnle changes. One way to interpret this is that Pontryagin classes are Chern classes of the complexified bundle $E \otimes \mathbb{C}$ so they are given by families of vector fields restricted to each $\mathbb{CP}^k \subset \mathbb{CP}^\infty$ as described previously.

Equivalently, take $r - 2k + 1$ sections of E and $p_k(E)$ is the set in M where these sections have rank at most $r - 2k - 1$. This characteristic class is a function on oriented $4k$-dimensional submanifolds of the oriented manifold M to the integers.

Let's calculate $p_1(T\mathbb{CP}^2)$. Take three vector fields given in terms of three elements of $su(3)$. We will choose

$$(ix_0, -ix_1, 0), (0, ix_1, -ix_2), (x_1, x_0, 0)$$

and these together with (x_0, x_1, x_2) have to have rank 2. This occurs on the intersection of two sets of three lines. That's 9 points. The answer should be 3 so this says that 6 points of intersection will be counted positively and 3 negatively. That takes some thought.

Conjecturally, Pontryagin classes can be defined by using quaternionic projective space $\mathbb{HP}^\infty$. This projective space uses the left action of the quaternions on themselves. Again there is a canonical bundle given by $\{(q, v) \in \mathbb{HP}^\infty \times \mathbb{H} \mid v = wq\}$ for some $w \in \mathbb{H}^*$. This definition would require a quaternionic structure on E which is a restriction on the real bundle.

Characteristic classes have applications in showing if a manifold can be embedded into Euclidean space, whether a manifold possesses an orientation reversing diffeomorphism, whether a manifold is the boundary of a manifold and whether a manifold is a product of manifolds.

7. Differential forms, connections and curvature

This section seems difficult to digest quickly. Perhaps we will introduce the concepts throughout earlier lectures.

An important object in differential geometry is a differential form. Let's start with usual integration. We can make sense of $\int_a^b f(t)dt$ ofr any function f defined on $[a, b]$. This can be thought of as integration over a manifold. We would like to integrate over submanifolds. Consider a curve in the plane. If we have a function in the plane can we integrate it over the curve? Really, integration needs $\mathbb{R}^n$. So, parametrise the curve. Notice, however, that the integral depends on the parametrisation:

$$\int f(g(s))dg = \int f(s)g'(s)ds \neq \int f(s)ds$$

for a change of parametrisation g. It ends up that we cannot integrate functions over submanifolds but we can integrate differential forms like $f(s)ds = f(g)dg/g'(s)$. In three dimensions, a 1-form is given by three functions

$$f_1(x_1, x_2, x_3)dx_1 + f_2(x_1, x_2, x_3)dx_2 + f_1(x_1, x_2, x_3)dx_3.$$

Similarly, a 2-form (in three dimensions) is given by three functions

$$f_{12}(x_1, x_2, x_3)dx_1 \wedge dx_2 + f_{13}(x_1, x_2, x_3)dx_1 \wedge dx_3 + f_{23}dx_2 \wedge dx_3$$

and it can be integrated over surfaces in the manifold. Notice that a 2-form becomes a well-defined function when restricted to a surface. Thus, another way to think of a 2-form on a three-manifold is as a function that makes sense when you specify a surface (or even a plane of tangent vectors at a point). The expression $\wedge$ is used to show that the definition is orientation sensitive which is necessary in changes of coordinates.

If $M^n = \bigcup U_i$ is a finite chart for M, so U_i maps diffeomorphically to a subset of $\mathbb{R}^n$ although we will abuse this and treat $U_i \subset \mathbb{R}^n$, then a differential 1-form is given by $\eta_1(x)dx_1 + ... + \eta_n(x)dx_n$. On an overlap $U_i \cap U_j$ with respect to the local coordinates $y = (y_1, ..., y_n)$ in U_j there is a change of coordinates $\phi = (x_1(y), ..., x_n(y))$ so the differential form changes by

$$\eta_1(x)dx_1 + ... + \eta_n(x)dx_n \mapsto \sum_{i,j} \eta_i(\phi(y))\partial_{y_j} x_i dy_j. \tag{1}$$

In other words a differential form is a section of a vector bundle.

The exterior derivative is an operator d that acts on differential forms. Expressed in local coordinates: $d\eta_1(x)dx_1 = \sum_j \partial_{x_j}\eta_1(x)dx_j \wedge dx_1$ and this extends linearly to $\eta_1(x)dx_1 + ... + \eta_n(x)dx_n$. A function f on M can be thought of as zero-form and its derivative df as a 1-form. (This suggests the alternative view that at each point $p \in M$ a differential form is a multilinear form on the tangent space T_pM.)

If $d\eta = 0$, we say η is closed, and when integrated over a submanifold without boundary, the integral is invariant under deformations. This follows from Stokes theorem: $0 = \int_{\Sigma\times I} d\eta = \int_{\Sigma\times\{0\}} - \int_{\Sigma\times\{1\}}$ where $\Sigma \times I \mapsto M$ gives the deformation of a submanifold $\Sigma \hookrightarrow M$. Thus we see that differential forms act a little like intersection numbers. We will see that characteristic classes can be represented as differntial forms.

Let η and ξ be two differential k-forms and suppose there is a differential $(k-1)$-form ν such that $\eta - \xi = d\nu$. Then Stokes theorem says that if $\Sigma \subset M$ has no boundary then $\int_\Sigma \eta = \int_\Sigma \xi$ so we say that η and ξ are equivalent when integrated, or equivalent in cohomology.

Suppose that isntead of a differential form transforming as in (1) it changes as:

$$\sum_i \eta_i(x)dx_i \mapsto \sum_{i,j} \eta_i(\phi(y)) + d\nu \tag{2}$$

where an extra term has been added. So rather than the differential forms being equal after a change of coordinates they are equivalent when integrated, or in cohomology.

A complex line bundle gives such a transformation rule, where $d\nu = d\ln\lambda_{ij}$ is related to the transition functions that define the bundle. A *connection* is a generalised differential form associated to a bundle that satisfies (2 and any two connections differ by a differential 1-form. The curvature of a connection is a 2-form given by the exterior derivative of the 1-form which *is* well-defined. Any two connections give equivalent 2-forms, so to each complex line bundle there is an equivalence class (a cohomology class) of 2-forms.

Remember that given a complex line bundle, the zero set of any generic section assigns to a dimension 2 submanifold a number—count zeros. Similarly, any connection gives rise to a 2-form which integrates over a dimension 2 submanifold to get a number. Theorem: the latter number is 2π times the former. (In particular, the latter gives 2π times an integer.) This 2-form gives the first Chern class of the bundle.

For a sum of complex line bundles, there are many combinations of the 2-forms associated to each of the line bundles. Any symmetric polynomial in the 2-forms gives rise to a chacteristic class.

8. Characteristic numbers

This section is based on a paper of Raoul Bott *Vector fields and Characteristic numbers* Michigan mathematical journal 14, p231-244. The zeros of any vector field, counted properly, give the Euler characteristic of the manifold. Generalising this, a special type of vector field has the property that local invariants of the vector field near its zeros determine the Pontryagin numbers of the tangent bundle of the manifold.

One satisfying aspect of this approach is the definition of Chern classes for an endomorphism $A : V \to V$ of a finite-dimensional complex vector space (which enables us to appreciate the word characteristic in characteristic classes.) We define $c_i(A)$ to be the ith coefficient of the characteristic polynomial A, so

$$\sum \lambda^i c_i(A) = \det(1 + \lambda A).$$

Let $\Phi(c) = \Phi(c_1, ...c_m)$ $(\dim M = 2m)$ be a polynomial in the indeterminates c_i with complex coefficients. By replacing c_i with $c_i(M)$ we obtain $\Phi\{c(M)\}$ which evaluates to numbers when the intersection of the classes is zero dimensional. So the monomials that will contribute are of the form $c_1^{a_1} c_2^{a_2} ... c_n^{a_n}$ such that $a_1 + 2a_2 + ... + na_n = m$. We also define Φ on an endomorphism A by $\Phi(A) = \Phi(c_1(A), ..., c_m(A))$.

There is a Lie bracket on vector fields defined as follows. When we think of vector fields as derivations acting on functions, then two vector fields u and u combine to give $[u, v]f = u(v(f)) - v(u(f))$. Amazingly, there exists a vector field u such that $uf = [v, w]f$ for all functions f so we put $w = [u, v]$. The action of a vector field u on other vector fields by $v \mapsto [u, v]$ is called the Lie derivative.

In general, as a function of v, the vector $[u, v](x)$ depends on v in a neighbourhood of x. Whereas, at the zeros of v, its Lie derivative is a zero order map. This means that $u(p) = 0$ implies that $[u, v]$ depends only on $v(p)$ and thus $[u, \cdot] : T_pM \to T_pM$. This is because $[u, v]f = u(v(f)) - v(u(f)) = u_p(v(f)) - v_p(u(f)) = -v_p(u(f))$ where we have put u_p, v_p for the vector fields evaluated at the point and the second equality uses $u_p = 0$.

A manifold M can be equipped with a Riemmannian metric $g = \langle \cdot, \cdot \rangle$—a symmetric positive definite inner product (continuously) associated to the tangent space at each point of M. So two vectors $u, v \in T_pM$ map to $\langle u, v \rangle \in \mathbb{R}$. Examples: S^2, S^4, $\mathbb{CP}^2$.

The Lie derivative on vector fields enables us to "differentiate" a metric g on the manifold. A vector field u is an infinitesimal isometry, or a Killing vector field, if it satisfies $\langle [u, v], w \rangle + \langle v, [u, w] \rangle = u \cdot \langle v, w \rangle$ for all vector fields v and w.

At the each zero p of an infinitesimal isometry u, denote the induced linear map on the tangent space by $L_p : T_pM \to T_pM$. With respect to the metric, L_p satisfies the skew symmetry property: $\langle L_p v, w \rangle + \langle v, L_p w \rangle = 0$. In other words, with respect to an orthonormal basis, so that the inner product looks like the standard dot product, the linear map satisfies $L_p = -L_p^T$. Thus in odd dimensions $\det L_p = 0$ and in even dimensions the determinant is a square.

Theorem 3 (Bott)**.** *If the vector field u is an infinitesimal isometry on a compact, even-dimensional Riemannian manifold M whose zeros are nondegenerate and $\Phi(c_1, ..., c_m)$ is any polynomial of weight not greater than $m/2$, then*

$$\sum_p \Phi(L_p) / \overset{1/2}{\det}(L_p) = \Phi(M)$$

where the sum is over the zeros of u.