SYMPLECTIC TORIC VARIETIES.

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ABSTRACT. These are the notes of two lectures given at the Mini Spring School *An introduction to the mathematics of string theory* held at Adelaide University in November 2002. It is a leisurely introduction to the mathematics surrounding toric varieties.

LECTURE I.

Aims of Lecture I.

(i) To contrast topological, Riemannian, symplectic and complex structures;

(ii) to set up various topological objects that will be given Riemannian, symplectic and complex structures in the next lecture.

Toric varieties: a topological construction.

Toric varieties are simple examples of Riemannian, symplectic and complex manifolds. There won't be a lot of toric varieties in the first lecture, since I am more concerned with the basic mathematics that surrounds them. For that reason, I will begin with a short description of toric varieties.

Consider the following two examples consisting of an interval and a triangle.

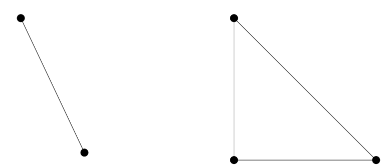

FIGURE 1. S^2 and $\mathbb{CP}^2$

Associate to each interior point of the interval a circle and to each vertex a point. Putting these together gives a two-sphere, S^2. The interval is the quotient of S^2 by rotations.

Associate to each interior point of the triangle a torus, T^2, and to each interior point of an edge a circle, and to each vertex a point. Putting these together gives the complex projective plane,

$$\mathbb{CP}^2 = \{(z_0, z_1, z_2) \in \mathbb{C}^3 - \{0\}\}/\{(z_0, z_1, z_2) \sim (\lambda z_0, \lambda z_1, \lambda z_2),\ \lambda \in \mathbb{C}^*\}.$$

We can see this via the map

$$(z_0, z_1, z_2) \mapsto \left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right)$$

which has image $\{(x, y)|x \geq 0,\ y \geq 0,\ x + y \leq 1\}$. The preimage of a point (x, y) is $|z_1|^2 = x,\ |z_2|^2 = y$ (when we set $|z_0|^2 + |z_1|^2 + |z_2|^2 = 1$) which is T^2 when (x, y)

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lies in the interior, and a circle or a point otherwise. The one-dimensional case lives inside, since the preimage of each edge is S^2. The image triangle is the quotient of $\mathbb{CP}^2$ by an action of the torus.

The construction generalises to higher dimensions. For any polyhedron, one would associate to each interior point a 3-torus, T^3, and then associate T^2, S^1, or a point to each of the points of a face as in the two-dimensional case. And so on, in higher dimensions.

Basic geometry.

Consider polygons in the plane. When are two polygons the same? Fix the number of sides. The answer is different depending on what structure we choose to put on the polygon. Topologically, every n-sided polygon looks the same, whereas if we can measure distance only triangles related by rigid motions look the same. The ability to measure area or angles allows the motions to have a flexibility somewhere between rigid and topological.

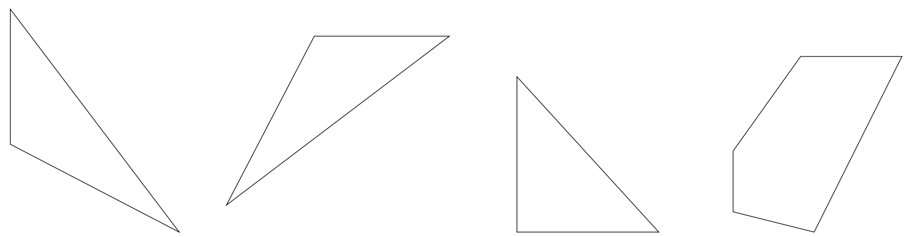

FIGURE 2. Polygons.

Rigid motions of triangles can move an edge to lie along the x-axis with one vertex at the orgin. The remaining freedom is the length of the edge and the coordinate of the vertex off the x-axis. This is a three-dimensional space of triangles, and we gain two-dimensions as each extra edge is added.

Linear area preserving maps are given by elements $g \in SL(2,\mathbb{R})$. In this case total area is the only freedom since under an element $g \in SL(2,\mathbb{R})$ any triangle can be moved to a right-angle isosceles triangle with edges along the axes. This gives a one-dimensional parameter space of triangles, and as we add sides this adds two dimensions to the parameter space. It is interesting that a linear map that preserves the total area necessarily preserves area locally.

Exercise 1. *Show that the area of a polygon with vertices on the integer lattice is given by $I+B/2-1$ for I and B the number of integer lattice points in the interior, respectively on the boundary, of the polygon.*

Angle preserving maps are governed by complex analysis. By the Riemann mapping theorem, there exists an angle-preserving map from any polygon to the unit disk in the plane, sending the vertices to points on the unit circle. (Angles are only preserved at interior points, so in particular the angles at the vertices are sent to 180 degrees.) Thus we may think of n-sided polygons as ordered n-tuples of points on the unit circle. For triangles, ordered triples of points on the unit circle are all equivalent under conformal, or angle-preserving, self maps of the disk, so any triangle is unique. As we add sides, or points on the unit circle, this adds one dimension to the parameter space. These results are summarised below.

topological
 0-dimensional family
distance preserving
 triangles $\Rightarrow$ 3-dimensional family
 quadrilaterals $\Rightarrow$ 5-dimensional family
area preserving and linear
 triangles $\Rightarrow$ 1-dimensional family
 quadrilaterals $\Rightarrow$ 3-dimensional family
angle preserving
 triangles $\Rightarrow$ 0-dimensional family
 quadrilaterals $\Rightarrow$ 1-dimensional family

We have just looked at simple examples of Riemannian, symplectic and complex geometry.

Definition 1. *A* Riemannian metric *is a smooth assignment of positive definite symmetric bilinear forms g_x to the tangent space of a manifold.*

Definition 2. *A* symplectic form *is a smooth assignment of non-degenerate skew-symmetric bilinear forms ω_x to the tangent space of a manifold.*

Definition 3. *An* almost-complex structure *is a smooth assignment of automorphisms J_x of the tangent space of a manifold that satisfy $J_x^2 = -I$.*

In these lectures, we will not be too concerned with the difference between an almost complex structure and a complex structure. The latter is an almost complex structure that arises from a complex manifold—a manifold with local charts given by open sets in $\mathbb{C}^n$ and complex coordinate charts.

Locally, a metric is more complicated than a symplectic form or a complex structure. At a point, after a change of basis, each of these can be put in standard form. Express the tangent space at a point as $V = W \oplus W$, since V is necessarily even-dimensional. Then the standard forms are

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{1}$$

respectively for a positive definite symmetric bilinear form, a non-degenerate skew-symmetric bilinear form, and an automorphism J, such that $J^2 = -I$.

Facts.

(i) If we take a smooth assignment from $U \subset \mathbb{R}^{2n}$ to the space of positive definite symmetric bilinear forms, then it is no longer true that the standard form can be realised continuously, whereas this can be done for symplectic and complex structures.
(ii) By combining a symplectic form ω and an almost complex structure J, we can get a metric $g(u, v) = \omega(u, Jv)$. We say that any two of these structures are *compatible.*
(iii) The first standard form in (1) is the product of the other two, which reflects fact (ii). In general, the metric cannot be locally constant, so this shows that the standard forms cannot usually hold concurrently.
(iv) Beginning with an almost complex structure, respectively a symplectic structure, by considering the space of all compatible metrics we can produce obstructions to two spaces being equivalent under complex, respectively symplectic, transformations. This is the idea behind conformal invariants, and Gromov's pseudoholomorphic curves. The metric serves an auxillary purpose.

(v) Both a symplectic form and a metric allow one to calculate the area of a submanifold surface. In two dimensions, they give the same thing. In higher dimensions this isn't the case. It is a little deceiving to say that a symplectic form calculates area. The boundary 2-sphere of a small 3-ball has non-zero area if one uses the metric and zero area if one uses the symplectic form. More generally, any zero-homologous surface has zero area using the symplectic form. When we combine symplectic, complex and Riemannian structures, the metric area dominates the symplectic area, with equality on a class of surfaces—symplectic submanifolds.

Next lecture we will want to put such geometric structures on the toric varieties that have been described only topological so far. Before that, we will describe another important topological construction.

Blow-ups.

Consider a point p on a surface Σ. Replace p with the family of tangent lines at p to get $\tilde{\Sigma}$. Is $\tilde{\Sigma}$ a manifold? Yes. In fact, it is homeomorphic to the manifold obtained by removing a disk from Σ and replacing the circle boundary with an $\mathbb{RP}^1$, i.e. identifying opposite points of the circle. The Euler characteristic (the quantity $F - E + V$ associated to a triangulation) changes by $\chi(\tilde{\Sigma}) = \chi(\Sigma) - 1$.

Recall the classification of surfaces without boundary. Essentially, there are two surfaces for even Euler characteristic and one surface for odd. Precisely, for each integer less than or equal to 1 there is a unique non-orientable surface with that Euler characteristic, and for each even integer less than or equal to 2 there is a unique orientable surface with that Euler characteristic.

Exercise 2. *The non-orientable surface of Euler characteristic -1 can be constructed by blowing up the torus at a point or blowing up the two-sphere at three points. Explicitly construct a homeomorphism between the two surfaces.*

One can alternatively describe the blow-up $\tilde{\Sigma}$ as the unique surface with a map $\pi : \tilde{\Sigma} \to \Sigma$ satisfying $\pi^{-1}(p) \cong \mathbb{RP}^1$ and the restriction $\pi : \tilde{\Sigma} - \pi^{-1}(p) \to \Sigma - \{p\}$ is a homeomorphism.

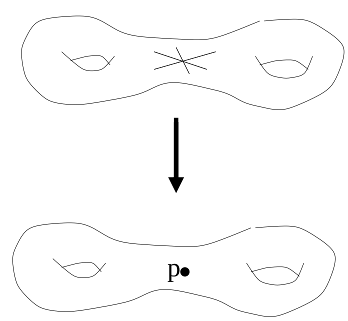

FIGURE 3. There is a map from the blow-up to the original surface.

This construction works in the complex world, and is better behaved. Notice in the real case we needed only define a blow-up at $0 \in \mathbb{R}^2$ since the construction is local, i.e. a point in a manifold is homeomorphic to $0 \in \mathbb{R}^2$. We will define the blow-up of $0 \in \mathbb{C}^2$ and this will apply to a point in any complex surface.

Define $\tilde{\mathbb{C}}^2 \subset \mathbb{C}^2 \times \mathbb{CP}^1$ by

$$\tilde{\mathbb{C}}^2 = \{(x, y, [z_0, z_1]) \in \mathbb{C}^2 \times \mathbb{CP}^1 | xz_1 = yz_0\}.$$

Projection onto the first coordinates gives the map $\pi : \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$ satisfying $\pi^{-1}(0) = \mathbb{CP}^1$ and the restriction $\pi : \tilde{\mathbb{C}}^2 - \pi^{-1}(0) \to \mathbb{C}^2 - \{0\}$ is an isomorphism. We have replaced $0 \in \mathbb{C}^2$ by the (complex) lines through 0, to get the blow-up of $\mathbb{C}^2$ at 0.
Facts:

(i) The blow-up is orientable.
(ii) When we blow up at $p \in X$ we call $\pi^{-1}(p) \subset \tilde{X}$ the exceptional divisor;
(iii) the construction can be described locally using two charts: replace the local coordinates (x, y) of a complex surface X by the coordinates $(x_0, y_0) = (x/y, y)$ in one chart and $(x_1, y_1) = (x, y/x)$ in another chart. The expressions $x = x_0 y_0$ and $y = y_0$ give the map from $\tilde{X}$ to X and the exceptional divisor is defined by $y_0 = 0$ (and the same can be done in the other chart.)
(iv) $S^2 \times S^2$ blown up at one point is isomorphic to $\mathbb{CP}^2$ blown up at two points.
(v) The blow-up operation works in all dimensions.

What do we do with the constructions of toric varieties and blow-ups? Put extra structure on them. For example, suppose we want to do a blow up in the presence of a metric or a symplectic structure. It seems a strange thing, the requirement that the exceptional divisor has non-trivial area.

Lecture II.

In the last lecture, I promised to put geometric structures on the topological constructions of symplectic varieties and blow-ups. Before I do that, I want to study a special class of algebraic varieties.

Fans.

A fan in the plane is a collection of rational convex cones in the plane. (A rational cone is spanned by rational slopes.) The integer lattice points in a fan represent monomials in $\mathbb{C}[x^{-1}, x, y^{-1}, y]$. The final three diagrams represent the rings

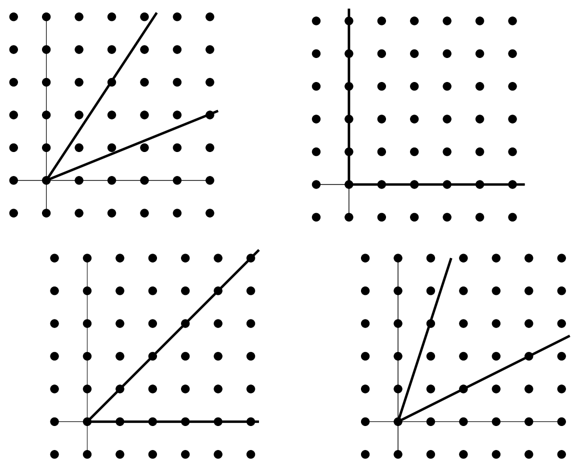

Figure 4. Cones.

$$\mathbb{C}[x, y],$$

$$\mathbb{C}[x, xy, xy^2] = \mathbb{C}[u, v, w]/\{uv - w^2\},$$

$$\mathbb{C}[xy, xy^2, xy^3, x^2y] = \mathbb{C}[t, u, v, w]/\{tv - u^2, t^3 - uw\}.$$

These are the ring of functions on affine varieties, or more accurately, local singularities.

Prove the statements in the following exercise, where we allow slopes of a cone to be irrational and rational.

Exercise 3. (i) *The integer lattice points in a fan form a semi-group.*
(ii) *The semi-group is finitely generated if and only if the slopes are rational.*
(iii) *Find the set of generators for the fan in the first diagram.*
(iv) *Calculate the number of generators of a fan with rational slopes p/q and r/s.*

The *dual* of a cone C is the cone

$$\hat{C} = \{v \in \mathbb{R}^2 | \langle u, v\rangle \geq 0, \ \forall u \in C\}.$$

The duals of the fans above are given in Figure 5. A collection of cones in the plane, a fan, produces a variety that is the union of affine varieties with rings of functions given by the dual cones. Consider the following three examples. The dual of a fan is the collection of dual cones. The dual fans for the first two fans

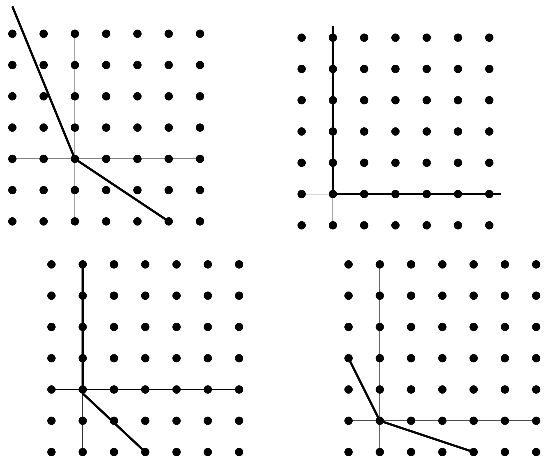

FIGURE 5. Dual cones.

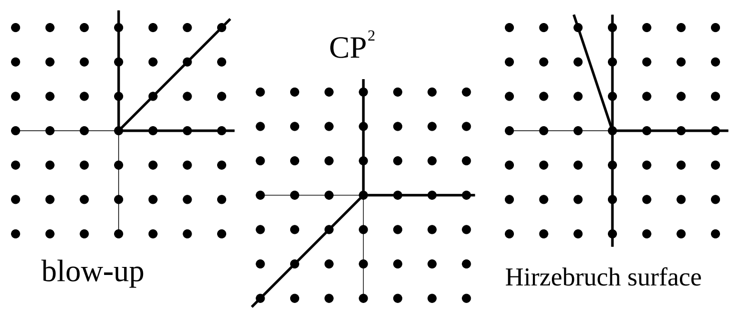

FIGURE 6. Fans.

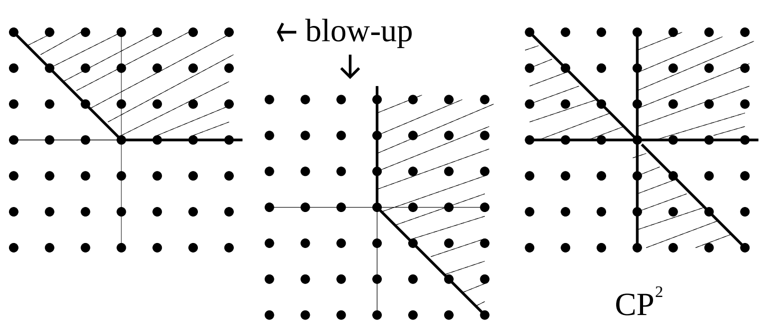

FIGURE 7. Dual fans.

in Figure 6 are given in Figure 7. The ring of functions is read off from these dual fans. In the first case, we have $\mathbb{C}[x, x^{-1}y] \cup \mathbb{C}[y, y^{-1}x]$. These both contain $\mathbb{C}[x, y]$ and give the two charts for the blow-up of $\mathbb{C}^2$ at 0. In the second case we have $\mathbb{C}[x, y] \cup \mathbb{C}[x^{-1}, x^{-1}y] \cup \mathbb{C}[y^{-1}, y^{-1}x]$ which gives the ring of functions on $\mathbb{CP}^2$ in its three charts.

Fans and polygons.

There is a relationship between polygons and fans. Let P be a polygon in the plane and Q be an edge or a vertex of P. Associate a cone to Q by:

$$\sigma_Q = \{v \in \mathbb{R}^2 | \langle u, v\rangle \leq \langle u', v\rangle,\ \forall u \in Q,\ u' \in P\}.$$

The union of these cones defines the fan associated to P. The polygons associated to the fans in Figure 6 are given below.

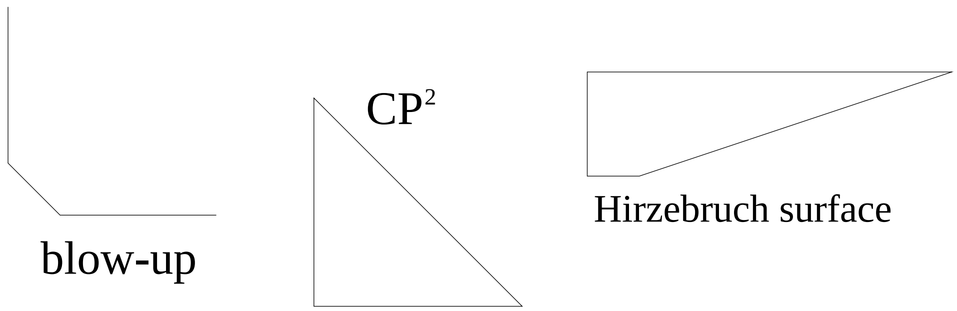

FIGURE 8. Polygons associated to fans.

Delzant polygons.

The polygons associated to fans are convex, the slopes of their boundaries are rational, and the slopes at each vertex satisfy an integrality and nondegeneracy condition. These properties are given below in the definition of a Delzant polygon.

We have mainly worked in the plane although almost everything generalises easily to all dimensions. In this section we will state results in full generality. One can always take the dimension to be 2 and polytopes to be polygons in each of the statements.

We need to describe the moment map on a symplectic manifold (M, ω) that admits an ω-preserving action of a group G.

Exercise 4. *Prove the following result due to Schur. Let M be an $n \times n$ Hermitian matrix with diagonal elements $\{a_1, a_2, \dots, a_n\}$ and eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Then $(a_1, a_2, \dots, a_n)$ lies in the convex hull of the points $(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \dots, \lambda_{\sigma(n)})$ where σ runs over all the permutations of n numbers.*

The group $U(n)$ acts on Hermitian matrices preserving eigenvalues. The orbits of the action are symplectic manifolds, and the diagonal elements give the moment map of the action of $T^n \subset U(n)$. Atiyah [2] and Guillemin and Sternberg [6] generalised this to any symplectic manifold.

Let G be a compact connected Lie group that acts on the symplectic manifold (M, ω) preserving the symplectic form. Any element $\xi \in \mathbf{g}$ defines a vector field on M, which in turn defines a Hamiltonian H_ξ that satisfies $dH_\xi(v) = \omega(\xi, v)$ for any $v \in T_pM$. Define the *moment map* $\mu : M \to \mathbf{g}^*$ by

$$(\mu(p), \xi) = H_\xi(p).$$

Theorem 1. *If $G = T^n$ acts on (M, ω) preserving the symplectic form then the image of μ is a convex polytope in $\mathbb{R}^n = \mathbf{t}^*$.*

We call such a symplectic manifold a *toric variety.* Delzant [3] completely characterised the convex polytopes arising from toric varieties. In fact, he showed that the polytopes and toric varieties determine each other.

Definition 4. *A convex polytope* Δ *in* $\mathbb{R}^n$ *is* Delzant *if:*

(1) *There are* n *edges meeting in each vertex* p.
(2) *The edges meeting in the vertex* p *are of the form* $p + tv_i, t \geq 0,\ v_i \in \mathbb{Z}^n$.
(3) *The* $v_1, \ldots, v_n$ *in (2) can be chosen to be a basis of* $\mathbb{Z}^n$.

Theorem 2. *Each Delzant polytope gives rise to a symplectic manifold* (M^{2n}, ω) *with an action of* T^n *that preserves* ω*, and all such symplectic manifolds arise this way.*

The proof of this result uses *symplectic quotients.* We will sketch it here.

Put $M_0 = \mu^{-1}(0) \subset M$. If G acts freely on M_0, then 0 is a regular point of μ, so M_0 is a closed submanifold of M of codimension $\dim G$. Put $B = M_0/G$. There exists a symplectic form ω_B on B satisfying $i^*\omega_M = \pi^*\omega_B$, where $\pi : M_0 \to B$ and $i : M_0 \to M$.

For our purposes, G will be a torus T^{d-n} and the symplectic manifold will be the space of λ_i's. Note that the torus should not be confused with the torus T^n acting on the toric variety.

Define a convex d-sided polygon P in the plane by

$$\langle \tilde{x}, u_i \rangle \geq \lambda_i$$

where $\tilde{x} = (x, y)$, $i = 1 \ldots d$, $u_i \in \mathbb{Z}^2$ is a primitive vector giving the inward pointing normal vector to the ith edge, and $\lambda_i \in \mathbb{R}$. Notice that the fan depends only on the u_i and not the λ_i. The symplectic structure does depend on the λ_i.

Consider the map

$$\pi : \mathbb{Z}^d \to \mathbb{Z}^n, \qquad e_i \mapsto u_i$$

that maps the standard basis of $\mathbb{Z}^d$ to the normal vectors in $\mathbb{Z}^n$. This extends to $\pi : \mathbb{R}^d \to \mathbb{R}^n$ and induces a map $\pi : T^d \to T^n$. The kernel of this map, T^{d-n} acts on $\mathbb{C}^d$ with moment map μ. In [5] it is proven that $\mu^{-1}(-\lambda_1, \ldots, -\lambda_n)/T^{d-n}$ is a symplectic manifold with an action of T^n. Furthermore, the polytope P is the image of its moment map.

Symplectic blow-ups.

The following definitions and results are taken from [5] where further references are given. Let X be the blow-up of $\mathbb{C}^n$ at the origin, described in the first lecture, given by $\pi : X \to \mathbb{C}^n$ and represent the exceptional divisor, $\pi^{-1}(0)$, by $i : \mathbb{CP}^{n-1} \to X$. Equip $\mathbb{C}^n$ with the standard symplectic form ω_0.

Definition 5. *A symplectic form* ω *on* X *is a* blow-up *of* ω_0 *if it is* $U(n)$*-invariant and* $\omega - \pi^*\omega_0$ *is compactly supported.*

Theorem 3. *Two blow-ups of* ω_0 *are equivalent under a* $U(n)$*-equivariant diffeomorphism if and only if* $i^*\omega_1 = i^*\omega_2$.

Theorem 4. *Given* $\delta > 0$ *there exists an* $\epsilon > 0$ *and an* ω *with the property that the volume of the exceptional curve is* ϵ *and* $\omega = \pi^*\omega_0$ *on the set* $|z| \geq \delta$.

It might seem disconcerting to have the symplectic structures agree off a small neighbourhood. The same type of thing happens with the Darboux theorem. One can see it explicitly in terms of Delzant polytopes. The blow-up $\tilde{P}$ of a Delzant polytope P replaces a vertex p by the n vertices $p + \epsilon v_i$ where the v_i are the basis of $\mathbb{Z}^n$ pointing along edges of P at p. Alternatively, one can use the moment map of a circle action ot get a one parameter family of symplectic manifolds giving ϵ blow-ups.

Kähler manifolds.

Kähler structures, the metric arising from compatible symplectic structures and complex structures, are of central importance to the subject.

A polytope determines a unique complex structure and a unique symplectic structure. It does not determine a unique Kähler structure. When we say that the complex structure is unique we mean *unique up to diffeomorphisms*. Similarly, the symplectic structure is unique up to diffeomorphisms. If we fix a complex structure, then equivalent symplectic structures gives rise to inequivalent Kähler structures.

There are two types of natural local coordinates on Kähler manifolds: complex (holomorphic) and symplectic (Darboux) coordinates. One can parametrise all compatible symplectic structures on a complex manifold via its potential whilst no such nice description exists for compatible complex structures. Abreu gave a nice description in the toric case [1], based on Guillemin's description of the Kähler potential of a toric variety in terms of combinatorial data in the polytope [4].

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