Moments of the Laguerre β Ensembles

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Abstract

Random matrix theory, in the most general sense, is the study of matrices with random entries, with some constraints on these entries. Carefully choosing 'nice' constraints leads to many types of random matrix theories, with some key cases having a rich theory that is deeply entrenched in the literature. In this thesis, we review the Gaussian β ensembles before moving onto the Laguerre β ensembles. The symbol β denotes a positive real number indexing the ensembles, with the further significance that the special cases $\beta = 1$, 2, and 4 correspond to the orthogonal, unitary, and symplectic symmetry respectively. Our review focuses on the cases $\beta = 1$, 2, and 4, with brief mention of the general case. In our treatment of the Laguerre β ensembles, we use the loop equation formalism to investigate the resolvents, which relate to the moments of the eigenvalue densities. We then investigate the special case $\beta = 2$ through differential equations for the densities and resolvents. The discussion draws on the literature where similar techniques have been applied to the Gaussian β ensembles.

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Chapter 1

Preliminaries

1.1 Introduction

This thesis is directed at an audience whose knowledge is akin to that of the author when he first embarked on this literary journey. That is, a rudimentary understanding of probability theory, combinatorics, complex analysis, and differential equations; and a vague grasp on concepts from statistical and quantum physics. Moreover, this thesis would be of interest to those who wish to see a demonstration of the interplay between these topics or are looking for a broad and simple introduction to random matrix theory – the author often details ideas related to the subject at hand. Those who are concerned with multivariate stochastics and statistics, mathematical physics, wireless communications, finance, and other forms of data analysis may also be interested in the content.

We begin with a discussion of ensembles. Consider a statistical system and let V be the set of all possible states that this system could be in. The traditional interpretation of a probability density function (p.d.f.) on such a statistical system is that, for any measurable subset U of V, the p.d.f. tells us the probability that the system is in a state that belongs to U. When the set of states is difficult to write down,

sometimes it is easier to instead consider a similar set called an ensemble. Like our set V above, an ensemble is a collection of all possible states that a statistical system could be in. The distinction is that every state in the ensemble is also assigned a weight which dictates how probable it is for the system to be in the given state. Physicists like to interpret the ensemble corresponding to a system as a virtual collection of infinitely many copies of the system, with copies in a more probable state appearing relatively more often than those that are in a less probable state. Sampling copies out of this virtual collection ad infinitum will reveal the underlying p.d.f.

Consider a statistical system S with corresponding ensemble A. When we say that X is an element of A, we could mean one of two things.

- Rarely, we may mean that X is an actual realisation of the system generated according to the appropriate p.d.f.

- Usually, X is instead taken to be a random variable which is an undetermined representative of the ensemble A – it has no fixed value. Given a measurable subset B of A, the probability that S is in a state that belongs to B is equal to the probability that $X \in B$. In this way, the random variable X is synonymous with the system S itself. In the following, we use this meaning.

The ensemble A has an associated measure $\mathrm{d}\mu : A \to \mathbb{R}$ such that, for a state $s \in A$, $\mathrm{d}\mu(s)$ is the weight of the state. Let B_1 and B_2 be measurable subsets of A such that the probability that $X \in B_1$ is equal to the probability that $X \in B_2$. Then, $\int_{B_1} \mathrm{d}\mu = \int_{B_2} \mathrm{d}\mu$. For this measure, there is a *partition function* given by

$$Z := \int_A \mathrm{d}\mu.$$

The partition function calculates the total measure of all of the states in A, and acts as a normalisation constant. It is of fundamental importance in physical applications which we shall remain ignorant of; it is not itself an observable. The partition

function allows us to form a *probability measure* $\mathrm{d}\nu = \frac{1}{Z}\mathrm{d}\mu$: If B is a measurable subset of A, then the probability that $X \in B$ is given by $\int_B \mathrm{d}\nu$. Of course, $\int_A \mathrm{d}\nu = 1$.

Let K be an operator dependent on the system S with corresponding ensemble A. For example, if the system involves only a single particle, K could be the position or momentum of the particle; if the system is a gas in a domain A, K could be the temperature or pressure of the gas. The *ensemble average* of K is

$$\langle K \rangle := \int_A K(X)\,\mathrm{d}\nu(X) = \frac{1}{Z}\int_A K(X)\,\mathrm{d}\mu(X).$$

In this setting, when K consists of scalar functions, we interpret them as the operation equivalent to multiplying by said scalar. For example, for $K = \frac{\partial}{\partial x} f(x)$, $K(X)$ means, "multiply X by $f(x)$, and then take the partial derivative of the resulting product with respect to x".

1.1.1 Random Matrix Ensembles

We now introduce random matrix ensembles, where the underlying systems are matrices.

Definition 1.1. A *random matrix ensemble* is a set of matrices which satisfy some constraints and are weighted according to some p.d.f. It is commonplace to define a random matrix ensemble by specifying a distribution on the entries of its representative random matrix. Then, the p.d.f. on the ensemble is equal to the joint probability density function (j.p.d.f.) formed by multiplying the p.d.f.s on the independent elements of said random matrix. The *independent elements* of a matrix are the real components of the entries which are independent of all other such components.

We are interested in real, complex, and quaternion random matrix ensembles because, according to the Frobenius theorem [Frobenius, 1878], [Artz, 2009, p. 26], every finite-dimensional associative division algebra over the real numbers is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Hence, only these types of random matrix ensembles correspond to irreducible matrix algebras over the real numbers [Dyson, 1962, p. 1202].

In the proceeding constructions, we shall treat real, complex, and quaternion random matrices simultaneously, by utilising the Dyson index β [Dyson, 1962, pp. 1214-1215]. Given a random matrix, the Dyson index counts the maximal number of real components in the entries. As complex numbers can be written in the form $x + y\mathrm{i}$ for $x, y \in \mathbb{R}$, and quaternions can be written in the form

$$a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \equiv \begin{bmatrix} a + b\mathrm{i} & c + d\mathrm{i} \\ -c + d\mathrm{i} & a - b\mathrm{i} \end{bmatrix} \text{ for } a, b, c, d \in \mathbb{R},$$

$\beta = 1$ corresponds to real matrices, $\beta = 2$ to complex matrices, and $\beta = 4$ to quaternion matrices. These β values come into play due to the fact that the p.d.f. of a given random matrix depends on the p.d.f.s of the independent elements. The Dyson index may be used as an adjective. e.g. β matrix, β ensemble.

A reasonable candidate for initial study is the random matrix whose entries are independent and identically distributed (i.i.d.) according to the standard Gaussian distribution.

Definition 1.2. The *real ($\beta = 1$), complex ($\beta = 2$), and quaternion ($\beta = 4$) $M \times N$ random matrix ensembles of standard Gaussian matrices* are represented by *$M \times N$ standard Gaussian β matrices G* with i.i.d. entries such that the real components of the entries are Gaussian with mean 0 and standard deviation $\frac{1}{\sqrt{\beta}}$. For each entry z, the j.p.d.f. of the entry is given by $\left(\frac{\beta}{2\pi}\right)^{\frac{\beta}{2}} \exp\left(-\frac{\beta}{2}|z|^2\right)$, where $|z|$ is the standard norm of z. In the case that $M = N$, these are called the *Ginibre β ensembles* [Ginibre, 1965].

The Ginibre β matrices are not self-adjoint. In the case $M = N$, the eigenvalues will typically be complex. Nonetheless, simple combinations of Ginibre β matrices, involving either a sum or a product, allows for the construction of self-adjoint matrices. These matrices lead to interesting ensembles, our first example of which are termed the Gaussian β ensembles.

1.2 The Gaussian β Ensembles

As already remarked, one may argue that it is intuitive to use the Ginibre β ensembles as stepping stones to create ensembles of self-adjoint random matrices. We first do this in possibly the most obvious way that also enables relatively simple yet insightful calculations.

Definition 1.3. Fixing β as 1, 2, or 4, let G be an element of the $N \times N$ Ginibre β ensemble as in definition 1.2. Then, let $X = \frac{1}{2}(G^\dagger + G)$, where $G^\dagger$ is the adjoint of G; in the real case, $G^\dagger$ is simply the transpose G^T of G, while in the complex case, $G^\dagger$ is the conjugate transpose $\overline{G}^T$ of G. In the quaternion case, we interpret G as a $2N \times 2N$ matrix with the quaternions as 2×2 blocks. Then $G^\dagger = Z_{2N} G^T Z_{2N}^{-1}$, where Z_{2N} is the $2N \times 2N$ block diagonal matrix with the 2×2 blocks

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

down the diagonal, and 0s everywhere else; if $N = 1$, then $G^\dagger G$ is a real number times the identity matrix I_2. The $N \times N$ *Gaussian β ensemble (GβE)* is the ensemble of self-adjoint random matrices represented by X [Forrester, 2010, pp. 1,11,12].

For fixed $\beta = 1$, 2, or 4, let X be an element of the Gaussian β ensemble. Then X is self-adjoint. Moreover, X has real diagonal entries drawn from the Gaussian distribution of mean 0 and standard deviation $\frac{1}{\sqrt{\beta}}$, and upper triangular entries consisting of β real components, each drawn from the Gaussian distribution of mean 0 and standard deviation $\frac{1}{\sqrt{2\beta}}$. Due to the self-adjoint nature of X, the lower triangular entries depend on the upper triangular entries, and X has real eigenvalues.

For $\beta = 1$, 2, or 4, the Gaussian β ensembles have specific names: As the ensembles consist of self-adjoint matrices, they can be diagonalised by unitary matrices. Let X be an element of the $N \times N$ Gaussian β ensemble. The aforementioned unitary matrices will have entries from the same field as X, and will thus form the matrices groups of orthogonal, unitary, and symplectic matrices, respectively. This explains

why the G1E is also called the *Gaussian orthogonal ensemble (GOE)*, the G2E is called the *Gaussian unitary ensemble (GUE)*, and the G4E is called the *Gaussian symplectic ensemble (GSE)*.

Considering the number of physical models that involve self-adjoint matrices, it isn't difficult to imagine the applications of the Gaussian β ensembles. For example, in the theory of quantum mechanics, all physical observables are self-adjoint operators. In particular, the GOE, GUE, and GSE are used to model quantum Hamiltonians that have time reversal symmetry with an even number of spin $\frac{1}{2}$ particles, no time reversal symmetry, and time reversal symmetry with an odd number of spin $\frac{1}{2}$ particles, respectively [Forrester, 2010, pp. 3,11,12]. Thus the existence of a time reversal symmetry restricts the generically complex entries of the matrix representation of the Hamiltonian. The simplest case is when the time reversal is represented by complex conjugation, telling us that the entries must be real and the relevant ensemble is the GOE. Related to this viewpoint is a celebrated application of the GUE to the statistics of the zeroes of the Riemann ζ function [Berry and Keating, 1999] (see subsection 1.2.2).

For another example, consider an ecology of N species such that in a pairing (i, j), species i and j are of equal detriment to each other. Then, in the style of [May, 1972] one may model the evolution of the ecology about equilibrium by constructing a symmetric $N \times N$ random matrix whose (i, j) element represents the stochastic effect of species j on species i.

It is moreover typical in statistical mechanics [Thomson, 1971] that there are physical regimes where all couplings (i, j), as above, can be weighted equally – this is referred to as a mean field approximation. The Gaussian β ensembles have this feature and can thus be thought of as mean field models. The main observables of interest involve the eigenvalues; for example, in the quantum mechanics applications, the eigenvalues represent the energy levels (see subsection 1.2.2).

These applications naturally motivate us to explore the eigenvalue densities of

the Gaussian β ensembles. Moreover, most applications of the Gaussian β ensembles reveal themselves upon calculation of the eigenvalue densities.

1.2.1 The Eigenvalue Densities of the Gaussian β Ensembles

Let X be a member of the $N \times N$ Gaussian β ensemble, for some $\beta = 1$, 2, or 4. Let $x_{i,j}$ denote the (i,j) entry of X, so that $X = [x_{i,j}]$. We denote the s^{th} real component of $x_{i,j}$ by $x_{i,j,s}$, with $s = 1$ corresponding to the real part. As the diagonal entries of X are real, $x_{i,i} = x_{i,i,1}$ for $1 \leqslant i \leqslant N$. Moreover, each $x_{i,i}$ has p.d.f. $\left(\frac{\beta}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{\beta}{2}x_{i,i}^2\right)$, and for $1 \leqslant i < j \leqslant N$, each $x_{i,j,s}$ has p.d.f. $\left(\frac{\beta}{\pi}\right)^{\frac{1}{2}} \exp\left(-\beta x_{i,j,s}^2\right)$. As X is self-adjoint, these are all of the independent elements of X. Hence, the j.p.d.f. of the independent elements of X is given by

$$p^{(G)}(X) := \left(\frac{\beta}{2\pi}\right)^{\frac{N}{2}} \left(\frac{\beta}{\pi}\right)^{\frac{N(N-1)\beta}{4}} \exp\left(-\frac{\beta}{2}\sum_{i=1}^{N} x_{i,i}^2 - \beta \sum_{1\leqslant j<k\leqslant N} \sum_{s=1}^{\beta} x_{j,k,s}^2\right). \tag{1.2.1}$$

This leads to the probability measure $p^{(G)}(X)(\mathrm{d}X)$ where

$$(\mathrm{d}X) := \prod_{i=1}^{N} \mathrm{d}x_{i,i} \prod_{1\leqslant j<k\leqslant N} \prod_{s=1}^{\beta} \mathrm{d}x_{j,k,s} \tag{1.2.2}$$

is the product of the independent elements of $\mathrm{d}X := [\mathrm{d}x_{i,j}]$. To obtain the eigenvalue density of the Gaussian β ensembles, we must change variables in $p^{(G)}(X)(\mathrm{d}X)$ to the eigenvalues and quantities related to the eigenvectors of X. We calculate the Jacobian via *metric forms* [Forrester, 2010, p. 9], [Hitchin, 2012, pp. 76-79].

Metric Forms

Let M be a smooth manifold (e.g. $\mathbb{R}^n$) with coordinates $(x_1, \ldots, x_n)$. At each point $p \in M$, there is a *tangent space* T_pM with basis $\left(\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\right)$. The *tangent bundle* is $TM = \coprod_{p\in M} T_pM$. Moreover, at each point $p \in M$, there is a *cotangent space* T_p^*M that is dual to the tangent space T_pM. The cotangent space has basis $(\mathrm{d}x_1|_p, \ldots, \mathrm{d}x_n|_p)$, where $\mathrm{d}x_i|_p\left(\frac{\partial}{\partial x_j}|_p\right) = \delta_{ij}$. The *cotangent bundle* is $T^*M = \coprod_{p\in M} T_p^*M$.

Definition 1.4. A *Riemannian metric* on M is a function $g : TM \times TM \to \mathbb{R}$ which assigns to each $p \in M$ an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$ on the tangent space at p. As g_p acts on $T_pM \times T_pM$, it lives in $T_p^*M \times T_p^*M$. Hence, $g_p = \sum_{i,j=1}^n g_{p_{ij}} \, \mathrm{d}x_i|_p \, \mathrm{d}x_j|_p$, where $g_{p_{ij}} = g_p\left(\frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p\right)$. Globally, $g = \sum_{i,j=1}^n g_{ij}(p) \, \mathrm{d}x_i \, \mathrm{d}x_j$, with smooth functions $g_{ij} : M \to \mathbb{R}$ given by $g_{ij}(p) = g_p\left(\frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p\right)$. The *metric tensor* corresponding to g is the $n \times n$ symmetric matrix $[g_{ij}]$.

Thus far we have been working locally, with our $\mathrm{d}x_i|_p$ being basis vectors of the tangent space at p. Henceforth, for a manifold M, we use the notation $\mathrm{d}x_i$ to represent a global notion of these basis vectors. The differential 1-form $\mathrm{d}x_i$ is defined smoothly over all of M and is interpreted locally as $\mathrm{d}x_i|_p$ at each point $p \in M$.

Definition 1.5. Let M be a manifold with coordinates $(x_1, \ldots, x_n)$, and let g be a Riemannian metric on M. Then the *volume form* on M induced by g is

$$\mathrm{d}V := \sqrt{|\det[g_{ij}]|}\mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

The volume form is usually defined as an exterior product using wedges $\wedge$ between the differentials $\mathrm{d}x_i$, in order to account for orientation. We drop these wedges, and instead choose to take the positive orientation in all cases. Let A be a measurable subset of M. Then $\int_A \mathrm{d}V$ is the volume of A, according to the metric g.

Change of Coordinates

We think of the $N \times N$ GβE as $\mathbb{R}^{N+\frac{\beta}{2}N(N-1)}$ with coordinates $x_{i,i}$ and $x_{j,k,s}$ in some unspecified ordering, with $1 \leqslant i \leqslant N$, $1 \leqslant j < k \leqslant N$, and $1 \leqslant s \leqslant \beta$. We give this space the Riemanninan metric

$$g := \sum_{i=1}^{N} (\mathrm{d}x_{i,i})^2 + \sum_{1 \leqslant j < k \leqslant N} \sum_{s=1}^{\beta} 2(\mathrm{d}x_{j,k,s})^2 \tag{1.2.3}$$

so that the induced volume form is

$$\mathrm{d}V = 2^{\frac{\beta}{4}N(N-1)} \prod_{i=1}^{N} \mathrm{d}x_{i,i} \prod_{1 \leqslant j < k \leqslant N} \prod_{s=1}^{\beta} \mathrm{d}x_{j,k,s}.$$

Referring back to (1.2.2), we see that $(\mathrm{d}X) = 2^{-\frac{\beta}{4}N(N-1)}\mathrm{d}V$. We choose this metric because we wish to take advantage of the fact that the above g is equal to $\mathrm{Tr}(\mathrm{d}X\,\mathrm{d}X^\dagger)$.

Proposition 1. *Let $X = [x_{i,j}]$ be an element of the $N \times N$ GβE. If an entry $x_{i,j}$ has multiple real components, we denote its s^{th} real component $x_{i,j,s}$. As X is self-adjoint, we may diagonalise it as $X = UDU^\dagger$, where D is the diagonal matrix of eigenvalues λ_j of X, and U is the matrix with columns $\mathbf{u_j} = [u_{i,j}]_{1\leqslant i\leqslant N}$, the corresponding eigenvectors of X; U is orthogonal in the real case, unitary in the complex case, and unitary symplectic in the quaternion case. We stipulate that the first component of each eigenvector be real and positive, and that the eigenvalues be ordered, so that the diagonalisation is a bijection. Then the Riemannian metric defined above* (1.2.3) *is given by*

$$g = Tr(\mathrm{d}X\,\mathrm{d}X^\dagger) = \sum_{i=1}^{N}(\mathrm{d}\lambda_i)^2 + \sum_{1\leqslant j<k\leqslant N}\sum_{s=1}^{\beta} 2(\lambda_k-\lambda_j)^2(\omega_{j,k,s})^2,$$

where the differential 1*-form $\omega_{i,j,s}$ is the s^{th} real component of the (i,j) entry of $U^\dagger\mathrm{d}U$.*

Proof. We differentiate both sides of $X = UDU^\dagger$ to obtain

$$\begin{aligned}\mathrm{d}X &= \mathrm{d}U\,DU^\dagger + U\mathrm{d}D\,U^\dagger + UD\mathrm{d}U^\dagger,\\ U^\dagger\mathrm{d}X\,U &= U^\dagger\mathrm{d}U\,D + \mathrm{d}D + D\mathrm{d}U^\dagger U\\ &= U^\dagger\mathrm{d}U\,D + \mathrm{d}D - DU^\dagger\mathrm{d}U,\end{aligned}$$

where we have used the fact that $U^\dagger U = UU^\dagger = I_N$, the $N \times N$ identity matrix, and thus $0 = \mathrm{d}(I_N) = \mathrm{d}(U^\dagger U) = U^\dagger\mathrm{d}U + \mathrm{d}U^\dagger\,U$. These matrices have the form $\mathrm{d}D = \mathrm{diag}(\mathrm{d}\lambda_1,\ldots,\mathrm{d}\lambda_N)$ and $\mathrm{d}U = [\mathrm{d}\mathbf{u_1}\cdots\mathrm{d}\mathbf{u_N}]$, so the (i,j) entry of $U^\dagger\mathrm{d}X\,U$ is given by $\left[(\lambda_j-\lambda_i)\mathbf{u_i}^\dagger\mathrm{d}\mathbf{u_j} + \delta_{i,j}\mathrm{d}\lambda_i\right]$. Moreover, $\left(U^\dagger\mathrm{d}X\,U\right)^\dagger = U^\dagger\mathrm{d}X^\dagger U$. As the trace is invariant under cyclic permutation of its arguments, we thus have that

$$\begin{aligned}\mathrm{Tr}(\mathrm{d}X\,\mathrm{d}X^\dagger) &= \mathrm{Tr}(U^\dagger\mathrm{d}X\,UU^\dagger\mathrm{d}X^\dagger\,U)\\ &= \mathrm{Tr}\left((U^\dagger\mathrm{d}X\,U)(U^\dagger\mathrm{d}X\,U)^\dagger\right).\end{aligned}$$

The i^{th} diagonal entry of $(U^\dagger \mathrm{d}X\, U)(U^\dagger \mathrm{d}X\, U)^\dagger$ is

$$(\mathrm{d}\lambda_i)^2 + \sum_{\substack{j=1 \\ j\neq i}}^{N} (\lambda_j - \lambda_i)^2 (\mathbf{u_i}^\dagger \mathrm{d}\mathbf{u_j})(\mathbf{u_i}^\dagger \mathrm{d}\mathbf{u_j})^\dagger.$$

As $\mathbf{u_i}^\dagger \mathrm{d}\mathbf{u_j}$ is the (i,j) entry of $U^\dagger \mathrm{d}U$, we have that $(\mathbf{u_i}^\dagger \mathrm{d}\mathbf{u_j})(\mathbf{u_i}^\dagger \mathrm{d}\mathbf{u_j})^\dagger = \sum_{s=1}^{\beta} (\omega_{i,j,s})^2$. Moreover, $(U^\dagger \mathrm{d}U)^\dagger = -U^\dagger \mathrm{d}U$, so $(\omega_{i,j,s})^2 = (\omega_{j,i,s})^2$ for $i \neq j$, $1 \leqslant s \leqslant \beta$. Hence,

$$\begin{aligned}
\mathrm{Tr}(\mathrm{d}X\, \mathrm{d}X^\dagger) &= \sum_{i=1}^{N} (\mathrm{d}\lambda_i)^2 + \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \sum_{s=1}^{\beta} (\lambda_j - \lambda_i)^2 (\omega_{i,j,s})^2 \\
&= \sum_{i=1}^{N} (\mathrm{d}\lambda_i)^2 + \sum_{1\leqslant j<k\leqslant N} \sum_{s=1}^{\beta} 2(\lambda_k - \lambda_j)^2 (\omega_{j,k,s})^2,
\end{aligned}$$

where we have grouped the upper and lower triangular terms. □

We've already seen that the volume form induced by g is $\mathrm{d}V = 2^{\frac{\beta}{4}N(N-1)}(\mathrm{d}X)$. Here, we see that we may assign new coordinates relating to the eigenvalues λ_i and eigenvectors $\mathbf{u}_i$ of X and write the Riemannian metric g in terms of these new coordinates. With respect to these new coordinates, the metric tensor is a diagonal matrix with N diagonal entries equal to 1, and the remaining $\frac{\beta}{2}N(N-1)$ diagonal entries given by $2(\lambda_k - \lambda_j)^2$ for all combinations $1 \leqslant j < k \leqslant N$ and $1 \leqslant s \leqslant \beta$. Hence, the volume form induced by g is

$$\mathrm{d}V = 2^{\frac{\beta}{4}N(N-1)} \prod_{i=1}^{N} \mathrm{d}\lambda_i \prod_{1\leqslant j<k\leqslant N} (\lambda_k - \lambda_j)^\beta \prod_{s=1}^{\beta} \omega_{j,k,s}.$$

Writing $(U^\dagger \mathrm{d}U)$ for the product of independent elements $\omega_{j,k,s}$ of $U^\dagger \mathrm{d}U$ and using absolute values so that, if required, we may relax the requirement of the eigenvalues being ordered, we thus have that

$$(\mathrm{d}X) = \prod_{i=1}^{N} \mathrm{d}\lambda_i \prod_{1\leqslant j<k\leqslant N} |\lambda_k - \lambda_j|^\beta (U^\dagger \mathrm{d}U). \tag{1.2.4}$$

It remains to change coordinates in $p^{(G)}(X)$ (1.2.1). Retaining our notation from proposition 1,

$$-\frac{\beta}{2}\sum_{i=1}^{N} x_{i,i}^2 - \beta \sum_{1\leqslant j<k\leqslant N}\sum_{s=1}^{\beta} x_{j,k,s}^2 = -\frac{\beta}{2}\mathrm{Tr}(XX^\dagger) = -\frac{\beta}{2}\mathrm{Tr}(D^2) = -\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i^2,$$

where we have used the facts that X is self-adjoint so $XX^\dagger = X^2$, that $U^\dagger U = I_N$, and that the trace is invariant under cyclic permutations of its arguments. Hence our probability measure is

$$p^{(G)}(X)(\mathrm{d}X) = \left(\frac{\beta}{2\pi}\right)^{\frac{N}{2}} \left(\frac{\beta}{\pi}\right)^{\frac{N(N-1)\beta}{4}} e^{-\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i^2} \prod_{1\leqslant j<k\leqslant N} |\lambda_k - \lambda_j|^\beta \prod_{i=1}^{N} \mathrm{d}\lambda_i (U^\dagger \mathrm{d}U). \tag{1.2.5}$$

Remark 1. The probability measure $p^{(G)}(X)(\mathrm{d}X)$ is invariant under orthogonal, unitary, and symplectic transformations, for $\beta = 1, 2$, and 4 respectively: Let Q be a fixed orthogonal, unitary, or symplectic matrix, whichever is appropriate. Let $Y = QXQ^\dagger$, where X is as above. Then, $Y = \tilde{U}D\tilde{U}^\dagger$ where $\tilde{U} = QU$. As $\tilde{U}^\dagger \mathrm{d}\tilde{U} = U^\dagger \mathrm{d}U$ and X and Y share eigenvalues, we immediately obtain that $p^{(G)}(Y)(\mathrm{d}Y) = p^{(G)}(X)(\mathrm{d}X)$.

In our expression for $p^{(G)}(X)(\mathrm{d}X)$ (1.2.5), we see that the eigenvalue and eigenvector dependence decouples into a product. Thus, we obtain our eigenvalue j.p.d.f. by simply integrating out the eigenvector dependence.

Definition 1.6. The $N \times N$ *GβE eigenvalue j.p.d.f.* is

$$p^{(G)}(\lambda_1, \ldots, \lambda_N; \beta) := \frac{1}{G_N(\beta)} \prod_{i=1}^{N} e^{-\beta\lambda_i^2/2} \prod_{1\leqslant j<k\leqslant N} |\lambda_k - \lambda_j|^\beta, \tag{1.2.6}$$

where $G_N(\beta) = \left(\frac{1}{2\pi}\right)^{N/2} \prod_{i=1}^{N} \frac{\Gamma(1+\beta/2)}{\Gamma(1+i\beta/2)}$ is a normalisation constant [Dumitriu and Edelman, 2002, p.5831]. Here, Γ is the Gamma function, and we no longer constrain the eigenvalues to be ordered, so that $\lambda_i \in \mathbb{R}$ for all $1 \leqslant i \leqslant N$.

While it is possible to calculate $G_N(\beta)$ by integrating $(U^\dagger \mathrm{d}U)$, we provide a computation via the Selberg integral in subsection 1.4.1. Indeed, this will give an indirect method for evaluating $\int (U^\dagger \mathrm{d}U)$, if required.

Definition 1.7. The $N \times N$ *GβE eigenvalue density* is

$$\rho_{(1)}^{(G)}(\lambda; \beta, N) := N \int_{\mathbb{R}^{N-1}} p^{(G)}(\lambda, \lambda_2, \ldots, \lambda_N)\, \mathrm{d}\lambda_2 \ldots \mathrm{d}\lambda_N, \tag{1.2.7}$$

with support $\mathbb{R}$. Let X be a member of the $N \times N$ GβE. By definition of the density, the expected number of eigenvalues of X in the interval $[a, b]$ is given by $\int_a^b \rho_{(1)}^{(G)}(\lambda; \beta, N)\, \mathrm{d}\lambda$.

Remark 2. Orthogonal polynomials are invaluable tools for working with the eigenvalue densities of random matrix ensembles; see remark 5 in subsection 1.4.1. For now, we note that for β even, the eigenvalue density of the $N \times N$ GβE is a polynomial multiplied by factors of $e^{-\beta\lambda_i^2/2}$, for each eigenvalue λ_i, $1 \leqslant i \leqslant N$. We recall that the Hermite polynomials are orthogonal with respect to the weight $e^{-x^2/2}$. It is for this reason that the Gaussian β ensembles are also called the *Hermite β ensembles*.

1.2.2 Applications of the Gaussian β Ensembles

We now present a few examples of the applications that the Gaussian β ensembles have enjoyed, extending the discussion at the beginning of this section. For now, we forego discussion of combinatorial interpretations, instead mentioning them in subsections 1.4.3 and 1.4.4.

Heavy Atom Spectra

In 1955, Eugene P. Wigner introduced random matrices to the field of nuclear physics in [Wigner, 1955]. This work was based off of [Lane et al., 1955], where the energy levels of heavy nuclei were investigated. The idea is to treat the Hamiltonians of the nuclei as real symmetric random matrices of large dimension, and to define the so-called *strength function* as a function of the eigenvectors. Under certain conditions, the integral of the strength function over an interval gives the average strength of absorption by all energy levels within said interval. Moreover, Wigner shows that the strength function is actually the eigenvalue density of the Hamiltonians.

The initial paper treated relatively simple random matrix ensembles. This raised the question of which type of random matrix ensemble could best model heavy atom spectra. Thus began the investigation of many types of ensembles, with the Gaussian β ensembles a favourite example. Foundational to this theme were a series of papers by Dyson, and Dyson and Mehta, written in the early 1960s. In particular, it was Dyson [Dyson, 1962] who introduced Gaussian ensembles in correspondence with global time reversal symmetry, as we already noted in subsection 1.1.1. Wigner provided a review [Wigner, 1967] which presents some of these advances, as applied to the statistics of the energy levels of U^{239}. Yet again, Hamiltonians are taken to be random matrices of large dimension, with the eigenvalues representing the energy levels of U^{239}.

Log-Gases

Consider N particles of charge $+1$ on an infinitely long and thin conducting line, say the x-axis, at positions $x_1, \ldots, x_N$. Suppose that they repel according to 2-dimensional electrostatics confined to the xy-plane. Equivalently, think of infinitely long parallel charged lines in the z-direction which intersect the xy-plane at $x_1, \ldots, x_N$; in this case, repulsion via traditional 3-dimensional electrostatics is treated as 2-dimensional due to symmetry. For the time being, we allow our particles to have unfixed y-ordinates. Then for $1 \leqslant i, j \leqslant N$, the electrostatic potential Φ_p at (x_i, y_i) due to the particle at (x_j, y_j) is given by the solution to the 2-dimensional Poisson equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Big|_{(x_i,y_i)} \Phi_p\left((x_i, y_i), (x_j, y_j)\right) = -2\pi\delta(x_i - x_j)\delta(y_i - y_j),$$

where δ is the Dirac delta function (distribution) [Forrester, 2010, pp.20-21]. By differentiating the result when $(x_i, y_i) \neq (x_j, y_j)$ and otherwise using the divergence theorem in the plane, it can be seen that $\Phi_p\left((x_i, y_i), (x_j, y_j)\right) = -\log|(x_i - x_j, y_i - y_j)|$ up to some scaling, which we choose to be trivial – hence the term *log-gas*. Then, by the superposition principle, the total potential energy of the gas due to

particle-particle interactions is given by

$$U_{p.p} = -\sum_{1\leqslant i<j\leqslant N} \log|x_i - x_j|,$$

where we've again restricted to the x-axis by setting $y_1 = \ldots = y_N = 0$.

To prevent the particles from repelling off to infinity, we trap them in a harmonic well so that the background potential Φ_b at x_i is given by $\Phi_b(x_i) = \frac{1}{2}x_i^2$. Thus the total potential energy of the gas is

$$U = \tfrac{1}{2}\sum_{i=1}^{N} x_i^2 + U_{p.p} = \tfrac{1}{2}\sum_{i=1}^{N} x_i^2 - \sum_{1\leqslant j<k\leqslant N} \log|x_j - x_k|.$$

A fundamental postulate of statistical mechanics is that the j.p.d.f. for the particles to be at positions $x_1, \ldots, x_N$ is $\hat{Z}_N^{-1}\exp\left(-\frac{1}{k_B T}U\right)$, where k_B is the Boltzmann constant, T is the temperature, and $\hat{Z}_N$ is a normalisation constant. But then our j.p.d.f. is exactly $p^{(G)}(x_1, \ldots, x_N; \beta)$ as given in (1.2.6), with $\beta = \frac{1}{k_B T}$ the inverse temperature.

This seems to suggest a meaning to general $\beta \in \mathbb{R}_{>0}$. After all, our eigenvalue j.p.d.f. (1.2.6) is sensible for $\beta \in \mathbb{R}_{>0}$. It turns out that there exists random matrix ensembles with eigenvalue j.p.d.f. (1.2.6) for $\beta \in \mathbb{R}_{>0}$. See [Forrester, 2010, pp.43-48] for a full account, and subsection 1.3.1 for the Laguerre analogue.

The Riemann Hypothesis

The Hilbert-Pólya conjecture states that the non-trivial zeroes of the Riemann ζ function correspond to the eigenvalues of $\frac{1}{2}\mathrm{id} + \mathrm{i}H$ for some unbounded self-adjoint operator H [Derbyshire, 2003, pp.277-278]. The proof of this conjecture would immediately prove the Riemann hypothesis, as such an H would admit only real eigenvalues. Montgomery's pair correlation conjecture states that the 2-point correlation function of the non-trivial zeroes of ζ is identical to that of the eigenvalues of the GUE [Montgomery, 1973, p.184]. It has been numerically shown that the spacing distribution of the non-trivial zeroes of ζ is statistically equal to the spacing distribution of the eigenvalues of the GUE [Odlyzko, 1987].

1.3 The Laguerre β Ensembles

In the previous section we studied Gaussian β ensembles whose elements were of the form $\frac{1}{2}(G^\dagger + G)$ for standard Gaussian β matrices G. We stated that this was an obvious method of introducing Hermitian structure to the ensemble of standard Gaussian matrices. However, there is an equally obvious method that we have circumvented thus far, due to the relative complexity of the computations involved. Now that we have familiarised ourselves with random matrices, and have beared witness to their applications, we construct the Laguerre β ensembles.

Definition 1.8. Fixing β as 1, 2, or 4, let G be an $M \times N$ standard Gaussian β matrix, as in definition 1.2, with $M \geqslant N$. Then, let $X = G^\dagger G$, where $G^\dagger$ is the adjoint of G, with the notion of adjoint being the same as the one given in definition 1.3. The *(a,N) Laguerre β ensemble (LβE)* is the ensemble of self-adjoint matrices represented by X, where $a := M - N$ is called the *exponent* [Forrester, 2010, pp.86,90]. Note that X is an $N \times N$ matrix.

Remark 3. The matrix X given above is a *Wishart matrix*, so these ensembles are also called the *Wishart β ensembles* or *Wishart-Laguerre β ensembles*. The name "Laguerre β ensembles" is due to the intimate connection they have with the Laguerre orthogonal polynomials; see remark 5 in subsection 1.4.1.

For fixed $\beta = 1$, 2, or 4, let X be an element of the (a, N) Laguerre β ensemble. Then like the Gaussian case, X is self-adjoint with real eigenvalues. However, unlike the Gaussian case, the entries are not drawn from Gaussian distributions. Indeed, write $M = a + N$, $X = [x_{i,j}]$, and $G = [g_{i,j}]$, where G is the $M \times N$ standard Gaussian β matrix used in the construction of X above. Then, for $1 \leqslant i \leqslant N$ and $i < j \leqslant N$,

$$x_{i,i} = \sum_{s=1}^{\beta} \sum_{k=1}^{M} g_{k,i,s}^2, \qquad\qquad x_{i,j} = \sum_{k=1}^{M} g_{k,i}^\dagger g_{k,j}.$$

Here, $g_{i,j,s}$ denotes the s^{th} real component of $g_{i,j}$. By considering the Chi-squared distribution, which is related to the sum of squares of independent Gaussian variables, and applying appropriate scaling, we see that the diagonal entries of X are

drawn from the Gamma distribution $\Gamma(\frac{\beta M}{2}, \frac{2}{\beta})$ with shape parameter $\frac{\beta M}{2}$ and scale parameter $\frac{2}{\beta}$; a variable x is distributed according to $\Gamma(k, \theta)$ if it has p.d.f.

$$\frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-x/\theta},$$

where here the Γ is the Gamma function. Thus, our diagonal terms have p.d.f.

$$\frac{(\frac{\beta}{2})^{\beta M/2}}{\Gamma(\frac{\beta M}{2})} x_{i,i}^{\beta M/2-1} e^{-\beta x_{i,i},/2}.$$

The off-diagonal terms are far more complicated, and do not beg discussion.

Like the Gaussian ensembles, the L1E is also called the *Laguerre orthogonal ensemble (LOE)*, the L2E is called the *Laguerre unitary ensemble (LUE)*, and the L4E is called the *Laguerre symplectic ensemble (LSE)*. These names are again due to invariance properties of the eigenvalue densities of these ensembles. Leaving motivations to subsection 1.3.2, we now calculate these eigenvalue densities.

1.3.1 The Eigenvalue Densities of the Laguerre β Ensembles

We will derive the eigenvalue densities of the Laguerre β ensembles in two ways. The first, via metric form methods, as seen in the Gaussian case in subsection 1.2.1. The second, via Householder transformations, in an effort to offer more insight.

Laguerre β Ensemble Eigenvalue Densities via Metric Form Methods

We begin by outlining a proof found in [Forrester, 2010, pp.92-97], [Muirhead, 1982, pp.63-67] of the following proposition.

Proposition 2. *Fix $\beta = 1$, 2, or 4. Let $X = G^\dagger G$ be a member of the (a, N) Laguerre β ensemble, with G an $M \times N$ standard Gaussian β matrix, where $M = a + N$. Then, the j.p.d.f. of the independent elements of X is*

$$p^{(L)}(X) := \frac{1}{C_{\beta,a,N}} \det(X)^{\frac{\beta}{2}(a+1)-1} \exp\left(-\tfrac{\beta}{2}\mathrm{Tr}(X)\right), \tag{1.3.1}$$

where $C_{\beta,a,N}$ is a normalisation constant that we are not interested in [Forrester, 2010, p.94].

Proof. Indeed, we need to show that the probability measure of X is $p^{(L)}(X)(\mathrm{d}X)$, where we recall that $(\mathrm{d}X)$ denotes the product of the independent elements of the matrix $\mathrm{d}X = [\mathrm{d}x_{i,j}]$, with $X =: [x_{i,j}]$.

As G is merely a standard Gaussian β matrix, we cannot necessarily diagonalise it. However, we can use Gram-Schmidt orthogonalisation to write $G = UT$, where $T = [t_{i,j}]$ is an $N \times N$ upper triangular β matrix with diagonal entries positive and real, and U is an $(M \times N)$ β matrix such that $U^\dagger U = I_N$. We can also extend U to an $M \times M$ orthogonal, unitary, or symplectic matrix $V := [U\, U']$ (depending on β) by defining an $(M \times a)$ β matrix U' whose columns are orthonormal to each other and to the columns of U. Then,

$$I_M = V^\dagger V = \begin{bmatrix} I_N & U^\dagger U' \\ (U')^\dagger U & (U')^\dagger U' \end{bmatrix},$$

so $U^\dagger U'$ and $(U')^\dagger U$ are both zero matrices, and $(U')^\dagger U' = I_a$. Using these facts, we may obtain that

$$(\mathrm{d}G) = \prod_{j=1}^{N} t_{j,j}^{\beta(M-j)+\kappa} (\mathrm{d}T)(V^\dagger \mathrm{d}U), \tag{1.3.2}$$

where $\kappa = 0$ in the real case, $\kappa = 1$ in the complex case, and $\kappa = 2$ in the quaternion case. We provide a proof of this fact in appendix A.

Moving on, we have $X = G^\dagger G = (UT)^\dagger (UT) = T^\dagger T$, so $\mathrm{d}X = \mathrm{d}T^\dagger\, T + T^\dagger \mathrm{d}T$ and the (i,j) entry of $\mathrm{d}X$ is $[\sum_{k=1}^{j} t_{k,j} \mathrm{d}t_{k,i}^\dagger + t_{k,i}^\dagger \mathrm{d}t_{k,j}]$. Since the diagonal terms of T are real and $\mathrm{d}X^\dagger = \mathrm{d}X$, the diagonal of $\mathrm{d}X$ consists of terms of the form $2t_{i,i}\mathrm{d}t_{i,i}$ for $1 \leqslant i \leqslant N$, and we may also ignore all lower triangular terms when extracting independent elements. Focusing on upper triangular terms, the (i,j) entry of $\mathrm{d}X$ contains the term $t_{i,i}^\dagger \mathrm{d}t_{i,j} = t_{i,i}\mathrm{d}t_{i,j}$. As this covers all of the differentials that appear in $\mathrm{d}X$, we have collected all of the independent elements of $\mathrm{d}X$. Multiplying them,

we have

$$(\mathrm{d}X) = 2^N \prod_{i=1}^{N} t_{i,i}^{\beta(N-i)+1} (\mathrm{d}T), \tag{1.3.3}$$

since the differentials $\mathrm{d}t_{i,i}$ are completely real, but all other differentials contain β independent real components.

Since the entries of $G =: [g_{i,j}]$ are all i.i.d. Gaussian variables, it immediately follows that the probability measure of G is $\prod_{i=1}^{M}\prod_{j=1}^{N}\left(\frac{\beta}{2\pi}\right)^{\frac{\beta}{2}} \exp\left(-\frac{\beta}{2}|g_{i,j}|^2\right)(\mathrm{d}G)$, from definition 1.2. Changing variables to that of X gives the probability measure of X. This just requires 3 facts: First, $\sum_{i=1}^{M}\sum_{j=1}^{N}|g_{i,j}|^2 = \mathrm{Tr}(G^\dagger G) = \mathrm{Tr}(A)$. Second, (1.3.2) and (1.3.3) combine to give

$$\begin{aligned}(\mathrm{d}G) &= 2^{-N}\prod_{j=1}^{N} t_{j,j}^{\beta a+\kappa-1}(\mathrm{d}X)(V^\dagger \mathrm{d}U) \\ &= 2^{-N}|\det(T)|^{\beta a+\kappa-1}(\mathrm{d}X)(V^\dagger \mathrm{d}U),\end{aligned}$$

where the last line is due to the fact that T is upper triangular. Third, since $X = T^\dagger T$, we have $|\det(X)| = |\det(T)|^2$. Substituting $\kappa = 0$ for $\beta = 1$, $\kappa = 1$ for $\beta = 2$ and $\kappa = 2$ for $\beta = 4$, and integrating out $(V^\dagger \mathrm{d}U)$ completes the proof. □

With this proposition in hand, we may compute the eigenvalue densities of the Laguerre β ensembles. We retain the notation of proposition 2 and denote the eigenvalues of X by $\lambda_1, \ldots, \lambda_N$. Then, $\det(X) = \prod_{i=1}^{N}\lambda_i$, and $\mathrm{Tr}(X) = \sum_{i=1}^{N}\lambda_i$. Moreover, since X is self-adjoint, we may use (1.2.4) to write $(\mathrm{d}X)$ as a product of $\prod_{i=1}^{N}\mathrm{d}\lambda_i \prod_{1\leqslant j<k\leqslant N}|\lambda_k - \lambda_j|^\beta$ and a term involving the eigenvectors of X. Integrating out the eigenvector terms, we find that the probability measure of X is proportional to

$$\prod_{i=1}^{N}\mathrm{d}\lambda_i\, \lambda_i^{\frac{\beta}{2}(a+1)-1} e^{-\beta\lambda_i/2} \prod_{1\leqslant j<k\leqslant N}|\lambda_k - \lambda_j|^\beta.$$

The resulting definitions are as follows.

Definition 1.9. The (a, N) *LβE eigenvalue j.p.d.f.* is

$$p^{(L)}(\lambda_1, \ldots, \lambda_N; \alpha_1, \beta) := \frac{1}{L_N(a, \beta)} \prod_{i=1}^{N} \lambda_i^{\alpha_1} e^{-\beta\lambda_i/2} \prod_{1 \leqslant j < k \leqslant N} |\lambda_k - \lambda_j|^{\beta}, \tag{1.3.4}$$

where $\alpha_1 := \frac{\beta}{2}(a+1) - 1$ is a notation we will use throughout the thesis, and $L_N(a, \beta) = 2^{-\frac{\beta}{2}MN} \prod_{j=1}^{N} \frac{\Gamma(1+\frac{\beta}{2})}{\Gamma(1+\frac{\beta}{2}j)\Gamma(\frac{\beta}{2}(a+j))}$ is a normalisation constant [Dumitriu and Edelman, 2002, p.5832]. Here, Γ is the Gamma function, and $\lambda_i \in [0, \infty)$ for all $1 \leqslant i \leqslant N$.

Like in the Gaussian case, the normalisation constant $L_N(a, \beta)$ will be independently derived in subsection 1.4.1.

Our LβE eigenvalue j.p.d.f. has support $[0, \infty)^N$ because our eigenvalues are non-negative: Let $X = G^{\dagger}G$ be a member of the (a, N) Laguerre β ensemble, for some $\beta = 1$, 2, or 4, and some $(a + N) \times N$ standard Gaussian β matrix G. Let λ be an eigenvalue of X with corresponding eigenvector $\mathbf{v}$. Then, we observe that

$$\lambda \mathbf{v}^{\dagger}\mathbf{v} = \mathbf{v}^{\dagger}X\mathbf{v} = (\mathbf{v}^{\dagger}G^{\dagger})(G\mathbf{v}) = (G\mathbf{v})^{\dagger}(G\mathbf{v}).$$

But $\mathbf{v}^{\dagger}\mathbf{v} = |\mathbf{v}|^2 \geqslant 0$ and $(G\mathbf{v})^{\dagger}(G\mathbf{v}) = |G\mathbf{v}|^2 \geqslant 0$, so it must be that $\lambda \geqslant 0$.

Definition 1.10. The (a, N) *LβE eigenvalue density* is

$$\rho_{(1)}^{(L)}(\lambda; \alpha_1, \beta, N) := N \int_{[0,\infty)^{N-1}} p^{(L)}(\lambda, \lambda_2, \ldots, \lambda_N; \alpha_1, \beta) \mathrm{d}\lambda_2 \ldots \mathrm{d}\lambda_N, \tag{1.3.5}$$

with support $[0, \infty)$. Let X be a member of the (a, N) LβE. For $t \geqslant s \geqslant 0$, the expected number of eigenvalues of X in the interval $[s, t]$ is given by $\int_s^t \rho_{(1)}^{(L)}(\lambda; \alpha_1, \beta, N)\, \mathrm{d}\lambda$.

Laguerre β Ensemble Eigenvalue Densities via Householder Transformations

We now derive the eigenvalue j.p.d.f.s and densities for the Laguerre β ensembles by utilising Householder transformations. Our discussion is based on the constructions given in [Dumitriu and Edelman, 2002, pp.5835-5842], [Forrester, 2010, pp.127-192].

Proposition 3. *Fix $\beta = 1$, 2, or 4. Let $X = G^\dagger G$ be a member of the (a, N) Laguerre β ensemble, with G an $M \times N$ standard Gaussian β matrix, where $M = a + N$. Then X is similar to the tridiagonal matrix $T := B^T B$, where*

$$B := \begin{bmatrix} x_M & y_{N-1} & 0 & \cdots & 0 \\ 0 & x_{M-1} & y_{N-2} & \ddots & \vdots \\ 0 & 0 & x_{M-2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & y_1 \\ \vdots & \ddots & \ddots & 0 & x_{a+1} \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix}\right\} a$$

is an $M \times N$ bidiagonal matrix with x_i and y_i real independent random variables drawn from the $\chi(\beta i)$ distribution, which we define as follows. A variable x is distributed according to $\chi(k)$ if x^2 is distributed according to the Gamma distribution $\Gamma(\frac{k}{2}, \frac{2}{\beta})$. i.e. if it has p.d.f.

$$\frac{2(\frac{\beta}{2})^{k/2}}{\Gamma(\frac{k}{2})} x^{k-1} e^{-\beta x^2/2}.$$

In particular, X and T have the same eigenvalues.

Proof. Write $G = [g_{i,j}]$, let $\mathbf{g_1}$ be the left-most column of G, and let $G^{(1)}$ be the $M \times (N-1)$ matrix that lies directly to the right of $\mathbf{g_1}$ in G.

Let $\mathbf{v_1} = \mathbf{g_1} - \sqrt{\mathbf{g_1}^\dagger \mathbf{g_1}}[1\,0\,\cdots\,0]^T$, and define the Householder transformation $L^{(1)} = I_M - \frac{2}{\mathbf{v_1}^\dagger \mathbf{v_1}} \mathbf{v_1} \mathbf{v_1}^\dagger$ [Golub and Van Loan, 1996, p.209]. Then $L^{(1)}$ is a left reflector such that $L^{(1)}\mathbf{g_1} = \sqrt{\mathbf{g_1}^\dagger \mathbf{g_1}}[1\,0\,\cdots\,0]^T$. Moreover, $L^{(1)}$ is an orthogonal, unitary, or symplectic matrix (depending on β). i.e. ${L^{(1)}}^\dagger L^{(1)}$ and $L^{(1)} {L^{(1)}}^\dagger$ are both equal to the identity.

Now, $\mathbf{g_1}^\dagger \mathbf{g_1}$ is a sum of the squares of all of the real components of $g_{1,1}, \ldots, g_{M,1}$. Hence, by considering the Chi-squared distribution, we see that $\mathbf{g_1}^\dagger \mathbf{g_1}$ is a random variable that is drawn from the $\Gamma(\frac{\beta N}{2}, \frac{2}{\beta})$ distribution. Thus $\sqrt{\mathbf{g_1}^\dagger \mathbf{g_1}}$ is drawn from

the $\chi(\beta M)$ distribution. Moreover, since $L^{(1)}$ and $G^{(1)}$ are independent of each other, $L^{(1)}G^{(1)}$ is an $M \times (N-1)$ standard Gaussian β matrix (see appendix B). Hence, $L^{(1)}G$ is an $M \times N$ matrix whose top left entry is drawn from $\chi(\beta M)$, all other entries in the left-most column are 0, and the remaining entries form an $M \times (N-1)$ standard Gaussian β matrix. Moreover, all of these entries are independently distributed.

Moving on, we may now repeat the exericse from the right. Let $\tilde{\mathbf{g}_1}$ be the top-most row of the $M \times (N-1)$ matrix $L^{(1)}G^{(1)}$, and let $\tilde{G}^{(1)}$ be the $(M-1) \times (N-1)$ matrix right below it. Let $\mathbf{u_1} = \tilde{\mathbf{g}_1} - \sqrt{\tilde{\mathbf{g}_1}\tilde{\mathbf{g}_1}^\dagger}[1\,0\,\cdots\,0]$, and define the Householder transformation $\tilde{R}^{(1)} = I_{N-1} - \frac{2}{\mathbf{u_1}\mathbf{u_1}^\dagger}\mathbf{u_1}^\dagger\mathbf{u_1}$. This matrix $\tilde{R}^{(1)}$ enjoys many of the properties of $L^{(1)}$, as the two are constructed in very much the same way. In particular, $\tilde{R}^{(1)}$ is orthogonal, unitary, or symplectic (depending on β), and $\tilde{R}^{(1)}$ and $\tilde{G}^{(1)}$ are independent of each other.

The top left entry of $\tilde{G}^{(1)}\tilde{R}^{(1)}$ is distributed according to $\Gamma(\frac{\beta(N-1)}{2}, \frac{2}{\beta})$ and every other entry in the top-most row of $\tilde{G}^{(1)}\tilde{R}^{(1)}$ is 0, and all remaining entries are standard Gaussian. All entries are of course independently distributed. The reasoning is the same as with $L^{(1)}$. Defining

$$R^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{R}^{(1)} \end{bmatrix},$$

we have that $L^{(1)}GR^{(1)}$ is an $M \times N$ matrix whose top left entry is distributed according to $\Gamma(\frac{\beta N}{2}, \frac{2}{\beta})$, the entry to its right is distributed according to $\Gamma(\frac{\beta(M-1)}{2}, \frac{2}{\beta})$, every other entry in the top-most row and left-most column is 0, and all other entries are standard Gaussian. We may now focus on the bottom right $(M-1) \times (N-1)$ block of $L^{(1)}GR^{(1)}$ and repeat this whole process. Naturally, we repeat this process a few more times, each time defining

$$R^{(n)} = \begin{bmatrix} I_n & 0 \\ 0 & \tilde{R}^{(n)} \end{bmatrix}, \quad L^{(n)} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & \tilde{L}^{(n)} \end{bmatrix}$$

for appropriate n. Eventually, we arrive at

$$B = L^{(N)} \cdots L^{(1)} G R^{(1)} \cdots R^{(N-1)},$$

where B is as given in the statement of the proposition. We note that the process ends when the last a entries of the N^{th} column are changed to 0 by $L^{(N)}$, as we do not need to follow this with $R^{(N)}$ (which cannot even be defined).

Now, since B is a real matrix, $T = B^T B = B^\dagger B$, so

$$\begin{aligned} T &= B^\dagger B \\ &= R^{(N-1)^\dagger} \cdots R^{(1)^\dagger} G^\dagger L^{(1)^\dagger} \cdots L^{(N)^\dagger} L^{(N)} \cdots L^{(1)} G R^{(1)} \cdots R^{(N-1)} \\ &= R^{(N-1)^\dagger} \cdots R^{(1)^\dagger} G^\dagger G R^{(1)} \cdots R^{(N-1)}, \end{aligned}$$

since for $1 \leqslant n \leqslant N$, $L^{(n)}$ is orthogonal, unitary, or symplectic, so $L^{(n)^\dagger} L^{(n)} = I_M$. Moreover, for $1 \leqslant n \leqslant N-1$, $R^{(n)}$ is also orthogonal, unitary, or symplectic. As these matrix groups are closed under multiplication, we have that $U := R^{(1)} \cdots R^{(N-1)}$ is orthogonal, unitary, or symplectic. Hence, as $X = G^\dagger G$, we have $X = UTU^\dagger$. Thus X is similar to T. □

Because of proposition 3, calculating the eigenvalue j.p.d.f.s and densities for the Laguerre β ensembles comes down to calculating the eigenvalue j.p.d.f.s and densities of the tridiagonal matrix defined in said proposition.

Proposition 4. *Let $\beta \in \mathbb{R}_{>0}$, $N \in \mathbb{N}_{\geqslant 1}$, and $a \in \mathbb{N}_0$. Let $M = a + N$ and define $T_\beta := B_\beta^T B_\beta$, where*

$$B_\beta = \begin{bmatrix} x_M & y_{N-1} & 0 & \cdots & 0 \\ 0 & x_{M-1} & y_{N-2} & \ddots & \vdots \\ 0 & 0 & x_{M-2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & y_1 \\ 0 & \cdots & \cdots & 0 & x_{a+1} \end{bmatrix}.$$

is an $N \times N$ bidiagonal matrix with x_i and y_i real independent random variables drawn from the $\chi(\beta i)$ distribution. i.e. x_i and y_i have p.d.f.

$$\frac{2(\frac{\beta}{2})^{\beta i}}{\Gamma(\frac{\beta i}{2})} z^{\beta i - 1} e^{-\beta z^2/2}.$$

where Γ is the Gamma function.

Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of T_β. Then the eigenvalue j.p.d.f. of T_β is

$$p^{(T)}(\lambda_1, \ldots, \lambda_N; a, \beta) := \frac{1}{T_{\beta,a,N}} e^{-\frac{\beta}{2}\sum_{i=1}^N \lambda_i} \prod_{i=1}^N \lambda_i^{\beta(a+1)/2-1} \prod_{1 \leqslant j < k \leqslant N} |\lambda_k - \lambda_j|^\beta,$$

where $T_{\beta,a,N} = 2^{-\frac{\beta}{2}MN} \prod_{j=1}^N \frac{\Gamma(1+\frac{\beta}{2})}{\Gamma(1+\frac{\beta}{2}j)\Gamma(\frac{\beta}{2}(a+j))}$ is a normalisation constant taken from definition 1.9. The eigenvalue density follows by definition 1.10.

Proof. First, we note that the j.p.d.f. of the independent elements of B_β is

$$p^{(T)}(B_\beta) := \frac{1}{B_{\beta,a,N}} \prod_{i=0}^{N-1} x_{M-i}^{\beta(M-i)-1} e^{-\beta x_{M-i}^2/2} \prod_{k=1}^{N-1} y_k^{\beta k - 1} e^{-\beta y_k^2/2}, \tag{1.3.6}$$

where $B_{\beta,a,N}$ is a normalisation constant. Writing $(\mathrm{d}B_\beta) = \mathrm{d}x_{a+1} \ldots \mathrm{d}x_M \mathrm{d}y_1 \ldots \mathrm{d}y_{N-1}$, the probability measure of B_β is then $p^{(T)}(B_\beta)(\mathrm{d}B_\beta)$. Next, we write

$$T_\beta = \begin{bmatrix} a_N & b_{N-1} & 0 & \cdots & 0 \\ b_{N-1} & a_{N-1} & b_{N-2} & \ddots & \vdots \\ 0 & b_{N-2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_2 & b_1 \\ 0 & \cdots & 0 & b_1 & a_1 \end{bmatrix},$$

so that $a_N = x_M^2$, and for $1 \leqslant i \leqslant N-1$, $a_i = y_i^2 + x_{M-N+i}^2$ and $b_i = y_i x_{M-N+i+1}$. Differentiating these relations, and computating the Jacobian, we have

$$(\mathrm{d}B_\beta) = \left(2^M x_{M-N+1} \prod_{i=0}^{N-2} x_{M-i}^2\right)^{-1} \mathrm{d}a_1 \ldots \mathrm{d}a_N \mathrm{d}b_1 \ldots \mathrm{d}b_{N-1}. \tag{1.3.7}$$

As T_β is a real symmetric matrix, we may diagonalise it as $T_\beta = PDP^T$, where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ and P is an $N \times N$ orthogonal matrix such that the j^{th} column

of P is the normalised eigenvector of T_β that corresponds to λ_j. For $1 \leqslant j \leqslant N$, let q_j be the first component of the j^{th} column of P. Each eigenvector has a choice in sign, so for uniqueness, we restrict each q_j to be positive. By substitution into $T_\beta = PDP^T$, we can see that the j^{th} eigenvector of T_β is completely determined by q_j and the eigenvalues and entries of T_β. In fact, since the rows of P must also be orthonormal, q_N is fully determined by $q_1, \ldots, q_{N-1}$. Hence, T_β is fully determined by the set $\{\lambda_1, \ldots, \lambda_N, q_1, \ldots, q_{N-1}\}$. By counting the number of independent variables on each side of the equation $B_\beta^T B_\beta = PDP^T$, we see that the set $\{\lambda_1, \ldots, \lambda_N, q_1, \ldots, q_{N-1}\}$ must be independent.

Porposition 1.9.3 of [Forrester, 2010, p.46] tells us that the Jacobian for changing variables in T_β from $\{a_1, \ldots, a_N, b_1, \ldots, b_{N-1}\}$ to $\{\lambda_1, \ldots, \lambda_N, q_1, \ldots, q_{N-1}\}$ is

$$J = \frac{1}{q_N} \frac{\prod_{i=1}^{N-1} b_i}{\prod_{i=1}^{N} q_i}, \quad q_N = \left(1 - \sum_{i=1}^{N-1} q_i^2\right)^{1/2}.$$

Hence by substituting into 1.3.6, and using 1.3.7, we arrive at the probability measure

$$\begin{aligned} p^{(T)}(B_\beta)(\mathrm{d}T_\beta) = \frac{1}{B_{\beta,a,N}} \exp\left\{-\tfrac{\beta}{2}\left(\sum_{i=0}^{N-1} x_{M-i}^2 + \sum_{k=1}^{N-1} y_k^2\right)\right\} \\ \times \frac{\prod_{i=0}^{N-1} x_{M-i}^{\beta(M-i)-2} \prod_{k=1}^{N-1} y_k^{\beta k}}{q_N \prod_{i=1}^{N} q_i} \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_N \mathrm{d}q_1 \ldots \mathrm{d}q_{N-1} \end{aligned} \tag{1.3.8}$$

Now,

$$\begin{aligned} \sum_{i=0}^{N-1} x_{M-i}^2 + \sum_{k=1}^{N-1} y_k^2 = \sum_{i=1}^{N} a_i = \mathrm{Tr}(T_\beta) = \sum_{i=1}^{N} \lambda_i, \\ \prod_{i=0}^{N-1} x_{M-i}^2 = \det(B_\beta)^2 = \det(T_\beta) = \prod_{i=1}^{N} \lambda_i, \\ \frac{\prod_{i=0}^{N-1} x_{M-i}^{\beta(M-i)-2} \prod_{k=1}^{N-1} y_k^{\beta k}}{\prod_{i=1}^{N} q_i} = \prod_{i=1}^{N} q_i^{\beta-1} \prod_{i=0}^{N-1} x_{M-i}^{\beta(a+1)-2} \prod_{1 \leqslant j < k \leqslant N} |\lambda_k - \lambda_j|^\beta, \end{aligned}$$

where we have used proposition 1.9.2 of [Forrester, 2010, pp.45,129] in the last line. Hence, after integrating out our q_i terms, 1.3.8 simplifies to give us the eigenvalue

j.p.d.f. of T_β as

$$p^{(T)}(\lambda_1, \ldots, \lambda_N; a, \beta) = \frac{1}{T_{\beta,a,N}} e^{-\frac{\beta}{2}\sum_{i=1}^N \lambda_i} \prod_{i=1}^N \lambda_i^{\beta(a+1)/2-1} \prod_{1 \leqslant j < k \leqslant N} |\lambda_k - \lambda_j|^\beta,$$

as required; we compute $T_{\beta,a,N}$ easily via a variant of the Selberg integral. □

Remark 4. In our constructions until now, we have determined the eigenvalue densities for the real, complex, and quaternion Gaussian and Laguerre ensembles. These have corresponded to the β values 1, 2, and 4. However, our eigenvalue j.p.d.f.s and densities given in definitions 1.6, 1.7, 1.9, and 1.10 do not have any qualms with general $\beta \in \mathbb{R}_{>0}$. To be able to call these functions eigenvalue j.p.d.f.s and densities for general β values, we desire random matrix ensembles whose eigenvalue j.p.d.f.s and densities are given by said functions. That is exactly what we have done here in the Laguerre case, as the random matrix ensemble represented by the T_β given in proposition 4 is valid for any $\beta \in \mathbb{R}_{>0}$. For this reason, the family of random matrix ensembles represented by T_β are called the *general-β Laguerre ensembles*, or similar. The same construction can be performed to define the *general-β Gaussian ensembles*.

We observe that in the cases $\beta = 1$, 2, or 4, B_β of proposition 4 is simply the B given in proposition 3 with the last a rows of 0s truncated off. Then, T_β of proposition 4 and the T given in proposition 3 are identical. Combining propositions 3 and 4, we see that for $\beta = 1$, 2, or 4, the eigenvalue j.p.d.f.s and densities of the (a, N) Laguerre β ensemble are as given in definitions 1.9 and 1.10.

1.3.2 Applications of the Laguerre β Ensembles

We now present some applications of the Laguerre β ensembles. Again, we forego discussion of combinatorics, instead visiting them in subsections 1.4.3 and 1.4.4.

Principal Component Analysis

We demonstrate the concept through an example. Consider a class of M students that must complete N tests throughout the semester. Let $X = [x_{i,j}]$ be the $M \times N$ matrix

such that $x_{i,j}$ is the mark that student i obtained on the j^{th} test. Let $\overline{x}_j = \frac{1}{M}\sum_{i=1}^{M} x_{i,j}$ be the empirical average of the class's marks on the j^{th} test, and let $\overline{\mathbf{x}}_\mathbf{j}$ be an $M \times 1$ column vector whose entries are all equal to $\overline{x}_j$. Let $\overline{X} = [\overline{\mathbf{x}}_\mathbf{1} \cdots \overline{\mathbf{x}}_\mathbf{N}]$, and let $Y = X - \overline{X}$. Let $\mathbf{e_1}, \ldots, \mathbf{e_N}$ be the standard basis on the space that Y acts on.

Let $\Sigma = Y^T Y$. The eigenvector $\mathbf{v_1}$ of Σ that corresponds to the largest eigenvalue λ_1 is called the *first principal component of* Y, and the eigenvector $\mathbf{v_2}$ that corresponds to the second largest eigenvalue λ_2 is the *second principal component of* Y, and so on [Jolliffe, 2002, pp.30-33]. These principal components can reveal a lot of information; for example, if the first principal component is mainly pointed in the directions of say $\mathbf{e_1}$ and $\mathbf{e_2}$, we can say that the first and second tests largely contribute to the variance of the data in X. i.e. that they're good tests. In formulating future tests, one would like to replicate what the first and second tests have in common. On the other hand, the data will barely vary in the direction of the N^{th} principal component. In practice, the dimension of the problem is reduced by successively projecting the data onto the hyperplane orthogonal to the N^{th} principal component, and then the $(N-1)^{\text{th}}$ and so on, since these components relatively offer little information. Indeed, $\Sigma = \sum_{i=1}^{N} \lambda_i \mathbf{v_i} \mathbf{v_i}^\dagger$; truncating this sum gives a low-rank approximation to Σ.

Now, assert that the j^{th} test's mark is normally distributed with mean μ_j and standard deviation 1, with no correlation between the tests. Then, Σ has eigenvalue j.p.d.f. given by the j.p.d.f. for the $(M-N, N)$ Laguerre orthogonal ensemble (1.9) [Gupta and Nagar, 2000, pp.92-93]. This gives us information about the eigenvalues and eigenvectors of Σ, allowing us to perform principal component analysis in a statistical sense. These concepts have applications in fields like image compression [Clausen and Wechsler, 2000] and financial correlations [Jolliffe, 2002, p.76].

Wireless Communications

We will give a largely heuristic description of a theory that is actually rather concrete. We again demonstrate the concepts through an example. Consider the information

transfer from a wireless modem to a laptop. Suppose that the modem has one transmitting antenna (transmitter) and that the laptop has one receiving antenna (receiver). In this example, the modem is trying to send binary data. Let $x(t)$ be the signal emitted by the wireless modem and let $z(t)$ be noise that is added onto $x(t)$ before it reaches the receiver, both in Volts. Then, the received signal $y(t)$ is given by $\sqrt{x(t)^2 + z(t)^2}$ due to conservation of power. The receiver assumes that $x(t)$ and $z(t)$ are real-valued and that there exist $X, Z \in \mathbb{R}_{>0}$ such that $|x(t)| < X$ and $|z(t)| < Z$ for all t. Then there exists $Y \in \mathbb{R}_{>0}$ such that $|y(t)| < Y$ and $\frac{Y}{Z} = \sqrt{1 + \frac{X}{Z}}$. Now, we subdivide the range $(-Y, Y)$ into 2^b subintervals of length $2Z$ so that when $y(t)$ is sampled, we can confidently assign it a binary value according to which of these subintervals $y(t)$ lies in. Then, $b = \frac{1}{2}\log_2(1 + \frac{X}{Z})$ is the number of bits that $y(t)$ can reliably represent. Now, if we further constrain $x(t)$ so that in the frequency domain, x has frequencies between $f_{\min}$ and $f_{\max}$, the sampling theorem tells us that we can take $2B$ good measurements of $y(t)$ over one second, where $B = f_{\max} - f_{\min}$. Hence, the number of bits we can transmit over one second is $2Bb = B\log_2(1 + \frac{X}{Z})$. The *information capacity* is defined as $I = \log_2(1 + \frac{X}{Z})$, and it specifies the number of bits that can be transmitted during one second, per frequency of bandwidth available. The ratio $\frac{X}{Z}$ is the *signal-to-noise ratio*. The formula $I = \log_2(1 + \frac{X}{Z})$ is called *Shannon's formula*.

For physical reasons, one antenna per side is not good enough for modern needs [Simon et al., 2001]. We now consider the situation where the wireless modem has m_T transmitting antennas and the laptop has m_R receiving antennas. We let $x_i(t)$ be the signal emitted by the i^{th} transmitter, and $y_j(t)$ be the signal received by the j^{th} receiver. Then, we form the column vectors $\mathbf{x}$ and $\mathbf{y}$ so that $\mathbf{y} = G\mathbf{x} + \mathbf{z}$, where G is an $m_R \times m_T$ *propagation matrix*, and $\mathbf{z}$ has i.i.d. entries which represent noise. The propagation matrix G describes how the signals x_i linearly combine through superposition before they arrive at the receivers. It accounts for factors such antenna placement, angle spread, and angle of arrival at the receivers due scattering of the electromagnetic waves by walls, desks, etc. [Ratnarajah et al., 2003]. We now consider

the case that our signals are complex-valued. The receivers have no prior knowledge of the x_i, other than the fact that they are Gaussian with mean 0. Moreover, we assume that $\mathbf{z}$ consists of standard Gaussian complex entries. Then, there is a well known generalisation of Shannon's formula [Foschini and Gans, 1998], where the information capacity of this system is given by

$$I = \log_2 \det\left(I_{m_R} + \frac{\rho}{m_T} G^\dagger G\right),$$

where ρ is the total power emitted by the modem. We now assume that G is a standard Gaussian complex matrix so that $G^\dagger G$ is a member of the Laguerre unitary ensemble. Then, we can use our knowledge of the eigenvalue density of the Laguerre unitary ensemble to compute statistics for the information capacity.

Quantum Entanglement

We highlight a recent application to a problem involving quantum entanglement [Chen et al., 2010]. We consider a bipartite quantum system Q consisting of the system of interest A, and the environment B. We let A have dimension N and B have dimension M, with $N \leqslant M$. Let $|e_i^A\rangle_{1\leqslant i\leqslant N}$ and $|e_j^B\rangle_{1\leqslant j\leqslant M}$ be the complete orthogonal bases for A and B, respectively. Then, any quantum state in Q can be written in the form $|\phi\rangle = \sum_{i=1}^N \sum_{j=1}^M x_{i,j}|e_i^A\rangle \otimes |e_j^B\rangle$, with $x_{i,j} \in \mathbb{C}$, so that $|\phi\rangle$ is represented by the matrix $X := [x_{i,j}]$. Chen et al. consider states $|\phi\rangle$ whose corresponding X matrix is a standard Gaussian matrix with the added constraint that $\mathrm{Tr}(XX^\dagger) = 1$. The j.p.d.f. of this so-called *fixed trace Laguerre ensemble* is

$$\rho^{(FT)}(\lambda_1, \ldots, \lambda_N) := \rho^{(L)}(\lambda_1, \ldots, \lambda_N; \alpha_1, 2)\, \delta\left(\sum_{i=1}^N \lambda_i - 1\right).$$

This density allows one to compute measures of entanglement. One such measure is the smallest eigenvalue and its p.d.f., which the authors compute. One of the conclusions of their work is that the global constraint of $\mathrm{Tr}(XX^\dagger) = 1$ does not influence local correlations, in the large N limit.

1.4 Further Prerequisites

We are required to explore some more ideas so that the discussion in the upcoming chapters flow smoothly.

1.4.1 The Selberg Integral

First, we take this opportunity to choose notation for the Vandermonde determinant.

Definition 1.11. Let $N \in \mathbb{N}_{\geqslant 1}$ and let $(x_1, \ldots, x_N)$ be a N-tuple of variables. Then the *Vandermonde determinant of order N in* $\mathbf{x}$ is

$$\Delta_N(\mathbf{x}) := \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{vmatrix} = \prod_{1 \leqslant j < k \leqslant N} (x_k - x_j). \tag{1.4.1}$$

We do not prove the right hand side of this equation; it is a famous result. The disbelieving reader may check that the right hand side satisfies properties of the determinant and that both sides are homogeneous monic polynomials of order $\frac{1}{2}N(N-1)$ which vanish at $x_j = x_k$ for all $1 \leqslant j, k \leqslant N$.

Remark 5. As determinants are invariant under addition of rows, $\Delta_N(\mathbf{x})$ is unchanged if each x_i^k is replaced with a monic orthogonal polynomial of order$k$ in x_i. This hints at how orthogonal polynomial theory interacts with random matrix theory.

Definition 1.12. Let $N \in \mathbb{N}_{>0}$ and let $\alpha_1, \alpha_2, \alpha \in \mathbb{C}$ such that $\mathrm{Re}(\alpha_1), \mathrm{Re}(\alpha_2) > 1$ and $\mathrm{Re}(\alpha) > -\min\{\frac{1}{N}, \frac{\mathrm{Re}(\alpha_1+1)}{N-1}, \frac{\mathrm{Re}(\alpha_2+1)}{N-1}\}$. The *Selberg integral* [Andrews et al., 1999, p.402] is

$$S_N(\alpha_1, \alpha_2, \alpha) := \int_{[0,1]^N} \prod_{i=1}^{N} \mathrm{d}x_i x_i^{\alpha_1} (1 - x_i)^{\alpha_2} |\Delta_N(\mathbf{x})|^{2\alpha}. \tag{1.4.2}$$

Theorem 1.1. *Selberg showed that for $N, \alpha_1, \alpha_2, \alpha$ as in the definition above,*

$$S_N(\alpha_1, \alpha_2, \alpha) = \prod_{j=0}^{N-1} \frac{\Gamma(\alpha_1 + 1 + j\alpha)\Gamma(\alpha_2 + 1 + j\alpha)\Gamma(1 + (j+1)\alpha)}{\Gamma(\alpha_1 + \alpha_2 + 2 + (N + j - 1)\alpha)\Gamma(1 + \alpha)}, \tag{1.4.3}$$

where Γ *is the Gamma function.*

Proof. We present a brief outline of Aomoto's method [Aomoto, 1987] of proving this identity, as presented in [Andrews et al., 1999, pp.402-406]. The idea is to define

$$I_k = \int_{[0,1]^N} \prod_{i=1}^{N} \mathrm{d}x_i x_i^{\alpha_1} (1-x_i)^{\alpha_2} \prod_{j=1}^{k} x_j |\Delta_N(\mathbf{x})|^{2\alpha}, \quad I_0 = S_N(\alpha_1, \alpha_2, \alpha),$$

and then to compute

$$J := \int_{[0,1]^N} \frac{\partial}{\partial x_1} \left\{ (1-x_1) \prod_{i=1}^{N} x_i^{\alpha_1} (1-x_i)^{\alpha_2} \prod_{j=1}^{k} x_j |\Delta_N(\mathbf{x})|^{2\alpha} \right\} \mathrm{d}x_1 \dots \mathrm{d}x_N$$

in two different ways. By the fundamental theorem of calculus, $J = 0$, while Leibniz's rule implies that

$$J = (\alpha_1 + 1) I_{k-1} - (\alpha_1 + \alpha_2 + 2) I_k + \alpha(N-k) I_{k-1} - \alpha(2N-k-1) I_k,$$

so that we obtain the recursion relation $I_k = \frac{\alpha_1 + 1 + (N-k)\alpha}{\alpha_1 + \alpha_2 + (2N-k-1)\alpha + 2} I_{k-1}$, which allows us to write I_k in terms of I_0. But this gives

$$\begin{aligned} S_N(\alpha_1 + 1, \alpha_2, \alpha) &= \prod_{j=1}^{N} \frac{\alpha_1 + (N-j)\alpha + 1}{\alpha_1 + \alpha_2 + (2N-j-1)\alpha + 2} S_N(\alpha_1, \alpha_2, \alpha) \\ \implies S_N(\alpha_1, \alpha_2, \alpha) &= \prod_{j=1}^{N} \frac{(\alpha_1 + \alpha_2 + (2N-j-1)\alpha + 2)_k}{(\alpha_2 + (N-j)\alpha + 1)_k} S_N(\alpha_1, \alpha_2 + k, \alpha), \end{aligned}$$

where $(\,\cdot\,)_k$ is the Pochhammer symbol and the last line is due to symmetry in α_1 and α_2 in the first line. On the right hand side of the above, change variables $x_i \mapsto x_i/k$ for $1 \leqslant i \leqslant N$, and take the limit as $k \to \infty$ to obtain

$$S_N(\alpha_1, \alpha_2, \alpha) = \prod_{j=1}^{N} \frac{\Gamma(\alpha_2 + (N-j)\alpha + 1)}{\Gamma(\alpha_1 + \alpha_2 + (2N-j-1)\alpha + 2)} \int_{[0,\infty)^N} \prod_{i=1}^{N} \mathrm{d}x_i x_i^{\alpha_1} e^{-x_i} |\Delta_N(\mathbf{x})|^{2\alpha}. \tag{1.4.4}$$

Writing $E_N(\alpha_1, \alpha)$ for the integral on the right hand side, we have by symmetry in α_1 and α_2 that $D_n(\alpha) := \frac{E_N(\alpha_1, \alpha)}{\prod_{j=1}^{N} \Gamma(\alpha_1 + (N-j)\alpha + 1)}$ is independent of α_1 and α_2 and that

$$S_N(\alpha_1, \alpha_2, \alpha) = \prod_{j=1}^{N} \frac{\Gamma(\alpha_1 + (N-j)\alpha + 1)\Gamma(\alpha_2 + (N-j)\alpha + 1)}{\Gamma(\alpha_1 + \alpha_2 + (2N-j-1)\alpha + 2)} D_N(\alpha). \tag{1.4.5}$$

Taking advantage of the symmetry between $x_1, \dots, x_N$, one has

$$\lim_{\alpha_1 \to -1^+} (\alpha_1 + 1) S_N(\alpha_1, \alpha_2, \alpha)$$
$$= N! \lim_{\alpha_1 \to -1^+} (\alpha_1 + 1) \int_0^1 \mathrm{d}x_N x_N^{\alpha_1} \left(\int_{x_N}^1 \cdots \int_{x_2}^1 \prod_{i=1}^{N-1} \mathrm{d}x_i x_i^{\alpha_1} (1 - x_i)^{\alpha_2} |\Delta_N(\mathbf{x})|^{2\alpha} \right).$$

Taking the limit by using the fact that $\lim_{\alpha_1 \to -1^+} (\alpha_1 + 1) \int_0^1 t^{\alpha_1} f(t) \mathrm{d}t = f(0)$ for f continuous, realising the resultant right hand side is also a Selberg integral, and substituting in (1.4.5) gives $D_N(\alpha) = \prod_{j=1}^N \frac{\Gamma(1+j\alpha)}{\Gamma(1+\alpha)}$. Substituting this back into (1.4.5) gives the stated result. □

The Normalisation Constants

Here, we use limiting procedures to calculate the normalisation constants for the Gaussian and Laguerre β ensemble eigenvalue j.p.d.f.s.

Theorem 1.2. *The normalisation constant for the $N \times N$ Gaussian β ensemble is*

$$G_N(\beta) := \int_{\mathbb{R}^N} \prod_{i=1}^N \mathrm{d}\lambda_i e^{-\beta \lambda_i^2/2} |\Delta_N(\lambda)|^\beta.$$

We claim that $G_N(\beta) = \left(\frac{1}{2\pi}\right)^{N/2} \prod_{i=1}^N \frac{\Gamma(1+\beta/2)}{\Gamma(1+i\beta/2)}$.

Proof. To obtain this result, change variables $x_i \mapsto \frac{x_i}{b} + \frac{1}{2}$ in the Selberg integral and set $\alpha_1 = \alpha_2 = -\frac{b^2\beta}{2}$, $\alpha = \frac{\beta}{2}$, then take the limit as $b \to \infty$.

Thus $x_i^{\alpha_1}(1 - x_i)^{\alpha_2} \mapsto (1 - x_i^2/b^2)^{-b^2\beta/2} \overset{b\to\infty}{\longrightarrow} e^{-\beta x_i^2/2}$. □

Theorem 1.3. *The normalisation constant for the (a, N) Laguerre β ensemble is*

$$L_N(a, \beta) := \int_{[0,\infty)^N} \prod_{i=1}^N \mathrm{d}\lambda_i \lambda_i^{\alpha_1} e^{-\beta\lambda_i/2} |\Delta_N(\lambda)|^\beta,$$

where $\alpha_1 = \frac{\beta}{2}(a+1) - 1$. *We claim that* $L_N(a, \beta) = 2^{-\frac{\beta}{2}MN} \prod_{j=1}^N \frac{\Gamma(1+\frac{\beta}{2})}{\Gamma(1+\frac{\beta}{2}j)\Gamma(\frac{\beta}{2}(a+j))}$.

Proof. Change variables $x_i \mapsto x_i/b$ in the Selberg integral and set $\alpha_2 = \frac{-b\beta}{2}$, $\alpha = \frac{\beta}{2}$, then take the limit as $b \to \infty$. □

1.4.2 Global Scaled Eigenvalue Densities

Our goal is to investigate the behaviour of the (a, N) Laguerre β ensembles in the large N limit. However, in this limit, the eigenvalue densities of these ensembles are unbounded both in range and domain; the largest eigenvalues move towards infinity, and integrals of the densities (1.10) themselves diverge. To avoid these issues and enable analysis, we introduce the global scaled eigenvalue densities.

Definition 1.13. The (a, N) Laguerre β ensemble *global scaled eigenvalue density* is

$$\tilde{\rho}^{(L)}_{(1)}(\lambda; \alpha_1, \beta, N) := \rho^{(L)}_{(1)}(N\lambda; \alpha_1, \beta, N), \tag{1.4.6}$$

where $\rho^{(L)}_{(1)}$ is as given in definition (1.10).

The global scaled density always integrates to 1 on its support. In the large N limit, the global scaled eigenvalue density has support $(0, 4)$. We remark that such a construction $\tilde{\rho}^{(G)}_{(1)}(\lambda; \beta, N)$ also exists for the Gaussian β ensembles, and draw attention to the much celebrated Wigner semicircle law [Wigner, 1958].

Theorem 1.4. *Let X be a symmetric random matrix with i.i.d. distributed entries, with the second moments of all entries being equal to say ν. Moreover, demand that all moments of the entries of X exist and have an upper bound independent of the entries' locations in X. Then, in the large N limit, the global scaled eigenvalue density of X is*

$$\tilde{\rho}_{(1)}(\lambda) = \frac{\sqrt{4\nu^2 - \lambda^2}}{2\pi\nu^2}\chi_{\lambda\in(-2\nu,2\nu)},$$

where $\tilde{\rho}_{(1)}(\lambda) = \frac{1}{\sqrt{N}}\rho_{(1)}(\sqrt{N}\lambda)$, and χ is the character function.

There is a similar result for the Laguerre β ensembles, called the Marchenko-Pastur law [Pastur and Shcherbina, 2011, p.188], a consequence of which is

$$\lim_{N\to\infty} \tilde{\rho}^{(L)}_{(1)}(\lambda; \alpha_1, \beta, N) =: \tilde{\rho}^{(L)}_{(1),0}(\lambda; \alpha_1, \beta) = \frac{1}{2\pi}\sqrt{\tfrac{4}{\lambda} - 1}\chi_{0<\lambda<4}.$$

A proof of this result will be provided in subsection 4.1.1. It is important to note that this result is very nice, but it only captures the large N data. In fact, a lot

of the intricacies and low-order details of the density have been scaled out. Such details are actually quite interesting. For example, the density has oscillations that become more numerous and violent as $N \to \infty$. The global scaled eigenvalue density $\tilde{\rho}_{(1)}^{(L)}(\lambda; \alpha_1, \beta, N)$ is effectively a smooth function after averaging. Thus, we often call $\tilde{\rho}_{(1)}^{(L)}(\lambda; \alpha_1, \beta, N)$ the *smoothed eigenvalue density* of the Laguerre β ensemble.

The definitions of the smoothed and regular eigenvalue densities depend on the eigenvalue j.p.d.f.s, so we define a measure related to the scaled Laguerre β ensembles. Henceforth, we will use LβE* to denote $\{\frac{1}{N}X \mid X \text{ a member of the } (a, N) \text{ L}\beta\text{E}\}$; the eigenvalues of $Y = \frac{1}{N}X \in \text{L}\beta\text{E}^*$ correspond to $\frac{1}{N}$ times the eigenvalues of X. This is the same correspondence between $\rho_{(1)}^{(L)}$ and $\tilde{\rho}_{(1)}^{(L)}$.

Definition 1.14. The *LβE* scaled measure* is

$$\mathrm{d}\mu_{L\beta E^*}(\lambda) := |\Delta_N(\lambda)|^\beta \prod_{i=1}^{N} \lambda_i^{\alpha_1} e^{-\beta N \lambda_i/2} \, \mathrm{d}\lambda_i, \quad \lambda_i \in [0, \infty)$$

where the LβE* denotes we are working with the scaled eigenvalue j.p.d.f.

Definition 1.15. The *LβE* partition function* is

$$Z_N^* := \int_{[0,\infty)^N} \mathrm{d}\mu_{L\beta E^*}(\lambda),$$

and can be calculated using the variant of the Selberg integral given in theorem 1.3. The partition function serves as our normalisation constant.

Definition 1.16. For an operator K, the *ensemble average of K with respect to LβE** is

$$\langle K \rangle^{L\beta E^*} := \frac{1}{Z_N^*} \int_{[0,\infty)^N} K \, \mathrm{d}\mu_{L\beta E^*}(\lambda).$$

In the definition of the eigenvalue density (1.10), we use symmetry between the eigenvalues to set λ as λ_1. We could've just as easily have chosen λ_2 instead. If we have a statistic $f(\lambda)$ that is dependent on the eigenvalues of the (a, N) Laguerre β ensemble, then

$$\langle f(\lambda) \rangle^* := \int_0^\infty f(\lambda) \tilde{\rho}_{(1)}^{(L)}(\lambda; \alpha_1, \beta, N) \, \mathrm{d}\lambda = \left\langle \frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) \right\rangle^{L\beta E^*}. \tag{1.4.7}$$

where here, the $*$ denotes an ensemble average with respect to the smoothed eigenvalue density.

1.4.3 The Moments

We analyse the smoothed eigenvalue densities of the (a, N) Laguerre β ensembles by investigating their moments, as the moments fully characterise the densities.

Definition 1.17. For $k \in \mathbb{N}_{>0}$, the k^{th} moment of $\tilde{\rho}_{(1)}^{(L)}(\lambda; \alpha_1, \beta, N)$ (1.4.6) is

$$\tilde{m}_k := \int_0^\infty \lambda^k \tilde{\rho}_{(1)}^{(L)}(\lambda; \alpha_1, \beta, N)\, \mathrm{d}\lambda. \tag{1.4.8}$$

The zeroth moment is the mean, while the second and third moments relate to the variance and skew, respectively. Moreover, suppose that we have a statistic $f(\lambda)$ that is dependent on the eigenvalues of the (a, N) Laguerre β ensemble. If f is analytic at 0, we may expand it as a Taylor series about 0, say $f(\lambda) = \sum_{k=0}^\infty f_k \lambda^k$ for some constants f_k. We can then calculate the ensemble average of $f(\lambda)$ w.r.t. $\tilde{\rho}_{(1)}^{(L)}$ term by term by using the moments of the density, given that the average exists:

$$\langle f(\lambda) \rangle^* := \int_0^\infty f(\lambda) \tilde{\rho}_{(1)}^{(L)}(\lambda; \alpha_1, \beta, N)\, \mathrm{d}\lambda = \sum_{k=0}^\infty f_k \tilde{m}_k.$$

Here, we use the $*$ to denote that the ensemble average is taken w.r.t. the smoothed eigenvalue density, rather than the true density (1.10). By a simple change of variables, for $k \in \mathbb{N}_{>0}$, the k^{th} moment of $\rho_{(1)}^{(L)}$ is given by

$$m_k := \int_0^\infty \lambda^k \rho_{(1)}^{(L)}(\lambda; \alpha_1, \beta, N)\, \mathrm{d}\lambda = N^{k+1} \tilde{m}_k, \tag{1.4.9}$$

so we may work with the smoothed eigenvalue density without loss of information.

Proposition 5. *There exist coefficients $a_i^{(k)}$ such that each moment of the smoothed eigenvalue density has the form*

$$\tilde{m}_k = \sum_{i=0}^k a_i^{(k)} N^{-i}. \tag{1.4.10}$$

This fact was shown in [Mezzadri and Reynolds, 2015] through considerations of Ferrers diagrams and Jack polynomials. This result has also been derived in the $\beta = 2$ case through a counting problem regarding the bicolorings of $2k$-gons [Di Francesco, 2003]. We remark that the moments of the Laguerre β ensemble are also linked to alternating Motzkin paths, as seen in chapters 5-6 of [Dumitriu, 2003].

1.4.4 The Resolvent and Related Correlators

In our goal to compute the moments of the smoothed eigenvalue density, we introduce the resolvent.

Definition 1.18. The *resolvent* of the smoothed eigenvalue density of the (a, N) Laguerre β ensemble is

$$W_1(s) := \left\langle \frac{N}{s-\lambda} \right\rangle^* = \left\langle \sum_{j=1}^{N} \frac{1}{s-\lambda_1} \right\rangle^{L\beta E^*}. \tag{1.4.11}$$

The resolvent $W_1(s)$ acts as a generating function for the moments of the smoothed eigenvalue density. As $s \to \infty$, $\frac{1}{s-\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{s^{k+1}}$, through consideration of the geometric series. Hence,

$$\frac{1}{N} W_1(s) \overset{s\to\infty}{\sim} \frac{1}{s} + \sum_{k=1}^{\infty} \frac{\tilde{m}_k}{s^{k+1}}. \tag{1.4.12}$$

We will need higher order versions of the resolvent, which we will simply refer to as correlators.

Definition 1.19. Consider the (a, N) Laguerre β ensemble. For $1 \leqslant n \leqslant N$, the *unconnected n-point correlator* is

$$U_n(s_1, \ldots, s_n) := \left\langle \sum_{j_1,\ldots,j_n=1}^{N} \frac{1}{(s_1-\lambda_{j_1}) \cdots (s_n-\lambda_{j_n})} \right\rangle^{L\beta E^*}. \tag{1.4.13}$$

We call these the unconnected correlators in order to distinguish them from similar quantities which we call the connected correlators.

Definition 1.20. In the setting above, let $1 \leqslant n \leqslant N$, and let $X_1, \ldots, X_n$ be some operators. Then, the *connected ensemble average* of the product $X_1 \cdots X_n$ is

$$\langle X_1 \cdots X_n \rangle_c^{L\beta E^*} := \sum_{m=1}^{n} \sum_{G} (-1)^{m-1} (m-1)! \prod_{j=1}^{m} \langle X_{g_j(1)} \cdots X_{g_j(|G_j|)} \rangle^{L\beta E^*}, \tag{1.4.14}$$

where the sum over G is over all subdivisions $G_1 \cup \cdots \cup G_m$ of $\{1, \ldots, n\}$ into m subsets, with the g_j defined by $G_j = \{g_j(1), \ldots, g_j(|G_j|)\}$ [Forrester, 2010, p.187].

Definition 1.21. Consider the (a, N) Laguerre β ensemble. For $1 \leqslant n \leqslant N$, the *connected n-point correlator* is

$$W_n(s_1, \ldots, s_n) := \left\langle \sum_{j_1, \ldots, j_n = 1}^{N} \frac{1}{(s_1 - \lambda_{j_1}) \cdots (s_n - \lambda_{j_n})} \right\rangle_c^{L\beta E^*}. \tag{1.4.15}$$

We now present some relations between the connected and unconnected correlators due to [Smith, 1995], as seen in [Witte and Forrester, 2015, pp.8-9].

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} W_n(s_1, \ldots, s_n) = \log \left(1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} U_n(s_1, \ldots, s_n) \right), \tag{1.4.16}$$

$$U_n(x_1, J_n) = W_n(x_1, J_n) + \sum_{\emptyset \neq J \subseteq J_n} W_{n-|J|}\left(x_1, J_n \setminus J\right) U_{|J|}(J), \quad J_n = (x_2, \ldots, x_n). \tag{1.4.17}$$

We observe that the connected correlators are defined through an inclusion-exclusion principle, in an effort to induce cancellations; connected correlators of higher order in n decay faster in N, compared to the unconnected versions. The probability theorist may prefer to think of moments and cumulants. We work with the connected correlators because of their faster decays in N. In fact, for $\beta, \alpha_1 \in \mathbb{R}_{>0}$, our connected correlators admit large N expansions of the form

$$W_n(s_1, \ldots, s_n) = N^{2-n} \kappa^{1-n} \sum_{l=0}^{\infty} \frac{W_n^l(s_1, \ldots, s_n)}{(N\sqrt{\kappa})^l}, \quad \kappa = \tfrac{\beta}{2}, \tag{1.4.18}$$

with W_n^l not dependent on N; the unconnected correlators do not admit such expansions. This large N expansion comes from the field of topological recursion, where such an expansion is proven to be valid if the resolvent satisfies certain hypotheses – hypotheses that our resolvent satisfies. See section 4 of [Borot and Guionnet, 2012].

This topological viewpoint also gives reasoning behind the terms "connected" and "unconnected". Indeed, these correlators are related to problems of enumerating maps, which are graphs embedded onto 2-dimensional manifolds [Zvonkin, 1997]. A reader versed in enumerative combinatorics may recognise that the relation between the generating functions of connected graphs and the generating functions of unconnected graphs is the exact same as the relation (1.4.16) given above.

1.4.5 The Stieltjes Transform

The resolvent is also given by the Stieltjes transform of $\tilde{\rho}_{(1)}^{(L)}$ [Wall, 1948, pp.247-250].

Definition 1.22. Let ρ be a density with support $I \subset \mathbb{R}$. Then, for all $s \in \mathbb{C} \setminus \mathbb{R}$, the *Stieltjes transform* of ρ is

$$W(s) = \mathcal{ST}\{\rho(\lambda); \lambda \to s\} := \int_I \frac{\rho(\lambda)}{s-\lambda}\,\mathrm{d}\lambda. \tag{1.4.19}$$

It is evident that $\frac{1}{N}W_1(s) = \mathcal{ST}\{\tilde{\rho}_{(1)}^{(L)}(\lambda; \alpha_1, \beta, N); \lambda \to s\}$. Moreover, we may extract the smoothed eigenvalue density from the resolvent via the Sokhotski-Plemelj theorem.

Proposition 6. *Let W be the Stieltjes transform of a function that is continuous over some interval $I \subset \mathbb{R}$. Then, for all $\lambda \in I$, the inverse Stieltjes transform of W is given by*

$$\rho(\lambda) = \mathcal{IST}\{W(z); z \to \lambda\} := \frac{1}{2\pi\mathrm{i}} \lim_{\epsilon \to 0} \left(W(\lambda - \mathrm{i}\epsilon) - W(\lambda + \mathrm{i}\epsilon)\right). \tag{1.4.20}$$

We remark that in the case that $\rho(\lambda)$ extends to an analytic function in the neighbourhood of $\mathbb{R}$, this is just a simple consequence of Cauchy's theorem.

It is relatively difficult to compute the inverse Stieltjes transform of a pole, so we give a result here.

Theorem 1.5. *Let $m \in \mathbb{N}_{>0}$ and let $\delta^{(m)}$ denote the m^{th} order derivative of the Dirac delta function (distribution). Then for $a \in \mathbb{R}$,*

$$\mathcal{IST}\{(z-a)^{-m}; z \to \lambda\} = \frac{(-1)^{m-1}}{(m-1)!}\delta^{(m-1)}(\lambda - a). \tag{1.4.21}$$

Proof. We prove by taking the Stieltjes transform of the right hand side. First, note that using the chain rule, $\frac{\mathrm{d}^{m-1}}{\mathrm{d}a^{m-1}}\delta(\lambda - a) = (-1)^{m-1}\frac{\mathrm{d}^{m-1}}{\mathrm{d}\lambda^{m-1}}\delta(\lambda - a)$. Thus,

$$\begin{aligned}
\frac{(-1)^{m-1}}{(m-1)!}\int_{-\infty}^{\infty} \frac{\delta^{(m-1)}(\lambda - a)}{z-\lambda}\,\mathrm{d}\lambda &= \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}a^{m-1}}\int_{-\infty}^{\infty}\frac{\delta(\lambda-a)}{z-\lambda}\,\mathrm{d}\lambda \\
&= \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}a^{m-1}}\frac{1}{z-a} \\
&= \frac{1}{(z-a)^m}.
\end{aligned}$$

□

Chapter 2

The Loop Equation Analysis

We adopt the loop equation formalism as seen in [Witte and Forrester, 2015], [Witte and Forrester, 2014], with $\beta \in \mathbb{R}_{>0}$ and $a \in \mathbb{N}_0$ such that $\alpha_1 = \frac{\beta}{2}(a+1) - 1 > 0$. We remark that the name *loop equation* has historical origins in quantum field theory [Brézin, 1993, pp.798-863].

2.1 Preliminaries Particular to the Loop Equations

Throughout this chapter, we fix a, β, α_1 and either fix N as a large integer, or consider the large N limit. Our analysis is valid for $\beta \in \mathbb{R}_{>0}$, so long as $\alpha_1 \in \mathbb{R}_{>0}$. We remind the reader that $\beta = 1$, 2, and 4 are special cases corresponding to the LOE, LUE, and LSE. Whenever we deal with the n-point connected or unconnected correlator, it will be assumed that $n \leqslant N$. As per definition 1.11 of the Vandermonde determinant in subsection 1.4.1, we will denote the product $\prod_{1 \leqslant j < k \leqslant N} (\lambda_k - \lambda_j)$ by $\Delta(\lambda)$, dropping the subscript N.

We are mainly concerned with the global scaled eigenvalue density and j.p.d.f. of the (a, N) Laguerre β ensemble. To this end, we let LβE* denote

$$\{\tfrac{1}{N} X \mid X \text{ a member of the } (a, N) \text{ L}\beta\text{E}\},$$

so that the eigenvalue density of the LβE* is given by $\tilde{\rho}^{(L)}_{(1)}(\lambda; \alpha_1, \beta, N)$ (1.13).

We provide a summary of the quantities from chapter 1 that we will be using in chapter 2.

Definition 2.1. For $1 \leqslant n \leqslant N$, the *connected n-point correlator* is

$$W_n(s_1, \dots, s_n) := \left\langle \sum_{j_1, \dots, j_n = 1}^{N} \frac{1}{(s_1 - \lambda_{j_1}) \cdots (s_n - \lambda_{j_n})} \right\rangle_c^{L\beta E^*},$$

while the *unconnected n-point correlator* is

$$U_n(s_1, \dots, s_n) := \left\langle \sum_{j_1, \dots, j_n = 1}^{N} \frac{1}{(s_1 - \lambda_{j_1}) \cdots (s_n - \lambda_{j_n})} \right\rangle^{L\beta E^*}.$$

Here, $\langle \cdot \rangle^{\mathrm{L\beta E^*}}$ denotes the ensemble average with respect to the $L\beta E^*$ *scaled measure* given in definition 1.14,

$$\mathrm{d}\mu_{L\beta E^*}(\lambda) := |\Delta(\lambda)|^\beta \prod_{i=1}^{N} \lambda_i^{\alpha_1} e^{-\beta N \lambda_i / 2} \, \mathrm{d}\lambda_i.$$

The identities (1.4.14) and (1.4.17) will be invaluable.

$$U_n(x_1, J_n) = W_n(x_1, J_n) + \sum_{\emptyset \neq J \subseteq J_n} W_{n-|J|}(x_1, J_n \setminus J) \, U_{|J|}(J), \quad J_n = (x_2, \dots, x_n).$$

$$W_n(s_1, \dots, s_n) = \sum_{m=1}^{n} \sum_{G} (-1)^{m-1} (m-1)! \prod_{j=1}^{m} U_{|G_j|}(s_{g_j(1)}, \dots, s_{g_j(|G_j|)}),$$

where the sum over G is over all subdivisions $G_1 \cup \cdots \cup G_m$ of $\{1, \dots, n\}$ into m subsets, with the g_j defined by $G_j = \{g_j(1), \dots, g_j(|G_j|)\}$.

We are interested in these correlators because they allow us to form loop equations that will be used to compute the resolvent $W_1(s_1)$. The resolvent is of interest because it acts as a moment generating function for the moments of the smoothed eigenvalue density of the (a, N) Laguerre β ensemble (1.4.12).

It is important to note that the connected correlators admit large N expansions of the form

$$W_n(s_1, \dots, s_n) = N^{2-n} \kappa^{1-n} \sum_{l=0}^{\infty} \frac{W_n^l(s_1, \dots, s_n)}{(N\sqrt{\kappa})^l}, \quad \kappa = \frac{\beta}{2},$$

with W_n^l not dependent on N. We now outline the loop equation formalism used to calculate these large N expansions.

2.2 Loop Equations for the Laguerre β Ensembles

In this section, we derive loop equations which allow us to recursively compute higher order terms of our $W_n(s_1, \ldots, s_n)$. We introduce new variables and define yet another measure in an effort to keep our calculations tidy.

Definition 2.2. The *LβE altered measure* is

$$\mathrm{d}\overline{\mu}_{L\beta E}(\gamma) := |\Delta(\gamma)|^{\beta} \prod_{i=1}^{N} \gamma_i^{\alpha_1} e^{-\gamma_i} \, \mathrm{d}\gamma_i.$$

For present purposes, we use

$$\begin{aligned}
Z_N &:= \int_{[0,\infty)^N} \mathrm{d}\overline{\mu}_{L\beta E}(\gamma), \\
\langle \cdot \rangle &:= \frac{1}{Z_N} \int_{[0,\infty)^N} \cdot \, \mathrm{d}\overline{\mu}_{L\beta E}(\gamma), \\
\overline{W}_n(x_1, \ldots, x_n) &:= \left\langle \sum_{j_1,\ldots,j_n=1}^{N} \frac{1}{(x_1 - \gamma_{j_1}) \cdots (x_n - \gamma_{j_n})} \right\rangle_c, \\
\overline{U}_n(x_1, \ldots, x_n) &:= \left\langle \sum_{j_1,\ldots,j_n=1}^{N} \frac{1}{(x_1 - \gamma_{j_1}) \cdots (x_n - \gamma_{j_n})} \right\rangle
\end{aligned}$$

to denote what we will call the *altered partition function*, *altered ensemble average*, *altered connected n-point correlator*, and *altered unconnected n-point correlator*, respectively. We retain our meaning of "connected" versus "unconnected", so that we have

$$\overline{W}_n(x_1, \ldots, x_n) = \sum_{m=1}^{n} \sum_{G} (-1)^{m-1} (m-1)! \prod_{j=1}^{m} \overline{U}_{|G_i|}(x_{g_j(1)}, \ldots, x_{g_j(|G_i|)}),$$

where the sum over G is over all subdivisions of $\{1, \ldots, n\}$ into m subsets, with the g_j defined by $G_j = \{g_j(1), \ldots, g_j(|G_j|)\}$.

We now use these definitions to derive our unscaled loop equations, and then scale them so that they yield relationships between our $W_n(s_1, \ldots, s_n)$.

Henceforth, $\kappa = \frac{\beta}{2}$, $h = \sqrt{\kappa} - \frac{1}{\sqrt{\kappa}}$, and χ is taken to be the character function.

2.2.1 Some Identities via Aomoto's Method

The derivation of our loop equations is quite involved, so we first present some identities. The proofs of these identities will be in the style of Aomoto's method; recall the proof in subsection 1.4.1. Thus, our first two identities will be derived by considering the ensemble averages of partial derivatives.

Loop Equation Identity 1. The first identity we will use is

$$\begin{aligned}
0 &= \left\langle \sum_{j_1,\dots,j_n=1}^{N} \frac{1}{(x_1-\gamma_{j_1})^2(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})} \right\rangle \\
&+ \sum_{k=2}^{n} \left\langle \sum_{j_1,\dots,\hat{j_k},\dots,j_n=1}^{N} \frac{\chi_{n\neq 1}}{(x_1-\gamma_{j_1})\cdots(x_{k-1}-\gamma_{j_{k-1}})(x_k-\gamma_{j_1})^2(x_{k+1}-\gamma_{j_{k+1}})\cdots(x_n-\gamma_{j_n})} \right\rangle \\
&+ \alpha_1 \left\langle \sum_{j_1,\dots,j_n=1}^{N} \frac{1}{\gamma_{j_1}(x_1-\gamma_{j_1})\cdots(x_n-\gamma_{j_n})} \right\rangle - \left\langle \sum_{j_1,\dots,j_n=1}^{N} \frac{1}{(x_1-\gamma_{j_1})\cdots(x_n-\gamma_{j_n})} \right\rangle \\
&+ \beta \left\langle \sum_{j_1,\dots,j_n=1}^{N} \frac{1}{(x_1-\gamma_{j_1})\cdots(x_n-\gamma_{j_n})} \sum_{\substack{p=1 \\ p\neq j_1}}^{N} \frac{1}{\gamma_{j_1}-\gamma_p} \right\rangle, \qquad (2.2.1)
\end{aligned}$$

where $j_1,\dots,\hat{j_k},\dots,j_n = j_1,\dots,j_{k-1},j_{k+1},\dots,j_n$ and we require $\alpha_1 > 0$.

Proof. Consider the ensemble average

$$I_1(x_1,\dots,x_n) := \left\langle \sum_{j_1=1}^{N} \frac{\partial}{\partial\gamma_{j_1}} \frac{1}{x_1-\gamma_{j_1}} \sum_{j_2,\dots,j_n=1}^{N} \frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})} \right\rangle.$$

We interchange the sum over j_1 with the integral over $[0,\infty)^N$, noting that we're allowed to do so because the sum is finite for fixed N. Then, for each j_1, we evaluate the integral that corresponds to γ_{j_1}. By the fundamental theorem of calculus, our ensemble average is then

$$\begin{aligned}
&I_1(x_1,\dots,x_n) \\
&= \frac{1}{\overline{Z}_N} \sum_{j_1=1}^{N} \int_{[0,\infty)^{N-1}} \prod_{\substack{i=1 \\ i\neq j_1}}^{N} \mathrm{d}\gamma_i \\
&\qquad \times \left[\frac{1}{x_1-\gamma_{j_1}} \sum_{j_2,\dots,j_n=1}^{N} \frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})} |\Delta(\gamma)|^\beta \prod_{i=1}^{N} \gamma_i^{\alpha_1} e^{-\gamma_i} \right]_{\gamma_{j_1}=0}^{\infty}.
\end{aligned}$$

For each value of j_1,

$$\left[\frac{1}{x_1-\gamma_{j_1}}\sum_{j_2,\ldots,j_n=1}^{N}\frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})}|\Delta(\gamma)|^\beta\prod_{i=1}^{N}\gamma_i^{\alpha_1}e^{-\gamma_i}\right]_{\gamma_{j_1}=0}^{\infty}$$
$$=\lim_{\gamma_{j_1}\to\infty}\left[\frac{1}{x_1-\gamma_{j_1}}\sum_{j_2,\ldots,j_n=1}^{N}\frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})}|\Delta(\gamma)|^\beta\prod_{i=1}^{N}\gamma_i^{\alpha_1}e^{-\gamma_i}\right]=0,$$

where the above limit converges to 0 because it is dominated by the $e^{-\gamma_{j_1}}$ term. The remaining factors in the integrand of $I_1(x_1,\ldots,x_n)$ are bounded on $[0,\infty)^{N-1}$, so the integrand reduces to 0 for each value of j_1. Hence, our ensemble average is zero:

$$I_1(x_1,\ldots,x_n)=\frac{1}{Z_N}\sum_{j_1=1}^{N}\int_{[0,\infty)^{N-1}}0\prod_{\substack{i=1\\i\neq j_1}}^{N}\mathrm{d}\gamma_i=0.$$

On the other hand, we may differentiate with respect to γ_{j_1} by applying Leibniz's rule. The integrand of $I_1(x_1,\ldots,x_n)$ is then given by

$$\begin{aligned}
&\sum_{j_1=1}^{N}\frac{\partial}{\partial\gamma_{j_1}}\left\{\frac{1}{x_1-\gamma_{j_1}}\sum_{j_2,\ldots,j_n=1}^{N}\frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})}|\Delta(\gamma)|^\beta\prod_{i=1}^{N}\gamma_i^{\alpha_1}e^{-\gamma_i}\right\}\\
&\quad=\sum_{j_1=1}^{N}\frac{1}{(x_1-\gamma_{j_1})^2}\sum_{j_2,\ldots,j_n=1}^{N}\frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})}|\Delta(\gamma)|^\beta\prod_{i=1}^{N}\gamma_i^{\alpha_1}e^{-\gamma_i}\\
&\qquad+\sum_{j_1=1}^{N}\frac{1}{x_1-\gamma_{j_1}}\frac{\partial}{\partial\gamma_{j_1}}\left\{\sum_{j_2,\ldots,j_n=1}^{N}\frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})}\right\}|\Delta(\gamma)|^\beta\prod_{i=1}^{N}\gamma_i^{\alpha_1}e^{-\gamma_i}\\
&\qquad+\sum_{j_1,\ldots,j_n=1}^{N}\frac{1}{(x_1-\gamma_{j_1})\cdots(x_n-\gamma_{j_n})}\frac{\partial}{\partial\gamma_{j_1}}\left\{|\Delta(\gamma)|^\beta\right\}\prod_{i=1}^{N}\gamma_i^{\alpha_1}e^{-\gamma_i}\\
&\qquad+\sum_{j_1,\ldots,j_n=1}^{N}\frac{1}{(x_1-\gamma_{j_1})\cdots(x_n-\gamma_{j_n})}|\Delta(\gamma)|^\beta\left(\frac{\alpha_1}{\gamma_{j_1}}-1\right)\prod_{i=1}^{N}\gamma_i^{\alpha_1}e^{-\gamma_i}.
\end{aligned}$$

We compute that for $n\geqslant 2$,

$$\begin{aligned}
&\frac{\partial}{\partial\gamma_{j_1}}\left\{\sum_{j_2,\ldots,j_n=1}^{N}\frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})}\right\}\\
&\quad=\sum_{j_n=1}^{N}\frac{1}{x_n-\gamma_{j_n}}\frac{\partial}{\partial\gamma_{j_1}}\left\{\sum_{j_2,\ldots,j_{n-1}=1}^{N}\frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_{n-1}})}\right\}\\
&\qquad+\frac{1}{(x_n-\gamma_{j_1})^2}\sum_{j_2,\ldots,j_{n-1}=1}^{N}\frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_{n-1}})}.
\end{aligned}$$

Hence, by induction on n, we have for $n \geqslant 2$,

$$\frac{\partial}{\partial \gamma_{j_1}} \left\{ \sum_{j_2,\dots,j_n=1}^{N} \frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})} \right\} = \sum_{k=2}^{n} \sum_{j_2,\dots,\hat{j_k},\dots,j_n=1}^{N} \frac{1}{(x_2-\gamma_{j_2})\cdots(x_{k-1}-\gamma_{j_{k-1}})(x_k-\gamma_{j_1})^2(x_{k+1}-\gamma_{j_{k+1}})\cdots(x_n-\gamma_{j_n})}.$$

Moreover, we compute that

$$\frac{\partial}{\partial \gamma_{j_1}} \left\{ |\Delta(\gamma)|^\beta \right\} = \frac{\partial}{\partial \gamma_{j_1}} \left\{ \prod_{\substack{p=1 \\ p\neq j_1}}^{N} |\gamma_p - \gamma_{j_1}|^\beta \right\} \prod_{\substack{1\leqslant j<k\leqslant N \\ j,k\neq j_1}}^{N} |\gamma_j-\gamma_k|^\beta = \prod_{\substack{p=1 \\ p\neq j_1}}^{N} \frac{\beta}{\gamma_{j_1}-\gamma_p} |\Delta(\gamma)|^\beta.$$

Thus, the integrand of $I_1(x_1,\dots,x_n)$ simplifies to

$$\begin{aligned} &\left\{ \sum_{j_1,\dots,j_n=1}^{N} \frac{1}{(x_1-\gamma_{j_1})^2(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})} \right. \\ &\quad + \sum_{k=2}^{n} \sum_{j_1,\dots,\hat{j_k},\dots,j_n=1}^{N} \frac{\chi_{n\neq 1}}{(x_1-\gamma_{j_1})\cdots(x_{k-1}-\gamma_{j_{k-1}})(x_k-\gamma_{j_1})^2(x_{k+1}-\gamma_{j_{k+1}})\cdots(x_n-\gamma_{j_n})} \\ &\quad \left. + \sum_{j_1,\dots,j_n=1}^{N} \frac{1}{(x_1-\gamma_{j_1})\cdots(x_n-\gamma_{j_n})} \left(\sum_{\substack{p=1 \\ p\neq j_1}}^{N} \frac{\beta}{\gamma_{j_1}-\gamma_p} + \frac{\alpha_1}{\gamma_{j_1}} - 1 \right) \right\} |\Delta(\gamma)|^\beta \prod_{i=1}^{N} \gamma_i^{\alpha_1} e^{-\gamma_i}, \end{aligned}$$

and $I_1(x_1,\dots,x_n)$ is simply equal to the right hand side of our identity (2.2.1). We've shown earlier that $I_1(x_1,\dots,x_n)=0$, so the identity is proven. □

We also utilise an identity that is essentially a simpler version of our first loop equation identity.

Loop Equation Identity 2. The second identity we will use is

$$\begin{aligned} 0 = &\sum_{k=2}^{N} \left\langle \sum_{j_1,\dots,\hat{j_k},\dots,j_n=1}^{N} \frac{\chi_{n\neq 1}}{(x_2-\gamma_{j_2})\cdots(x_{k-1}-\gamma_{j_{k-1}})(x_k-\gamma_{j_1})^2(x_{k+1}-\gamma_{j_{k+1}})\cdots(x_n-\gamma_{j_n})} \right\rangle \\ &+ \alpha_1 \left\langle \sum_{j_1=1}^{N} \frac{1}{\gamma_{j_1}} \sum_{j_2,\dots,j_n=1}^{N} \frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})} \right\rangle - N \left\langle \sum_{j_2,\dots,j_n=1}^{N} \frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})} \right\rangle, \end{aligned} \tag{2.2.2}$$

where we recall that $j_1,\dots,\hat{j_k},\dots,j_n = j_1,\dots,j_{k-1},j_{k+1},\dots,j_n$ and we retain our requirement that $\alpha_1 > 0$.

Proof. This time, we begin by considering the ensemble average

$$I_2(x_1,\dots,x_n) := \left\langle \sum_{j_1=1}^{N} \frac{\partial}{\partial \gamma_{j_1}} \sum_{j_2,\dots,j_n=1}^{N} \frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})} \right\rangle.$$

We immediately obtain $I_2(x_1,\dots,x_n)=0$ by the fundamental theorem of calculus, as in the preceding proof.

Applying Leibniz's rule, we observe that

$$\begin{aligned}
&\sum_{j_1=1}^{N} \frac{\partial}{\partial \gamma_{j_1}} \left\{ \sum_{j_2,\dots,j_n=1}^{N} \frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})} |\Delta(\gamma)|^{\beta} \prod_{i=1}^{N} \gamma_i^{\alpha_1} e^{-\gamma_i} \right\} \\
&= \left\{ \sum_{k=2}^{n} \sum_{j_1,\dots,\hat{j_k},\dots,j_n=1}^{N} \frac{\chi_{n\neq 1}}{(x_2-\gamma_{j_2})\cdots(x_{k-1}-\gamma_{j_{k-1}})(x_k-\gamma_{j_1})^2(x_{k+1}-\gamma_{j_{k+1}})\cdots(x_n-\gamma_{j_n})} \right. \\
&\quad \left. + \sum_{j_1,\dots,j_n=1}^{N} \frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})} \left(\sum_{\substack{p=1 \\ p\neq j_1}}^{N} \frac{\beta}{\gamma_{j_1}-\gamma_p} + \frac{\alpha_1}{\gamma_{j_1}} - 1 \right) \right\} |\Delta(\gamma)|^{\beta} \prod_{i=1}^{N} \gamma_i^{\alpha_1} e^{-\gamma_i},
\end{aligned}$$

using calculations from the preceding proof. Furthermore, we note that

$$\sum_{j_1,\dots,j_n=1}^{N} \frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})} \sum_{\substack{p=1 \\ p\neq j_1}}^{N} \frac{\beta}{\gamma_{j_1}-\gamma_p} = 0,$$

since interchanging variables $j_1 \leftrightarrow p$ and then extracting a minus sign yields

$$\sum_{\substack{p,j_1=1 \\ p\neq j_1}}^{N} \frac{\beta}{\gamma_{j_1}-\gamma_p} = \sum_{\substack{p,j_1=1 \\ p\neq j_1}}^{N} \frac{\beta}{\gamma_p-\gamma_{j_1}} = -\sum_{\substack{p,j_1=1 \\ p\neq j_1}}^{N} \frac{\beta}{\gamma_{j_1}-\gamma_p}.$$

This then completes the proof. □

Our first two identities are all that are necessary to derive our loop equations via Aomoto's method. However, we supplement them with a few more identities that will aide in our calculations.

Loop Equation Identity 3. We present our third loop equation identity without proof,

$$\frac{\partial}{\partial x_1} \overline{U}_n(x_1,\dots,x_n) = -\left\langle \sum_{j_1,\dots,j_n=1}^{N} \frac{1}{(x_1-\gamma_{j_1})^2(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})} \right\rangle. \tag{2.2.3}$$

□

Loop Equation Identity 4. For $2 \leqslant k \leqslant n$, our fourth loop equation identity is

$$\begin{aligned}
&\frac{\partial}{\partial x_k}\left\{\frac{\overline{U}_{n-1}(x_1,\dots,\hat{x}_k,\dots,x_n)-\overline{U}_{n-1}(x_2,\dots,x_n)}{x_k-x_1}\right\}\\
&=\frac{\partial}{\partial x_k}\left\langle \sum_{j_1,\dots,\hat{j_k},\dots,j_n=1}^{N}\frac{1}{(x_1-\gamma_{j_1})\cdots(x_{k-1}-\gamma_{j_{k-1}})(x_k-\gamma_{j_1})(x_{k+1}-\gamma_{j_{k+1}})\cdots(x_n-\gamma_{j_n})}\right\rangle\\
&=-\left\langle \sum_{j_1,\dots,\hat{j_k},\dots,j_n=1}^{N}\frac{1}{(x_1-\gamma_{j_1})\cdots(x_{k-1}-\gamma_{j_{k-1}})(x_k-\gamma_{j_1})^2(x_{k+1}-\gamma_{j_{k+1}})\cdots(x_n-\gamma_{j_n})}\right\rangle.
\end{aligned} \tag{2.2.4}$$

Proof. The first equality follows from the fact that

$$\sum_{j_1=1}^{N}\frac{\frac{1}{(x_1-\gamma_{j_1})}-\frac{1}{(x_k-\gamma_{j_1})}}{x_k-x_1}=\sum_{j_1=1}^{N}\frac{1}{(x_1-\gamma_{j_1})(x_k-\gamma_{j_1})}$$

and that

$$\begin{aligned}
&\overline{U}_{n-1}(x_1,\dots,\hat{x}_k,\dots,x_n)-\overline{U}_{n-1}(x_2,\dots,x_n)\\
&=\sum_{j_1=1}^{N}\left(\frac{1}{(x_1-\gamma_{j_1})}-\frac{1}{(x_k-\gamma_{j_k})}\right)\\
&\qquad\times\sum_{j_2,\dots,\hat{j_k},\dots,j_n=1}^{N}\frac{1}{(x_2-\gamma_{j_2})\cdots(x_{k-1}-\gamma_{j_{k-1}})(x_{k+1}-\gamma_{j_{k+1}})\cdots(x_n-\gamma_{j_n})}.
\end{aligned}$$

The second equality is a simple derivative. □

Loop Equation Identity 5. Our fifth loop equation identity is

$$\begin{aligned}
&\frac{\alpha_1}{x_1}\overline{U}_n(x_1,\dots,x_n)+\frac{N}{x_1}\overline{U}_{n-1}(x_2,\dots,x_n)+\frac{1}{x_1}\sum_{k=2}^{n}\frac{\partial}{\partial x_k}\overline{U}_{n-1}(x_2,\dots,x_n)\chi_{n\neq 1}\\
&=\alpha_1\left\langle\sum_{j_1=1}^{N}\frac{1}{\gamma_{j_1}(x_1-\gamma_{j_1})}\sum_{j_2,\dots,j_n=1}^{N}\frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})}\right\rangle.
\end{aligned} \tag{2.2.5}$$

Proof. First, we observe that by loop equation identity (2.2.2), the left hand side of identity (2.2.5) is equal to

$$\frac{\alpha_1}{x_1}\overline{U}_n(x_1,\dots,x_n)+\frac{\alpha_1}{x_1}\left\langle\sum_{j_1=1}^{N}\frac{1}{\gamma_{j_1}}\sum_{j_2,\dots,j_n=1}^{N}\frac{1}{(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})}\right\rangle,$$

which is seen to be equal to the right hand side of identity (2.2.5) upon realisation of the fact that $\sum_{j_1=1}^{N}\left(\frac{1}{x_1(x_1-\gamma_{j_1})}+\frac{1}{x_1\gamma_{j_1}}\right)=\sum_{j_1=1}^{N}\frac{1}{\gamma_{j_1}(x_1-\gamma_{j_1})}$. □

Loop Equation Identity 6. Our sixth and final loop equation identity is

$$\begin{aligned}\kappa\overline{U}_{n+1}(x_1,x_1,\dots,x_n)+\kappa\frac{\partial}{\partial x_1}\overline{U}_n(x_1,\dots,x_n)\\ =\beta\left\langle\sum_{j_1,\dots,j_n=1}^{N}\frac{1}{(x_1-\gamma_{j_1})\cdots(x_n-\gamma_{j_n})}\sum_{\substack{p=1\\p\neq j_1}}^{N}\frac{1}{\gamma_{j_1}-\gamma_p}\right\rangle,\end{aligned}\tag{2.2.6}$$

where we recall that $\kappa=\frac{\beta}{2}$.

Proof. Interchanging variables $j_1\leftrightarrow p$ gives us

$$\sum_{\substack{p,j_1=1\\p\neq j_1}}\frac{1}{(x_1-\gamma_{j_1})(\gamma_{j_1}-\gamma_p)}=\sum_{\substack{p,j_1=1\\p\neq j_1}}\frac{1}{(x_1-\gamma_p)(\gamma_p-\gamma_{j_1})},$$

hence

$$\begin{aligned}2\sum_{\substack{p,j_1=1\\p\neq j_1}}\frac{1}{(x_1-\gamma_{j_1})(\gamma_{j_1}-\gamma_p)}&=\sum_{\substack{p,j_1=1\\p\neq j_1}}\frac{1}{\gamma_{j_1}-\gamma_p}\left(\frac{1}{x_1-\gamma_{j_1}}-\frac{1}{x_1-\gamma_p}\right)\\&=\sum_{\substack{p,j_1=1\\p\neq j_1}}\frac{1}{(x_1-\gamma_{j_1})(x_1-\gamma_p)}.\end{aligned}$$

Substituting this into the right hand side of our identity (2.2.6), we obtain

$$\begin{aligned}&\frac{\beta}{2}\left\langle\sum_{j_2,\dots,j_n=1}^{N}\frac{1}{(x_1-\gamma_{j_1})\cdots(x_n-\gamma_{j_n})}\sum_{\substack{p,j_1=1\\p\neq j_1}}^{N}\frac{1}{(x_1-\gamma_{j_1})(x_1-\gamma_p)}\right\rangle\\&=\kappa\left\langle\sum_{p,j_1,\dots,j_n=1}^{N}\frac{1}{(x_1-\gamma_p)(x_1-\gamma_{j_1})\cdots(x_n-\gamma_{j_n})}\right\rangle\\&\quad-\kappa\left\langle\sum_{j_1,\dots,j_n=1}^{N}\frac{1}{(x_1-\gamma_{j_1})^2(x_2-\gamma_{j_2})\cdots(x_n-\gamma_{j_n})}\right\rangle\\&=\kappa\overline{U}_{n+1}(x_1,x_1,\dots,x_n)+\kappa\frac{\partial}{\partial x_1}\left\langle\sum_{j_1,\dots,j_n=1}^{N}\frac{1}{(x_1-\gamma_{j_1})\cdots(x_n-\gamma_{j_n})}\right\rangle\\&=\kappa\overline{U}_{n+1}(x_1,x_1,\dots,x_n)+\kappa\frac{\partial}{\partial x_1}\overline{U}_n(x_1,\dots,x_n),\end{aligned}$$

which is equal to the left hand side of loop equation identity (2.2.6). □

We are now well-equipped to derive our loop equations.

2.2.2 The Unscaled Loop Equations

We now use our loop equation identities to derive unscaled loop equations for our altered connected correlators. We present an inductive construction.

Theorem 2.1. *Our first unscaled loop equation is*

$$0 = \kappa \overline{W}_2(x_1, x_1) + \kappa \left(\overline{W}_1(x_1)\right)^2 + (\kappa - 1)\overline{W}_1'(x_1) + \left(\frac{\alpha_1}{x_1} - 1\right)\overline{W}_1(x_1) + \frac{N}{x_1}. \quad (2.2.7)$$

Proof. For $n = 1$, our loop equation identity (2.2.1) reads

$$0 = \left\langle \sum_{j_1=1}^{N} \frac{1}{(x_1 - \gamma_{j_1})^2} \right\rangle + \alpha_1 \left\langle \sum_{j_1=1}^{N} \frac{1}{\gamma_{j_1}(x_1 - \gamma_{j_1})} \right\rangle - \left\langle \sum_{j_1=1}^{N} \frac{1}{x_1 - \gamma_{j_1}} \right\rangle + \beta \left\langle \sum_{\substack{p, j_1 = 1 \\ p \neq j_1}}^{N} \frac{1}{(x_1 - \gamma_{j_1})(\gamma_{j_1} - \gamma_p)} \right\rangle.$$

In the above, the first term is simply $-\frac{\partial}{\partial x_1}\overline{U}_1(x_1)$ by loop equation identity (2.2.3), the second term is $\frac{\alpha_1}{x_1}\overline{U}_1(x_1) + \frac{N}{x_1}$ by loop equation identity (2.2.5), and the last term is $\kappa\overline{U}_2(x_1, x_1) + \kappa\frac{\partial}{\partial x_1}\overline{U}_1(x_1)$ by loop equation identity (2.2.6). Hence, the above equation simplifies to

$$0 = -\overline{U}_1'(x_1) + \frac{\alpha_1}{x_1}\overline{U}_1(x_1) + \frac{N}{x_1} - \overline{U}_1(x_1) + \kappa\overline{U}_2(x_1, x_1) + \kappa\overline{U}_1'(x_1).$$

Noting that $\overline{U}_1(x_1) = \overline{W}_1(x_1)$ and $\overline{U}_2(x_1, x_1) = \overline{W}_2(x_1, x_1) + \left(\overline{W}_1(x_1)\right)^2$ completes the proof. □

Theorem 2.2. *Our second unscaled loop equation is*

$$0 = (\kappa - 1)\frac{\partial}{\partial x_1}\overline{W}_2(x_1, x_2) + \frac{\partial}{\partial x_2}\left\{\frac{\overline{W}_1(x_1) - \overline{W}_1(x_2)}{x_1 - x_2}\right\} + \left(\frac{\alpha_1}{x_1} - 1\right)\overline{W}_2(x_1, x_2) + \kappa\left[\overline{W}_3(x_1, x_1, x_2) + 2\overline{W}_2(x_1, x_2)\overline{W}_1(x_1)\right] + \frac{1}{x_1}\frac{\partial}{\partial x_2}\overline{W}_1(x_2). \quad (2.2.8)$$

Proof. For $n = 2$, our loop equation identity (2.2.1) reads

$$0 = -\frac{\partial}{\partial x_1}\overline{U}_2(x_1, x_2) + \left\langle \sum_{j_1=1}^{N} \frac{1}{(x_1 - \gamma_{j_1})(x_2 - \gamma_{j_1})^2} \right\rangle + \alpha_1 \left\langle \sum_{j_1, j_2 = 1}^{N} \frac{1}{\gamma_{j_1}(x_1 - \gamma_{j_1})(x_2 - \gamma_{j_2})} \right\rangle - \overline{U}_2(x_1, x_2) + \beta \left\langle \sum_{j_1, j_2 = 1}^{N} \frac{1}{(x_1 - \gamma_{j_1})(x_2 - \gamma_{j_2})} \sum_{\substack{p=1 \\ p \neq j_1}}^{N} \frac{1}{\gamma_{j_1} - \gamma_p} \right\rangle,$$

which, upon substitution of our remaining loop equation identities, becomes

$$0 = -\frac{\partial}{\partial x_1}\overline{U}_2(x_1,x_2) + \frac{\partial}{\partial x_2}\left\{\frac{\overline{U}_1(x_1)-\overline{U}_1(x_2)}{x_1-x_2}\right\} + \frac{\alpha_1}{x_1}\overline{U}_2(x_1,x_2) + \frac{N}{x_1}\overline{U}_1(x_2) + \frac{1}{x_1}\frac{\partial}{\partial x_2}\overline{U}_1(x_2) - \overline{U}_2(x_1,x_2) + \kappa\overline{U}_3(x_1,x_1,x_2) + \kappa\frac{\partial}{\partial x_1}\overline{U}_2(x_1,x_2).$$

Substituting

$$\begin{aligned}
\overline{U}_1(x_i) &= \overline{W}_1(x_i), \quad i=1,2,\\
\overline{U}_2(x_1,x_2) &= \overline{W}_2(x_1,x_2) + \overline{W}_1(x_1)\overline{W}_1(x_2),\\
\overline{U}_3(x_1,x_1,x_2) &= \overline{W}_3(x_1,x_1,x_2) + 2\overline{U}_1(x_1)\overline{U}_2(x_1,x_2) + \overline{U}_1(x_2)\overline{U}_2(x_1,x_1)\\
&\quad - 2\left(\overline{U}_1(x_1)\right)^2\overline{U}_1(x_2)
\end{aligned}$$

into the above, we obtain

$$\begin{aligned}
0 &= (\kappa-1)\frac{\partial}{\partial x_1}\overline{W}_2(x_1,x_2) + (\kappa-1)\overline{W}_1'(x_1)\overline{W}_1(x_2) + \frac{\partial}{\partial x_2}\left\{\frac{\overline{W}_1(x_1)-\overline{W}_1(x_2)}{x_1-x_2}\right\}\\
&+ \left(\frac{\alpha_1}{x_1}-1\right)\left[\overline{W}_2(x_1,x_2)+\overline{W}_1(x_1)\overline{W}_1(x_2)\right] + \frac{N}{x_1}\overline{W}_1(x_2) + \frac{1}{x_1}\overline{W}_1'(x_2)\\
&+ \kappa\left[\overline{W}_3(x_1,x_1,x_2) + 2\overline{W}_1(x_1)\overline{W}_2(x_1,x_2) + \overline{W}_1(x_2)\overline{W}_2(x_1,x_1) + \left(\overline{W}_1(x_1)\right)^2\overline{W}_1(x_2)\right].
\end{aligned}$$

Subtracting the right hand side of our first unscaled loop equation (2.2.7) multiplied by $\overline{W}_1(x_2)$ completes the proof. □

Theorem 2.3. *Define J_n to be the ordered set $(x_2,\ldots,x_n)$. Then, for $n \geqslant 2$, the n^{th} unscaled loop equation is*

$$\begin{aligned}
0 &= \left[(\kappa-1)\frac{\partial}{\partial x_1} + \left(\frac{\alpha_1}{x_1}-1\right)\right]\overline{W}_n(x_1,J_n)\\
&+ \sum_{k=2}^{n}\frac{\partial}{\partial x_k}\left\{\frac{\overline{W}_{n-1}(x_1,\ldots,\hat{x}_k,\ldots,x_n)-\overline{W}_{n-1}(J_n)}{x_1-x_k} + \frac{1}{x_1}\overline{W}_{n-1}(J_n)\right\}\\
&+ \kappa\left[\overline{W}_{n+1}(x_1,x_1,J_n) + \sum_{J\subseteq J_n}\overline{W}_{|J|+1}(x_1,J)\overline{W}_{n-|J|}(x_1,J_n\setminus J)\right]. \qquad (2.2.9)
\end{aligned}$$

Proof. First, we present the equivalent equation for the unconnected correlators. For $n \geqslant 2$,

$$\begin{aligned} 0 = &\left[(\kappa-1)\frac{\partial}{\partial x_1} + \left(\frac{\alpha_1}{x_1} - 1\right)\right] \overline{U}_n(x_1, J_n) + \frac{N}{x_1}\overline{U}_{n-1}(J_n) \\ &+ \sum_{k=2}^{n} \frac{\partial}{\partial x_k}\left\{\frac{\overline{U}_{n-1}(x_1,\ldots,\hat{x}_k,\ldots,x_n) - \overline{U}_{n-1}(J_n)}{x_1 - x_k} + \frac{1}{x_1}\overline{U}_{n-1}(J_n)\right\} \\ &+ \kappa\overline{U}_{n+1}(x_1, x_1, J_n). \end{aligned} \tag{2.2.10}$$

This equation can be derived by substituting loop equation identities (2.2.3) to (2.2.6) into loop equation identity (2.2.1) in the same way as in the derivation of the first two unscaled loop equations.

We prove by induction on n, with $n = 2$ serving as our base case. For this purpose, let $I_3(x_1, \ldots, x_n)$ denote the right hand side of our n^{th} unscaled loop equation (2.2.9) and, for our induction hypothesis, assume that $I_3(x_1, \ldots, x_m) = 0$ for all $1 \leqslant m \leqslant n-1$. We rewrite identity (1.4.17) in terms of the altered correlators,

$$\overline{U}_n(x_1, J_n) = \overline{W}_n(x_1, J_n) + \sum_{\emptyset \neq J \subseteq J_n} \overline{W}_{n-|J|}\left(x_1, J_n \setminus J\right)\overline{U}_{|J|}(J), \quad J_n = (x_2, \ldots, x_n).$$

Subtracting $\overline{U}_{n-1}(J_n)I_3(x_1)$ from the right hand side of the loop equation for unconnected correlators (2.2.10) then leaves us with

$$\begin{aligned} 0 = &\left[(\kappa-1)\frac{\partial}{\partial x_1} + \left(\frac{\alpha_1}{x_1} - 1\right)\right] \left\{\overline{W}_n(x_1, J_n) + \sum_{\emptyset \neq J \subset J_n} \overline{W}_{n-|J|}\left(x_1, J_n \setminus J\right)\overline{U}_{|J|}(J)\right\} \\ &+ \sum_{k=2}^{n} \frac{\partial}{\partial x_k}\left\{\frac{\overline{U}_{n-1}(x_1,\ldots,\hat{x}_k,\ldots,x_n) - \overline{U}_{n-1}(J_n)}{x_1 - x_k} + \frac{1}{x_1}\overline{U}_{n-1}(J_n)\right\} \\ &+ \kappa\overline{W}_{n+1}(x_1, x_1, J_n) + \kappa \sum_{\emptyset \neq J \subset J_n} \overline{W}_{n+1-|J|}(x_1, x_1, J_n \setminus J)\overline{U}_{|J|}(J) \\ &+ \kappa \sum_{J \subseteq J_n} \overline{W}_{n-|J|}(x_1, J_n \setminus J)\overline{U}_{|J|+1}(x_1, J) - \kappa\overline{W}_1(x_1)\overline{W}_1(x_1)\overline{U}_{n-2}(J_n), \end{aligned} \tag{2.2.11}$$

where $\subset$ denotes a strict or proper subset, and $\subseteq$ denotes otherwise. Here, we have used the relation

$$\overline{U}_{n+1}(x_1, x_1, J_n) = \overline{W}_{n+1}(x_1, x_1, J_n) + \sum_{\emptyset \neq J' \subseteq (x_1, J_n)} \overline{W}_{n-|J'|}(x_1, (x_1, J_n) \setminus J')\overline{U}_{|J'|}(J')$$

and the fact that $\{\emptyset \neq J' \subseteq (x_1, J_n)\} = \{\emptyset \neq J' \subseteq J_n\} \cup \{J' = (x_1, J'') \mid J'' \subseteq J_n\}$.

Continuing on, we note that for any $J \subseteq J_n$,

$$\overline{U}_{|J|}(J)\frac{\partial}{\partial x_1}\overline{W}_{n-|J|}(x_1, J_n \setminus J) = \frac{\partial}{\partial x_1}\left\{\overline{W}_{n-|J|}(x_1, J_n \setminus J)\overline{U}_{|J|}(J)\right\},$$

$$\begin{aligned}&\overline{U}_{|J|}(J)\frac{\partial}{\partial x_k}\frac{\overline{W}_{n-1-|J|}(x_1, (J_n \setminus J) \setminus (x_k)) - \overline{W}_{n-1-|J|}(J_n \setminus J)}{x_1 - x_k}\\ &= \frac{\partial}{\partial x_k}\frac{\overline{W}_{n-1-|J|}(x_1, (J_n \setminus J) \setminus (x_k))\overline{U}_{|J|}(J) - \overline{W}_{n-1-|J|}(J_n \setminus J)\overline{U}_{|J|}(J)}{x_1 - x_k}, \quad x_k \in (J_n \setminus J),\end{aligned}$$

$$\overline{U}_{|J|}(J)\frac{\partial}{\partial x_k}\left\{\frac{1}{x_1}\overline{W}_{n-1-|J|}(J_n \setminus J)\right\} = \frac{\partial}{\partial x_k}\left\{\frac{1}{x_1}\overline{W}_{n-1-|J|}(J_n \setminus J)\overline{U}_{|J|}(J)\right\}, \quad x_k \in (J_n \setminus J),$$

as the relevant partial derivatives do not depend on any of the variables present in our $\overline{U}$ multipliers. Hence, subtracting $\sum_{\emptyset \neq J \subset J_n} I_3(x_1, J_n \setminus J)\overline{U}_{|J|}(J)$ from our equation (2.2.11) results in

$$\begin{aligned}0 = &\left[(\kappa - 1)\frac{\partial}{\partial x_1} + \left(\frac{\alpha}{x_1} - 1\right)\right]\overline{W}_n(x_1, J_n) + \kappa\overline{W}_{n+1}(x_1, x_1, J_n)\\ &+ \sum_{k=2}^{n}\frac{\partial}{\partial x_k}\left\{\frac{\overline{W}_{n-1}(x_1, \ldots, \hat{x}_k, \ldots, x_n) - \overline{W}_{n-1}(J_n)}{x_1 - x_k} + \frac{1}{x_1}\overline{W}_{n-1}(J_n)\right\}\\ &+ \kappa\sum_{J \subseteq J_n}\overline{W}_{n-|J|}(x_1, J_n \setminus J)\overline{U}_{|J|+1}(x_1, J) - \kappa\overline{W}_1(x_1)\overline{W}_1(x_1)\overline{U}_{n-2}(J_n)\\ &- \kappa\sum_{\emptyset \neq J \subset J_n}\sum_{K \subseteq J_n \setminus J}\overline{W}_{|K|+1}(x_1, K)\overline{W}_{n-|K|}(x_1, (J_n \setminus J) \setminus K)\overline{U}_{|J|}(J) \qquad (2.2.12)\end{aligned}$$

The last two lines of the above (2.2.12) then simplify as follows:

$$\begin{aligned}&\kappa\sum_{J \subseteq J_n}\overline{W}_{n-|J|}(x_1, J_n \setminus J)\overline{U}_{|J|+1}(x_1, J) - \kappa\overline{W}_1(x_1)\overline{W}_1(x_1)\overline{U}_{n-2}(J_n)\\ &\quad - \kappa\sum_{\emptyset \neq J \subset J_n}\sum_{K \subseteq J_n \setminus J}\overline{W}_{|K|+1}(x_1, K)\overline{W}_{n-|K|}(x_1, (J_n \setminus J) \setminus K)\overline{U}_{|J|}(J)\\ &= \kappa\sum_{J \subseteq J_n}\overline{W}_{n-|J|}(x_1, J_n \setminus J)\overline{U}_{|J|+1}(x_1, J) + \kappa\sum_{K \subseteq J_n}\overline{W}_{|K|+1}(x_1, K)\overline{W}_{n-|K|}(x_1, J_n \setminus K)\\ &\quad - \kappa\sum_{J \subseteq J_n}\sum_{K \subseteq J_n \setminus J}\overline{W}_{|K|+1}(x_1, K)\overline{W}_{n-|K|}(x_1, (J_n \setminus J) \setminus K)\overline{U}_{|J|}(J),\end{aligned}$$

where we absorb the $\kappa\overline{W}_1(x_1)\overline{W}_1(x_1)\overline{U}_{n-2}(J_n)$ term into the latter sum, and extract the $J = \emptyset$ terms from it. Furthermore, interchanging the order of summation, the last two lines of (2.2.12) become

$$\begin{aligned}
&\kappa \sum_{J\subseteq J_n} \overline{W}_{n-|J|}(x_1, J_n \setminus J)\overline{U}_{|J|+1}(x_1, J) + \kappa \sum_{K\subseteq J_n} \overline{W}_{|K|+1}(x_1, K)\overline{W}_{n-|K|}(x_1, J_n \setminus K) \\
&\quad - \kappa \sum_{K\subseteq J_n} \overline{W}_{|K|+1}(x_1, K) \sum_{J\subseteq J_n\setminus K} \overline{W}_{n-|J|-|K|}(x_1, (J_n \setminus K) \setminus J)\overline{U}_{|J|}(J) \\
&= \kappa \sum_{J\subseteq J_n} \overline{W}_{n-|J|}(x_1, J_n \setminus J)\overline{U}_{|J|+1}(x_1, J) + \kappa \sum_{K\subseteq J_n} \overline{W}_{|K|+1}(x_1, K)\overline{W}_{n-|K|}(x_1, J_n \setminus K) \\
&\quad - \kappa \sum_{K\subseteq J_n} \overline{W}_{|K|+1}(x_1, K)\overline{U}_{n-|K|}(x_1, J_n \setminus K),
\end{aligned}$$

which, after writing $L = J_n \setminus K$, simplifies to

$$\begin{aligned}
&\kappa \sum_{J\subseteq J_n} \overline{W}_{n-|J|}(x_1, J_n \setminus J)\overline{U}_{|J|+1}(x_1, J) + \kappa \sum_{K\subseteq J_n} \overline{W}_{|K|+1}(x_1, K)\overline{W}_{n-|K|}(x_1, J_n \setminus K) \\
&\quad - \kappa \sum_{L\subseteq J_n} \overline{W}_{n-|L|}(x_1, J_n \setminus L)\overline{U}_{|L|+1}(x_1, L) \\
&= \kappa \sum_{K\subseteq J_n} \overline{W}_{|K|+1}(x_1, K)\overline{W}_{n-|K|}(x_1, J_n \setminus K).
\end{aligned}$$

Replacing the last two lines of (2.2.12) by this result then completes the proof. □

2.2.3 The Scaled Loop Equations

Thus far, we have derived unscaled loop equations which provide relationships between our $\overline{W}_n(x_1, \ldots, x_n)$, which are the n-point connected correlators with respect to our altered measure $\mathrm{d}\overline{\mu}_{L\beta E}(\gamma)$. We now derive loop equations for our $W_n(s_1, \ldots, s_n)$, the n-point connected correlators with respect to our scaled measure $\mathrm{d}\mu_{L\beta E^*}(\lambda)$. This is easily done by simply scaling the $\overline{W}_n(x_1, \ldots, x_n)$ to the $W_n(s_1, \ldots, s_n)$.

We write $\lambda_i = \frac{1}{N\kappa}\gamma_i$ as a change of variables. Then,

$$\begin{aligned}
&\overline{W}_n(x_1,\dots,x_n) \\
&= \left\langle \sum_{j_1,\dots,j_n=1}^{N} \frac{1}{(x_1-\gamma_{j_1})\cdots(x_n-\gamma_{j_n})} \right\rangle_c \\
&= \frac{\int_{[0,\infty)^N} \sum_{j_1,\dots,j_n=1}^{N} \frac{1}{(x_1-\gamma_{j_1})\cdots(x_n-\gamma_{j_n})} |\Delta(\gamma)|^\beta \prod_{i=1}^N \gamma_i^{\alpha_1} e^{-\gamma_i}\,\mathrm{d}\gamma_i}{\int_{[0,\infty)^N} |\Delta(\gamma)|^\beta \prod_{i=1}^N \gamma_i^{\alpha_1} e^{-\gamma_i}\,\mathrm{d}\gamma_i} \\
&= \frac{\int_{[0,\infty)^N} \sum_{j_1,\dots,j_n=1}^{N} \frac{1}{(x_1-N\kappa\lambda_{j_1})\cdots(x_n-N\kappa\lambda_{j_n})} |\Delta(N\kappa\lambda)|^\beta (N\kappa)^{N(\alpha_1+1)} \prod_{i=1}^N \lambda_i^{\alpha_1} e^{-N\kappa\lambda_i}\,\mathrm{d}\lambda_i}{\int_{[0,\infty)^N} |\Delta(N\kappa\lambda)|^\beta (N\kappa)^{N(\alpha_1+1)} \prod_{i=1}^N \lambda_i^{\alpha_1} e^{-N\kappa\lambda_i}\,\mathrm{d}\lambda_i},
\end{aligned}$$

where we pick up the factor of $(N\kappa)^{N(\alpha_1+1)}$ from the γ_i and $\mathrm{d}\gamma_i$. Recalling the definition of the Vandermonde determinant, we have

$$|\Delta(N\kappa\lambda)| = \prod_{1\leqslant j<k\leqslant N} (N\kappa)|\lambda_i-\lambda_j| = (N\kappa)^{N(N-1)/2}|\Delta(\lambda)|.$$

Hence, noting that the numerator and denominator both contain the same factor of $(N\kappa)^{N(\alpha_1+\frac{N}{2}+\frac{1}{2})}$, we find that

$$\begin{aligned}
&\overline{W}_1(x_1,\dots,x_n) \\
&= \frac{\int_{[0,\infty)^N} \sum_{j_1,\dots,j_n=1}^{N} \frac{1}{(x_1-N\kappa\lambda_{j_1})\cdots(x_n-N\kappa\lambda_{j_n})} |\Delta(\lambda)|^\beta \prod_{i=1}^N \lambda_i^{\alpha_1} e^{-N\kappa\lambda_i}\,\mathrm{d}\lambda_i}{\int_{[0,\infty)^N} |\Delta(\lambda)|^\beta \prod_{i=1}^N \lambda_i^{\alpha_1} e^{-N\kappa\lambda_i}\,\mathrm{d}\lambda_i} \\
&= \frac{1}{Z_N^*} \int_{[0,\infty)^N} \frac{1}{(N\kappa)^n} \frac{1}{((N\kappa)^{-1}x_1-\lambda_{j_1})\cdots((N\kappa)^{-1}x_n-\lambda_{j_n})}\,\mathrm{d}\mu_{L\beta E^*}(\lambda) \\
&= \frac{1}{(N\kappa)^n} \left\langle \sum_{j_1,\dots,j_n=1}^{N} \frac{1}{((N\kappa)^{-1}x_1-\lambda_{j_1})\cdots((N\kappa)^{-1}x_n-\lambda_{j_n})} \right\rangle_c^{L\beta E^*} \\
&= \frac{1}{(N\kappa)^n} W_1\left(\frac{x_1}{N\kappa},\dots,\frac{x_n}{N\kappa}\right)
\end{aligned}$$

It is at this point that we are lead to write $s_i = \frac{x_i}{N\kappa}$ in order to obtain

$$\overline{W}_n(x_1,\dots,x_n) = \frac{1}{(N\kappa)^n} W_n(s_1,\dots,s_n).$$

Substituting these relations between the unscaled and scaled connected correlators into our unscaled loop equations (2.2.7) to (2.2.9) give us loop equations for our $W_n(s_1,\dots,s_n)$.

Theorem 2.4. *Our first loop equation is*

$$0 = \frac{1}{N}W_2(s_1,s_1) + \frac{1}{N}\left(W_1(s_1)\right)^2 + \frac{1}{N}\left(1-\frac{1}{\kappa}\right)W_1'(s_1) + \left(\frac{\alpha_1}{N\kappa s_1}-1\right)W_1(s_1) + \frac{N}{s_1}. \tag{2.2.13}$$

Theorem 2.5. *For $n \geqslant 2$, our n^{th} loop equation is*

$$\begin{aligned} 0 = &\left[\frac{1}{N}\left(1-\frac{1}{\kappa}\right)\frac{\partial}{\partial s_1} + \left(\frac{\alpha_1}{N\kappa s_1}-1\right)\right]W_n(s_1,\ldots,s_n) \\ &+ \frac{1}{N\kappa}\sum_{k=2}^{n}\frac{\partial}{\partial s_k}\left\{\frac{W_{n-1}(s_1,\ldots,\hat{s}_k,\ldots,s_n) - W_{n-1}(s_2,\ldots,s_n)}{s_1-s_k} + \frac{1}{s_1}W_{n-1}(s_2,\ldots,s_n)\right\} \\ &+ \frac{1}{N}\left[W_{n+1}(s_1,s_1,s_2,\ldots,s_n) + \sum_{J\subseteq(s_2,\ldots,s_n)} W_{|J|+1}(s_1,J)W_{n-|J|}(s_1,(s_2,\ldots,s_n)\setminus J)\right]. \end{aligned} \tag{2.2.14}$$

2.3 Calculating the Connected Correlators

2.3.1 Methodology for Calculating the Resolvent $W_1(s_1)$

With our loop equations in hand, we may now calculate our resolvent $W_1(s_1)$ up to any order in N. We begin by substituting the series expansions from (1.4.18),

$$W_1(s_1) = N\sum_{l=0}^{\infty}\frac{W_1^l(s_1)}{(N\sqrt{\kappa})^l},$$
$$W_2(s_1,s_2) = \frac{1}{\kappa}\sum_{l=0}^{\infty}\frac{W_2^l(s_1,s_2)}{(N\sqrt{\kappa})^l}$$

into our first loop equation (2.2.13), resulting in

$$\begin{aligned} 0 = &\frac{1}{N\kappa}\sum_{l=0}^{\infty}\frac{W_2^l(s_1,s_1)}{(N\sqrt{\kappa})^l} + N\sum_{l,k=0}^{\infty}\frac{W_1^l(s_1)W_1^k(s_1)}{(N\sqrt{\kappa})^{l+k}} + \left(1-\frac{1}{\kappa}\right)\sum_{l=0}^{\infty}\frac{W_1^{l'}(s_1)}{(N\sqrt{\kappa})^l} \\ &+ \left(\frac{\alpha_1}{\kappa s_1}-N\right)\sum_{l=0}^{\infty}\frac{W_1^l(s_1)}{(N\sqrt{\kappa})^l} + \frac{N}{s_1}. \end{aligned} \tag{2.3.1}$$

This enables us to form loop equations for our resolvent coefficients $W_1^l(s_1)$, as we shall now demonstrate. Comparing terms of order N in (2.3.1), we obtain

$$0 = \left(W_1^0(s_1)\right)^2 - W_1^0(s_1) + \frac{1}{s_1}.$$

A simple application of the quadratic formula then reveals that

$$W_1^0(s_1) = \tfrac{1}{2}\left(1 - \sqrt{1 - \tfrac{4}{s_1}}\right). \tag{2.3.2}$$

Here, we have had to make a choice in sign when applying the quadratic formula. This choice has been made definite by requiring that $W_1^0(s_1)$ behaves as $\frac{1}{s_1}$ for $|s_1|$ large, in keeping with the roles played by our resolvent.

Moving on, comparing terms of order 1 in (2.3.1), we have that

$$0 = 2W_1^0(s_1)W_1^1(s_1) + hW_1^{0'}(s_1) + \frac{\alpha_1}{\sqrt{\kappa}s_1}W_1^0(s_1) - W_1^1(s_1), \tag{2.3.3}$$

where we recall that $h = \sqrt{\kappa} - \frac{1}{\sqrt{\kappa}}$. Substituting in our expression for $W_1^0(s_1)$ and rearranging (2.3.3), we find that

$$W_1^1(s_1) = \frac{\alpha_1}{2\sqrt{\kappa}}\left(\frac{1}{t_1} - \frac{1}{s_1}\right) - \frac{h}{t_1^2}, \tag{2.3.4}$$

where we take this opportunity to define

$$t_i := s_i\sqrt{1 - \frac{4}{s_i}}, \quad i \in \mathbb{N}_{>0},$$

as this square root appears frequently from here on out.

In general, for $l \geqslant 2$, the first loop equation presents us with the following loop equation for our resolvent coefficients.

$$0 = W_2^{l-2}(s_1, s_1) + \sum_{k=0}^{l} W_1^k(s_1)W_1^{l-k}(s_1) + hW_1^{l-1'}(s_1) + \frac{\alpha_1}{\sqrt{\kappa}s_1}W_1^{l-1}(s_1) - W_1^l(s_1) \tag{2.3.5}$$

We shall call this the *loop equation for* W_1^l. We realise that in order to calculate $W_1(s_1)$ to higher orders in N, we need to calculate our 2-point connected correlator $W_2(s_1, s_2)$ up to a similar order in N. Indeed, when attempting to do so, we shall see that we require $W_3(s_1, s_2, s_3)$, and so on. In fact, we have a triangular recursive system, the meaning of which will be made precise in the next subsection.

2.3.2 Loop Equations for the Correlator Coefficients W_n^l

We expand our n-point connected correlators in large N as

$$W_n(s_1,\dots,s_n) = N^{2-n}\kappa^{1-n}\sum_{l=0}^{\infty}\frac{W_n^l(s_1,\dots,s_n)}{(N\sqrt{\kappa})^l}$$

and substitute them into our general loop equation (2.2.14) to obtain

$$\begin{aligned}0 = &\left[\frac{1}{N}\left(1-\frac{1}{\kappa}\right)\frac{\partial}{\partial s_1}+\left(\frac{\alpha_1}{N\kappa s_1}-1\right)\right]\sum_{l=0}^{\infty}\frac{W_n^l(s_1,\dots,s_n)}{(N\sqrt{\kappa})^l}\\
&+\sum_{k=2}^{n}\frac{\partial}{\partial s_k}\left\{\frac{\sum_{l=0}^{\infty}\frac{W_{n-1}^l(s_1,\dots,\hat{s}_k,\dots,s_n)}{(N\sqrt{\kappa})^l}-\sum_{l=0}^{\infty}\frac{W_{n-1}^l(s_2,\dots,s_n)}{(N\sqrt{\kappa})^l}}{s_1-s_k}+\frac{1}{s_1}\sum_{l=0}^{\infty}\frac{W_{n-1}^l(s_2,\dots,s_n)}{(N\sqrt{\kappa})^l}\right\}\\
&+\frac{1}{N^2\kappa}\sum_{l=0}^{\infty}\frac{W_{n+1}^l(s_1,s_1,s_2,\dots,s_n)}{(N\sqrt{\kappa})^l}\\
&+\sum_{J\subseteq(s_2,\dots,s_n)}\sum_{l,k=0}^{\infty}\frac{W_{|J|+1}^l(s_1,J)W_{n-|J|}^k(s_1,(s_2,\dots,s_n)\setminus J)}{(N\sqrt{\kappa})^{l+k}}.\end{aligned}\tag{2.3.6}$$

Then, for $n\geqslant 2$, we may extract three types of loop equations for our $W_n^l(s_1,\dots,s_n)$, in analogue with the equivalent equations for $W_1(s_1)$ presented in the preceding subsection. Extracting the order 1 terms from the above equation gives

$$\begin{aligned}0 = &-W_n^0(s_1,\dots,s_n)+\sum_{J\subseteq(s_2,\dots,s_n)}W_{|J|+1}^0(s_1,J)W_{n-|J|}^0(s_1,(s_2,\dots,s_n)\setminus J)\\
&+\sum_{k=2}^{n}\frac{\partial}{\partial s_k}\left\{\frac{W_{n-1}^0(s_1,\dots,\hat{s}_k,\dots,s_n)-W_{n-1}^0(s_2,\dots,s_n)}{s_1-s_k}+\frac{1}{s_1}W_{n-1}^0(s_2,\dots,s_n)\right\},\end{aligned}\tag{2.3.7}$$

which we will call our *first seed equation*. We note that in order to calculate $W_n^0(s_1,\dots,s_n)$, we only require terms relating to $W_m^0(s_1,\dots,s_m)$ for $m<n$.

For the second type of equation, we extract order $\frac{1}{N}$ terms from (2.3.6) to form

our *second seed equation*

$$\begin{aligned}0 = h\frac{\partial}{\partial s_1}W_n^0(s_1,\dots,s_n) + \frac{\alpha_1}{\sqrt{\kappa}s_1}W_n^0(s_1,\dots,s_n) - W_n^1(s_1,\dots,s_n)\\ + \sum_{k=2}^{n}\frac{\partial}{\partial s_k}\left\{\frac{W_{n-1}^1(s_1,\dots,\hat{s}_k,\dots,s_n) - W_{n-1}^1(s_2,\dots,s_n)}{s_1-s_k} + \frac{1}{s_1}W_{n-1}^1(s_2,\dots,s_n)\right\}\\ + \sum_{J\subseteq(s_2,\dots,s_n)}\sum_{k=0}^{1}W_{|J|+1}^k(s_1,J)W_{n-|J|}^{1-k}(s_1,(s_2,\dots,s_n)\setminus J). \qquad (2.3.8)\end{aligned}$$

In general, we extract order $\frac{1}{N^l}$ terms from (2.3.6) for $l \geqslant 2$ to obtain

$$\begin{aligned}0 = h\frac{\partial}{\partial s_1}W_n^{l-1}(s_1,\dots,s_n) + \frac{\alpha_1}{\sqrt{\kappa}s_1}W_n^{l-1}(s_1,\dots,s_n) - W_n^l(s_1,\dots,s_n)\\ + \sum_{k=2}^{n}\frac{\partial}{\partial s_k}\left\{\frac{W_{n-1}^l(s_1,\dots,\hat{s}_k,\dots,s_n) - W_{n-1}^l(s_2,\dots,s_n)}{s_1-s_k} + \frac{1}{s_1}W_{n-1}^l(s_2,\dots,s_n)\right\}\\ + W_{n+1}^{l-2}(s_1,s_1,s_2,\dots,s_n)\\ + \sum_{J\subseteq(s_2,\dots,s_n)}\sum_{k=0}^{l}W_{|J|+1}^k(s_1,J)W_{n-|J|}^{l-k}(s_1,(s_2,\dots,s_n)\setminus J), \qquad (2.3.9)\end{aligned}$$

which we shall refer to as the *loop equation for* W_n^l. We observe that in order to calculate W_n^l from the above equation, we require W_m^k for $m < n$ and $k < l$. Overall, our strategy for calculating $W_1(s_1)$ can be outlined in 3 steps: First, we initialise the recursion with our expressions for $W_1^0(s_1)$ (2.3.2) and $W_1^1(s_1)$ (2.3.4). Then, we calculate in order, $W_2^0(s_1,s_2)$, $W_1^2(s_1)$, $W_2^1(s_1,s_2)$, $W_1^3(s_1)$, and so on.Following this pattern, when we get to calculating $W_1^l(s_1)$, we will have already calculated many other correlator coefficients, and will only need to compute a few more.

If l is even, we use our first seed equation (2.3.7) to compute $W_{\frac{l}{2}+1}^0(s_1,\dots,s_{\frac{l}{2}+1})$, and then the second seed equation (2.3.8) to compute $W_{\frac{l}{2}}^1(s_1,\dots,s_{\frac{l}{2}})$. Finally, we successively use the loop equation for W_n^l (2.3.9) above to compute W_n^l with l increasing by 2 and n decreasing by 1 at each step.

If l is odd, we use our second seed equation (2.3.8) to compute $W_{\frac{l+1}{2}}^1(s_1,\dots,s_{\frac{l+1}{2}})$ and then use the loop equation for W_n^l (2.3.9) above to compute W_n^l with l increasing by 2 and n decreasing by 1 at each step. We present a table whose entries describe the order in which the first ten coefficients of W_1 are calculated in thirty six steps.

n \ l / W_n^l	0	1	2	3	4	5	6	7	8	9	10
1	1	2	4	6	9	12	16	20	25	30	36
2	3	5	8	11	15	19	24	29	35		
3	7	10	14	18	23	28	34				
4	13	17	22	27	33						
5	21	26	32								
6	31										

To better see the order of recursion, one may like to draw arrows starting at each entry in the first two columns, pointing to the top right at a gradient of $\frac{1}{2}$. There is a simple algorithm for generating this table: Assume that the last entry entered was in the top row. If the bottom row is of length one, the next entry is added to the end of that row. Otherwise, the next entry is placed in a new row below the current bottom row. Then, move up one row at a time, adding an entry to the end of each row.

Remark 6. While performing the recursion, one will need to compute terms of the form $W_{n+1}^{l-2}(s_1, s_1, s_2, \ldots, s_n)$, for example in (2.3.9). When computing these terms, the second line in each of (2.3.7), (2.3.8), and (2.3.9) may then raise alarms, as we cannot simply set $s_2 = s_1$, due to the singularity. However, this is a removable singularity:

$$\begin{aligned}
&\lim_{s_1 \to s_2} \frac{\partial}{\partial s_2} \left\{ \frac{W_{n-1}^l(s_1, s_3, \ldots, s_n) - W_{n-1}^l(s_2, \ldots, s_n)}{s_1 - s_2} \right\} \\
&\quad = \lim_{s_1 \to s_2} \frac{(s_2 - s_1)\frac{\partial}{\partial s_2} W_{n-1}^l(s_2, \ldots, s_n) + W_{n-1}^l(s_1, s_3, \ldots, s_n) - W_{n-1}^l(s_2, \ldots, s_n)}{(s_1 - s_2)^2} \\
&\quad = \lim_{s_1 \to s_2} \frac{1}{s_2 - s_1} \left\{ \frac{\partial}{\partial s_2} W_{n-1}^l(s_2, \ldots, s_n) - \frac{W_{n-1}^l(s_1, s_3, \ldots, s_n) - W_{n-1}^l(s_2, \ldots, s_n)}{s_1 - s_2} \right\}.
\end{aligned}$$

The term in the bracket tends to 0 as $s_1 \to s_2$. Thus, via L'Hôpital's rule, we find that the limit is actually equal to $\frac{1}{2}\frac{\partial^2}{\partial s_2^2} W_{n-1}^l(s_2, \ldots, s_n)$, which is well defined. In practice, the author used Taylor expansions.

2.3.3 The Resolvent Up to Order 3 in $\frac{1}{N}$ and Related Correlators

We now use the recursion outlined above to compute a few resolvent coefficients, and the correlators we had to compute along the way. We present our results in order of computation. We recall that $t_i = s_i\sqrt{1-4/s_i}$ for $i \in \mathbb{N}_{>0}$, $\kappa = \frac{\beta}{2}$, and $h = \sqrt{\kappa} - \frac{1}{\sqrt{\kappa}}$.

$$W_1^0(s_1) = \frac{1}{2}\left(1 - \sqrt{1-\tfrac{4}{s_1}}\right) \tag{2.3.10}$$

$$W_1^1(s_1) = \frac{\alpha_1}{2\sqrt{\kappa}}\left(\frac{1}{t_1} - \frac{1}{s_1}\right) - \frac{h}{t_1^2} \tag{2.3.11}$$

$$W_2^0(s_1,s_2) = \frac{s_1s_2 - 2s_1 - 2s_2}{2t_1t_2(s_1-s_2)^2} - \frac{1}{2(s_1-s_2)^2} \tag{2.3.12}$$

$$W_2^0(s_1,s_1) = \frac{1}{t_1^4} \tag{2.3.13}$$

$$W_1^2(s_1) = \frac{s_1}{t_1^5} + \frac{\alpha_1^2}{\kappa t_1^3} - \frac{h\alpha_1}{2\sqrt{\kappa}}\left(\frac{4-s_1}{t_1^3} + \frac{s_1^2}{t_1^4}\right) + h^2\frac{2s_1^2-3s_1}{t_1^5} \tag{2.3.14}$$

$$\begin{aligned} W_2^1(s_1,s_2) = {} & \frac{\alpha_1}{\sqrt{\kappa}}\frac{s_1s_2}{t_1^3t_2^3} - \frac{h}{s_1-s_2}\left(\frac{1}{t_1t_2^2}\right) + \frac{1}{(s_1-s_2)^2\sqrt{\kappa}}\left(\frac{s_1^2(2-s_2)+2s_1s_2}{t_1^4t_2}\right) \\ & - \frac{h}{(s_1-s_2)^2}\left(\frac{4s_1s_2-s_2^3}{t_1t_2^4} + \frac{s_1^2(s_2-6)+2s_1(8-s_2)}{t_1^4t_2}\right) \\ & + \frac{h}{(s_1-s_2)^3}\left(\frac{s_1}{t_1} + \frac{s_1^4(2-s_2)+2s_1^3(3s_2-5)+8s_1^2(2-s_2)+2s_1(s_2^2-8s_2)}{t_1^4t_2}\right) \\ & + \frac{1}{(s_1-s_2)^3\sqrt{\kappa}}\left(\frac{s_1^3(s_2-2)-s_1^2s_2^2+2s_1s_2^2}{t_1^4t_2}\right) \end{aligned} \tag{2.3.15}$$

$$W_2^1(s_1,s_1) = \frac{\alpha_1}{\sqrt{\kappa}}\left(\frac{s_1^2}{t_1^6}\right) - h\left(\frac{s_1^3+2s_1^2-2s_1}{t_1^7}\right) \tag{2.3.16}$$

$$\begin{aligned} W_1^3(s_1) = {} & -h\left(\frac{s_1^4+6s_1^3-6s_1^2}{t_1^8}\right) - h^3\left(\frac{6s_1^4-12s_1^3+12s_1^2}{t_1^8}\right) + \frac{\alpha_1}{\sqrt{\kappa}}\left(\frac{s_1^3+s_1^2}{t_1^7}\right) \\ & - \frac{h^2\alpha_1}{\sqrt{\kappa}}\left(\frac{s_1^3-6s_1^2+8s_1}{t_1^6} - \frac{s_1^4+3s_1^3-3s_1^2}{t_1^7}\right) + \frac{h\alpha_1^2}{2\kappa}\left(\frac{1}{t_1^3} - \frac{s_1^3+6s_1^2-8s_1}{t_1^6}\right) \\ & + \frac{\alpha_1^3}{\kappa\sqrt{\kappa}}\left(\frac{s_1}{t_1^5}\right) \end{aligned} \tag{2.3.17}$$

This data gives us our resolvent up to order 3 in $\frac{1}{N}$ using the large N expansion $W_1(s_1) = N\sum_{l=0}^{\infty} \frac{W_1^l(s_1)}{(N\sqrt{\kappa})^l}$ from (1.4.18).

2.4 The First 3 Moments of the Laguerre β Ensembles

We now use the data given above to compute the moments $\tilde{m}_k$ of the scaled Laguerre β ensemble, as defined in subsection 1.4.3. We know from (1.4.12) that for large $|s|$,

$$\frac{1}{N}W_1(s) = \frac{1}{s} + \sum_{k=1}^{\infty} \frac{\tilde{m}_k}{s^{k+1}},$$

where the equality is in an asymptotic sense. From (1.4.10), we know that there exist coefficients $a_i^{(k)}$ such that

$$\tilde{m}_k = \sum_{i=0}^{k} a_i^{(k)} N^{-i}.$$

Combining these two formulae, we obtain

$$\begin{aligned}\frac{1}{N}W_1(s) &= \sum_{k=0}^{\infty}\sum_{i=0}^{k} \frac{a_i^{(k)}N^{-i}}{s^{k+1}}, \quad a_0^{(0)} = 1 \\ &= \sum_{i=0}^{\infty} \frac{1}{N^i}\sum_{k=i}^{\infty} \frac{a_i^{(k)}}{s^{k+1}},\end{aligned}$$

where we're allowed to interchange the order of summation due to convergence in the large $|s|$, large N limit. Now, we compare with the large N expansion for $W_1(s)$ given by (1.4.18),

$$\frac{1}{N}W_1(s) = \sum_{l=0}^{\infty} \frac{W_1^l(s)}{(N\sqrt{\kappa})^l}.$$

Equating terms of equal order in N, we have

$$\begin{aligned}\sum_{l=0}^{\infty} \frac{W_1^l(s)}{(N\sqrt{\kappa})^l} = \frac{1}{N}W_1(s) = \sum_{i=0}^{\infty} \frac{1}{N^i}\sum_{k=i}^{\infty} \frac{a_i^{(k)}}{s^{k+1}} \\ \implies W_1^l(s) = \kappa^{l/2}\sum_{k=l}^{\infty} \frac{a_l^{(k)}}{s^{k+1}}.\end{aligned}$$

As our coefficients $a_i^{(k)}$ are independent of N and s, this immediately tells us that for $l \in \mathbb{N}_0$, $W_1^l(s)$, is of order at most $-(l+1)$ in s. Referring back to (1.4.12), this tells us that to compute $\tilde{m}_k$ ($k \in \mathbb{N}_{>0}$), we only need $W_1^0(s), \ldots, W_1^l(s)$. We now present our resolvent coefficients as Taylor expansions in s about 0, up to order 4 in s.

$$W_1^0(s) = \frac{1}{s} + \frac{1}{s^2} + \frac{2}{s^3} + \frac{5}{s^4} + O(\tfrac{1}{s^5}) \tag{2.4.1}$$

$$W_1^1(s) = \frac{1-\kappa+\alpha_1}{\sqrt{\kappa}s^2} + \frac{4-4\kappa+3\alpha_1}{\sqrt{\kappa}s^3} - \frac{2(8\kappa-5\alpha_1-8)}{\sqrt{\kappa}s^4} + O(\tfrac{1}{s^5}) \tag{2.4.2}$$

$$W_1^2(s) = \frac{2-4\kappa+2\kappa^2+3\alpha_1-3\kappa\alpha_1+\alpha_1^2}{\kappa s^3} + \frac{17-33\kappa+17\kappa^2+21\alpha_1-21\kappa\alpha_1+6\alpha_1^2}{\kappa s^4} + O(\tfrac{1}{s^5}) \tag{2.4.3}$$

$$W_1^3(s) = \frac{6-17\kappa+17\kappa^2-6\kappa^3+11\alpha_1-21\kappa\alpha_1+11\kappa^2\alpha_1+6\alpha_1^2-6\kappa\alpha_1^2+\alpha_1^3}{\kappa^{3/2}s^4} + O(\tfrac{1}{s^5}) \tag{2.4.4}$$

Remark 7. We point out that the coefficients in the Taylor expansion of $W_1^0(s)$ are the Catalan numbers. This is expected, and is a well known result in the Gaussian case [Witte and Forrester, 2014].

Now, extracting the coefficients of $\frac{1}{s^2}$ from $W_1^0(s)$ and $W_1^1(s)$, we obtain that

$$\tilde{m}_1 = 1 + \frac{1-\kappa+\alpha_1}{N\kappa}. \tag{2.4.5}$$

Extracting the coefficients of $\frac{1}{s^3}$ from $W_1^0(s)$, $W_1^1(s)$, and $W_1^2(s)$, we obtain that

$$\tilde{m}_2 = 2 + \frac{4-4\kappa+3\alpha_1}{N\kappa} + \frac{2-4\kappa+2\kappa^2+3\alpha_1-3\kappa\alpha_1+\alpha_1^2}{N^2\kappa^2}. \tag{2.4.6}$$

Likewise,

$$\begin{aligned}\tilde{m}_3 = 5 - \frac{2(8\kappa-5\alpha_1-8)}{N\kappa} + \frac{17-33\kappa+17\kappa^2+21\alpha_1-21\kappa\alpha_1+6\alpha_1^2}{N^2\kappa^2} \\ + \frac{6-17\kappa+17\kappa^2-6\kappa^3+11\alpha_1-21\kappa\alpha_1+11\kappa^2\alpha_1+6\alpha_1^2-6\kappa\alpha_1^2+\alpha_1^3}{N^3\kappa^3}.\end{aligned} \tag{2.4.7}$$

We recall that from equation (1.4.9), we can easily retreieve the moments of the Laguerre β ensemble using the relation $m_k = N^{k+1}\tilde{m}_k$. These moments agree with existing results, as we will see in subsection 4.1.2.

Chapter 3

The Differential Equations

3.1 The Hypergeometric Function Approach

We now restrict to $\beta = 2$, and approach the problem via differential equations. Our discussion is heavily reliant on identities [4.127], [4.133], [13.12], [13.25], [13.44] and [13.45] from [Forrester, 2010, pp.165-168,594-606], which we present here for completeness.

First, note that for even β, equation [13.45] tells us that the eigenvalue density of the $(\frac{2}{\beta}(\alpha_1+1)-1, N+1)$ Laguerre β ensemble is given by

$$\rho^{(L)}_{(1)}(x;\alpha_1,\beta,N+1) = \frac{(N+1)\,L_N(a,\beta)}{L_{N+1}(a+\frac{2}{\beta},\beta)}\, x^{\alpha_1} e^{-\beta x/2}\, {}_1F_1{}^{(\beta/2)}(-N;a+2;(x)^\beta), \quad (3.1.1)$$

where $a = \frac{2\alpha_1}{\beta}$, $(x)^\beta$ is the β-tuple of x's, and the L_N normalisation constants are specified in subsection 1.4.1. We warn that this a is not our usual exponent. This identity may be retrieved from our expression for $\rho^{(L)}_{(1)}(x;\alpha_1,\beta,N+1)$ given in subsection 1.3.1 by using identity [13.44] with $m=\beta$ and $t_1 = \ldots = t_\beta = x =: x_{N+1}$ to obtain

$$\begin{aligned} &{}_1F_1^{(\beta/2)}(-N;a+2;(x)^\beta) \\ &\qquad = \frac{1}{W_{a,\beta,N}} \int_{[0,\infty)^N} \prod_{j=1}^N \Big[\mathrm{d}x_j\, x_j^{\alpha_1} e^{-\beta x_j/2} (x_j - x)^\beta \Big] \prod_{1\leqslant j<k\leqslant N} |x_k - x_j|^\beta. \quad (3.1.2) \end{aligned}$$

for some normalisation constant $W_{a,\beta,N}$. Next, we take identity [13.25] with $\alpha = \frac{\beta}{2}$, $\alpha_1 = a\alpha$, $\alpha_2 = b\alpha$, and $m = \beta$, to see that

$$\begin{aligned} &{}_2F_1^{(\beta/2)}(-N, a+b+\tfrac{2}{\beta}+N+1; a+2; (\tfrac{x}{b})^\beta) \qquad (3.1.3)\\ &= \frac{1}{S_N(\alpha_1+\beta,\alpha_2,\alpha)} \int_{[0,1]^N} \prod_{j=1}^N \Big[\mathrm{d}x_j\, x_j^{\alpha_1}(1-x_j)^{\alpha_2}(x_j-\tfrac{x}{b})^\beta \Big] \prod_{1\leqslant j<k\leqslant N} |x_k-x_j|^\beta, \end{aligned}$$

where the S_N normalisation constant has been specified in subsection 1.4.1. This quantity is of interest to us because, using this expression, it is easy to see that

$${}_1F_1^{(\beta/2)}(-N; a+2; (x)^\beta) = \lim_{b\to\infty} {}_2F_1^{(\beta/2)}(-N, a+b+\tfrac{2}{\beta}+N+1; a+2; (\tfrac{x}{b})^\beta).$$

We may now exploit a duality between β and N through identity [13.12] to write the quantity given in (3.1.3) above as a β-dimensional integral,

$$\begin{aligned} &{}_2F_1^{(\beta/2)}(-N, \lambda_1-\tfrac{2}{\beta}+3; \lambda_1+\lambda_2-\tfrac{4}{\beta}+6; (\tfrac{x}{b})^\beta)\\ &= \frac{1}{S_\beta(\lambda_1,\lambda_2,\frac{2}{\beta})} \int_{[0,1]^\beta} \prod_{j=1}^\beta \Big[\mathrm{d}x_j\, x_j^{\lambda_1}(1-x_j)^{\lambda_2}(1-\tfrac{xx_j}{b})^N \Big] \prod_{1\leqslant j<k\leqslant \beta} |x_k-x_j|^{4/\beta}. \qquad (3.1.4) \end{aligned}$$

To compare this with (3.1.3), let $\lambda_1 = a+b+N+\frac{4}{\beta}-2$ and $\lambda_2 = -b-N-2$.

We have a related quantity given in [4.127], which we may use by substituting $N = \beta$ and $\lambda = \frac{2}{\beta}$ to obtain

$$I_p^{(\alpha)}(x) = \int_{[0,1]^\beta} \prod_{j=1}^\beta \Big[\mathrm{d}x_j x_j^{\lambda_1}(1-x_j)^{\lambda_2}(x_j-x)^{\alpha-\chi_{j>p}} \Big] \prod_{1\leqslant j<k\leqslant\beta} |x_k-x_j|^{4/\beta}. \qquad (3.1.5)$$

This object is accompanied by differential-difference equation [4.133],

$$\begin{aligned} &(\beta-p)E_p I_{p+1}^{(\alpha)}(x)\\ &\qquad = -(A_p x+B_p) I_p^{(\alpha)}(x) + x(x-1)\frac{\mathrm{d}}{\mathrm{d}x} I_p^{(\alpha)}(x) + D_p x(x-1) I_{p-1}^{(\alpha)}(x), \qquad (3.1.6)\\ &A_p = (\beta-p)(\lambda_1+\lambda_2+4-\tfrac{4}{\beta}(p+1)+2\alpha),\\ &B_p = (p-\beta)(\lambda_1+\alpha+2-\tfrac{2}{\beta}(p+1)),\\ &D_p = p(2-\tfrac{2}{\beta}p+\alpha),\\ &E_p = \lambda_1+\lambda_2+5-\tfrac{2}{\beta}(p+2)+\alpha. \end{aligned}$$

To relate this information to the object given in (3.1.4), we define an auxillary function,

$$\begin{aligned} J_p^{(\alpha)}(x) :&= (-\tfrac{x}{b})^{\beta(\alpha-1)+p} I_p^{(\alpha)}(\tfrac{b}{x}) \\ &= \int_{[0,1]^\beta} \prod_{j=1}^{\beta} \left[\mathrm{d}x_j x_j^{\lambda_1} (1-x_j)^{\lambda_2} (1-\tfrac{x x_j}{b})^{\alpha-\chi_{j>p}} \right] \prod_{1\leqslant j<k\leqslant\beta} |x_k - x_j|^{4/\beta}. \end{aligned} \tag{3.1.7}$$

Then, with the help of (3.1.6), we obtain a differential-difference equation for $J_p^{(\alpha)}(x)$,

$$\begin{aligned} b(\beta-p)E_p J_{p+1}^{(\alpha)}(x) = &\left[bA_p + B_p x + (\beta(\alpha-1)+p)(x-b)\right] J_p^{(\alpha)}(x) \\ &+ x(b-x)\frac{\mathrm{d}}{\mathrm{d}x} J_p^{(\alpha)}(x) + D_p(b-x) J_{p-1}^{(\alpha)}(x), \end{aligned} \tag{3.1.8}$$

where the constants A_p, B_p, C_p, and D_p retain their definitions as given in (3.1.6).

3.2 The LUE Eigenvalue Density Differential Equation

We now let $\beta = 2$. Substituting $p = 0, 1$, and 2 into (3.1.8) yields

$$\begin{aligned} 2bE_0 J_1^{(\alpha)}(x) &= \left[bA_0 + B_0 x + 2(\alpha-1)(x-b)\right] J_0^{(\alpha)}(x) + x(b-x)\frac{\mathrm{d}}{\mathrm{d}x} J_0^{(\alpha)}(x), \\ bE_1 J_2^{(\alpha)}(x) &= \left[bA_1 + B_1 x + (2\alpha-1)(x-b)\right] J_1^{(\alpha)}(x) + x(b-x)\frac{\mathrm{d}}{\mathrm{d}x} J_1^{(\alpha)}(x) \\ &\quad + D_1(b-x) J_0^{(\alpha)}(x), \\ 0 &= 2\alpha(x-b) J_2^{(\alpha)}(x) + x(b-x)\frac{\mathrm{d}}{\mathrm{d}x} J_2^{(\alpha)}(x) + 2\alpha(b-x) J_1^{(\alpha)}(x). \end{aligned} \tag{3.2.1}$$

Substituting the second of these equations into the third then gives

$$\begin{aligned} 0 = &D_1 \left[2\alpha(x-b) - x + x(b-x)\frac{\mathrm{d}}{\mathrm{d}x} \right] J_0^{(\alpha)}(x) + x^2(b-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} J_1^{(\alpha)}(x) \\ &+ x\left[b(A_1+2) + (B_1-3)x + 4\alpha(x-b) \right] \frac{\mathrm{d}}{\mathrm{d}x} J_1^{(\alpha)}(x) \\ &+ \Big[(B_1-1)x + 4\alpha^2(b-x) - 2\alpha\left(b(A_1 - E_1 + 1) + (B_1-2)x \right) \Big] J_1^{(\alpha)}(x). \end{aligned}$$

We intend to take the limit as $b \to \infty$, so we discard terms of low order in b to obtain

$$\begin{aligned} 0 = &D_1 \left[-2\alpha + x\frac{\mathrm{d}}{\mathrm{d}x} \right] J_0^{(\alpha)}(x) + x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} J_1^{(\alpha)}(x) + x(a-x-2\alpha)\frac{\mathrm{d}}{\mathrm{d}x} J_1^{(\alpha)}(x) \\ &+ \Big(2\alpha^2 + 2\alpha(x+1) - x \Big) J_1^{(\alpha)}(x). \end{aligned}$$

Substituting the first equation of the set (3.2.1) into this result then yields

$$\begin{aligned} 0 = {} & x^2 \frac{\mathrm{d}^3}{\mathrm{d}x^3} J_0^{(N+1)}(x) - x(3x - 3a - 4) \frac{\mathrm{d}^2}{\mathrm{d}x^2} J_0^{(N+1)}(x) \\ & + \left[4Nx + 2x^2 - 4(1+a)x + (2a+1)(a+2) \right] \frac{\mathrm{d}}{\mathrm{d}x} J_0^{(N+1)}(x) \\ & - 2N(2x - 2a - 1) J_0^{(N+1)}(x), \end{aligned} \tag{3.2.2}$$

where we have again discarded terms of low order in b, and substituted $\alpha = N + 1$. Throughout, we have been using the expressions $\lambda_1 = a + b + N + \frac{4}{\beta} - 2$ and $\lambda_2 = -b - N - 2$. With these values, we note that

$$J_0^{(N+1)}(x) = {}_2F_1^{(1)}(-N, a + b + N + 2; a + 2; (\tfrac{x}{b})^2).$$

Taking the limit as $b \to \infty$ in (3.2.2) then gives us the differential equation

$$\begin{aligned} 0 = {} & x^2 \frac{\mathrm{d}^3}{\mathrm{d}x^3} F(x) - x(3x - 3a - 4) \frac{\mathrm{d}^2}{\mathrm{d}x^2} F(x) \\ & + \left[4Nx + 2x^2 - 4(1+a)x + (2a+1)(a+2) \right] \frac{\mathrm{d}}{\mathrm{d}x} F(x) \\ & - 2N(2x - 2a - 1) F(x), \end{aligned} \tag{3.2.3}$$

where we define

$$\begin{aligned} F(x) :&= {}_1F_1^{(1)}(-N; a + 2; (x)^2) \\ &= \frac{L_{N+1}(a+1, 2)}{(N+1) L_N(a, 2)} x^{-a} e^x \rho_{(1)}^{(L)}(x; a, 2, N+1). \end{aligned}$$

Finally, replacing N with $N - 1$, we arrive at our desired differential equation,

$$\begin{aligned} 0 = {} & x^3 \frac{\mathrm{d}^3}{\mathrm{d}x^3} \rho_{(1)}^{(L)}(x; a, 2, N) + 4x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \rho_{(1)}^{(L)}(x; a, 2, N) \\ & - x \left[x^2 - 2(2N + a)x + a^2 - 2 \right] \frac{\mathrm{d}}{\mathrm{d}x} \rho_{(1)}^{(L)}(x; a, 2, N) \\ & + \left[(2N + a)x - a^2 \right] \rho_{(1)}^{(L)}(x; a, 2, N). \end{aligned} \tag{3.2.4}$$

3.3 The LUE Resolvent Differential Equation

We now use the results from the previous section to derive a differential equation for the resolvent $W_1(s)$. With the resulting differential equation, we will compute the

large N expansion of $W_1(s)$ independent to the methods used in chapter 2. We recall that the resolvent is given by the Stieltjes transform of the smoothed eigenvalue density $\tilde{\rho}^{(L)}_{(1)}$, which is given in definition 1.13. We refer the reader to subsection 1.4.5. Hence, we scale equation (3.2.4) given in the previous section to obtain a differential equation for $\tilde{\rho}^{(L)}_{(1)}$. This is simply achieved by changing variables $x \mapsto Nx$. The resultant differential equation is

$$\begin{aligned} 0 = {} & x^3 \frac{\mathrm{d}^3}{\mathrm{d}x^3}\tilde{\rho}^{(L)}_{(1)}(x;a,2,N) + 4x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\tilde{\rho}^{(L)}_{(1)}(x;a,2,N) \\ & - x\left[N^2x^2 - 2N(2N+a)x + a^2 - 2\right]\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}^{(L)}_{(1)}(x;a,2,N) \\ & + \left[N(2N+a)x - a^2\right]\tilde{\rho}^{(L)}_{(1)}(x;a,2,N). \end{aligned} \tag{3.3.1}$$

We remark that, up to scaling, this differential equation has been stated without proof in [Adachi et al., 2011]. We refer the reader so subsection 4.1.3. We wish to take the Stieltjes transform of equation (3.3.1) term by term. First, we present some identities. In the following six identities, we will represent $\tilde{\rho}^{(L)}_{(1)}(x;a,2,N)$ simply by $\tilde{\rho}(x)$. Then, $\frac{1}{N}W_1(s) = \mathcal{ST}\{\tilde{\rho}(x); x \to s\}$.

We use the fact that $\partial_s(s-x)^{-1} = -\partial_x(s-x)^{-1}$, and that our density has support $[0,\infty)$ with $\tilde{\rho}^{(L)}_{(1)}(0;a,2,N) = 0$ and $\tilde{\rho}^{(L)}_{(1)}(x;a,2,N) \xrightarrow{x\to 0} 0$ exponentially. Then, through integration by parts and differentiation under the integral sign, we obtain

$$\begin{aligned} \mathcal{ST}\left\{x\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}(x); x \to s\right\} &= -s\int_0^\infty \frac{\tilde{\rho}(x)}{(s-x)^2}\,\mathrm{d}x \\ &= \frac{s}{N}\frac{\mathrm{d}}{\mathrm{d}s}W_1(s), \end{aligned} \tag{3.3.2}$$

$$\begin{aligned} \mathcal{ST}\left\{x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\tilde{\rho}(x); x \to s\right\} &= s^2\int_0^\infty \frac{2\tilde{\rho}(x)}{(s-x)^3}\,\mathrm{d}x \\ &= \frac{s^2}{N}\frac{\mathrm{d}^2}{\mathrm{d}s^2}W_1(s), \end{aligned} \tag{3.3.3}$$

$$\begin{aligned} \mathcal{ST}\left\{x^3\frac{\mathrm{d}^3}{\mathrm{d}x^3}\tilde{\rho}(x); x \to s\right\} &= -s^3\int_0^\infty \frac{6\tilde{\rho}(x)}{(s-x)^4}\,\mathrm{d}x \\ &= \frac{s^3}{N}\frac{\mathrm{d}^3}{\mathrm{d}s^3}W_1(s). \end{aligned} \tag{3.3.4}$$

We also present three more identities, this time using tricks where we add and

subtract terms to and from the numerator, or split the numerator otherwise.

$$\mathcal{ST}\left\{x\tilde{\rho}(x); x \to s\right\} = \int_0^\infty \frac{s\tilde{\rho}(x)}{s-x}\,\mathrm{d}x - \int_0^\infty \tilde{\rho}(x)\,\mathrm{d}x = \frac{s}{N}W_1(s) - 1, \tag{3.3.5}$$

where we have used the fact that by definition 1.13 $\tilde{\rho}(x)$, is normalised to integrate to 1 on its support. Using this, we may compute

$$\begin{aligned}
\mathcal{ST}\left\{x^2\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}(x); x \to s\right\} &= -\int_0^\infty \frac{2sx - x^2}{(s-x)^2}\tilde{\rho}(x)\,\mathrm{d}x \\
&= -s\int_0^\infty \frac{x}{(s-x)^2}\tilde{\rho}(x)\,\mathrm{d}x - \int_0^\infty \frac{x}{s-x}\tilde{\rho}(x)\,\mathrm{d}x \\
&= s\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{N}W_1(s) - 1\right) + 1 - \frac{s}{N}W_1(s) \\
&= \frac{s^2}{N}\frac{\mathrm{d}}{\mathrm{d}s}W_1(s) + 1.
\end{aligned} \tag{3.3.6}$$

Our sixth and last identity is

$$\begin{aligned}
\mathcal{ST}\left\{x^3\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}(x); x \to s\right\} &= -\int_0^\infty \frac{sx^2 + 2x^2(s-x)}{(s-x)^2}\tilde{\rho}(x)\,\mathrm{d}x \\
&= -s\int_0^\infty \frac{x^2}{(s-x)^2}\tilde{\rho}(x)\,\mathrm{d}x - 2\int_0^\infty \frac{x^2}{s-x}\tilde{\rho}(x)\,\mathrm{d}x \\
&= \left[s\frac{\mathrm{d}}{\mathrm{d}s} - 2\right]\int_0^\infty \frac{x^2 - s^2 + s^2}{s-x}\tilde{\rho}(x)\,\mathrm{d}x \\
&= \left[s\frac{\mathrm{d}}{\mathrm{d}s} - 2\right]\int_0^\infty \left(\frac{s^2}{s-x} - s - x\right)\tilde{\rho}(x)\,\mathrm{d}x \\
&= \left[s\frac{\mathrm{d}}{\mathrm{d}s} - 2\right]\left(\frac{s^2}{N}W_1(s) - s - \tilde{m}_1\right) \\
&= \frac{s^3}{N}\frac{\mathrm{d}}{\mathrm{d}s}W_1(s) + s + 2\tilde{m}_1,
\end{aligned} \tag{3.3.7}$$

where we recall that $\tilde{\rho}(x)$ integrates to 1 on its support, and $\tilde{m}_1$ is the first moment (and mean) of $\tilde{\rho}$; see subsection 1.4.3. While it is possible to calculate $\tilde{m}_1$ from the very definition of $\tilde{\rho}(x)$, we use the value (2.4.5) computed in section 2.4. That is, for $\beta = 2 \implies \kappa = 1$, and $\alpha_1 = a$, $\tilde{m}_1 = 1 + \frac{a}{N}$. Hence, substituting these six identities into the Stieltjes transform of our differential equation for $\tilde{\rho}(x)$ (3.3.1), we obtain a

differential equation for our resolvent:

$$0 = \frac{s^3}{N}\frac{\mathrm{d}^3}{\mathrm{d}s^3}W_1(s) + \frac{4s^2}{N}\frac{\mathrm{d}^2}{\mathrm{d}s^2}W_1(s) - s\left[Ns^2 - 2(2N+a)s + \frac{a^2-2}{N}\right]\frac{\mathrm{d}}{\mathrm{d}s}W_1(s) + \left[(2N+a)s - \frac{a^2}{N}\right]W_1(s) - N^2 s - Na \tag{3.3.8}$$

We note that this differential equation is an inhomogenous version of equation (3.3.1); the only difference is a factor of N and the $-N^2 s - Na$ term seen above.

3.3.1 The LUE Resolvent Coefficients via Differential Equations

In this section, we have $\kappa = 1$ and $\alpha_1 = a$. We use the differential equation (3.3.8) given above to compute the coefficients of the large N expansion (1.4.18) of $W_1(s)$ given in subsection 1.4.4,

$$\frac{1}{N}W_1(s) = \sum_{l=0}^{\infty}\frac{W_1^l(s)}{N^l}.$$

We substitute this expansion into equation (3.3.8) to obtain

$$\begin{aligned} 0 = {} & s^3\frac{\mathrm{d}^3}{\mathrm{d}s^3}\sum_{l=0}^{\infty}\frac{W_1^l(s)}{N^l} + 4s^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}\sum_{l=0}^{\infty}\frac{W_1^l(s)}{N^l} \\ & - s\left[N^2s^2 - 2N(2N+a)s + a^2 - 2\right]\frac{\mathrm{d}}{\mathrm{d}s}\sum_{l=0}^{\infty}\frac{W_1^l(s)}{N^l} \\ & + \left[N(2N+a)s - a^2\right]\sum_{l=0}^{\infty}\frac{W_1^l(s)}{N^l} - N^2 s - Na. \end{aligned} \tag{3.3.9}$$

From this, we extract order N^2 terms to obtain

$$0 = (4s - s^2)\frac{\mathrm{d}}{\mathrm{d}s}W_1^0(s) + 2W_1^0(s) - 1. \tag{3.3.10}$$

This has the general solution

$$W_1^0(s) = \tfrac{1}{2} + c\sqrt{\pm\left(1 - \tfrac{4}{s}\right)}, \tag{3.3.11}$$

for some constant c. Requiring our resolvent to behave as $\frac{1}{s}$ for large $|s|$, we immediately obtain that $W_1^0(s) = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{s}}\right)$, agreeing with our result in subsection

2.3.3. Extracting the order N terms from (3.3.9) yields

$$0 = (4s^2 - s^3)\frac{\mathrm{d}}{\mathrm{d}s}W_1^1(s) + 2sW_1^1(s) + 2as^2\frac{\mathrm{d}}{\mathrm{d}s}W_1^0(s) + asW_1^0(s) - a. \tag{3.3.12}$$

This equation is satisfied by the coefficients listed in subsection 2.3.3.

For $l \geqslant 0$, extracting order N^{-l} terms from (3.3.9) gives us the general differential difference equation

$$\begin{aligned} 0 = {} & s^3\frac{\mathrm{d}^3}{\mathrm{d}s^3}W_1^l(s) + 4s^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}W_1^l(s) + (4s^2 - s^3)\frac{\mathrm{d}}{\mathrm{d}s}W_1^{l+2}(s) + 2as^2\frac{\mathrm{d}}{\mathrm{d}s}W_1^{l+1}(s) \\ & + (2 - a^2)s\frac{\mathrm{d}}{\mathrm{d}s}W_1^l(s) + 2sW_1^{l+2}(s) + asW_1^{l+1}(s) - a^2W_1^l(s). \end{aligned} \tag{3.3.13}$$

This equation is satisfied by the resolvent coefficients given in subsection 2.3.3 for $l = 0, 1$. These are all of the resolvent coefficients we have at hand for checking purposes.

We note that this family of differential difference equations is the source of a recursive system, not unlike the recursion derived in chapter 2. Indeed, we can use the above to calculate $W_1^k(s)$ by rearranging equation (3.3.13) (with $l = k - 2$) to a form where the left hand side is $(4s^2 - s^3)\frac{\mathrm{d}}{\mathrm{d}s}W_1^k(s) + 2sW_1^k(s)$ and the right hand side contains terms dependent on W_1^m for $m < l$. Then, it is a simple matter of integration by parts, or perhaps series solutions.

Chapter 4

Conclusion

4.1 Comparisons with Existing Literature

4.1.1 The Smoothed Eigenvalue Density From the Resolvent

In this section, we compute our smoothed eigenvalue density up to $O(\frac{1}{N^2})$, by applying the inverse Stieltjes transform to the resolvent coefficients that we have given in subsection 2.3.3. We compare our results to the smoothed density given in [Forrester et al., 2006]. We shall see that our results match.

From definition 1.13 and proposition 6, we know that

$$\tilde{\rho}^{(L)}_{(1)}(x;\alpha_1,\beta,N) = \frac{1}{2\pi \mathrm{i} N} \lim_{\epsilon\to 0} \left[W_1(x-\mathrm{i}\epsilon) - W_1(x+\mathrm{i}\epsilon)\right]. \tag{4.1.1}$$

As this limit is independent of N, and we wish to work in the large N limit, we express $\tilde{\rho}^{(L)}_{(1)}(x;\alpha_1,\beta,N)$ as a large N expansion,

$$\tilde{\rho}^{(L)}_{(1)}(x;\alpha_1,\beta,N) = \sum_{l=0}^{\infty} \frac{\tilde{\rho}_{(1),l}(x;\alpha_1,\beta,N)}{(N\sqrt{\kappa})^l}, \quad \kappa = \tfrac{\beta}{2}. \tag{4.1.2}$$

Then it is immediate that

$$\tilde{\rho}_{(1),l}(x;\alpha_1,\beta,N) = \frac{1}{2\pi \mathrm{i}} \lim_{\epsilon\to 0} \left[W_1^l(x-\mathrm{i}\epsilon) - W_1^l(x+\mathrm{i}\epsilon)\right]. \tag{4.1.3}$$

Before proceeding, we prove a lemma.

Lemma 1. *For $n \in \mathbb{N}_0$, the inverse Stieltjes transform of $\left(1-\frac{4}{s}\right)^{\frac{1}{2}-n}$ is given by*

$$\frac{(-1)^{n+1}}{\pi}\left(\frac{4}{x}-1\right)^{\frac{1}{2}-n}\chi_{x\in(0,4)}, \tag{4.1.4}$$

where χ is the character function.

Proof. We begin by considering each term in the limit separately.

$$\lim_{\epsilon\to 0}\left(1-\frac{4}{x-\mathrm{i}\epsilon}\right)^{\frac{1}{2}-n} = \begin{cases} \left|1-\frac{4}{x}\right|^{\frac{1}{2}-n}, & x\in\mathbb{R}\setminus(0,4) \\ \left|1-\frac{4}{x}\right|^{\frac{1}{2}-n} e^{-\mathrm{i}\pi(\frac{1}{2}-n)}, & 0<x<4 \end{cases}$$

$$\lim_{\epsilon\to 0}\left(1-\frac{4}{x+\mathrm{i}\epsilon}\right)^{\frac{1}{2}-n} = \begin{cases} \left|1-\frac{4}{x}\right|^{\frac{1}{2}-n}, & x\in\mathbb{R}\setminus(0,4) \\ \left|1-\frac{4}{x}\right|^{\frac{1}{2}-n} e^{\mathrm{i}\pi(\frac{1}{2}-n)}, & 0<x<4 \end{cases}$$

We calculate $\left(1-\frac{4}{x\pm\mathrm{i}\epsilon}\right) = \left(\frac{x^2+\epsilon^2-4x\pm 4\mathrm{i}\epsilon}{x^2+\epsilon^2}\right)$ so $\left|1-\frac{4}{x\pm\mathrm{i}\epsilon}\right| \xrightarrow{\epsilon\to 0} \left|1-\frac{4}{x}\right|$. The argument is a little more complicated: For $x\in\mathbb{R}\setminus(0,4)$, $\left(\frac{x^2+\epsilon^2-4x\pm 4\mathrm{i}\epsilon}{x^2+\epsilon^2}\right)$ has positive real part, while for $0<x<4$, $\left(\frac{x^2+\epsilon^2-4x\pm 4\mathrm{i}\epsilon}{x^2+\epsilon^2}\right)$ has negative real part. Hence, $\operatorname{Arg}\left(\frac{x^2+\epsilon^2-4x\pm 4\mathrm{i}\epsilon}{x^2+\epsilon^2}\right) \xrightarrow{\epsilon\to 0} 0$ for $x\in\mathbb{R}\setminus(0,4)$ while for $0<x<4$, $\left(\frac{x^2+\epsilon^2-4x\pm 4\mathrm{i}\epsilon}{x^2+\epsilon^2}\right)$ tends towards the branch cut on the negative real line. Now $\left(\frac{x^2+\epsilon^2-4x+4\mathrm{i}\epsilon}{x^2+\epsilon^2}\right)$ has a positive imaginary part, so $\operatorname{Arg}\left(\frac{x^2+\epsilon^2-4x+4\mathrm{i}\epsilon}{x^2+\epsilon^2}\right) \xrightarrow{\epsilon\to 0} \pi$ for $0<x<4$. Likewise, $\operatorname{Arg}\left(\frac{x^2+\epsilon^2-4x-4\mathrm{i}\epsilon}{x^2+\epsilon^2}\right) \xrightarrow{\epsilon\to 0} -\pi$ for $0<x<4$.

Moving on, we see that

$$\begin{aligned} &\frac{1}{2\pi\mathrm{i}}\lim_{\epsilon\to 0}\left[\left(1-\frac{4}{x-\mathrm{i}\epsilon}\right)^{\frac{1}{2}-n}-\left(1-\frac{4}{x+\mathrm{i}\epsilon}\right)^{\frac{1}{2}-n}\right] \\ &\qquad = \frac{1}{2\pi\mathrm{i}}\begin{cases} 0, & x\in\mathbb{R}\setminus(0,4) \\ -2\mathrm{i}e^{\mathrm{i}\pi n}\left(\frac{4}{x}-1\right)^{\frac{1}{2}-n}, & 0<x<4 \end{cases} \\ &\qquad = \frac{(-1)^{n+1}}{\pi}\left(\frac{4}{x}-1\right)^{\frac{1}{2}-n}\chi_{x\in(0,4)} \end{aligned}$$

□

Using this lemma, and theorem 1.5, we find that

$$\tilde{\rho}_{(1),0}(x;\alpha_1,\beta,N) = \mathcal{IST}\left\{W_1^0(s); s\to x\right\} = \mathcal{IST}\left\{\frac{1}{2}\left(1-\sqrt{1-\tfrac{4}{s}}\right); s\to x\right\}$$
$$= -\frac{1}{2}\mathcal{IST}\left\{\left(1-\tfrac{4}{s}\right)^{\frac{1}{2}}; s\to x\right\} = \frac{1}{2\pi}\sqrt{\tfrac{4}{x}-1}\,\chi_{x\in(0,4)} \qquad (4.1.5)$$

$$\begin{aligned}
&\tilde{\rho}_{(1),1}(x;\alpha_1,\beta,N)\\
&= \mathcal{IST}\left\{W_1^1(s); s\to x\right\}\\
&= \mathcal{IST}\left\{\frac{-h}{s^2-4s} + \frac{\alpha_1}{2\sqrt{\kappa}}\left(\frac{1}{s}\left(1-\tfrac{4}{s}\right)^{-\frac{1}{2}} - \frac{1}{s}\right); s\to x\right\}\\
&= -\frac{h}{4}\mathcal{IST}\left\{\frac{1}{s-4} - \frac{1}{s}; s\to x\right\} + \frac{\alpha_1}{2\sqrt{\kappa}}\mathcal{IST}\left\{\frac{1}{s}\left(1-\tfrac{4}{s}\right)^{-\frac{1}{2}} - \frac{1}{s}; s\to x\right\}\\
&= \frac{h}{4}\left[\delta(x)-\delta(x-4)\right] + \frac{\alpha_1}{2\sqrt{\kappa}}\left[\frac{1}{\pi x}\left(\tfrac{4}{x}-1\right)^{-\frac{1}{2}}\chi_{x\in(0,4)} - \delta(x)\right]
\end{aligned} \qquad (4.1.6)$$

$$\begin{aligned}
&\tilde{\rho}_{(1),2}(x;\alpha_1,\beta,N)\\
&= \mathcal{IST}\left\{W_1^2(s)\right\}\\
&= \mathcal{IST}\left\{\frac{1}{s^4}\left(1-\tfrac{4}{s}\right)^{-\frac{5}{2}} + \frac{{\alpha_1}^2}{s^3\kappa}\left(1-\tfrac{4}{s}\right)^{-\frac{3}{2}}; s\to x\right\}\\
&\quad + h^2\mathcal{IST}\left\{\frac{2}{s^3}\left(1-\tfrac{4}{s}\right)^{-\frac{3}{2}} + \frac{5}{s^4}\left(1-\tfrac{4}{s}\right)^{-\frac{5}{2}}; s\to x\right\}\\
&\quad + \frac{h\alpha_1}{2\sqrt{\kappa}}\mathcal{IST}\left\{\frac{1}{s^2}\left(1-\tfrac{4}{s}\right)^{-\frac{3}{2}} - \frac{4}{s^3}\left(1-\tfrac{4}{s}\right)^{-\frac{3}{2}} - \frac{1}{s^2-4s} - \frac{4}{s(s-4)^2}; s\to x\right\}\\
&= \frac{1}{\pi x^4}\left(\tfrac{4}{x}-1\right)^{-\frac{5}{2}}\chi_{x\in(0,4)} - \frac{{\alpha_1}^2}{\pi x^3\kappa}\left(\tfrac{4}{x}-1\right)^{-\frac{3}{2}}\chi_{x\in(0,4)}\\
&\quad + h^2\left[\frac{5}{\pi x^4}\left(\tfrac{4}{x}-1\right)^{-\frac{5}{2}}\chi_{s\in(0,4)} - \frac{2}{\pi x^3}\left(\tfrac{4}{x}-1\right)^{-\frac{3}{2}}\chi_{x\in(0,4)}\right]\\
&\quad + \frac{h\alpha_1}{2\sqrt{\kappa}}\left[\frac{4-x}{\pi x^3}\left(\tfrac{4}{x}-1\right)^{-\frac{3}{2}}\chi_{x\in(0,4)} + \delta'(x-4)\right]
\end{aligned} \qquad (4.1.7)$$

As a check, we apply the Stieltjes transform to these, and retrieve our resolvent coefficients W_1^0, W_1^1, W_1^2. Moving on, we compare with the relevant quantities given

in [Forrester et al., 2006]. In this work, Forrester et al. treats the Laguerre unitary ensemble. Thus, we compare by setting $\kappa = 1$ and $h = 0$ in our density coefficients, and then writing $X = \frac{x}{4}$ and $\alpha = \alpha_1$. Moreover, because we are dealing with the smoothed eigenvalue density, we must ignore the oscillatory terms in [Forrester et al., 2006]. Finally, note that we cannot simply compare our $\tilde{\rho}_{(1),l}$ to their $\rho_{(1),l}$, as we need to account for the change in measure due to our notational change. With equality given by the change in notation, the $O(1)$ non-oscillatory term in (2.7) of [Forrester et al., 2006] corresponds to

$$\frac{2}{\pi}\sqrt{\frac{1}{X}-1}\,\chi_{X\in(0,1)}\mathrm{d}X = \frac{1}{2\pi}\sqrt{\frac{4}{x}-1}\,\chi_{x\in(0,4)}\mathrm{d}x.$$

The $O(\frac{1}{N})$ non-oscillatory term in (2.7) of [Forrester et al., 2006] corresponds to

$$\frac{\alpha}{2\pi X\sqrt{\frac{1}{X}-1}}\,\chi_{X\in(0,1)}\mathrm{d}X = \frac{\alpha_1}{2\pi x\sqrt{\frac{4}{x}-1}}\,\chi_{x\in(0,4)}\mathrm{d}x.$$

The $O(\frac{1}{N^2})$ non-oscillatory term in (2.15) of [Forrester et al., 2006] corresponds to

$$\begin{aligned}\frac{1+4(-1+X)\alpha^2}{64\pi(1-X)^{5/2}X^{3/2}}\,\chi_{X\in(0,1)}\mathrm{d}X &= \frac{1+(s-4)\alpha_1^2}{256\pi\left(\frac{4}{x}-1\right)\left(\frac{x}{4}\right)^4}\,\chi_{x\in(0,4)}\mathrm{d}x \\ &= \left[\frac{1}{\pi x^4\left(\frac{4}{x}-1\right)^{5/2}} - \frac{\alpha_1^2}{\pi x^3\left(\frac{4}{x}-1\right)^{3/2}}\right]\chi_{x\in(0,4)}\mathrm{d}x.\end{aligned}$$

These expressions are exactly the terms in equations (4.1.5), (4.1.6) and (4.1.7) that are not accompanied by factors of h.

4.1.2 Comparing The Moments

In section 2.4, we computed the first few moments of the Laguerre β ensemble. We check these against two resources.

In subsection 1.4.3, we mentioned that [Mezzadri and Reynolds, 2015] developed methods of computing these moments via Jack polynomial theory. The moments of the Laguerre β ensemble presented in [Mezzadri and Reynolds, 2015] agree

completely with our results, after taking care to substitute $m_k = N^{k+1}\tilde{m}_k$, and then scaling N so that our eigenvalue density agrees with the eigenvalue density in [Mezzadri and Reynolds, 2015].

The second resource is the Maple package titled Multivariate Orthogonal Polynomials (Symbolically), or MOPS. This package is documented in [Dumitriu et al., 2007]. Using the package, we computed the moments of the Laguerre β ensemble. Again, the results agreed with our computations in section 2.4.

4.1.3 Comparing The Eigenvalue Density Differential Equation

We present some objects seen in [Forrester, 2010, pp.189-200], with $N \in \mathbb{N}_{>0}$ and $n \in \mathbb{N}_0$.

$$K_N(x,x) = \frac{w_2(x)}{(p_{N-1}, p_{N-1})_2}\left(p'_N(x)p_{N-1}(x) - p'_{N-1}(x)p_N(x)\right), \tag{4.1.8}$$

$$(p_n, p_n)_2 = \Gamma(n+1)\Gamma(a+n+1), \tag{4.1.9}$$

$$p_n(x) = (-1)^n n! L_n^a(x), \tag{4.1.10}$$

$$L_n^a(x) = \sum_{m=0}^{n}(-1)^m \binom{n+a}{n-m}\frac{x^m}{m!}. \tag{4.1.11}$$

The $K_N(x,x)$ given here, with the appropriate substitutions, gives an expression for the eigenvalue density of the Laguerre β ensemble. We find that this satisfies equation (3.2.4) for $N = 2$ and 3, which raises our confidence in the validity of our differential equation.

For another check of equation (3.2.4), we compare to the differential equation given in [Adachi et al., 2011]. To make the comparison, we replace our x with $(N+a)\xi$, and their ϵ with $\frac{1}{N^2}$ and their λ with $\frac{a}{N}$. Then, the two differential equations are identical.

4.2 Moving Forward

One of the advantages of the loop equation formalism is that the method used to construct the recursion can be applied to more general cases. The ideas presented in chapter 2 could very well be generalised to general potentials $V(x)$. By this, we mean that one could replace the factors of $e^{-\gamma_i}$ with $e^{-V(\gamma_i)}$ in the scaled eigenvalue j.p.d.f.s, to obtain a more general result. Also, it may be worthwile investigating the structure of the loop equations if α_1 were to be taken to be of order N. Our work was in the regime $\alpha_1 > 0$. However, we would like to allow $\alpha_1 > -1$, so as to allow cases like the $(0, N)$ Laguerre unitary ensemble. We believe that we should be able to extend our result to the $\alpha_1 > -1$ regime via analytic continuation of one form or another.

Looking at the resolvent coefficients presented in subsection 2.3.3, we suspect that there might be nicer forms for them. For example, a lot of our coefficients are of the form $\left(\frac{\alpha_1}{\sqrt{\kappa}}\right)^k$ for integers k; perhaps the α_1 parameter could be redefined in a more natural way. Also, it would be interesting to search for structure by setting certain terms to 0 or 1. We suspect a duality between the cases $\beta = 1$ and $\beta = 4$.

Regarding chapter 3, there are a few directions that the work could be continued in. First, in the stylings of [Witte and Forrester, 2014], one could further expand the resolvent coefficients in large x to derive expansions for the resolvent coefficients themselves. On the other hand, the information presented in section 3.1 is valid for even β, so can be applied to $\beta = 4$ immediately. Moreover, once the $\beta = 4$ results are obtained, one may exploit a duality between the $\beta = 1$ and $\beta = 4$ cases to obtain similar results for the $\beta = 1$ case.

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Appendices

Appendix A

A Fact Used in Proposition 2

Fix $\beta = 1$, 2, or 4. Let G be an $M \times N$ standard Gaussian β matrix, and use Gram-Schmidt orthogonalisation to write $G = UT$, where $T = [t_{i,j}]$ is an $N \times N$ upper triangular β matrix with diagonal entries positive and real, and U is an $(M \times N)$ β matrix such that $U^\dagger U = I_N$. Extend U to an $(M \times M)$ β matrix $V := [U\, U']$ by defining an $(M \times a)$ β matrix U' such that $V^\dagger V = I_M$, where $a = M - N$. We show that

$$(\mathrm{d}G) = \prod_{j=1}^{N} t_{j,j}^{\beta(M-j)+\kappa} (\mathrm{d}T)(V^\dagger \mathrm{d}U),$$

where for a matrix of differentials $\mathrm{d}Y$, $(\mathrm{d}Y)$ denotes the product of all independent elements of $\mathrm{d}Y$. We recall that the independent elements are the real components of the entries that do not depend on any other such components.

Proof. Since

$$I_M = V^\dagger V = \begin{bmatrix} I_N & U^\dagger U' \\ (U')^\dagger U & (U')^\dagger U' \end{bmatrix},$$

we have that $U^\dagger U'$ and $(U')^\dagger U$ are both zero matrices, and $(U')^\dagger U' = I_a$. Moreover, $G = UT$ implies that $\mathrm{d}G = \mathrm{d}U T + U\mathrm{d}T$, so

$$V^\dagger \mathrm{d}G = \begin{bmatrix} U^\dagger(\mathrm{d}UT + U\mathrm{d}T) \\ (U')^\dagger(\mathrm{d}UT + U\mathrm{d}T) \end{bmatrix} = \begin{bmatrix} U^\dagger \mathrm{d}U\, T + \mathrm{d}T \\ (U')^\dagger \mathrm{d}U\, T \end{bmatrix}. \tag{A.0.1}$$

Now, in the style of proposition 1, we denote the j^{th} columns of U and U' by $\mathbf{u_j}$ for $1 \leqslant j \leqslant N$ and $\mathbf{u'_j}$ for $1 \leqslant j \leqslant a$, respectively. Following the proof of proposition 1, we note that since T is an upper triangular matrix, the (i,j) entry of $U^\dagger \mathrm{d}U\, T + \mathrm{d}T$ is $\left[\mathrm{d}t_{i,j} + \sum_{k=1}^{j} t_{k,j} \mathbf{u_i}^\dagger \mathrm{d}\mathbf{u_k}\right]$. Since $U^\dagger U = I_N$, we have that $U^\dagger \mathrm{d}U = -\mathrm{d}U^\dagger U = -\left(U^\dagger \mathrm{d}U\right)^\dagger$, so $\mathbf{u_i}^\dagger \mathrm{d}\mathbf{u_j} = -\mathbf{u_j}^\dagger \mathrm{d}\mathbf{u_i}$ for all $1 \leqslant i, j \leqslant N$. Thus, to extract the independent elements of $U^\dagger \mathrm{d}U\, T + \mathrm{d}T$, it is sufficient to collect all of $(\mathrm{d}T)$ and then look for terms of the form $\mathbf{u_i}^\dagger \mathrm{d}\mathbf{u_j}$ for all $1 \leqslant j \leqslant i \leqslant N$. For every such combination, the (i,j) entry of $U^\dagger \mathrm{d}U\, T + \mathrm{d}T$ contains the term $t_{j,j} \mathbf{u_i}^\dagger \mathrm{d}\mathbf{u_j}$. Thus, we arrive at a collection of independent elements of $U^\dagger \mathrm{d}U\, T + \mathrm{d}T$,

$$\{\mathrm{d}t_{i,j} \,|\, 1 \leqslant i, j \leqslant N\} \cup \{t_{j,j} \mathbf{u_i}^\dagger \mathrm{d}\mathbf{u_j} \,|\, 1 \leqslant j \leqslant i \leqslant N\}.$$

In fact, these are all of the independent elements of $U^\dagger \mathrm{d}U\, T + \mathrm{d}T$; any other independent term would have to be of the form $\mathbf{u_i}^\dagger \mathrm{d}\mathbf{u_j}$ for $1 \leqslant i < j \leqslant N$, but we already explained that such a term would be dependent on our collection. Thus,

$$(U^\dagger \mathrm{d}U\, T + \mathrm{d}T) = \prod_{j=1}^{N} t_{j,j}^{\beta(N-j)+\kappa} (\mathrm{d}T)(U^\dagger \mathrm{d}U), \tag{A.o.2}$$

where $\kappa = 0$ if $\beta = 1$, $\kappa = 1$ if $\beta = 2$, and $\kappa = 2$ if $\beta = 4$; the κ is necessary because $U^\dagger \mathrm{d}U = -(U^\dagger \mathrm{d}U)^\dagger$ implies that for $1 \leqslant i \leqslant N$, $\mathbf{u_i}^\dagger \mathrm{d}\mathbf{u_i}$ is 0 in the real case, has only one independent real component in the complex case, and has two independent real components the quaternion case.

If $\beta = 2$, we write $v_{i,j,1} + \mathrm{i} v_{i,j,2}$ for the (i,j) entry of $V^\dagger$, $\mathrm{d}x_{i,j} + \mathrm{i}\,\mathrm{d}y_{i,j}$ for the (i,j) entry of $V^\dagger \mathrm{d}G$, and $\mathrm{d}g_{i,j,1} + \mathrm{i}\,\mathrm{d}g_{i,j,2}$ for the (i,j) entry of $\mathrm{d}G$. Then,

$$\begin{aligned}
\mathrm{d}x_{i,j} &= \sum_{k=1}^{M} \left(v_{k,i,1} \mathrm{d}g_{k,j,1} - v_{k,i,2} \mathrm{d}g_{k,j,2}\right), \\
\mathrm{d}y_{i,j} &= \sum_{k=1}^{M} \left(v_{k,i,1} \mathrm{d}g_{k,j,2} + v_{k,i,2} \mathrm{d}g_{k,j,1}\right)
\end{aligned}$$

for $1 \leqslant i \leqslant M$ and $1 \leqslant j \leqslant N$. For any column $1 \leqslant k \leqslant N$, the Jacobian matrix for the change of variables from $(\mathrm{d}g_{1,k,1}, \ldots, \mathrm{d}g_{M,k,1}, \mathrm{d}g_{1,k,2}, \ldots, \mathrm{d}g_{M,k,2})$ to

$(\mathrm{d}x_{1,k}, \ldots, \mathrm{d}x_{M,k}, \mathrm{d}y_{1,k}, \ldots, \mathrm{d}y_{M,k})$, in the given orders, is

$$\begin{bmatrix} [v_{i,j,1}] & -[v_{i,j,2}] \\ [v_{i,j,2}] & [v_{i,j,1}] \end{bmatrix}.$$

This has determinant

$$\begin{aligned} \begin{vmatrix} [v_{i,j,1}] & -[v_{i,j,2}] \\ [v_{i,j,2}] & [v_{i,j,1}] \end{vmatrix} &= \begin{vmatrix} [v_{i,j,1}] + \mathrm{i}[v_{i,j,2}] & \mathrm{i}[v_{i,j,1}] - [v_{i,j,2}] \\ [v_{i,j,2}] & [v_{i,j,1}] \end{vmatrix} \\ &= \begin{vmatrix} [v_{i,j,1}] + \mathrm{i}[v_{i,j,2}] & 0 \\ [v_{i,j,1}] & [v_{i,j,1}] - \mathrm{i}[v_{i,j,2}] \end{vmatrix} \\ &= \det(V^\dagger)\det(V). \end{aligned}$$

Thus, the Jacobian for changing variables from $\mathrm{d}g_{i,j,s}$ to $\mathrm{d}x_{i,j}, \mathrm{d}y_{i,j}$ for $1 \leqslant i \leqslant M$, $1 \leqslant j \leqslant N$, and $s = 1, 2$ is $|\det(V)|^{2N}$, as there are N columns in $\mathrm{d}G$, thus N sets of changes as given above. If $\beta = 1$, we simply remove all imaginary components, so that the Jacobian matrix is $[v_{i,j}]$, and our Jacobian is $|\det(V)|^N$. If $\beta = 4$, we use the matrix interpretation of quaternions to repeat the above with two sets of real and imaginary parts for $\mathrm{d}G$ and $V^\dagger \mathrm{d}G$, and find the Jacobian to be $|\det(V)|^{4N}$. Hence, $(V^\dagger \mathrm{d}G) = |\det(V)|^{\beta N}(\mathrm{d}G)$. However, V is either orthogonal, unitary, or symplectic, so its determinant is ± 1, and we have

$$(V^\dagger \mathrm{d}G) = (\mathrm{d}G). \tag{A.0.3}$$

By a similar argument, we note that $(U')^\dagger \mathrm{d}U\, T$ may be interpreted as a change of variables from $U^\dagger \mathrm{d}U$ to $(U')^\dagger \mathrm{d}U\, T$ – the only difference here is that we have T acting from the right instead of $V^\dagger$ acting from the left. Thus, since T is upper triangular,

$$\left((U')^\dagger \mathrm{d}U\, T\right) = |\det(T)|^{\beta a} \left((U')^\dagger \mathrm{d}U\right) = \prod_{i=1}^{N} t_{i,i}^{\beta a} \left((U')^\dagger \mathrm{d}U\right). \tag{A.0.4}$$

Now, combining (A.0.2) and (A.0.4), we have that the product of the independent elements of the right hand side of (A.0.1) is

$$\prod_{j=1}^{N} t_{j,j}^{\beta(M-j)+\kappa} (\mathrm{d}T)(V^\dagger \mathrm{d}U),$$

because $(V^\dagger \mathrm{d}U)$, by definition, is a product of the independent elements of $U^\dagger \mathrm{d}U$ and $(U')^\dagger \mathrm{d}U$. Here, we remark that our notation is different to that of the cited references, but note that by a statement of [Muirhead, 1982, p.69], $(V^\dagger \mathrm{d}U)$ is not dependent on the choice of U', so it is equivalent to $(U^\dagger \mathrm{d}U)$ in some sense. Thus, (A.0.1) with (A.0.3) and the above tells us that

$$(\mathrm{d}G) = \prod_{j=1}^{N} t_{j,j}^{\beta(M-j)+\kappa} (\mathrm{d}T)(V^\dagger \mathrm{d}U),$$

as required. □

Appendix B

A Fact Used in Proposition 3

Fix $\beta = 1$, 2, or 4. Let $G = [g_{i,j}]$ be an $M \times N$ standard Gaussian β matrix. Let $\mathbf{g_1}$ be the left-most column of G and let $G^{(1)}$ be the $M \times (N-1)$ matrix such that $G = [\mathbf{g_1}\, G^{(1)}]$.

Let $\mathbf{v_1} = \mathbf{g_1} - \sqrt{\mathbf{g_1}^\dagger \mathbf{g_1}}[1\,0\,\cdots\,0]^T$, and define the Householder transformation $L^{(1)} = I_M - \frac{2}{\mathbf{v_1}^\dagger \mathbf{v_1}} \mathbf{v_1}\mathbf{v_1}^\dagger$. Then $L^{(1)}\mathbf{g_1} = \sqrt{\mathbf{g_1}^\dagger \mathbf{g_1}}[1\,0\,\cdots\,0]^T$. Moreover, $L^{(1)}$ is an orthogonal, unitary, or symplectic matrix (depending on β). We show that $L^{(1)}G^{(1)}$ is in fact an $M \times (N-1)$ standard Gaussian β matrix.

Proof. Let $g^{(1)}_{i,j}$ denote the (i,j) entry of $L^{(1)}G^{(1)}$, with $2 \leqslant j \leqslant N$, and let $l_{i,k}$ denote the (i,k) entry of $L^{(1)}$. Then $g^{(1)}_{i,j} = \sum_{k=1}^{M} l_{i,k} g_{k,j}$.

- If $\beta = 1$, $L^{(1)}$ is an orthogonal matrix that does not depend on $G^{(1)}$. Thus, all of the $g_{k,j}$ are independent of the $l_{i,k}$ and each other. Hence, the mean of $g^{(1)}_{i,j}$ is 0, and the variance of $g^{(1)}_{i,j}$ is $\sum_{k=1}^{M} l_{i,k}^2 = 1$.

- If $\beta = 2$, we write $l_{i,k} = l_{i,k,1} + \mathrm{i} l_{i,k,2}$ and $g_{k,j} = g_{k,j,1} + \mathrm{i} g_{k,j,2}$. Then, let

$$g^{(1)}_{i,j,1} = \sum_{k=1}^{M} (l_{i,k,1} g_{k,j,1} - l_{i,k,2} g_{k,j,2}), \qquad g^{(1)}_{i,j,2} = \sum_{k=1}^{M} (l_{i,k,1} g_{k,j,2} + l_{i,k,2} g_{k,j,1}),$$

  so that $g^{(1)}_{i,j} = g^{(1)}_{i,j,1} + \mathrm{i} g^{(1)}_{i,j,2}$. Now, for each $1 \leqslant k \leqslant M$, $\pm g_{k,j,1}$ and $\pm g_{k,j,2}$ are Gaussian with mean 0 and variance $\frac{1}{2}$ from definition 1.2. Since $L^{(1)}$ is a unitary

matrix, $\sum_{k=1}^{M} l_{i,k,1}^2 + l_{i,k,2}^2 = 1$. As $L^{(1)}$ does not depend on $G^{(1)}$, $g_{i,j,1}^{(1)}$ and $g_{i,j,2}^{(1)}$ are hence both Gaussian with mean 0 and variance $\frac{1}{2}$.

- If $\beta = 4$, we write

$$l_{i,k} = \begin{bmatrix} l_{i,k,1} + \mathrm{i} l_{i,k,2} & l_{i,k,3} + \mathrm{i} l_{i,k,4} \\ -l_{i,k,3} + \mathrm{i} l_{i,k,4} & l_{i,k,1} - \mathrm{i} l_{i,k,2} \end{bmatrix}, \quad g_{k,j} = \begin{bmatrix} g_{k,j,1} + \mathrm{i} g_{k,j,2} & g_{k,j,3} + \mathrm{i} g_{k,j,4} \\ -g_{k,j,3} + \mathrm{i} g_{k,j,4} & g_{k,j,1} - \mathrm{i} g_{k,j,2} \end{bmatrix}.$$

Then, let

$$\begin{aligned} g_{i,j,1}^{(1)} &= \sum_{k=1}^{M} (l_{i,k,1} g_{k,j,1} - l_{i,k,2} g_{k,j,2} - l_{i,k,3} g_{k,j,3} - l_{i,k,4} g_{k,j,4}), \\ g_{i,j,2}^{(1)} &= \sum_{k=1}^{M} (l_{i,k,1} g_{k,j,2} + l_{i,k,2} g_{k,j,1} + l_{i,k,3} g_{k,j,4} - l_{i,k,4} g_{k,j,3}), \\ g_{i,j,3}^{(1)} &= \sum_{k=1}^{M} (l_{i,k,1} g_{k,j,3} - l_{i,k,2} g_{k,j,4} + l_{i,k,3} g_{k,j,1} + l_{i,k,4} g_{k,j,2}), \\ g_{i,j,4}^{(1)} &= \sum_{k=1}^{M} (l_{i,k,1} g_{k,j,4} + l_{i,k,2} g_{k,j,3} - l_{i,k,3} g_{k,j,2} + l_{i,k,4} g_{k,j,1}), \end{aligned}$$

so that

$$g_{i,j}^{(1)} = \begin{bmatrix} g_{i,j,1}^{(1)} + \mathrm{i} g_{i,j,2}^{(1)} & g_{i,j,3}^{(1)} + \mathrm{i} g_{i,j,4}^{(1)} \\ -g_{i,j,3}^{(1)} + \mathrm{i} g_{i,j,4}^{(1)} & g_{i,j,1}^{(1)} - \mathrm{i} g_{i,j,2}^{(1)} \end{bmatrix}.$$

Since $L^{(1)}$ is symplectic, we have that $\sum_{k=1}^{M} l_{i,k}^{\dagger} l_{i,k} = I_2$. Thus,

$$\sum_{k=1}^{M} (l_{i,k,1}^2 + l_{i,k,2}^2 + l_{i,k,3}^2 + l_{i,k,4}^2) = 1.$$

From definition 1.2, we know that for each $1 \leqslant k \leqslant N$, $\pm g_{k,j,1}, \pm g_{k,j,2}, \pm g_{k,j,3}, \pm g_{k,j,4}$ are Gaussian with mean 0 and variance $\frac{1}{4}$. Hence, as $L^{(1)}$ does not depend on $G^{(1)}$, $g_{i,j,s}^{(1)}$ has mean 0 and variance $\frac{1}{4}$ for each $1 \leqslant s \leqslant 4$.

In all 3 cases, we thus have that $L^{(1)} G^{(1)}$ is an $M \times (N-1)$ standard Gaussian β matrix, by the very definition of a standard Gaussian β matrix. $\square$