

# Zero and infinite curvature limits of hyperbolic monopoles.

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## Abstract

We show that the zero curvature limit of the space of hyperbolic monopoles gives the Euclidean monopoles, settling a conjecture of Atiyah. We also study the infinite curvature limit of the space of hyperbolic monopoles and show that the associated rational maps appear explicitly here.

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## 1 Introduction.

A monopole is a pair  $(A, \Phi)$  consisting of a connection and a section of the adjoint bundle of a trivial  $SU(2)$  bundle  $E$  over  $\mathbf{R}^3$  satisfying the Bogomolny equation

$$d_A \Phi = *F_A$$

where  $d_A$  is the covariant derivative defined by  $A$  and  $F_A$  is the curvature of  $A$ . We also require that  $(A, \Phi)$  have finite energy and  $\|\Phi\| \rightarrow 1$  as  $r \rightarrow \infty$ .

The Hodge star uses a metric on  $\mathbf{R}^3$ . We will consider hyperbolic and Euclidean metrics. Atiyah [1, 2] conjectured that the zero curvature limit of the space of hyperbolic monopoles yields the space of Euclidean monopoles. Chakrabarti [8] confirmed Atiyah's conjecture in the cases where he has explicit formulae for monopoles.

There are various ways of interpreting how the limit of the moduli space of hyperbolic monopoles should converge to the moduli space of Euclidean monopoles. One natural way is to use the fact that each of the moduli spaces is diffeomorphic to the space of rational maps from the two-sphere to itself [1, 10]. By fixing a rational map we can ask if the path of associated monopoles—one for each metric—is continuous. This is essentially what we do, except that we use a different correspondence between the moduli space of monopoles and the space of rational maps. Instead of framing the bundle at a point at infinity, we frame the bundle at  $0 \in \mathbf{R}^3$  or in other words require that gauge transformations are the identity at 0. The

dimension of this moduli space is larger than the dimension of the usual moduli space since there are no preferred frames at 0. With this condition on the gauge transformations we can associate to a monopole a rational map via radial scattering from the centre of hyperbolic or Euclidean space rather than scattering from infinity. In previous work [14, 15] it was shown that these moduli spaces correspond to the space of *unbased* rational maps from the two-sphere to itself which contrasts with the more usual space of based maps. This reflects the fact that there is no longer a distinguished direction in  $\mathbf{R}^3$ . We can now state the main result of this paper.

**Theorem 1** *The sequence of hyperbolic monopoles obtained by fixing a rational map and sending the curvature of hyperbolic space to zero converges to the corresponding Euclidean monopole.*

More precisely, we prove that any subsequence of this convergent sequence of hyperbolic monopoles possesses a further subsequence that converges uniformly on compact subsets.

Our approach uses a non-linear heat flow to construct the monopoles as well as to supply estimates for the monopoles. It ends up that the initial data for the flow corresponds in some sense to the infinite curvature limit of hyperbolic monopoles. The infinite curvature limit, or equivalently the renormalised zero mass limit, of hyperbolic monopoles is interesting in itself due to the work of Atiyah and Murray [4, 5] relating this limit to solutions of the Yang-Baxter equations.

In the next four sections we will summarise the results from [14, 16] regarding the construction of monopoles using the non-linear heat flow equation. We will only consider  $SU(2)$  monopoles here, although the method of proof works for any compact group  $G$ .

## 2 Monopoles and rational maps.

In this paper we will consider the following family of metrics:

$$\begin{aligned} ds_\kappa^2 &= \frac{\sinh^2(\kappa r)}{\kappa^2} \frac{4d\bar{w}dw}{(1+|w|^2)^2} + dr^2 \\ ds_0^2 &= r^2 \frac{4d\bar{w}dw}{(1+|w|^2)^2} + dr^2 \end{aligned}$$

where  $r$  gives the radial distance outwards from the centre and  $w$  is a holomorphic coordinate for the spheres of constant radius. The metric  $ds_\kappa^2$  is the hyperbolic metric of curvature  $-\kappa$  and  $ds_0^2$  is the Euclidean metric. Notice that as  $\kappa \rightarrow 0$ ,  $ds_\kappa^2$  converges to  $ds_0^2$  uniformly on compact sets.

Put  $d_A = \partial_w^A \cdot dw + \partial_{\bar{w}}^A \cdot d\bar{w} + \partial_r^A \cdot dr$ . Locally, the Bogomolny equation can be expressed by the following two equations:

$$[\partial_{\bar{w}}^A, \partial_r^A - i\Phi] = 0 \tag{1}$$

$$[\partial_{\bar{w}}^A, \partial_w^A] = \frac{-2i \sinh^2(\kappa r)}{\kappa^2(1 + |w|^2)^2} \partial_r^A \Phi \quad (2)$$

and in the Euclidean case the second equation is given by

$$[\partial_{\bar{w}}^A, \partial_w^A] = \frac{-2ir^2}{(1 + |w|^2)^2} \partial_r^A \Phi. \quad (3)$$

The asymptotic conditions on the monopole ensure along each radial geodesic the existence of a bounded solution of the scattering equation

$$(\partial_r^A - i\Phi)s = 0 \quad (4)$$

which is unique up to a constant factor. We can choose a family of solutions to (4) that depend smoothly on  $w$ . It follows from (1) that  $(\partial_r^A - i\Phi)\partial_{\bar{w}}^A s = 0$  so  $\partial_{\bar{w}}^A s = \xi(w)s$  for some  $\xi(w) \in \mathbf{C}$ . In particular,  $\partial_{\bar{w}} s(0) = \xi(w)s(0)$  since by the choice of coordinate system  $\partial_{\bar{w}}^A = \partial_{\bar{w}}$  at  $r = 0$ . In other words,

$$s(0) : \mathbf{CP}^1 \rightarrow \mathbf{CP}^1$$

is a holomorphic map.

### 3 Hermitian metrics.

We can use the scattering equation to produce a unique smooth frame  $g$  of  $E$  satisfying

$$(\partial_r^A - i\Phi)g = 0, \quad g(0) = I. \quad (5)$$

Since gauge transformations are required to be the identity at 0, they preserve these properties of  $g$ . A consequence of (1) is that  $g^{-1}\partial_{\bar{w}}^A g$  is independent of  $r$  and furthermore, since  $\partial_{\bar{w}}^A = \partial_{\bar{w}}$  at  $r = 0$  in any gauge, we have  $\partial_{\bar{w}}^A g = 0$ .

Define the Hermitian metric  $H = \bar{g}^T g$ . Notice that  $H$  is independent of the choice of gauge. By expressing the monopole in the gauge defined by  $g$  the monopole can be retrieved from  $H$  as follows:

$$\begin{aligned} \partial_r^A - i\Phi &= \partial_r, \quad \partial_{\bar{w}}^A = \partial_{\bar{w}}, \\ \partial_r^A + i\Phi &= \partial_r + H^{-1}\partial_r H, \quad \partial_w^A = \partial_w + H^{-1}\partial_w H. \end{aligned} \quad (6)$$

We will find it convenient to work with  $H$  instead of the monopole. We think of  $H$  as taking its values in the space  $SL(2, \mathbf{C})/SU(2)$  which comes equipped with the complete metric  $d$  given locally by  $tr(H^{-1}\delta H)$ . It is also useful to use

$$\sigma(H_1, H_2) = tr(H_1^{-1}H_2) + tr(H_1H_2^{-1}) - 4 \quad (7)$$

which satisfies  $c_1 d^2 \leq \sigma \leq c_2 d^2$  for positive constants  $c_1$  and  $c_2$ .

For any Hermitian metric  $H$ , define its Bogomolny tensor as follows:

$$\begin{aligned} B_\kappa(H) &= \partial_r(H^{-1}\partial_r H) + \frac{\kappa^2(1+|w|^2)^2}{\sinh^2(\kappa r)}\partial_{\bar{w}}(H^{-1}\partial_w H), \\ B_0(H) &= \partial_r(H^{-1}\partial_r H) + \frac{(1+|w|^2)^2}{r^2}\partial_{\bar{w}}(H^{-1}\partial_w H). \end{aligned}$$

If  $H$  arises from a monopole using the construction described above, then  $B_\kappa(H) = 0$  by (2) or (3).

In order to show that each rational map gives rise to a monopole we use the non-linear heat flow equation

$$H^{-1}\partial H/\partial t = B_\kappa(H), \quad H(w, r, 0) = \mathcal{H}_\kappa(w, r)$$

for any metric  $\mathcal{H}_\kappa(w, r)$  defined over  $\mathbf{R}^3$ . The non-compactness of  $\mathbf{R}^3$  and the fact that  $B_\kappa(H)$  blows up at  $r = 0$  forces us to instead consider a sequence of boundary value problems defined on  $\{r_0 \leq r \leq R\}$ .

$$\left. \begin{aligned} H^{-1}\partial H/\partial t &= B_\kappa(H) \\ H(w, r, t) &= \mathcal{H}_\kappa(w, r), \quad \{r = r_0\} \cup \{r = R\} \cup \{t = 0\} \end{aligned} \right\} \quad (8)$$

where the second condition includes the boundary condition throughout the flow as well as the initial state of  $H$ .

**Proposition 1** ([14, 16]) *There is a unique solution  $H_\kappa(w, r, t)$  of (8) converging to a metric  $H_\kappa(w, r, \infty)$  that is smooth on the interior  $\{r_0 < r < R\}$  and satisfies  $B_\kappa(H_\kappa(w, r, \infty)) = 0$ . Furthermore,*

$$d(H_\kappa(w, r, t), H_\kappa(w, r, 0)) \leq \int_0^\infty \min\{r, s\} |\mathcal{H}_\kappa(w, r)| ds.$$

The right choice of  $\mathcal{H}_\kappa(w, r)$  will guarantee that as we let  $r_0 \rightarrow 0$  and  $R \rightarrow \infty$  the sequence of limiting metrics will converge to a metric defined over all of  $\mathbf{R}^3$ .

## 4 Hyperbolic monopoles.

Let  $u : S^2 \rightarrow SU(2)$  be a lift of the rational map  $f : S^2 \rightarrow S^2$ . The lift is only defined up to a right action of  $U(1)$ . This ambiguity is used to patch together  $f$  across  $S^2$ . Nevertheless,

$$\mathcal{H}_\kappa(w, r) = u(w) \begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix} u^{-1}(w)$$

is well-defined. Now,  $|B_\kappa(\mathcal{H}_\kappa)| \leq C\kappa^2 r / \sinh^2(\kappa r)$  for some constant  $C$ , independent of  $\kappa$ . The factor of  $r$  comes from the fact that  $|B_\kappa(\mathcal{H}_\kappa)|$

vanishes at  $r = 0$  and is differentiable there. The other factors are made clear by the term

$$\mathcal{H}_\kappa^{-1} \partial_w \mathcal{H}_\kappa = u \begin{pmatrix} e^{2r} & 0 \\ 0 & e^{-2r} \end{pmatrix} u^{-1} \partial_w u \begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix} u^{-1} - \partial_w u \cdot u^{-1}$$

which is bounded since  $u^{-1} \partial_w u$  is lower triangular (by holomorphicity). So we get

$$d(H(w, r, t), H(w, r, 0)) \leq \int_0^\infty C \kappa^2 s \min\{r, s\} / \sinh^2(\kappa s) ds \quad (9)$$

and the right hand side is dominated by  $C \int_0^\infty \ln(1 - e^{-s}) ds / \kappa = C_1 / \kappa$  and vanishes as  $r \rightarrow 0$ .

Define

$$D_\kappa = -\partial_r^2 - \frac{\kappa^2(1 + |w|^2)^2}{\sinh^2(\kappa r)} \partial_{\bar{w}} \partial_w. \quad (10)$$

If  $H^{r_1, R}$  and  $H^{r_2, R}$  are the respective solutions of (8) for  $r_0 = r_1$  and  $r_0 = r_2$ , then  $\sigma(H^{r_1, R}, H^{r_2, R})$ , defined in (7) satisfies  $\sigma \geq 0$  and  $D_\kappa \sigma \leq 0$  on  $r_1 < r < R$  (for  $r_1 > r_2$ ). Since  $D_\kappa$  is elliptic,  $\sigma$  takes its maximum value on the boundary. At  $r = R$ ,  $\sigma = 0$  since  $H^{r_1, R}$  and  $H^{r_2, R}$  have the same boundary condition there. The maximum at  $r = r_1$  is given by a constant times (9), since the boundary value of  $H^{r_1, R}$  gives the initial metric for the flow used to produce  $H^{r_2, R}$ . For fixed  $R$ , as  $r_0 \rightarrow 0$  the expression in (9) tends to 0 so we have a Cauchy sequence that converges to a metric  $H_\kappa^R$  on the ball  $r \leq R$ .

Now we will show that as  $R \rightarrow \infty$ , the metrics  $H_\kappa^R$  converge uniformly on compact subsets to a metric  $H_\kappa$  that satisfies  $B_\kappa(H_\kappa) = 0$ . For  $R_1 > R$  consider  $\sigma(H_\kappa^R, H_\kappa^{R_1})$ . Notice that  $D(\sigma(r) - cr) \leq 0$  for any constant  $c$ . If we choose  $c = \sup_{r=R} \sigma / R$ , then  $\sigma - cr \leq 0$  at  $r = 0$  and  $r = R$ . By the maximum principle,  $\sigma(r) \leq r \sup_{r=R} \sigma / R$  for  $r \leq R$ . Again (9) shows that  $\sup_{r=R} \sigma$  is bounded as  $R \rightarrow \infty$ . Thus

$$d(H_\kappa^R(w, r), H_\kappa^{R_1}(w, r)) \leq Cr/R$$

so as  $R \rightarrow \infty$ ,  $\{H_\kappa^R\}$  converges uniformly on compact subsets. The convergence can be improved to  $C^\infty$  [9, 18] so the limiting metric has vanishing Bogomolny tensor. This produces a hyperbolic monopole defined on  $\mathbf{R}^3 - 0$ . A removable singularities theorem [17] enables us to conclude that the connection is smooth on all of  $\mathbf{R}^3$ .

Notice that as  $\kappa \rightarrow \infty$ , the metrics  $H_\kappa(w, r)$  converge uniformly to the initial metric,  $\mathcal{H}_\kappa(w, r)$ . This is the infinite curvature limit. The metric  $\mathcal{H}_\kappa(w, r)$  is associated to a ‘‘monopole’’ on hyperbolic space of infinite curvature as follows. For a lift  $u$  of the rational map, put

$$u^{-1} \partial_{\bar{w}} u = \begin{pmatrix} \lambda(w) & \mu(w) \\ 0 & -\lambda(w) \end{pmatrix}$$

and define  $(A, \Phi)$  by

$$A_{\bar{w}} = u \cdot \begin{pmatrix} \lambda(w) & e^{-2r} \mu(w) \\ 0 & -\lambda(w) \end{pmatrix}, \quad \Phi = u \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad (11)$$

$$A_w = -\bar{A}_{\bar{w}}^T, \quad A_r = 0$$

where  $A$  and  $\Phi$  transform under  $u$  respectively as a connection and an endomorphism. We need to include the transformation  $u$  for two reasons. It is not the identity at  $r = 0$  so it doesn't qualify as a gauge transformation and, although the lift  $u$  is not unique, (11) only depends on the rational map. The pair  $(A, \Phi)$  comes from (6) for  $H = \mathcal{H}_\kappa(w, r)$  and satisfies (1) and

$$\partial_r^A \Phi = 0 \quad (12)$$

which is the limit of (2) as  $\kappa \rightarrow \infty$ . We have not shown that in the infinite curvature limit the monopoles converge to pairs of the form (11). Rather, we have proven a weak analogue of Theorem 1.

**Theorem 2** *Consider the sequence of hyperbolic monopoles obtained by fixing a rational map and sending the curvature of hyperbolic space to infinity. Then the corresponding Hermitian metrics converge uniformly to the Hermitian metric corresponding to the infinite curvature “monopole”.*

## 5 Euclidean limit.

Consider the sequence of hyperbolic monopoles obtained by fixing a rational map and sending the curvature of hyperbolic space to zero. In order to show that this sequence converges to the Euclidean monopole corresponding to the rational map, we will begin with a result that gives a uniform bound on the curvature of a monopole depending only on its charge. It will follow that any sequence of hyperbolic monopoles possesses a subsequence that converges to a Euclidean monopole.

**Lemma 1** *Any hyperbolic monopole  $(A, \Phi)$  satisfies*

$$|F_A(w, r)|, |d_A \Phi(w, r)| < C \cosh(\kappa r)$$

where  $r$  is the hyperbolic distance from the centre of  $H^3$  and  $C$  is a constant depending only on the charge of  $(A, \Phi)$ .

**Proof.** We can rescale the hyperbolic monopole by setting

$$(\tilde{A}_w, \tilde{A}_{\bar{w}}, \tilde{A}_r, \tilde{\Phi})(w, r) = (A_w, A_{\bar{w}}, A_r/\kappa, \Phi/\kappa)(w, r/\kappa)$$

so  $(\tilde{A}, \tilde{\Phi})$  is a hyperbolic monopole of mass  $1/\kappa$  over hyperbolic space of curvature  $-1$ . The monopole  $(\tilde{A}, \tilde{\Phi})$  defines a  $U(1)$ -invariant instanton on  $S^4 - S^2$  with respect to the round metric on  $S^4$  of charge  $2k/\kappa$  where  $k$  is the charge of the monopole. The conformal change from the product

metric on  $H^3 \times S^1$  to the round metric changes the length of each  $S^1$  in the product from  $2\pi$  to  $2\pi/\cosh(\kappa r)$ . This yields the fact that the charge over a ball in  $S^4$  centred at  $(w, r, 0) \in H^3 \times S^1$  of radius  $\delta = \epsilon\kappa\pi/2k \cosh(\kappa r)$  is less than  $\epsilon$ . Choosing  $\epsilon$  to be that required by Uhlenbeck's theorem [11] we get

$$|F_{\tilde{A}}(w, r)| < c/\delta^2 = c' \cosh(\kappa r)/\kappa^2$$

for a constant  $c'(k)$ . Thus

$$|F_A(w, r)| = \kappa^2 |F_{\tilde{A}}(w, r/\kappa)| < c' \cosh(\kappa r) .$$

This gives uniform bounds on the curvature of monopoles at interior points as  $\kappa \rightarrow 0$ .  $\square$

**Remark.** This uniform bound on the curvature contrasts with a similar result for Euclidean monopoles [13]. In that case, the bound is uniform in  $r$  but depends on the monopole whereas in this case it is uniform over the space of monopoles but depends on  $r$ .

The uniform bound on the curvature of a sequence of monopoles combined with the fact that the sequence of hyperbolic metrics converges uniformly on compact subsets to the Euclidean metric implies that there is a subsequence of connections that converges smoothly on compact subsets to a Euclidean monopole. It may be the case, however, that the charge of the sequence of monopoles tends towards the sphere at infinity so in the limit we get the trivial monopole. This seems likely since we obtain monopoles over hyperbolic space of small curvature by renormalising monopoles of large mass over usual hyperbolic space. The normalisation involves a scaling which pushes the charge of each monopole out towards the sphere at infinity. In other words, if the limit is to converge to a non-trivial Euclidean monopole, then the unrenormalised limit should concentrate at  $0 \in \mathbf{R}^3$ . This should not be too surprising since if we were to calculate the centre of any Euclidean monopole using the hyperbolic construction [6] it would be  $0 \in \mathbf{R}^3$ .

We will show that for a fixed rational map  $\{H_\kappa\} \rightarrow H_0$  where  $H_\kappa$  is the Hermitian metric associated to the monopole over hyperbolic space of curvature  $\kappa$  and  $H_0$  is the (non-trivial) Hermitian metric associated to the Euclidean monopole. The fact that the underlying monopoles converge smoothly to a Euclidean monopole implies that the sequence  $\{H_\kappa\}$  converges uniformly. A priori we could get convergence of  $\{H_\kappa\}$  to the trivial Hermitian metric  $H \equiv I$  (corresponding to the trivial monopole) but in the rest of this section we will prove that this cannot in fact arise.

In the previous section we defined  $H_\kappa^R$  to be the unique Hermitian metric over the ball of radius  $R$  in  $\mathbf{H}^3$  that satisfies  $B_\kappa(H) = 0$ ,  $H(0) = I$  and the boundary condition given in (8). No such problems with the zero curvature limit on this compact set occur so we get immediately that  $\{H_\kappa^R\} \rightarrow H_0^R$  uniformly because the underlying connections converge

smoothly since the curvature is uniformly bounded and hence cannot concentrate. Now  $H_\kappa^R \rightarrow H_\kappa$  uniformly on compact subsets as  $R \rightarrow \infty$ . If this convergence is *uniform in  $\kappa$*  then by taking a diagonal sequence we see that  $\{H_\kappa\} \rightarrow H_0$  as required. To appreciate the issue here we will describe what it would mean for the charge of the connections to move out towards infinity. In that case, each sequence  $H_\kappa^R \rightarrow H_\kappa$  would converge uniformly on compact subsets as  $R \rightarrow \infty$ , however the convergence would get slower as  $\kappa \rightarrow 0$ . Thus a diagonal sequence could easily degenerate to a trivial metric.

We have reduced the problem to showing that given  $\epsilon > 0$ , there exists an  $R_\epsilon$  large enough so that for  $R > R_\epsilon$ ,  $d(H_\kappa^R, H_\kappa) < \epsilon$  for all  $\kappa$ . Unfortunately the initial conditions in the heat flow used in Section 4 supply estimates that do not satisfy this uniform bound. We have to specify better initial conditions for the heat flow for  $H_\kappa^R$ . These are given to us from a combination of the Euclidean monopole heat flow and the following explicit formula for symmetric monopoles [7] in the radially free gauge:

$$\begin{aligned} A_w &= \frac{\kappa \sinh((\kappa+2)r) - (\kappa+2) \sinh(\kappa r)}{\kappa \sinh((\kappa+2)r)(1+|w|^2)} u(w) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} u^{-1}(w), \\ \Phi &= \frac{1}{2} \left( \frac{\kappa+2}{\tanh((\kappa+2)r)} - \frac{\kappa}{\tanh(\kappa r)} \right) u(w) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} u^{-1}(w), \\ A_{\bar{w}} &= -\bar{A}_w^T, \quad A_r = 0, \end{aligned} \quad (13)$$

where  $-\kappa$  is the curvature of hyperbolic space and

$$u(w) = \frac{1}{(1+|w|^2)^{1/2}} \begin{pmatrix} w & -1 \\ 1 & \bar{w} \end{pmatrix}.$$

Notice that in the limit  $\kappa \rightarrow \infty$ , away from  $r = 0$ , (13) converges uniformly to (11). As  $\kappa \rightarrow 0$ , (13) converges uniformly on compact sets to the Prasad-Sommerfield monopole.

For the Euclidean monopole heat flow, put

$$\mathcal{H}_0 = u(w) \begin{pmatrix} e^{-2r} r^k \chi^k(r) & 0 \\ 0 & e^{2r} r^{-k} \chi^{-k}(r) \end{pmatrix} u^{-1}(w)$$

where  $\chi(r)$  is a smooth cut-off function satisfying  $r\chi(r) = 1$  near  $r = 0$  and  $\chi(r) = 1$  near  $r = \infty$ . Then  $|B_0(\mathcal{H}_0)| \leq C(r)/r^2$  where  $C(r) = O(r)$  as  $r \rightarrow 0$  and  $C(r) = O(1)$  as  $r \rightarrow \infty$ . For the solution of (8) using the Euclidean metric we get

$$d(H(w, r, t), H(w, r, 0)) \leq \int_0^\infty C(s) \min\{r, s\} / s^2 ds$$

and this vanishes as  $r \rightarrow 0$  and is  $O(\ln(r)) = o(r)$  as  $r \rightarrow \infty$ . Thus, we again get a Cauchy sequence when we fix  $R$  and let  $r_0 \rightarrow 0$ . As  $R \rightarrow \infty$  we also get a Cauchy sequence since for  $R_1 > R$

$$d(H^R(w, r), H^{R_1}(w, r)) \leq Cr \ln(R)/R$$



so convergence is uniform on compact subsets. Nontriviality of the limit follows from the fact that the limiting metric is at most  $\ln(r)$  away from the initial metric, so in particular, unbounded.

We are now in a position to give a better set of initial conditions for hyperbolic monopoles. Put

$$\mathcal{H}_\kappa(w, r) = u(w) \begin{pmatrix} \xi(r) & 0 \\ 0 & \xi^{-1}(r) \end{pmatrix} u^{-1}(w)$$

for  $\xi(r) = e^{-(2+\kappa k)r} (\chi_\kappa(r) \sinh(\kappa r) / \kappa)^k$ . The cut-off function satisfies

$$\chi_\kappa(r) e^{-\kappa r} \sinh(\kappa r) / \kappa = 1, \quad r \leq 1/2$$

$$\chi_\kappa(r) = 1, \quad r \geq 3/2$$

and is smooth in between. Also,  $\chi_\kappa(r) \leq \chi_{\kappa'}(r)$  for  $\kappa > \kappa'$ . The two inequalities

$$e^{-\kappa r} \sinh(\kappa r) / \kappa \leq r \quad \text{and} \quad \partial_r^2 \ln(e^{-\kappa r} \sinh(\kappa r) / \kappa) \leq \partial_r^2 \ln(r)$$

guarantee that the convergence  $H_\kappa^R$  as  $R \rightarrow \infty$  is dominated by the convergence in the Euclidean case and hence is uniform in  $\kappa$ .

It follows that the limit of the sequence of hyperbolic monopoles with fixed rational map tends to the corresponding Euclidean monopole as the curvature of hyperbolic space tends to zero. We have shown only that there exists a subsequence that satisfies this. It then follows that the entire sequence converges to the corresponding Euclidean monopole since any subsequence possesses a further subsequence that converges to this unique limit. Thus, Theorem 1 is proven.

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