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Combinatorial aspects of juggling

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Abstract

This paper examines the relationship between the practice of juggling and mathematics. First an introduction and brief history of juggling is presented, where we highlight some of the notable events and personalities in juggling and mathematical juggling.

In **Section 2** we discuss some of the ways that mathematical ideas have been used to solve juggling problems. The main contribution in this field was the development of siteswap, a compact and powerful notation for describing juggling patterns, that readily lends itself to generalisation. We prove some simple theorems involving this notation. Some other systems of notation are introduced and we discuss some juggling theorems of Claude Shannon.

Section 3 looks at how juggling concepts have been used to prove theorems that are not strictly juggling-related. We develop a theorem for counting the number of periodic juggling patterns with a fixed number of balls, and use this result to prove an identity relating the number of *drops* of a permutation of a set to the chromatic polynomial of a graph of the set. We also use this result to develop juggling concepts in the context of q-binomial numbers, by introducing *juggling cards* which we can use to create arbitrary juggling patterns. This development allows us to easily calculate the Poincaré series of the affine Weyl group $\tilde{A}_{d-1}$. We also find that we can apply these ideas to prove a theorem involving q-Stirling numbers. Vector compositions can also be naturally described by juggling patterns and we prove some identities involving unitary vector compositions before broadening the discussion to include more general vector compositions.

Acknowledgments

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On a technical level, I would like to thank Jeff Briffa, Moshe Sniedovich and Patrick Desrosiers for their assistance with the diagrams and graphs that were needed.

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Contents

NOTATION

$\boldsymbol{\delta}_{d-1} = (0, 1, ..., d-1)$

$[n] = (1 + q + q^2 + ... + q^{n-1})$

$df(t) = f(t) - t$

$\beta_n(k) =$ the number of permutations of $\boldsymbol{\delta}_{n-1}$ that have k drops.

$\Pi_k n =$ the set of all partitions of $(1, 2, ..., n)$ into k blocks.

$\|\boldsymbol{\alpha}\| = \sum_{i=1}^{d} \alpha_i$

$\boldsymbol{z}(\boldsymbol{\alpha}) = (\underbrace{1, 1, ..., 1}_{\alpha_1}, \underbrace{2, 2, ..., 2}_{\alpha_2}, ..., \underbrace{d, d, ..., d}_{\alpha_d})$

$\mathbf{0}_k = (\underbrace{0, 0, .., 0}_{k})$

$(\boldsymbol{d} + \mathbf{1})_k = (\underbrace{d+1, d+1, ..., d+1}_{k})$

1 Juggling history and practice

What may be surprising to non-jugglers is that the definition of the word *juggling* is not agreed upon. Questions that must be answered before a definition can be made include:

- How many objects do we need?
- Must they be certain types of objects? (as an extreme case, can you juggle nebulous things like steam and air?)
- What must be done with these objects? (must they be thrown, or can we bounce them? What about *contact juggling*, where the ball is always in contact with the 'juggler'?)

One definition that is oft quoted is the following: manipulating more objects than you have hands, as long as those objects are not constantly in contact with the 'juggler'. This definition captures what most people would consider juggling; the question of whether juggling two balls is really juggling will of course determine if you would replace word *more* with the words *at least* for your boundary condition. However, for the purposes of this paper, I will be using the following definition.

Definition 1.1. *Juggling* is the act of manipulating objects in a way that can be described by a system of juggling notation.

This definition may sound circular, but in **Section 2** we will examine several well-defined systems of juggling notation. One trivial consequence of this definition is that it is possible to 'juggle' one or even zero balls. Which has the immediate implication that everybody in the world is quite an experienced juggler; who knew the world was so full of talent?

As for the types of objects juggled, I should guess that almost any object that can be lifted has been juggled at some stage in history. This includes the obvious: balls, rings, clubs, scarves, knives, rocks, sticks, food, etc... And the not-so-obvious: cannonballs, bowling balls, swords, chainsaws, water, ice-cream, children,[1] and pretty much anything else. However, the type of prop used makes very little difference to the types of patterns that can be attempted (in most cases). For ease of writing, in this paper I will assume that the prop is a ball.

The standard way of juggling an odd number of balls is to have them follow each other in a figure 8 pattern. This pattern is called a *cascade* (Figure 1(a)). For an even number of balls, you juggle half the balls in one hand and the other half in the other hand, in circles (Figure 1(b)). This pattern is called a *fountain*; it can be juggled in two ways, where the hands are throwing synchronously, or asynchronously. However, the common way for non-jugglers to picture juggling is to see the balls following each other in a circle, with a high throw from (say) the right hand, and a quick throw back from the left hand (Figure 1(c)). Indeed this is how practically everybody goes about learning to juggle, and it is the standard way for non-jugglers to juggle two balls. This pattern is called a *shower*. A shower pattern is much less efficient, and more difficult to juggle than the cascade or fountain with the same number of balls.

In the basic concept of juggling, we imagine the hands throwing and catching a single ball at time. These patterns are called *simple* juggling patterns. However, there are also patterns where multiple balls are thrown and caught at the same time, the patterns of which are called *multiplex* juggling patterns.

[1] There is a pattern called the 'Baby Juggling Pattern', in the language of siteswap (which will be described in **Section 2**) this pattern is 52512. The baby, corresponding to the '1' and the '2's, is passed from arm to arm, while the free hand throws two balls.

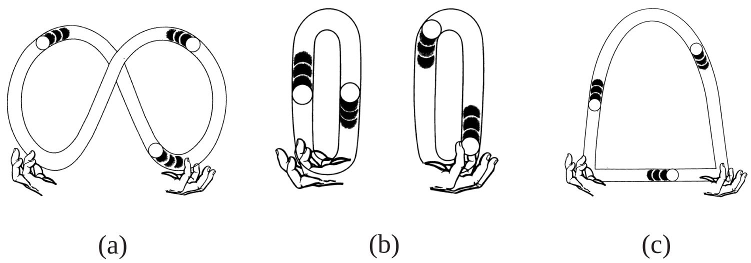

Figure 1: (a) cascade, (b) fountain, (c) shower. Source: [4].

Since two jugglers, who are juggling the same pattern, should be juggling at much the same rate, it is possible that they can exchange balls between them. This is quite common, and it is most visually effective when juggling clubs. Surprisingly, it is not much more difficult that juggling clubs by oneself. There is no limit to the number of people that can be involved in juggling a single pattern.

Juggling—in its various forms, and by the various definitions—has a long history, which is not surprising since it requires no complicated or specialised apparatus. The oldest record of the practice seems to be a painting in an Egyptian tomb, dating from sometime between 1994–1781 BCE, see [13]. A copy of this painting is contained in Figure 2.

Figure 2: Juggling women painted on the wall of an Egyptian tomb, dating from 1994–1781 BCE, see [13].

It is known that juggling was performed in Europe during Roman times and in the middle ages. The performers were known as *joculatores* in Latin, and became *jongleurs* in French and jugglers and jesters in English. In these performances, juggling was mixed with singing, dancing, joke-telling and general buffonery for a whole entertainment package. It is easy to see that the English words *joke* and *jocular* share the same root. Historical records have claimed the following feats[2]:

- A Chinese man named Lie Zhi, who lived some time between 770 and 476 BCE, could juggle seven swords.
- It is mentioned in the Talmud that a Rabbi Shimon ben Gamaliel juggled eight flaming torches.
- An Irish man called Cuchulainn, from some time in the fifth century CE, could juggle nine apples.

[2]The historical information provided here has been sourced from [19], [23] and [27]

- At a banquet around 400 BCE, Socrates observed a woman juggling 12 rings.

There are also accounts of juggling in the Americas, Japan, India and the Middle East. An interesting aspect in the historical development of juggling is that presented by some Pacific Islanders. Tonga apparently has a strong tradition of juggling amongst young girls. Several European explorers have recorded that many girls can juggle six (and more) objects, often fruit, nuts or balls, usually in a shower pattern. This represents quite an amazing concentration of juggling talent.

In Western culture, juggling became linked to vaudeville, with performers such as W. C. Fields, Jimmy Savo and Fred Allen. Enrico Rastelli is often credited as the greatest juggler of all time. He was said to practice for up to ten hours a day, and could perform an amazing range of feats: juggling ten balls, eight clubs or eight plates; continually bouncing three balls on top of his head; and juggling three balls in one hand at the same time as rotating a cylinder on his feet, all while doing a one-armed handstand. Modern jugglers of note include Sergei Ignatov (who was with the Moscow Circus in the 1970s; he could juggle 11 rings), Albert Lucas (who could juggle nine rings, balance a ball on a stick in his mouth, rotate another ring around his leg, all while ice-skating. He also holds the world record for juggling 13 rings.), Bruce Sarafian (who holds the record of juggling 12 balls), Bruce Tiemann (who holds the equal club juggling record of 9 clubs), Anthony Gatto and Thomas Dietz (who each hold records for various juggles). (For information on juggling records see [17].)

It is commonly remarked that juggling seems to attract those people who have an interest in mathematics, physics and computer science (see [23]). Famous jugglers of this sort include Allen Knutson and Jack Boyce of University of California, Bengt Magnusson of CalTech, Colin Wright and Ed Carstens. Burkard Polster, of Monash University, published *The mathematics of juggling*. Probably the most famous jugglers in this category though are Ron Graham and Claude Shannon who worked at Bell Laboratories, New Jersey. They are both well-known in mathematics and computer science circles for major contributions to their fields. They are also quite well-known for their juggling talents.

This link has unsurprisingly led to analysis of the relationship between mathematics and juggling; in **Sections 2** and **3** this analysis is investigated. Another area of research is that of the physics of juggling: throw heights and times, rotation of objects, angles of throws, bounce juggling, and so on. See [21] and [23]. In their 1989 paper (which has become a seminal work in the area of juggling mathematics, for their presentation of 'siteswap' notation to be explained in **Section 2**) Bengt Magnusson and Bruce Tiemann [26] look at the effect of errors in throwing angles and how this affects the difficulty of juggling large numbers of balls. They also examine the physics behind throwing clubs, which rotate in the air and must perform whole number rotations before being caught again.

This sort of analysis is becoming quite important in applications of robotics and studies of human movement; several articles have been written on these subjects. See [1] for an introductory overview, while [6] is an example of the type of serious papers on the topic. Incidentally, the first juggling robot was built by Claude Shannon. The robot could not juggle in the usual way of throwing and catching balls, rather it dropped the balls and they rebounded into the other hand. This is a type of juggling called 'bounce juggling', which is generally believed to be much easier than toss juggling. Shannon's robot could juggle three balls indefinitely. A similar robot has since been built that can juggle five balls.

2 Mathematics in Juggling

2.1 Early applications of mathematics to juggling

The 1980s saw the first investigations of the mathematical aspects of juggling. The first known juggling-related theorems were in a paper by Claude Shannon [23] (this paper was written sometime around 1980, although it wasn't published until 1993). These theorems have become known as Shannon's theorems. The theorems use several juggling concepts. *Flight time (F)* is the time that the ball spends in the air between being thrown and then caught again. *Dwell time (D)* is the time that a ball is held between throws. *Vacant time (V)* is the time that a hand is vacant between throwing and then catching a ball.

Definition 2.1. A *uniform juggle* is a simple juggling pattern where D, F and V are all constant throughout the pattern.

This definition is quite appropriate to real juggling as it includes all the standard cascades and fountains, as well as many passing patterns.

Theorem 2.2. Shannon's first theorem
For a uniform juggle

$$\frac{F+D}{V+D} = \frac{B}{H}$$

where B is the number of balls and H is the number of hands.

The following example will help in understanding the proof. Assume that $B = 3$ and $H = 2$, so we are juggling the standard three ball cascade. One complete cycle of the pattern is depicted in Figure 3. Note that the dwell time is coloured in black, the ball flight time is gray, and the hand vacant time is white.

Figure 3: Diagram illustrating Shannon's Thereom for the case $B = 3$, $H = 2$. Source: [1].

The time taken for a cycle must be the same regardless of whether we view it from the point of view of a hand or a ball. For the hand, the cycle is broken down into $B = 3$ groups of $D + V$; for the ball the cycle consists of $H = 2$ groups of $D + F$. So $B(D + V) = H(D + F)$ and we have the result for this case.

Proof: We pick a particular ball at the moment it is caught by a hand and watch it until it has been caught at least H more times. Now it must have come into contact with a hand at least $H + 1$ times (including the first hand it was thrown from). Given that there are only H hands, using the pigeonhole principle, the ball must have visited at least one hand at least twice. So now we concentrate on a hand that was visited more than once. Between the first and second catches of the ball, the hand made x catches of other balls, so time

$x(V+D)$ has passed. During the same time, the ball in question was caught y times by other hands, so time $y(F+D)$ has passed. Therefore

$$x(V+D) = y(F+D) \iff \frac{F+D}{V+D} = \frac{x}{y}$$

Let $x = pg, y = qg$ where $g = gcd(x,y)$ and so $\frac{x}{y} = \frac{p}{q}$ where p and q are relatively prime. So we see that $p(V+D)$ is the first time that the balls and hands are back in coincidence. Now we watch the pattern for a time $q(F+D)$ where no ball is being caught at the beginning of the interval. Balls will be caught during this interval at times $t_1, t_2, t_3, ...$, and at each of these times s_i balls will be caught by s_i hands. This follows from the simpleness of the juggling pattern (that is each hand can only catch and throw one ball at a time). By the uniformity of the juggle: every ball is caught q times (since F and D are common to all balls); and every hand makes p catches (since V and D are common to all hands). So

$$\sum s_i = qb = ph$$

By combining this with the equation above, we have the result.

□

One corollary is that for actual juggling, trying to juggle more and more balls with the same flight time leaves less and less room for corrections. Since, from Newtonian mechanics the flight time is proportional to the square of the height the balls are thrown, we end up with the (perhaps obvious) conclusion that to juggle more balls we will probably need to throw them higher. The details are contained in [23], p859–860 and [21], p102. See these references for statements and proofs of Shannon's Second and Third theorems.

2.2 Siteswap

Other mathematically minded jugglers were working on related aspects around the same time. The main development of their research was a form of notation, commonly known as **siteswap**. There is some confusion and debate about who was first to develop siteswap but it is generally agreed that several groups developed similar systems simultaneously around 1985. Arthur Lewbel [18] says that the first published paper to present a version of siteswap was by Bruce Tiemann and Bengt Magnusson [26]. They in turn credit Paul Klimek, University of California, with developing the original idea. Other people given credit for the creation of siteswap are Adam Chalcraft, Mike Day, Jim Mellor and Colin Wright at Cambridge, and Charles Brookman from Edinburgh.

In this notation we need to assume that time is broken up into a sequence of discrete 'beats'. Note that it is not necessary that the beats are equally or even regularly spaced in time. There are also some things to be juggled, which (as mentioned in **Section 1**) I will call balls for convenience, although clearly the notation will apply to any prop. We also assume that the balls are caught and then thrown immediately, that is the *dwell time* D is 0 so both the throw and catch happen on the same beat. The pattern is described by assigning a non-negative integer H to each throw (this is not to be confused with the number of hands used in Shannon's Theorem above); this integer is the number of beats before the ball is thrown again. We call this integer the *height* of the throw. While there are clearly finite juggling patterns, for siteswap notation we assume that the juggler has been juggling forever, and will continue to do so. This gives a sequence $(..., H_{-1}, H_0, H_1, H_2, ...)$. Although this sequence will be infinitely long, we usually think of juggling patterns as periodic, for example

$(..., 5, 3, 1, 5, 3, 1, 5, 3, 1, ...)$, and so we can express the same information by just writing the shortest repeating sequence of heights. The length of this sequence is d, the period of the pattern. The convention is that we then write each of these heights in a sequence, without bothering about commas or brackets, $H_0H_1H_2...H_{d-1}$. For example $(..., 5, 3, 1, 5, 3, 1, 5, 3, 1, ...) := 531$. This sequence then describes all the information that we need to replicate the pattern. In actual juggling, we can also assume that any cyclic permutations of a pattern are still the same pattern, for instance 441 is the same as 144 and 414; however the differences between these patterns have very important implications in some applications that will be explored later. An obvious ambiguity arises with this notation; for example, the pattern 111 could either be $11, 1$ (the six-ball shower, which is very difficult) or $1, 1, 1$ (the simplest possible juggling pattern where one ball just moves from hand to hand). To get around this, we make a further assumption to avoid confusion, that $H \leq 9$. Often this is not a restriction, since the bulk of jugglers are attempting patterns with $H \leq 7$. However if the pattern requires throws of $H \geq 10$, then we can just insert commas to separate the throws.[3]

We can graph these sequences in a straightforward manner. Refer to Figure 4 for the graph of the pattern 441.

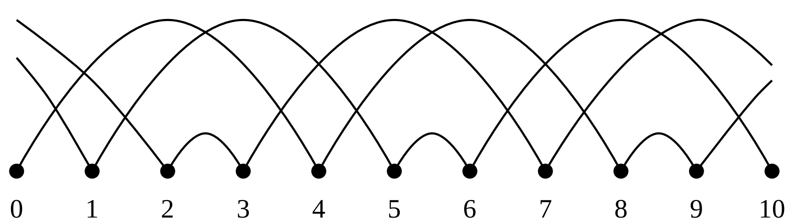

Figure 4: Graph of the siteswap pattern 441.

We note that this notation abstracts away from almost all the aspects of real-world juggling. For example, patterns would normally be juggled by two hands however this requirement is unnecessary as you can juggle the pattern $2 \equiv (2)$ with one ball in each hand, or two balls in one hand, and the other hand doing nothing. You could imagine that a pattern is juggled by multiple people: as long as the balls are thrown in the correct sequence; who is catching and throwing, and with which hand, makes no difference. Also, we think of the balls being thrown, although really this is not required. All we need is an operation performed on some objects (not necessarily physical objects) that is repeated periodically. Polster [21] applies a similar notation to bell ringing. Nonetheless, in the juggling context the usual interpretation is that the pattern is juggled with two hands, and so $H_i \equiv 0 \bmod 2$ is a throw to the same hand, and $H_i \equiv 1 \bmod 2$ is a throw to the opposite hand. This means that a pattern $h := (h)$ is a fountain for h even and a cascade for h odd.

For this paper I define the non-standard term *pattern vector* to be $\mathbf{H} := (H_0, H_1, ..., H_{d-1})$, to refer to the sequence of heights of throws to avoid confusion with other vectors to be defined later. From this point on I will write all siteswap patterns in this way, and I will call them *juggling patterns* or just *patterns*. We will only be using patterns of finite period and bounded height.

For this discussion I will assume that we are only dealing with simple juggling patterns. This leads to the following definition.

[3]Another convention is to use letters when the throws are of height greater than 10, ie. $[10, 11, 12, ...] = [a, b, c, ...]$

Definition 2.3. A pattern vector $\mathbf{H} = (H_0, H_1, ..., H_{d-1})$ is a *juggling sequence* if no two throws land on the same beat, which is equivalent to saying $\{i + H_{i \bmod d} : i \in \mathbb{Z}\}$ is a permutation of $\mathbb{Z}$.

Siteswap has proven itself useful for three reasons: communication of juggling patterns; development of new juggling patterns; and increasing the ability of jugglers to learn new tricks. Until the development of siteswap (and their associated diagrams), communicating juggling patterns could only be done by video or a large number of still photographs, or complicated written descriptions. Burkard Polster ([21] pg. 137) provides a nice example of this; the magazine *Juggler's World* showed how difficult it is to communicate juggling patterns in their publication by printing six photographs of a person juggling a particular pattern, and, I would assume, that it was still quite difficult to learn the trick from these photos. Video is a much better as a tutor, but it can still be quite difficult to translate what you see on the screen into your own movements, while still catching all the balls. However, in siteswap notation, the trick was simply 5623.

New juggling patterns have since been discovered and/or popularised by using siteswap. The canonical example is that of the pattern 441, which is extremely easy to learn, but until the development of siteswap, was practically unknown. There is also a related advantage: a lot of jugglers have reported that the abstraction of siteswap has increased their ability to learn new juggling tricks, by seeing similarities between patterns that were previously not known to be related.

One of the first theorems to be proved about this notation, and certainly the most well-known is that the sum of the throws divided by the period is equal to the number of balls.

Theorem 2.4. ***Average Theorem***

If $\boldsymbol{a} = (a_0, a_1 ..., a_{d-1})$ is a pattern vector for a simple juggling pattern with period d then:

$$ball(a) = \frac{\sum_{i=0}^{d-1} a_i}{d}$$

We omit the proof since this theorem is just a particular case of Theorem 2.12, which is proved later. However, we can provide a quick heuristic argument that captures the spirit of the theorem. The number of balls is equal to the number of edges passing through the interval between any two vertices. For example, in Figure 4 we see that if we draw a vertical line anywhere between vertices 1 and 2, the vertical line will intersect three edges. This is true for a vertical line between any pair of adjacent vertices. Every d beats throw i will contribute a ball to the next a_i intervals. So on average throw i contributes $\frac{a_i}{d}$ balls. For example, say $a_0 = 2$ and $d = 3$, then we have a ball in intervals $(0, 1)$ and $(1, 2)$, no ball in $(2, 3)$, balls in $(3, 4)$ and $(4, 5)$, no ball in $(5, 6)$ etc. The average contribution here is $\frac{2}{3}$ balls per interval.

The converse to Theorem 2.4 is false however. For example, the pattern $(3, 2, 1)$ has average 2, but is not a (simple) juggling pattern since all of the throws will land on the same beat. There does exist a partial converse to the Average Theorem, Theorem 2.6, which is actually a special case of a theorem about Abelian groups due to Marshall Hall [15].

Definition 2.5. A **qualifying sequence** is a sequence $\boldsymbol{a} = (a_0, a_1, ..., a_{d-1}) \in \mathbb{Z}^d$ such that $\frac{\sum_{i=0}^{d-1} a_i}{d}$ is an integer.

Theorem 2.6 (From [21] page 30)**. *Partial Converse to the Average Theorem***

Given a qualifying sequence $\boldsymbol{a} = (a_0, a_1, ..., a_{d-1}) \in \mathbb{Z}^d$ there is a permutation of $\boldsymbol{a}$ that is a juggling sequence.

To prove Theorem 2.6 we need the following lemma.

Lemma 2.7. *Let $\boldsymbol{b} = (b_0, b_1, ..., b_{d-1}) \in \mathbb{Z}^d$ be a qualifying sequence such that $\boldsymbol{b}$ can be reordered into a juggling sequence. Replace any two entries in $\boldsymbol{b}$ such that it is still a qualifying sequence. Then the resulting sequence can also be rearranged into a juggling sequence.*

The proof of this lemma is not difficult to understand, but it is tedious and unenlightening. The details can be found in Polster [21], p 31–34.

Proof of Theorem 2.6: Take any sequence $\boldsymbol{a} = (a_0, a_1, ..., a_{d-1})$ of integers. Now if we start with the trivial juggling sequence $(0, 0, ..., 0)$ then this can be transformed into our sequence $\boldsymbol{a}$ in no more than $d - 1$ steps, where each of the intermediate sequences is a qualifying sequence. All we have to do is start with a_0, check if $a_0 = 0$, if not then change position 0 in the trivial sequence to a_0. Next, we change position 1 in the trivial sequence to a value such that it is still a qualifying sequence, ie. $\frac{\sum_{i=0}^{d-1} a_i}{d}$ is an integer. We keep repeating this for all $a_i, i \leq d$, and we end up with our sequence. For example, let $\boldsymbol{a} = (0, 3, 2, 5, 0, 8)$, which is not a juggling sequence since throws $a_2 = 3$ and $a_3 = 2$ will both land on the same beat, however it is still a qualifying sequence. We start with (0,0,0,0,0,0), and we see that $a_0 = 0$ so we don't need to do anything. $a_1 = 3$ is the first non-zero entry, so we get $(0, 3, 3, 0, 0, 0)$, where position 2 has been changed so that the average of the sequence is an integer. We continue:

$$(0, 3, 3, 0, 0, 0) \to (0, 3, 2, 1, 0, 0) \to (0, 3, 2, 5, 2, 0)$$

$$\to (0, 3, 2, 5, 0, 2) \to (0, 3, 2, 5, 0, 8)$$

So, given that the trivial sequence is a juggling sequence, because all the following sequences, which are qualifying sequences by the above construction, differ by the replacement of at most two entries, Lemma 2.7 implies that the final sequence in the process can be rearranged into a juggling sequence, and we are done.

□

Using siteswap notation, Shannon's Theorem (Theorem 2.2) has been generalised to include non-uniform juggling patterns. Yeung Yam and Jinjang Song in their 1998 paper [29] combine finite dwell times with the siteswap description of a juggling pattern to conclude that Shannon's Theorem is satisfied by a general juggling pattern if we look at the total dwell time, vacant time and flight time over a full period of the pattern.

2.3 Creating new juggling patterns from existing patterns

Although the notation described above is commonly called siteswap, the term actually refers to a method for creating new juggling patterns from existing ones, where the landing sites of two throws are 'swapped'. Polster [21] provides a good description of this method. Let $\boldsymbol{a} = (a_0, a_1, ..., a_{d-1})$ be a sequence of integers, where $d \geq 2$, not necessarily a juggling sequence. Now we create a new sequence $\boldsymbol{a}^{j,k}$, called the *siteswap of $\boldsymbol{a}$ at beats j and k*, where $0 \leq j < k \leq d-1$ by interchanging the landing sites of throws a_j and a_k. We do this by letting

$$\begin{aligned}
\boldsymbol{a}_j^{j,k} &= a_k + k - j \\
\boldsymbol{a}_k^{j,k} &= a_j + j - k \\
\boldsymbol{a}_i^{j,k} &= a_i \qquad\qquad \forall\, 0 \le i \le d-1, \text{ where } i \ne i, j
\end{aligned}$$

For example, $(4,4,1)$ becomes $(5,3,1)$ by interchanging beats 0 and 1 (Figure 5), and it becomes $(4,5,0)$ by interchanging beats 1 and 2 (Figure 6).

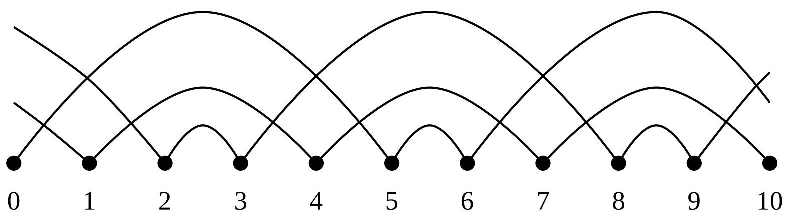

Figure 5: Graph of $(4,4,1)^{0,1} = (5,3,1)$.

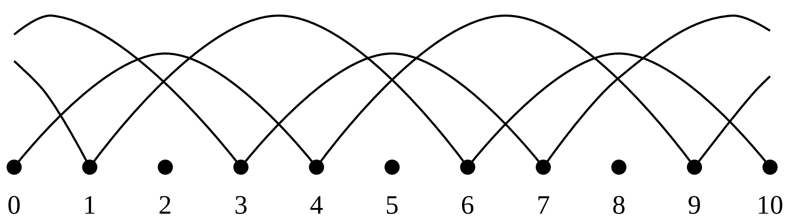

Figure 6: Graph of $(4,4,1)^{1,2} = (4,5,0)$.

Using this method we find that the following properties hold:

S1 $\quad \boldsymbol{a}$ is a juggling sequence $\iff$ $\boldsymbol{a}^{j,k}$ is a juggling sequence.

S2 $\quad \frac{\sum_{i=0}^{d-1} a_i}{d} = \frac{\sum_{i=0}^{d-1} a_i^{j,k}}{d}$

S3 $\quad$ If $\boldsymbol{a}$ is a juggling sequence then $ball(\boldsymbol{a}) = ball(\boldsymbol{a}^{j,k})$

Proof: S1 can be seen in Figures 5 and 6. By interchanging the sites, we just alter the pattern at the sites $j + nd$ and $k + nd$, $n \in \mathbb{Z}$. From the diagram we see that the new pattern still forms a permutation of the integers.

S2 is obvious and S3 follows from S2 and Theorem 2.4, the Average Theorem. $\square$

We have already remarked that cyclic permutations of juggling patterns, are still the same pattern from a juggling perspective. Let $\boldsymbol{a} = (a_0, a_1, ..., a_{d-1})$ be a sequence of integers, where $d \ge 2$. Let $\boldsymbol{a}^c$ be the *cyclic shift of* $\boldsymbol{a}$, where $\boldsymbol{a}^c = (a_{d-1}, a_0, a_1, ..., a_{d-2})$. The following properties hold, the proofs of which are obvious:

C1 $\quad \boldsymbol{a}$ is a juggling sequence $\iff$ $\boldsymbol{a}^c$ is a juggling sequence.

C2 $\quad \frac{\sum_{i=0}^{d-1} a_i}{d} = \frac{\sum_{i=0}^{d-1} a_i^c}{d}$

C3 $\quad$ If $\boldsymbol{a}$ is a juggling sequence then $ball(\boldsymbol{a}) = ball(\boldsymbol{a}^c)$

Now we can use these properties to generate all juggling sequences via the following algorithm, which was developed by Allen Knutson (see [21], p20):

Flattening Algorithm Let $\boldsymbol{a} = (a_0, a_1, ..., a_{d-1})$ be a sequence of non-negative integers.

1. If $\boldsymbol{a}$ is a constant sequence, that is $a_0 = a_1 = ... = a_{d-1}$ then stop and output this sequence. Otherwise go to 2.
2. Let $m = \max(a_i)$. Cyclic shift $\boldsymbol{a}$ until $a_0^c = m$ and $a_1^c < m$. If a_0^c and a_1^c differ by 1, then stop and output this sequence. Otherwise redefine $\boldsymbol{a}^c$ to be $\boldsymbol{a}$ and go to step 3.
3. Create the siteswap of $\boldsymbol{a}$ at beats 0 and 1, $\boldsymbol{a}^{0,1}$. Redefine $\boldsymbol{a}^{0,1}$ to be $\boldsymbol{a}$ and go to 1.

Step 3 of the algorithm lowers the maximum entry of the sequence. This can be seen easily; $a_0 = m$ is greater than both $a_0^{0,1}$ and $a_1^{0,1}$ (remember that $a_1 \leq a_0 - 2$ at step 2 in order to get to step 3):

$$a_0^{0,1} = a_1 + 1 < a_0$$
$$a_1^{0,1} = a_0 - 1 < a_0$$

So we are effectively taking 1 away from the highest value in the sequence, and adding 1 to a value that is at least 2 less than the highest value. So this algorithm must terminate in a finite number of steps. By properties S1 and C1, if the first sequence is a juggling sequence, then all the following sequences are also juggling sequences. So in that case we will never terminate at step 2, since $a_0 - 1 = a_1$ would lead to a collision, and so it is not a juggling sequence. In the case of a juggling sequence, the end result is the constant sequence.

Some examples will illustrate the algorithm:

$$441 \to 414 \to 234 \to 423 \to 333$$
$$164 \to 524 \to 434$$

In the second example, 164 is not a juggling sequence, since after following the algorithm we have a collision of the 4 and 3 throws. Of course, in this case the average is not an integer, and so it couldn't have been a juggling sequence anyway.

We can now make the following statement:

Theorem 2.8. *The constant sequence* $\boldsymbol{a} = (a_0, a_1, ..., a_{d-1}) = (b, b, ..., b)$ *can be transformed into any juggling sequence of period d such that* $\frac{\sum_{i=0}^{d-1} a_i}{d} = b = ball(\boldsymbol{a})$ *through siteswaps and cyclic shifts.*

Proof: As previously remarked, the flattening algorithm, when applied to a juggling sequence, results in the constant sequence. Since we can just reverse the cyclic shifts and siteswaps, we can start with the constant sequence and reverse the flattening algorithm, ending up at any sequence we like.

□

2.4 Functional definition of a juggling pattern

Siteswap is a practical notation for communicating and learning new juggling tricks, but for our purposes a more useful definition of a juggling sequence is that provided by Buhler *et al* [4].

Definition 2.9. A *juggling pattern* is a function $f : \mathbb{Z} \to \mathbb{Z}$ such that f is a permutation of $\mathbb{Z}$ and $f(t) \geq t \quad \forall\, t \in \mathbb{Z}$. Define the *height* of the pattern as $df(t) = f(t) - t$.

The form of the function we can keep in mind is:

$$f(x) = \begin{cases} y & \text{if the ball thrown at time } x \text{ is thrown again at time } y \\ x & \text{if there is no throw at time } x \end{cases}$$

The pattern (3), which is the three ball cascade has the function:

$$f(x) = x + 3$$

The four ball fountain (4):

$$f(x) = x + 4$$

The three ball shower (5,1):

$$f(x) = \begin{cases} x+5 & \text{if x} \equiv 0 \text{ mod } 2 \\ x+1 & \text{if x} \equiv 1 \text{ mod } 2 \end{cases}$$

Note that these functions partition $\mathbb{Z}$; the elements of the partition are called *orbits* or *paths*. For example $f(x) = x + 3$ forms the following partition $\{\{..., -6, -3, 0, 3, 6, ...\}, \{..., -5, -2, 1, 4, 7, ...\}, \{..., -4, -1, 2, 5, 8, ...\}\}$ and the three ball shower gives $\{\{..., -6, -1, 0, 5, 6, 11, ...\}, \{..., -5, -4, 1, 2, 7, 8...\},$ $\{..., -3, -2, 3, 4, 9, 10, ...\}\}$.

Most often, juggling orbits are periodic, like the examples above, with some period d, and so we make the following definition relating to juggling patterns.

Definition 2.10. A *periodic juggling pattern of period d* is a juggling pattern f such that

$$i \equiv j \text{ mod } d \Rightarrow df(i) = df(j) \;\; \forall\, i, j \in \mathbb{Z}$$

This definition is identical to the siteswap notation: the siteswap sequences are just lists of the successive heights ($df(t)$) of throws in a pattern—the *height* in Definition 2.9 is identical to that that used in siteswap. Note that by the bijective nature of f we have only one ball landing on each beat, so this system is limited to simple juggling patterns.

We can represent these functions as directed graphs on the integers, where every edge points from left to right. These graphs are virtually identical to the graphs of the siteswap patterns. Note that the edges form infinite disjoint paths. If the edges didn't form paths then f would not be a permutation of the integers: that is, the map would no longer be bijective; and given that $f(t) \geq t$ the paths must be infinitely long. We can think of the partitions as lists of the times that each ball is thrown; this means that the pattern uses as many balls as there are sets in the partition, which is the same as the number of disjoint paths. See Figures 7, 8 and 9 for example graphs.

Definition 2.11. Let f be a juggling function. The number of balls of f, *ball*(f), is the number of infinite, disjoint paths in the juggling diagram corresponding to f.

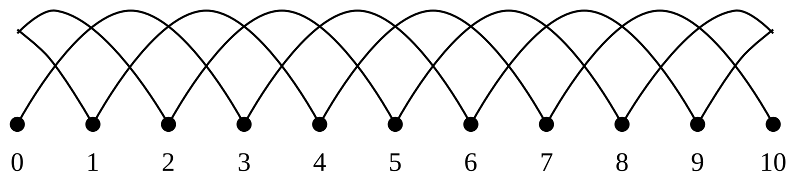

Figure 7: Graph of three ball cascade.

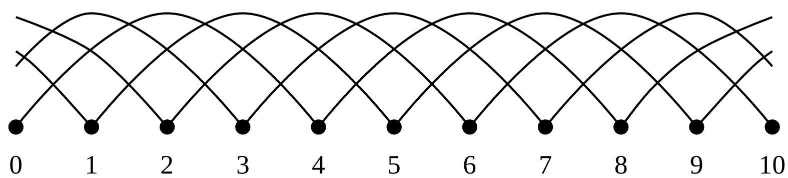

Figure 8: Graph of four ball fountain.

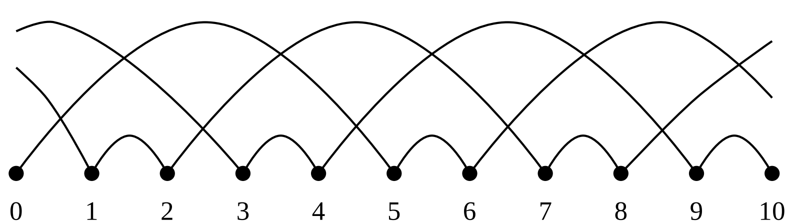

Figure 9: Graph of three ball shower.

Using Definitions 2.9 and 2.11 we can prove the following theorem, of which Theorem 2.4 is a special case.

Theorem 2.12 (From [4], page 511 and [21], page 15)**.** *Let f be the function in* Definition 2.9*, and suppose $df(t)$ is non-negative and bounded. Then*

$$ball(f) = lim_{|I|\to\infty} \frac{\sum_{x\in I} df(x)}{|I|}$$

where the limit is over all intervals $I = \{a, a+1, ..., b\} \subset \mathbb{Z}$.

Proof: Since $df(t)$ is bounded, $M = \max\{df(t) : t \in \mathbb{Z}\}$ exists. Take I such that $|I| > M$. Then every path in the graph of the function must land on at least one of the vertices inside I. Since each path corresponds to a ball, and there is at most one ball per vertex (recall that these are simple juggling patterns), the number of balls is finite. Now we claim that for each path P

$$|I| + 1 - M \leq \sum_{i\in I_P} df(i) \leq |I| - 1 + M$$

where the sum is over the vertices $i \in I_P \subset I$ that P intersects with. We can see this is true from a diagram of a single path (see Figure 10). If P lands on vertices a and b we have $\sum_{i\in P} df(i) + 1 = |I| + df(b)$.

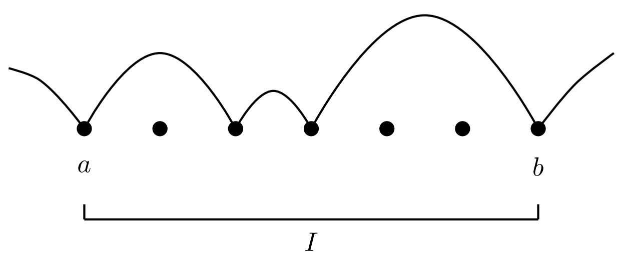

Figure 10: Graph of a single path, where the path lands on the first and last vertices in the interval I.

Each $df(i)$ counts the $df(i)$ vertices immediately following vertex i, which are all in I except for those counted by $df(b)$. Vertex i will be counted by the previous edge, except for vertex a so we add 1 to the sum to account for it. We know that $df(b) \leq M$, so we claim that $\sum_{i\in P} df(i) = |I| - 1 + M$ is an upper bound. If P does not land on b, and t is instead the last vertex in I that P lands on (see Figure 11), then clearly we cannot obtain the suggested upper bound, since the throw height is bounded. If P does not land on a and instead lands on s, then the vertices before s will not be counted at all, so again we cannot equal the above sum.

So we have the upper bound. For the lower bound we see that the maximum number of vertices that will not be counted before s is M. There are two ways to minimise the number of vertices counted at the right hand end of I. Either P lands on $t < b$, and then jumps to $b+1$ which counts the vertices greater than t plus 1, or P lands on b and we know from the definition of f that this implies $df(b) \geq 1$. In either case we have $\sum_{i\in I_P} df(i) = |I|+1-M$ as the lower bound, thus establishing the sought inequality. Summing the terms in the inequality over the different paths (that is, the different balls) and dividing by $|I|$ gives

$$\frac{ball(f)(|I| + 1 - M)}{|I|} \leq \frac{\sum_{i\in I} df(i)}{|I|} \leq \frac{ball(f)(|I| - 1 + M)}{|I|}$$

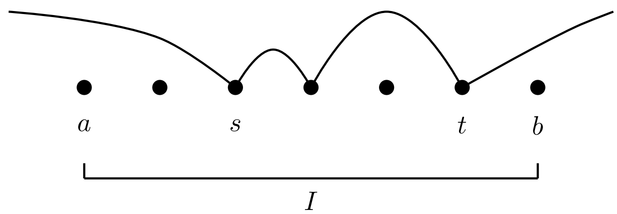

Figure 11: Graph of a single path, where the path does not land on the first and last vertices in the interval I.

Taking the limit as $|I| \to \infty$ means the upper and lower bounds both tend to $ball(f)$.

$\square$

Clearly Theorem 2.12 implies Theorem 2.4 in the case when f is periodic with period d.

2.5 Generalisations and extensions of siteswap and other juggling notations

So far in this chapter we have only been discussing simple juggling patterns, however siteswap can be generalised to multiplex patterns quite easily. If we are throwing multiple balls on a single beat, then we enclose those throws inside square brackets. For example, [33]33 is a four ball pattern. It is essentially the three ball cascade, but with two balls thrown together on every third beat. Note that while the three ball cascade had period 1, [33]33 has period 3. Other examples include [32] and [43]0323. We can graph these patterns in the same way as for simple patterns. We can identify the number of balls as the number of infinite paths in the diagram, although these paths are no longer unique since we can have multiple paths incident with any vertex. The same average theorem holds for multiplex patterns as for simple patterns, that is the total of the digits divided by the period is the number of balls required for the pattern. See Polster ([21], Section 2.1) for the details of the theorem.

We can further expand the notation to account for the number of hands being used to juggle pattern. In this case we use matrices, where the number of rows corresponds to the number of hands being used, and the number of columns corresponds to the period of the pattern. In [9] Ed Carstens presents a notation he developed called Multi-Hand Notation (MHN). Each entry (i, j) in the matrix prescribes the height that hand i must throw at time j, with a subscript describing which hand it throws to. For example the following matrix describes a 3 hand, 7 ball, period 4 juggling pattern

$$\begin{pmatrix} [42]_0 2_2 & 1_0 & [21]_0 1_2 & 3_0 1_1 \\ [31]0 & 0 & 0 & 0 \\ 4_0 & 0 & 1_2 & 1_1 1_2 \end{pmatrix}$$

So at time 0 hand 0 throws two balls, both of which will land back in the same hand, one at time 2 and the other at time 4. Hand 1 will also throw two balls, which land in hand 0 at times 1 and 3. In this way we can read off the juggling pattern from the matrix. We can graph these matrices as a set of multiplex graphs. The graph of the example above is contained in Figure 12.

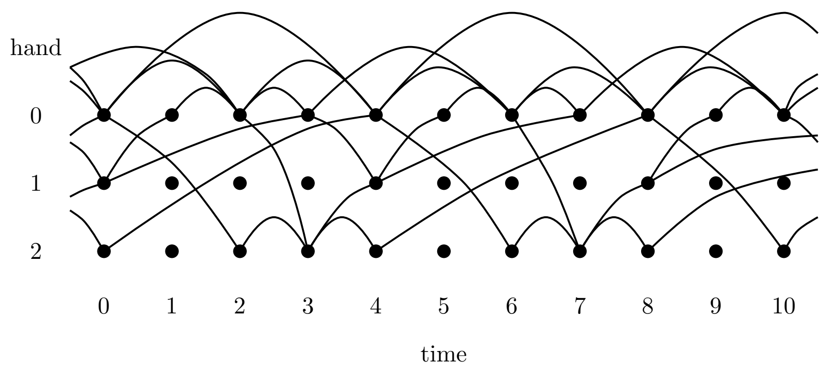

Figure 12: Graph of a juggling matrix. Source: [21].

This graph can be collapsed into a standard multiplex graph, which does not record the number of hands, see Figure 13.

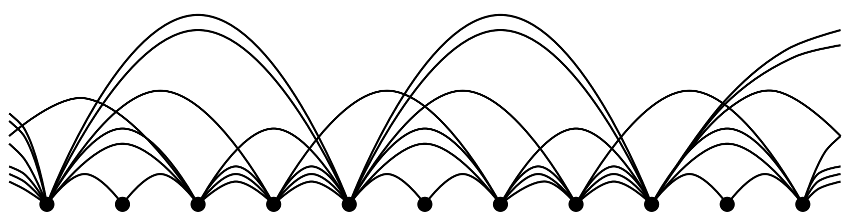

Figure 13: Graph of a juggling matrix that has been collapsed into a standard multiplex graph. Source: [21].

Since these graphs are equivalent, the number of distinct paths in the MHN graph is equal to the number of distinct paths in the corresponding multiplex graph. Recall that the average theorem holds for multiplex siteswap, so it must hold for MHN as well: the sum of the integers in the matrix divided by the period is the number of balls required to juggle the pattern. That is, the sum of the integers in the matrix divided by the period gives the number of balls required to juggle the pattern. This notation is quite useful for describing passing patterns (where objects are juggled between two or more people). For example, the two most common passing patterns are

$$\begin{pmatrix} 3_3 & 0 & 3_1 & 0 \\ 0 & 3_0 & 0 & 3_0 \\ 3_1 & 0 & 3_3 & 0 \\ 0 & 3_2 & 0 & 3_2 \end{pmatrix} \qquad\qquad \begin{pmatrix} 3_3 & 0 \\ 0 & 3_0 \\ 3_1 & 0 \\ 0 & 3_2 \end{pmatrix}$$

each of which involves 2 people juggling 3 balls. In the pattern on the left, on every 4th beat each person passes with their right hand to the other person's left hand (note that this is only by convention, it could just as easily be done the other way, with each person passing from their left hand to the other's right hand). The pattern on the right is similar except that the passes are made on every 2nd beat. The patterns are called *4 count* and *2 count* respectively. The matrices have been further adapted to enable description of different types of throws and catches (eg. behind the back, under the leg, arms crossed) and other tricks that may be performed (see [2]). Accordingly, these matrices are often used in juggling animators, which accept a valid juggling pattern as input and generate an animated character juggling the pattern.

Another common way of representing juggling patterns involves what are known as *juggling states*. In this representation, we think of the pattern as being in a particular state at each discrete time point. The state is described by a string of binary symbols: one symbol representing a ball landing in the hand, and the other symbol representing no ball landing. Each position i in the string corresponds to time point i. The string is infinitely long, but since we will assume only a finite number of balls, there must be only a finite number of ball-landing symbols. For example, if we let 1 represent a ball landing on that beat and 0 represent no ball landing, then the sequence 101101 tells us that at time 0 (ie. now) we have a ball in the hand, at time 1, no ball will land, but at times 2 and 3 we will have a ball landing in our hand, and so on. We assume that all values to the right of time 5 are 0. Now we specify some rules and from this we can establish a juggling pattern. For example, a simple set of rules is the following:

1. We can only catch one ball at a time, so we can't throw a ball such that it will land on the same beat as another ball.
2. If we catch a ball on a beat, we throw the ball again immediately.
3. We can only catch a ball in a hand if that hand is empty, and thus we can only make throws of one ball at a time.

With these rules, we see that in our example we have just caught a ball, and so we must decide what to do with it. We are not expecting a ball to land on the next beat, or the 4th beat from now, or any beat after the 5th, so we could fill any of these positions with the current ball. If we throw the ball to land 4 beats later, then on the next beat we have the following state, 01111. Unsurprisingly, we call this throw a 4. So by picking a fixed number of balls b, and a maximum throw height h we can specify every possible state (of which there must clearly be $\binom{h}{b}$), and all the connections between them. We draw a graph G by taking the states as the vertex set of G and the possible throws as the edge set. For example, the 3-ball state graph with maximum height throws of 5 is in Figure 14. By following the edges we can construct all 3 ball patterns where the maximum throw height is 5. This idea can be generalised to multiplex graphs in a straightforward way, see [21] Section 3.4 for the details.

Using this concept, Gregory Warrington [28] calculates the fraction of time spent in any particular state, if the transitions between states are random (that is, we are taking a random walk on the juggling state graph). We let $\boldsymbol{s}_{b,h} = s_0 s_1 \ldots s_{h-1}$ be a juggling state with b balls and maximum throw height h.

Definition 2.13. For a juggling state $\boldsymbol{s}_{b,h}$ define $\phi_t(\boldsymbol{s}_{b,h})$ for $0 \leq t \leq h-1$ by

$$\phi_t(\boldsymbol{s}_{b,h}) = \begin{cases} |\{j : t < j \leq h-1, s_j = 0\}| & \text{if } s_t = 1 \\ 0 & \text{if } s_t = 0 \end{cases}$$

Also, let

$$\Lambda(\boldsymbol{s}_{b,h}) = \prod_{t=0}^{h-1} (1 + \phi_t(\boldsymbol{s}_{b,h}))$$

So $\phi_t(\boldsymbol{s}_{b,h})$ counts the number of possible throws we can make at time t, based on the current sequence. The main result from Warrington's paper is

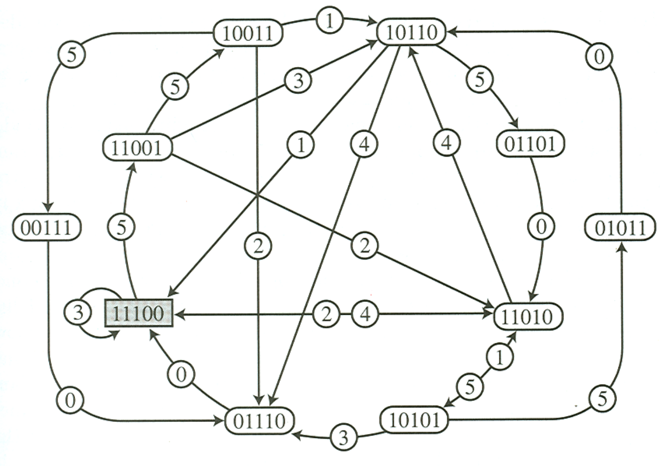

Figure 14: 3 ball state diagram with maximum throw height 5. Source: [21], pg 45.

that the fraction of time spent in state $\boldsymbol{s}_{b,h}$ is

$$\frac{\Lambda(\boldsymbol{s}_{b,h})}{\left\{ \begin{array}{c} h+1 \\ h-b+1 \end{array} \right\}}$$

where $\left\{ \begin{array}{c} x \\ y \end{array} \right\}$ is the *Stirling number of the second kind.* The proof of this uses Markov chains; the details are contained in [28].

In their paper of 2005 Jean Cardinal, Steve Kremer and Stefan Langerman [7] examine the transitions between possible juggling states. They define a fast algorithm for finding a transition between states, and they also give a formula for calculating the number of transitions of a fixed length between two states.

There has also been some investigation of juggling patterns in the context of braid groups.

Definition 2.14. The braid group B_n is generated by $\{\sigma_1, \sigma_2, ..., \sigma_n\}$ with the relations:

R1 $\quad \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1}$

R2 $\quad \sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i \quad \text{if } |i-j| > 1$

Let w be a word in B_n. With the following relation, the braid group is known as a *solid torus braid*:

R3 $\quad w = \sigma_i \cdot w \cdot \sigma_i^{-1}$

With the following relation the solid torus braid is known as a *link*:

R4 $\quad w = w \cdot \sigma_n \quad \text{where } \sigma_n \in B_{n+1}$

In [11] Satyan Devadoss and John Mugno show that there is a surjection from juggling sequences to links, that is, all links can be juggled. Their concept of juggling sequences is the same as that used in Definition 2.9. Polster [21] (section 7.2) also discusses this issue from a slightly different viewpoint. Matthew Macauley, in his 2003 thesis [20], looks at how braids may be juggled, and the types of graphs used to represent these jugglable braids.

3 Juggling in Mathematics

We have seen how juggling can be usefully described using mathematics—now we look at ways we can use juggling to develop some interesting mathematics. The theorems we will prove have no necessary link to juggling, and have previously been proved in quite different contexts.

In Section 2 Buhler *et al*'s definition of juggling patterns was presented; the main point of their paper [4] is Theorem 3.12, which gives the number of juggling patterns for a fixed number of balls and a fixed period. (While this might strictly be considered as a theorem with more applicability to juggling than maths, I have included it in this section since it is a key result that is used in the mathematical applications that follow this paper.) In order to prove this theorem Buhler *et al* develop the idea of *drops* and *descents* of permutations. Polster [21] gives a simpler, more direct proof, which relies on the concept of juggling cards. Juggling cards were first presented by Ehrenborg and Readdy in [12]. I will present Buhler *et al*'s proof since it is interesting and it is the same method of proof used by Buhler and Graham in [5], which I will also discuss.

As mentioned above, Ehrenborg and Readdy introduced juggling cards in their 1996 paper. In this paper they build on the results of Buhler *et al* to explore properties of q-enumeration formulas, developing remarkable proofs of identities involving Poincaré series (of affine Weyl groups), q-Stirling numbers of the second kind, and unitary vector compositions. Several of these identities were proved by Haglund [14] in the context of rook placements. We will conclude this section by discussing Jonathan Stadler's paper [25], in which he generalises Ehrenborg and Readdy's discussion to provide proofs of several other similar identities.

3.1 Criterion for a juggling pattern

From Definition 2.10 clearly if $i \equiv j \bmod d$ then $f(i) \equiv f(j) \bmod d$, since $df(i) = df(j)$ and $f(t) = t + df(t)$. This implies that each juggling pattern of period d induces a well-defined permutation on $\boldsymbol{\delta}_{d-1} := (0, 1, ..., d-1)$ that is defined by

$$\pi_f(t) \equiv f(t) \bmod d \quad t \in \boldsymbol{\delta}_{d-1}$$

This gives us a way to check that an arbitrary sequence is a juggling pattern, which is the content of the next theorem.

Theorem 3.1. *Let $\boldsymbol{a} = (a_0, a_1, ..., a_{d-1})$ be a sequence of non-negative integers. Then $\boldsymbol{a}$ satisfies $df(t) = a_t$, where f is a period d juggling pattern, if and only if $\{a_t + t \bmod d : t \in \boldsymbol{\delta}_{d-1}\} = (a_0 + 0 \bmod d, a_1 + 1 \bmod d, ..., a_{d-1} + (d-1) \bmod d)$ is a permutation of $\boldsymbol{\delta}_{d-1}$.*

Proof: Let f be a juggling pattern and $\boldsymbol{a}$ a sequence of non-negative integers such that $df(t) = a_t$. Then by the comment preceeding the statement of the theorem, that $f(t) \equiv \pi_f(t) \bmod d$, we have

$$f(t) = \pi_f(t) + d \cdot g(t)$$

where g is an integer valued function. And so

$$a_t + t = df(t) + t = f(t) = \pi_f(t) + d \cdot g(t) \equiv \pi_f(t) \bmod d$$

and we have $a_t + t \bmod d$ is a permutation of $\boldsymbol{\delta}_{d-1}$.

Now let $(a_0, a_1, ..., a_{d-1})$ be a sequence such that $a_t + t \bmod d$ is a permutation of $\boldsymbol{\delta}_{d-1}$. Extend the sequence periodically so that we have an a_t value for all $t \in \mathbb{Z}$. Define $f(t) = a_t + t$ to create the required juggling sequence. f is injective since if $f(x) = f(y)$ then $x \equiv y \bmod d$ as f is a permutation on $\boldsymbol{\delta}_{d-1}$ and so $a_x = a_y$. Then $x = y$. f is surjective since for any $z \in \mathbb{Z}$ we can find a t such that $f(t) = a_t + t \equiv z \bmod d$. This is true because $a_t + t \bmod d$ is a permutation of $\boldsymbol{\delta}_{d-1}$. So then we can find $s = t + \alpha d$, $\alpha \in \mathbb{Z}$ such that $f(s) = z$. The last condition for a juggling pattern, that $f(t) \geq t$ is obvious from the fact that a_t is a non-negative integer. So f is a period d juggling pattern.

□

For example, take the pattern vector $(4, 1, 4)$, then:

$$f(t) = \begin{cases} t+4 & \text{if } t \equiv 0, 2 \bmod 3 \\ t+1 & \text{if } t \equiv 1 \bmod 3 \end{cases}$$

and so

$$\pi_f(t) = \begin{cases} 1 & t \equiv 0 \bmod 3 \\ 2 & t \equiv 1 \bmod 3 \\ 0 & t \equiv 2 \bmod 3 \end{cases}$$

then $f(0) = 4, f(1) = 2, f(2) = 6$ and so $g(0) = 1, g(1) = 0, g(2) = 2$. $(a_0 \bmod 3, a_1 + 1 \bmod 3, a_2 + 2 \bmod 3) = (1, 2, 0)$, which is a permutation of $\boldsymbol{\delta}_2$.

However, if instead we take the pattern vector $(5, 4, 0)$, which is not a valid juggling pattern (even though its average is an integer), we find that $(a_0 \bmod 3, a_1 + 1 \bmod 3, a_2 + 2 \bmod 3) = (2, 2, 2)$, clearly not a permutation of $\boldsymbol{\delta}_2$.

3.2 Counting the number of juggling patterns

Definition 3.2. Define $N(b, n)$ to be the number of juggling patterns of period n with exactly b balls.

The method we use to count this number is to pick a permutation $\pi \in S_n$ and then count the number of corresponding juggling patterns. So using the formula from the proof of Theorem 3.1

$$f(t) = \pi_f(t) + n \cdot g(t) = \pi(t) + n \cdot g(t) \quad 0 \leq t \leq n-1$$

Counting $N(b, n)$ amounts to counting the number of functions g such that (i) $df(t) \geq 0$ and (ii) $ball(f) = b$.

The second criteria is easy to deal with. From Theorem 2.12 for a periodic juggling pattern of period n

$$ball(f) = \frac{\sum_{t=0}^{n-1} df(t)}{n} = \frac{1}{n} \sum_{t=0}^{n-1} \pi(t) - t + n \cdot g(t)$$

But $\sum_{t=0}^{n-1} \pi(t) - t = 0$ because $\pi(t)$ is a permutation of $\boldsymbol{\delta}_{n-1}$ so the condition becomes

$$ball(f) = \sum_{t=0}^{n-1} g(t) = b$$

For the first requirement, that $df(t) \geq 0$, we first need a definition

Definition 3.3. A *drop* of the permutation $\pi \in S_n$ is an integer $t \in \boldsymbol{\delta}_{n-1}$ such that $\pi(t) < t$. Let $d_\pi(t)$ be the corresponding indicator function

$$d_\pi(t) = \begin{cases} 1 & \text{if } t \text{ is a drop of } \pi \\ 0 & \text{if } t \text{ is not a drop of } \pi \end{cases}$$

Define $G(t) = g(t) - d_\pi(t)$ so that

$$f(t) = \pi(t) + n \cdot d_\pi(t) + n \cdot G(t)$$

Note that since $df(t) = \pi(t) - t + n \cdot g(t)$, this implies $g(t) \geq 0$, and when $\pi(t) < t$ then $g(t)$ must be strictly positive. This is encapsulated by taking $G(t)$ non-negative.

Lemma 3.4. *Let $f(t) = \pi(t) + n \cdot d_\pi(t) + n \cdot G(t)$ be a juggling pattern, where k is the number of drops of π. Then*

$$ball(f) = b \iff \sum_{t=0}^{n-1} G(t) = b - k$$

Proof: Assume $ball(f) = b = \frac{\sum_{t=0}^{n-1} df(t)}{n}$. Then

$$nb = \sum_{t=0}^{n-1} f(t) - t = \sum_{t=0}^{n-1} (\pi(t) + nd_\pi(t) + nG(t) - t) = n \sum_{t=0}^{n-1} (d_\pi(t) + G(t))$$

since $\sum_{t=0}^{n-1} \pi(t) - t = 0$. Cancelling n gives

$$b = \sum_{t=0}^{n-1} d_\pi(t) + G(t)$$

Since the number of drops is k it follows that

$$b - k = \sum_{t=0}^{n-1} G(t)$$

Now assume that the sum of the $G(t)$ values is $b - k$. We then have

$$\begin{aligned} ball(f) &= \frac{\sum_{t=0}^{n-1} df(t)}{n} \\ &= \frac{\sum_{t=0}^{n-1} \pi(t) + nd_\pi(t) + nG(t) - t}{n} \\ &= \frac{n(k + b - k)}{n} \\ &= b \end{aligned}$$

□

So we can restate the conditions above: $N(b, n)$ is equal to the sum over all permutations of the number of (i) non-negative, integer valued functions $G(t)$ such that (ii) $\sum_{t=0}^{n-1} G(t) = b - k$. The non-negativeness of $G(t)$ ensures that $df(t) \geq 0$ while the sum of G equalling $b - k$ ensures that $ball(b) = b$. The following lemma will allow us to count these functions.

Lemma 3.5. *Let $\boldsymbol{a} = (a_0, a_1, ..., a_{n-1})$ be a sequence of non-negative integers. The number of such sequences where $\sum_{i=0}^{n-1} a_i = x$ is given by*

$$\binom{x + n - 1}{n - 1}$$

Proof: The number of these sequences is the same as the number of ways of arranging x '1's and $n-1$ '+'s in a row. We think of the '1's as representing the components of the sequence and the '+'s as addition signs. The number of contiguous '1's occurring after the ith '+' sign is the ith element of the sequence. For example, $4 = 11+1+1$ represents the sequence $(2, 1, 1)$, and $5 = 111++1+1$ represents the sequence $(3, 0, 1, 1)$. So the number of these sequences is the same as the number of ways to choose the positions of the $n-1$ '+' signs from $x+n-1$ total positions, ie

$$\binom{x+n-1}{n-1}$$

□

Hence if we let $\beta_n(k)$ be the number of permutations of $\boldsymbol{\delta}_{n-1}$ that have k drops, then, by Lemma 3.5 we have

$$N(b,n) = \sum_{k=0}^{n-1} \beta_n(k) \binom{b-k+n-1}{n-1}$$

A quantity that will be more important later is $N_<(b,n)$, the number of juggling patterns with fewer than b balls.

$$\begin{aligned} N_<(b,n) &= \sum_{a=0}^{b-1} N(a,n) = \sum_{a=0}^{b-1}\sum_{k=0}^{n-1} \beta_n(k) \binom{a-k+n-1}{n-1} \\ &= \sum_{k=0}^{n-1} \beta_n(k) \sum_{a=0}^{b-1} \binom{a-k+n-1}{n-1} \end{aligned}$$

In fact the second summation can be evaluated.

Lemma 3.6.

$$\binom{t}{z} = \sum_{s=0}^{t-1} \binom{s}{z-1}$$

Proof: Let $T = \{\tau_1, \tau_2, ..., \tau_t\}$ be a set containing t elements. Now if we want to choose z elements from T, where τ_1 is one of the elements chosen, then there are $\binom{t-1}{z-1}$ possibilities. If instead we want τ_2, then we must exclude τ_1 to avoid over counting, and there are $\binom{t-2}{z-1}$. We continue in this way until we have exhausted all the possibilities, and we find that we have the result.

□

Using this lemma, with $t = b-k+n-1$, $z = n$ together with the fact that $\binom{p}{n} = 0$ for $p < n$ we have

$$N_<(b,n) = \sum_{k=0}^{n-1} \beta_n(k) \binom{b-k+n-1}{n}$$

This equality has a key role in the proof of Theorem 3.12.

To proceed further, we link the number of drops of a permutation with the number of *descents*.

Definition 3.7. A *descent* of a permutation $\pi \in S_n$ is an integer $i \in \boldsymbol{\delta}_{n-1}$ such that $\pi(i) > \pi(i+1)$. The number of such permutations with k descents is

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$$

For example, $(1,2,0,3)$ has one descent since $0 = \pi(2) < \pi(1) = 2$. The number

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$$

is called an Eulerian number. Note that

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \left\langle \begin{matrix} n \\ n-1-k \end{matrix} \right\rangle$$

Now we describe a unique way to write each permutation, which will be useful for the next step.

Definition 3.8. Let $\pi \in S_n$. Let $\hat{\pi}$ be constructed as follows: Write π in cycle decomposition form, including the singleton cycles. Write each of the cycles such that the largest element in each cycle is the left most element. Arrange the cycles such that the leading elements are in ascending order. Lastly, remove the brackets. This resulting permutation is $\hat{\pi}$.

For example

$$\pi = 1206453 \to (120)(63)(4)(5) \to (201)(63)(4)(5)$$

$$\to (201)(4)(5)(63) \to 2014563 = \hat{\pi}$$

The map $\pi \to \hat{\pi}$ is bijective since it is completely reversible, ie. π can be uniquely recreated from $\hat{\pi}$ by putting a left bracket at the far left of the permutation, and between each ascent. Then matching them with right brackets.

We can use this to prove Lemma 3.9

Lemma 3.9. *The number of permutations of $\boldsymbol{\delta}_{n-1}$ with k drops is equal to the number of permutations of $\boldsymbol{\delta}_{n-1}$ with k descents, that is*

$$\beta_n(k) = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$$

Proof: We will show that the number of drops of π equals the number of descents of $\hat{\pi}$. Since the map $\pi \to \hat{\pi}$ is bijective, the equation stated in the Lemma follows.

Each descent of $\hat{\pi}$ must be inside a cycle, since by the construction of $\hat{\pi}$ the right-most element in a cycle is followed by a larger element in the next cycle. Now we recall the meaning of a cycle, that is that each element of the cycle is mapped to the next element in the cycle. So a descent in $\hat{\pi}$, is $\hat{\pi}(i) > \hat{\pi}(i+1) \iff \pi(\hat{\pi}(i)) > \hat{\pi}(i)$, which is a drop of π.

Drops of π must also occur within cycles, since we are comparing a value of the sequence with its image under π, which, by definition, means they are in the same cycle. This drop, $\pi(j) < j$ will occur in the order $...j\pi(j)...$ in $\hat{\pi}$, so it is a descent of $\hat{\pi}$.

We conclude that the number of permutations with k descents equals the number with k drops, as required.

$\square$

Using the example from before

$$\begin{array}{ll} \pi = 1206453 & \text{has 2 drops, at positions } 2,6 \\ \hat{\pi} = 2014563 & \text{has 2 descents, between positions } 0,1 \text{ and } 5,6 \end{array}$$

Lemma 3.10.

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = (k+1)\left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle + (n-k)\left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle$$

Proof: We isolate the element that has value $n-1$, which is the maximum of the set, and call it M. When we insert it into the permutation we have two choices: either the addition of the new element creates a new descent, or it does not.

Let us say that it does, then before the insertion of M there must have been $k-1$ descents spread amongst $n-1$ objects. Now if M is inserted between the elements of an existing descent, then no new descent is created. For example, in the sequence 2304 (which has one descent), if we insert 5 between 3 and 0, yielding 23504, then we haven't created a new descent, and we still only have one. So that means that we can only insert M in one of the $n-k$ places remaining. And so we have the term

$$(n-k)\left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle$$

If the insertion of M does not create a new descent, then there must have been k descents amongst the $n-1$ objects. By the same reasoning above we can insert M between any descent and not create a new one. We could also place it at the extreme right of the sequence. In either case we still only have k descents. These $(k+1)$ possibilities are the only allowable cases; since M is the maximum of the set, if we place it anywhere else then a new descent will be created. So we have the additional term

$$(k+1)\left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle$$

$\square$

One more identity is presented before we can prove Theorem 3.12. This identity was first used in 1881 by Worpitzky.

Theorem 3.11.

$$x^n = \sum_{k=0}^{n-1} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \binom{x+k}{n}$$

Proof: We use an inductive proof. The base case, where $n=1$, is easily established. Now we write

$$x = A\frac{x+k-n}{n+1} + B\frac{x+k+1}{n+1}$$

By equating coefficients of x, we see that we need

$$\frac{A+B}{n+1} = 1$$

from which we can find that $B = n-k$ and $A = k+1$. Recall that

$$\binom{x+k}{n} = \frac{(x+k)(x+k-1)...(x+k-n+1)}{n!}$$

so then

$$x\binom{x+k}{n} = (k+1)\binom{x+k}{n+1} + (n-k)\binom{x+k+1}{n+1}$$

By the induction hypothesis

$$
\begin{aligned}
x^{n+1} &= x\textstyle\sum_{k=0}^{n-1}\left\langle {n \atop k} \right\rangle \tbinom{x+k}{n} \\
&= \textstyle\sum_{k=0}^{n-1}(k+1)\left\langle {n \atop k} \right\rangle \tbinom{x+k}{n+1} + (n-k)\left\langle {n \atop k} \right\rangle \tbinom{x+k+1}{n+1} \\
&= \left\langle {n \atop n-1} \right\rangle \tbinom{x+n}{n+1} + \textstyle\sum_{k=0}^{n-1}(k+1)\left\langle {n \atop k} \right\rangle \tbinom{x+k}{n+1} \\
&\quad +(n-k+1)\left\langle {n \atop k-1} \right\rangle \tbinom{x+k}{n+1}
\end{aligned}
$$

where, for the last equality, we have replaced k by $k-1$ and readjusted the terminals of the sum.

On the other hand

$$
\begin{aligned}
x^{n+1} &= x\textstyle\sum_{k=0}^{n}\left\langle {n+1 \atop k} \right\rangle \tbinom{x+k}{n+1} \\
&= \left\langle {n+1 \atop n} \right\rangle \left({x+n \atop n+1} \right) \tbinom{x+n}{n+1} + \textstyle\sum_{k=0}^{n-1}\left\langle {n+1 \atop k} \right\rangle \tbinom{x+k}{n+1} \\
&= \left\langle {n+1 \atop n} \right\rangle \tbinom{x+n}{n+1} + \textstyle\sum_{k=0}^{n-1}((k+1)\left\langle {n+1 \atop n} \right\rangle \\
&\quad +(n-k-1)\left\langle {n \atop k-1} \right\rangle)\tbinom{x+k}{n+1}
\end{aligned}
$$

Now we can see from the definition or from Lemma 3.10 that

$$\left\langle {n+1 \atop n} \right\rangle = 1 = \left\langle {n \atop n-1} \right\rangle$$

and so we have the result.

□

Theorem 3.12.

$$N_<(b,n) = b^n$$

Proof: From the equation developed earlier using Lemma 3.5 we have

$$N_<(b,n) = \sum_{k=0}^{n-1} \beta_n(k) \binom{b-k+n-1}{n} = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{b-k+n-1}{n}$$

Now we can replace k with $n-k-1$, which will just count the terms in reverse order, to give

$$N_<(b,n) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{b+k}{n}$$

Using Worpitzky's identity gives the result.

□

The immediate consequence of this theorem is of course that we can now count the number of juggling patterns of period n with exactly b balls, which I will state as a corollary for ease of reference.

Corollary 3.13.

$$N(b,n) = (b+1)^n - b^n$$

3.3 Partially ordered sets and the chromatic polynomial

Joe Buhler and Ron Graham produced a second paper [5] not long after [4] was published. In this new paper Buhler and Graham apply the same method of proof to show a more general result on partially ordered sets (posets). This result is an identity linking the number of drops in a permutation of a poset to the chromatic polynomial of a graph of the set.

Let $(P, \leq)$ be a partially ordered set, where $|P| = n$ and $\leq$ is a transitive, irreflexive relation. Also let π be a permutation of the set P. In this context we define a drop to be when $\pi(x) \leq x$, and $\beta_P(k)$ is the number of permutations of P that have k drops.

Definition 3.14. Define $\Delta_P(x)$ to be the *drop polynomial of* P by

$$\Delta_P(x) \equiv \sum_{k=0}^{n-1} \beta_P(k) \binom{x+k}{n}$$

Definition 3.15. Let $(P, \leq)$ be a poset. Define $I(P)$ to be the *incomparability graph of* P as follows. Let the vertex set of $I(P)$ be the elements of P. The edge set is given by the set of all pairs (x, y) where $x, y \in P$, such that x and y are incomparable, that is $x \not\leq y$ and $y \not\leq x$.

See Figure 15 for an example.

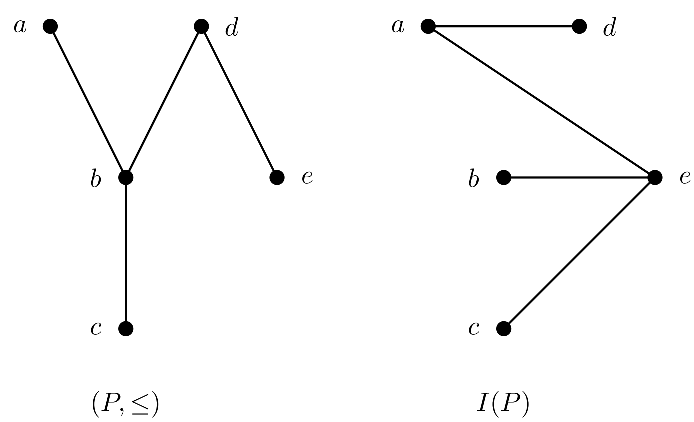

Figure 15: The graph of a poset, $(P, \leq)$, with it's incomparability graph $I(P)$.

Definition 3.16. Let G be a graph. Define a λ-*vertex colouring* to be a colouring of the vertex set $V(G)$ using up to λ colours, such that if $(x, y) \in E(G)$, then x and y are different colours. Let $\chi_G(\lambda)$ be the *chromatic polynomial* of G, which counts the number of λ-vertex colourings of G. (See Part Two in [3].)

This leads us to the following theorem.

Theorem 3.17.

$$\Delta_P(\lambda) \equiv \sum_{k=0}^{n-1} \beta_P(k) \binom{x+k}{n} = \chi_{I(P)}(\lambda)$$

Outline of proof: Let $\eta : \{1, 2, ..., n\} \to P$ be a *numbering* of P (recall that $|P| = n$). Define descents as earlier: η has a *descent* at i if $\eta(i+1) \leq \eta(i)$. It can be seen that there are $\beta_P(k)$ numberings of $(P, \leq)$ that have k descents.

For any numbering η, let $C(\eta)$ be the set of λ-vertex colourings $\alpha : P \to \{1, 2, ..., \lambda\}$ such that $\alpha(\eta(i)) \leq \alpha(\eta(i+1))$, with equality only when η has a descent at i. There is always some numbering that will satisfy this property and it is unique.

We can now claim that the number of λ-vertex colourings of $I(P)$ is given by $\sum_\eta |C(\eta)|$, that is

$$\sum_\eta |C(\eta)| = \chi_{I(P)}(\lambda)$$

If η has k descents, then

$$|C(\eta)| = \binom{\lambda + k}{n}$$

And so

$$\begin{aligned}
\chi_{I(P)}(\lambda) &= \sum_\eta |C(\eta)| \\
&= \sum_{k=0}^{n-1} \sum_{\substack{\eta \text{ has k} \\ \text{descents}}} |C(\eta)| \\
&= \sum_{k=0}^{n-1} \beta_P(k) \binom{\lambda + k}{n} \equiv \Delta_P(\lambda)
\end{aligned}$$

Note that if P is linearly ordered, for example $P = \{1, 2, ..., n\}$, then $I(P)$ is the graph on P with no edges so $\chi_{I(P)} = x^n$. Also $\beta_P(k)$ is just the Eulerian number from earlier

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$$

And so Theorem 3.17 reduces to Worpitkzky's identity in Theorem 3.11

$$x^n = \sum_{k=0}^{n-1} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \binom{x + k}{n}$$

3.4 q-binomial coefficients

Theorem 3.17 is a good example of how juggling ideas have been applied to develop mathematics with no necessary link to juggling. In their 1996 paper [12] Ehrenborg and Readdy make another contribution to non-juggling mathematics from a juggling perspective. Using a definition similar to that of Buhler *et al* in [4] they prove several theorems related to q-binomial numbers, q-Stirling numbers, affine Weyl groups and vector compositions.

Before we proceed to look at their results, we should first examine their discussion of q-binomial coefficients ([12], pg 111), which are q-analogues of binomial coefficients.

Definition 3.18. Let $[n] = 1 + q^1 + ... + q^{n-1}$ and $[n]! = [n] \cdot [n-1] \cdots [2] \cdot [1]$ The *Gausssian coefficient* or *q-binomial coefficient* is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}$$

The similarity to binomial numbers is clearly seen from this definition. There is a useful combinatorial interpretation of the Gaussian coefficient, which follows from the following identity due to Marcel-Paul Schutzenberger (this identity appeared in [22]).

Theorem 3.19. *Let x and y be non-commutative, obeying $yx = qxy$, then*

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \cdot x^{n-k} y^k$$

Proof: We show that by using the identity in the theorem, we can derive the Gaussian coefficient $\begin{bmatrix} n \\ m \end{bmatrix}$ as in Definition 3.18. We do this by establishing two recurrences. For the first we note that

$$(x+y)^{n+1} = (x+y)^n (x+y)$$

and so

$$\sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix} \cdot x^{n+1-k} y^k = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \cdot x^{n-k} y^k (x+y)$$

Because $y^k x = q^k x y^k$, then

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} q^k + \begin{bmatrix} n \\ k-1 \end{bmatrix}$$

which is the first recurrence.

Now, in the same way, we use

$$(x+y)^{n+1} = (x+y)(x+y)^n$$

to give us the second recurrence

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}$$

By subtracting the recurrences, we find

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{1 - q^{n+1-k}}{1 - q^k} \begin{bmatrix} n \\ k-1 \end{bmatrix}$$

If we iterate this process, by taking successively smaller values of k we are left with

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1 - q^{n-(k-1)}) \cdots (1 - q^n)}{(1 - q^k) \cdots (1 - q)} \begin{bmatrix} n \\ 0 \end{bmatrix}$$

But from the identity in the theorem, it can be seen that $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$. By multiplying top and bottom of the right hand side by $(1-q)(1-q^2)...(1-q^{n-k})$, and noting that $[n]! = (1-q)(1-q^2)...(1-q^n) \cdot (1-q)^{-n}$, we have the result.

□

We think of this combinatorially as follows. When we multiply out the left hand side of the equation, we have a monomial in x's and y's. Using the relation $yx = qxy$, we rewrite the monomials with the x's appearing before the y's. The power of q in each term will be the number of times that an x and a y had to be interchanged to write it in the final form. The Gaussian coefficients are then the coefficients of the resulting terms.

3.5 Juggling triples

Definition 3.20. Two vectors $\boldsymbol{u} = (u_1, u_2, ..., u_m)$ and $\boldsymbol{v} = (v_1, v_2, ..., v_m)$ are *similar* if $\boldsymbol{u}$ is a permutation of $\boldsymbol{v}$, in which case we write $\boldsymbol{u} \sim \boldsymbol{v}$.

Definition 3.21. Let $(d, \boldsymbol{x}, \boldsymbol{a})$ be a *juggling triple*, where d is a positive integer, $\boldsymbol{x} = (x_1, x_2, ..., x_m) \in \mathbb{Z}^m$ and $\boldsymbol{a} = (a_1, a_2, ..., a_m) \in \mathbb{N}^m$ such that:

- $0 \leq x_i \leq d - 1$ for all $i = 1, 2, ..., m$
- $(\boldsymbol{x} + \boldsymbol{a}) \bmod d \sim \boldsymbol{x}$

Call d the *period*, $\boldsymbol{x}$ the *base vector* and $\boldsymbol{a}$ the *throw vector*. Let m be the cardinality of the base and throw vectors.

(Note that I will use the term juggling pattern to loosely refer to a juggling triple and its associated graph when no confusion is likely.)

Definition 3.22. A juggling triple $(d, \boldsymbol{x}, \boldsymbol{a})$ is *simple* if $\boldsymbol{x} = \boldsymbol{\delta}_{d-1}$. Otherwise it is called *multiplex*.

This definition is similar to siteswap and Definition 2.9 except that, unlike in Definition 2.9, multiplex throws are allowed. The throw vector $\boldsymbol{a}$ is just the sequence of throw heights, where throw a_i occurs on beat x_i. So, for example, the three ball cascade is written $(1, (0), (3))$ by this definition. The pattern 441 is written as $(3, (0, 1, 2), (4, 4, 1))$. An example of a multiplex pattern is $(2, (0, 0, 1), (1, 4, 1))$, where two balls are thrown on the first beat: one is a small throw to the opposite hand, and the other is a higher throw to the same hand. On the second beat, the ball that was thrown to the opposite hand is quickly returned to the first hand again. (This pattern is not very easy or natural to juggle.) Note that in the case of a multiplex juggling pattern there is some non-uniqueness in the juggling triple. In the example above, we could have written the same pattern as $(2, (0, 0, 1), (4, 1, 1))$ since we still have 2 throws (one of height 1 and the other of height 4) occurring on beat 0. While this makes no difference to the physical act of juggling, it has important consequences for the mathematics of the construction as we shall see.

We can define a directed multigraph G of the juggling triple that is similar to the graphs in Figure 7, 8 and 9. The vertex set is $\mathbb{Z}$ and the edge set is given explicitly by

$$E(G) = \{(x_i + k \cdot d, x_i + a_i + k \cdot d) : 1 \leq i \leq m, k \in \mathbb{Z})$$

Note that all the edges are directed forward in time.

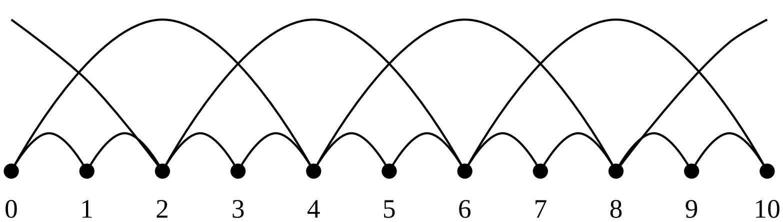

Figure 16: The graph of $(2, (0, 0, 1), (1, 4, 1))$ and $(2, (0, 0, 1), (4, 1, 1))$.

Each of the vertices has the same number of edges entering it (from the left) as there are edges leaving it (toward the right), that is the indegree and outdegree are equal. This is guaranteed by the second condition in Definition 3.21. For example say there are two edges in $E(G)$ that both terminate at a vertex z

$$x_1 + a_1 + id = z$$
$$x_2 + a_2 + jd = z$$

and let $z \bmod d = \bar{z} \in \boldsymbol{\delta}_{d-1}$. Since $(\boldsymbol{x} + \boldsymbol{a}) \bmod d \sim \boldsymbol{x}$, $\bar{z}$ appears in $\boldsymbol{x}$ twice. Because the pattern has period d, there will be two balls thrown at all beats $\bar{z} + \alpha \cdot d$, and so z must have outdegree two. So we can view the graph as being the composition of a finite number of edge-disjoint paths. The actual composition is not unique, since for any multiplex throw we have a choice of options for assigning labels to the edges, but the number of edges is unique, and hence the number of paths is unique. We now take this number to be the definition of the number of balls.

Definition 3.23. Let $ball(d, \boldsymbol{x}, \boldsymbol{a})$ be the number of edge-disjoint paths in the graph of the juggling triple $(d, \boldsymbol{x}, \boldsymbol{a})$. Call $ball(d, \boldsymbol{x}, \boldsymbol{a})$ the number of balls of $(d, \boldsymbol{x}, \boldsymbol{a})$.

Since $\boldsymbol{a}$ is just the sequence of throw heights, Theorem 2.4, extended to multiplex patterns, says that

$$ball(d, \boldsymbol{x}, \boldsymbol{a}) = \frac{1}{d} \sum_{i=1}^{m} a_i$$

Let α_i be the in/outdegree of vertex i in the graph of a juggling triple, that is $\alpha_i = |\{j : x_j = i\}|$. Note that when we are dealing with a simple juggling triple (recall Definition 3.22) $\alpha_i = 1$ for all $i \in \boldsymbol{\delta}_{d-1}$. This means that there can only be one ball thrown at a time. It also means that zero throws are not allowed in simple patterns (unlike in siteswap and Definition 2.9). Simple juggling triples will be denoted by $(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a})$. For example, in the diagrams above, $(1, (0), (3))$ $(3, (0, 1, 2), (4, 4, 1))$ are both simple juggling triples, although $(2, (0, 0, 1), (1, 4, 1))$ and $(2, (0, 0, 1), (4, 1, 1))$ are multiplex juggling triples.

We now define crossings of the graph of a juggling triple.

Definition 3.24. A *crossing* of G is a pair of edges $((x, y), (u, v)) \in (E(G))^2$ such that $x < u < y < v$. Two crossings are equivalent if their position differs by a multiple of d; explicitly if

$$x_1 = x_2 + k \cdot d, y_1 = y_2 + k \cdot d, u_1 = u_2 + k \cdot d, v_1 = v_2 + k \cdot d$$

where k is an integer, then $((x_1, y_1), (u_1, v_1))$ is equivalent to $((x_2, y_2), (u_2, v_2))$. Let the number of equivalent crossings be the called the number of *external crossings.*

An *internal crossing* is a pair (i, j) such that $1 \leq i < j \leq m$, $x_i = x_j$ and $a_i > a_j$.

The number of crossings of a juggling triple is the sum of the external and internal crossings, which is written $cross(d, \boldsymbol{x}, \boldsymbol{a})$.

So external crossings are the crossings between edges that start and terminate on different edges, and are easily seen in the graph of a juggling triple. Internal crossings are those crossings between edges that start on the same vertex but terminate on different vertices. The order of the edges as they leave the first vertex determines if a crossing occurs, so if we assign an order to these edges then we can see the crossings in a graph. If there is some $x_i = x_j$ where $i < j$, then let the edge corresponding to x_i be above the edge corresponding to x_j. To illustrate this we use the example from earlier. According to this ordering,

Figure 16 is the graph of $(2, (0, 0, 1), (1, 4, 1))$, with no internal crossings, but $(2, (0, 0, 1), (4, 1, 1))$ does have internal crossings, and these show up in Figure 17.

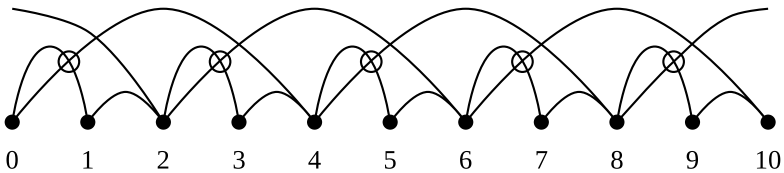

Figure 17: The graph of $(2, (0, 0, 1), (4, 1, 1))$ showing the internal crossings.

Note that $cross(d, \boldsymbol{x}, \boldsymbol{a})$ counts the number of crossings of the graph between vertices v and $v + d$, where d is the period of the pattern.

We need to make a distinction between crossings from the *inside* and crossings from the *outside*. Refer to Figure 18. For external crossings, the edge (a, b) is crossed from the *inside* by the edge (c, d) if $a < c < b < d$. Whereas (c, d) is crossed from the *outside* by (a, b). For internal crossings the edge (e, f) is crossed from the *inside* by edge (e, g), while (e, g) is crossed from the *outside* by (e, f).

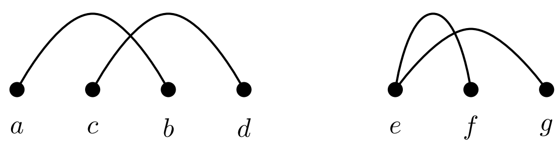

Figure 18: Crossings from the inside and outside.

These distinctions become important for reconstructing juggling triples from the diagrams. The following definition will prove useful.

Definition 3.25. The *weight* of a juggling triple $(d, \boldsymbol{x}, \boldsymbol{a})$ is $q^{cross(d,\boldsymbol{x},\boldsymbol{a})}$

3.6 Juggling cards

Now we introduce *juggling cards*. We will only discuss juggling cards of simple juggling patterns here; later we will generalise these cards to multiple throws per beat. First we pick some number n of balls. Let C^i be the card where the $(i+1)$'th ball (counted from the bottom) is caught; note that card C^i will have i crossings on it, since the ball that is caught must cross the paths of the i balls underneath it. Also note that all crossings of an edge (x, y) from the *inside* will occur on card y, that is when the ball corresponding to the edge (x, y) is being caught. The set of all cards with $n = 3$ is contained in Figure 19.

We are now in a position to prove the following theorem.

Theorem 3.26.

$$\sum_{(d,\boldsymbol{\delta}_{d-1},\boldsymbol{a})} q^{cross(d,\boldsymbol{\delta}_{d-1},\boldsymbol{a})} = [n]^d$$

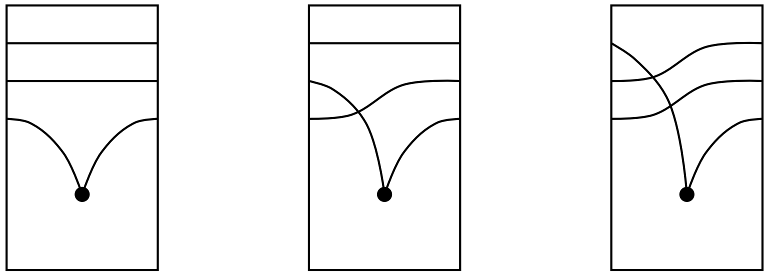

Figure 19: The set of all juggling cards with $n = 3$. Source: [12].

where the sum is over all simple juggling triples with period d and at most n balls.

Proof: The method of this proof is that we first find the weight of a juggling triple if it can be expressed as a sequence of juggling cards, and then show that all juggling triples can be expressed as a juggling card sequence.

Using cards $C^0, C^1, ..., C^{n-1}$ we can construct juggling patterns that contain up to n balls. Clearly there are n^d patterns of period d that can be made with these cards. Also, it is clear from the definition that the sum of the weights of the n different cards is $1 + q + q^2 + ... + q^{n-1}$. So $(1 + q + q^2 + ... + q^{n-1})^d := [n]^d$ is the sum of the weights of all the different patterns.

To complete the proof, we need to show that the cards $C^0, C^1, ..., C^{n-1}$ can be used to construct all simple juggling patterns with at most n balls. Define a map Φ from the juggling triples to $\mathbb{N}^d$ by $\Phi(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a}) = (\phi_1, \phi_2, ..., \phi_d)$ where $\phi_i = |\{(u, v) \in E(G) : i - 1 < u < i - 1 + a_i < v\}|$. That is, ϕ_i counts the number of crossings of the edge $(i-1, i-1+a_i)$ *from the inside* by other edges (u, v) (recalling that these are simple patterns and so we only have external crossings). Now let $\mu_j = \phi_i$ where $i - 1 + a_i \equiv j \bmod d$ for $0 \leq j \leq d - 1$. So this means that μ_j counts the number of crossings of the edge that terminates at vertex j. Recall that all crossings from the *inside* of an edge (x, y) occur on the card corresponding to the vertex y, where the edge terminates. So any graph of a juggling triple can be constructed using the cards $C^{\mu_0}, C^{\mu_1}, ..., C^{\mu_{d-1}}$.

$\square$

3.7 Poincaré series of affine Weyl groups

Theorem 3.26 has the unexpected consequence that we can now quite easily calculate the Poincaré series for the affine Weyl group $\tilde{A}_{d-1}$.

Definition 3.27. Let $\tilde{A}_{d-1}$ be the group of permutations $\sigma : \mathbb{Z} \to \mathbb{Z}$ with composition, where the permutations satisfy the criteria:

1. $\sigma(i + d) = \sigma(i) + d$ for all i
2. $\sum_{i=0}^{d-1}(\sigma(i) - i) = 0$

The set of permutations corresponding to the juggling sequences of the type

$(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a})$ is precisely $\tilde{A}_{d-1}$. To see this, define the permutation $\sigma : \mathbb{Z} \to \mathbb{Z}$ by

$$i \mapsto a_{i \bmod d} + i - \frac{1}{d}\sum_{j=0}^{d-1} a_j$$

Which will be a permutation because $(a + \boldsymbol{\delta}_{d-1}) \bmod d \sim \boldsymbol{\delta}_{d-1}$. This means that $\sigma : (0, 1, ..., d-1) \mapsto (0, 1, ..., d-1)$ is a bijection, and then this reordering is repeated for all other sets $(0 + kd, 1 + kd, ..., d-1+kd)$, where $k \in \mathbb{Z}$. The two required properties of the permutations are immediate.

It is known that the group $\tilde{A}_{d-1}$ can be generated by simple reflections [16]:

$$s_i(k) = \begin{cases} k+1 & \text{if } k \equiv i \bmod d \\ k-1 & \text{if } k \equiv i+1 \bmod d \\ k & \text{if } k \not\equiv i, i+1 \bmod d \end{cases}$$

So any element of $\tilde{A}_{d-1}$ can be written as a composition of these simple reflections.

Definition 3.28. Let $l(\sigma)$ be the *length* of an element σ, defined as the smallest number of simple reflections necessary to write σ as their composition.

This allows the following theorem.

Theorem 3.29. *Let $\sigma \in \tilde{A}_{d-1}$, fix $n \in \mathbb{N}$ such that $n > i - \sigma(i)\ \ \forall\ i = 0, 1, ..., d-1$. Let $\boldsymbol{a} = (a_0, a_2, ..., a_{d-1})$ where $a_i = \sigma(i) - i + n$. Then $(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a})$ is a simple juggling triple with $ball(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a}) = n$ and $cross(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a}) = (n-1) \cdot d - l(\sigma)$.*

Proof: $n > i - \sigma(i)$ and so $a_i = \sigma(i) - i + n > 0$, which is a necessary condition for $(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a})$ to be a juggling triple. Note that $\boldsymbol{a} + \boldsymbol{\delta}_{d-1} = (\sigma(0) + n, \sigma(1) + n, ..., \sigma(d-1) + n)$. We know that σ permutes the set $\boldsymbol{\delta}_{d-1}$ so $(\boldsymbol{a} + \boldsymbol{\delta}_{d-1}) \sim \boldsymbol{\delta}_{d-1} \bmod d$. So $(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a})$ is a juggling triple.

We can calculate $ball(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a})$:

$$ball(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a}) = \frac{1}{d}\sum_{j=0}^{d-1}(\sigma(i) - i + n) = n$$

So it remains to show that $cross(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a}) = (n-1) \cdot d - l(\sigma)$. An induction argument is used. For the base case, assume that σ is the identity element so that $l(\sigma) = 0$, then $\boldsymbol{a} = (n, n, ..., n)$ since $\sigma(i) = i$. For this pattern we can count the crossings easily. Each time a ball is caught, its edge must cross $(n-1)$ other edges on the way down, and these are the only crossings that we see. There are d catches between vertices 0 and d. So $cross(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a}) = (n-1) \cdot d$.

Now assume that $\sigma = \tau \circ s_i$ where $l(\sigma) = l(\tau) + 1$ and $cross(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a}) = (n-1) \cdot d - l(\xi)\ \forall\ \xi$ such that $l(\xi) < l(\sigma)$. We need the result of Shi [24] (Corollary 4.2.3) here, that this implies $\tau(i) < \tau(i+1)$. So $\sigma(i) > \sigma(i+1)$ since $\sigma(i) = \tau(i+1)$ and $\sigma(\mathrm{i}+1) = \tau(\mathrm{i})$. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the juggling sequences derived from σ and τ respectively, and let A be the graph corresponding to $\boldsymbol{a}$ and B the graph corresponding to $\boldsymbol{b}$. Since the permutations σ and τ only differ at points i and $i+1$, $\boldsymbol{a}$ and $\boldsymbol{b}$ also only differ that these points. Explicitly $b_i = a_{i+1} + 1$,$b_{i+1} = a_i - 1$ and $b_j = a_j$ for $j \neq i, i+1$. So $((i, i + a_i), (i+1, i+1+a_{i+1})) = ((i, i + b_{i+1} + 1), (i+1, i+b_i))$ is not a crossing in A since $b_i < b_{i+1}$, that is the arc from vertex i jumps over the arc from vertex $i+1$. But $(i, i+b_i), (i+1, i+1+b_i)$ is a crossing B. This can be seen in Figure 20.

Now we look at all the other possible crossings of the graphs. Take $x \not\equiv i, i+1 \bmod d$ and $y > x$. If $((i, i+a_i), (x, y))$ is a crossing in A then $((i+1, i+b_{i+1}), (x, y))$ is also a crossing in B since $i + a_i = i + 1 + b_{i+1}$. If $((i+$

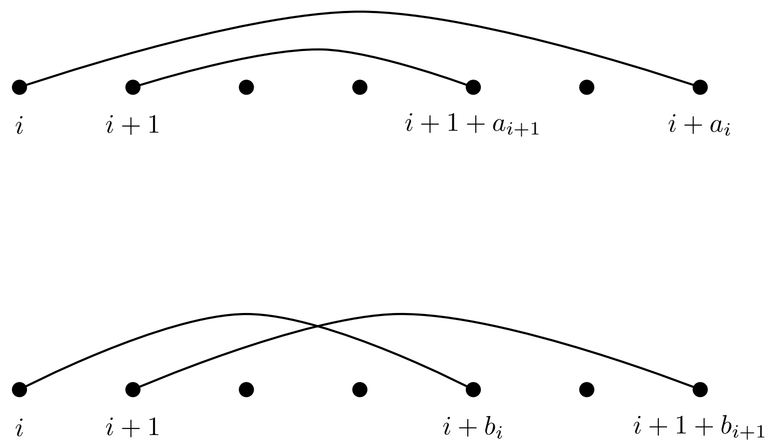

Figure 20: Graph showing that the graph B (bottom) has one extra crossing than graph A (top).

$1, i+1+a_{i+1}), (x,y))$ a crossing in A then $((i, i+b_i), (x,y))$ is a crossing in B since $i+1+a_{i+1} = i+b_i$. For any other crossing $((x,y),(u,v))$, where $u \not\equiv i, i+1 \bmod d$ and $v > u$, the graphs of A and B co-incide. So there is one less crossing in A than B, that is $cross(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a}) = cross(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{b}) - 1$ and we are done.

$\square$

So we can now prove the following.

Corollary 3.30. *The Poincaré series of $\tilde{A}_{d-1}$, defined by the following sum, has the evaluation*

$$\sum_{\sigma \in \tilde{A}_{d-1}} q^{l(\sigma)} = \frac{1-q^d}{(1-q)^d}$$

Proof: Let $P_n \subset \tilde{A}_{d-1}$ be $P_n = \{\sigma \in \tilde{A}_{d-1} : n > \max(i - \sigma(i))\}$. Then by Theorems 3.26 and 3.29

$$\sum_{\sigma \in P_n} q^{(n-1)\cdot d - l(\sigma)} = (1+q+...+q^{n-1})^d - (1+q+...+q^{n-2})^d$$

This is true since if $\sigma \in P_n$, then by Theorem 3.29 $cross(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a}) = (n-1)\cdot d - l(\sigma)$ where $(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a})$ is the simple juggling triple corresponding to σ, as defined in Theorem 3.29. Also, from the same theorem, $ball(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a}) = n$. We know from Theorem 3.26 that $(1+q+...+q^{n-1})^d$ counts the crossings in all juggling triples of at most n balls , and $(1+q+...+q^{n-2})^d$ counts the crossings in juggling triples of at most $n-1$ balls, so the right hand side counts the crossings in juggling triples with exactly n balls. By replacing q by $\frac{1}{q}$ and simplifying, we arrive at

$$\begin{aligned}\sum_{\sigma\in P_n} q^{l(\sigma)} &= (1+q+...+q^{n-1})^d - (q+...+q^{n-1})^d \\ &= \left(\frac{1-q^n}{1-q}\right)^d - q^d\left(\frac{1-q^{n-1}}{1-q}\right)^d\end{aligned}$$

Now note that $\tilde{A}_{d-1} = \cup_{n\geq 1} P_n$, and so by letting $n \to \infty$ we have the result. □

3.8 Multiplex juggling triples

Now we generalise the discussion of juggling triples to include multiplex patterns, that is we remove the restriction that the base vector $\boldsymbol{x} = \boldsymbol{\delta}_{d-1}$, and so α_i the in/outdegree of time point i is not necessarily equal to 1. Note that we have not changed Condition 2 in Definition 3.21, so we still have the requirement that the in and out-degrees must be equal. The first result is a generalisation of Theorem 3.26.

Theorem 3.31.

$$\sum_{(d,\boldsymbol{x},\boldsymbol{a})} q^{(d,\boldsymbol{x},\boldsymbol{a})} = \begin{bmatrix} n \\ \alpha_0 \end{bmatrix}\begin{bmatrix} n \\ \alpha_1 \end{bmatrix}\cdots\begin{bmatrix} n \\ \alpha_{d-1} \end{bmatrix}$$

where α_i is the in/out-degree of time point i, and where the sum is over all juggling triples with period d and at most n balls.

Proof: We use the same method of proof as for Theorem 3.26—that is we first calculate the weight of the triple if it has a corresponding sequence of juggling cards, and then show that all triples have such a corresponding sequence.

This time we use a different deck of cards for each time point. Let D_k be the set of cards that show k balls being caught from a total of n balls, and those k balls being thrown into the lowest orbits. It is obvious that there must be $\binom{n}{k}$ cards in this deck. For example, when $n = 4$, the deck D_2 is shown in Figure 21 (the 'x's and 'y's will be explained shortly). Label the cards in this deck with a set of size k, whose elements are the number of crossings for each ball that is caught. The set is arranged so that the lowest ball occurs first, then the second lowest ball, and so on. Call this set $M = \{m_1, m_2, ..., m_k\}$, and label the card thusly C^M. For example, in Figure 21 we have, from left to right, $C^{\{0,0\}}, C^{\{0,1\}}, C^{\{0,2\}}, C^{\{1,1\}}, C^{\{1,2\}}, C^{\{2,2\}}$. So the weight of card C^M is $q^{\sum_{m\in M}}$.

Now we recall the combinatorial interpretation of Theorem 3.19. We think of the balls that are being caught as 'x's and the rest as 'y's. Then we see that on the right side of every card the 'x's are grouped at the bottom of the card and the 'y's are grouped at the top. However, on the left hand side, the order is permuted. See Figure 21 for an example. So we think about interchanging the positions of the 'x's and 'y's on the left to order them as on the right of each card. Now each interchange of an 'x' and a 'y' will add or remove a crossing, depending on whether a 'y' is shifted below an 'x' or above an 'x'. If we count these crossings by the powers of q, then this is identical to the combinatorial interpretation of Schutzenberger's identity, where we are using

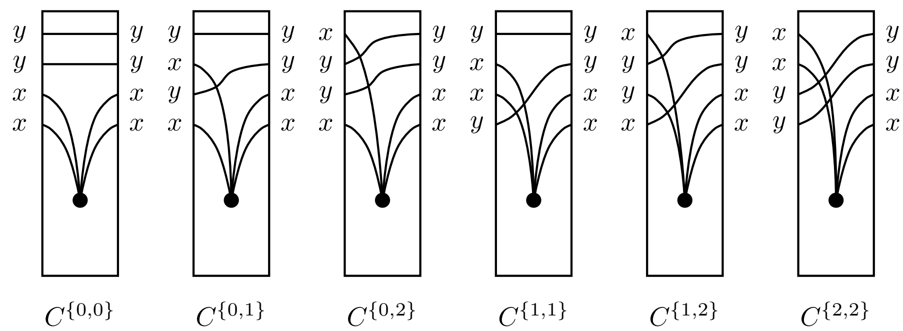

Figure 21: The deck D_2 with $n = 4$. We can understand the combinatorial interpretation of Schutzenberger's identity on these cards by transforming the order of 'x's and 'y's on the left of each card to match the order on the right of each card, using the relation $yx = qxy$.

the rule $yx = qxy$. So the sum of the weights of the cards in deck D_k is $\begin{bmatrix} n \\ k \end{bmatrix}$ as given by the identity.

At time point i we use deck D_{α_i}, and so the sum of the weights of these patterns is

$$\begin{bmatrix} n \\ \alpha_0 \end{bmatrix} \begin{bmatrix} n \\ \alpha_1 \end{bmatrix} \cdots \begin{bmatrix} n \\ \alpha_{d-1} \end{bmatrix}$$

Now we need to establish a bijection between the set of juggling triples and the sequences of juggling cards. Let $\Psi(d, \boldsymbol{x}, \boldsymbol{a}) = (\psi_1, \psi_2, ..., \psi_m)$ where

$$\psi_i = |\{(u, v) \in E(G) : x_i < u < x_i + a_i < v\}| + |\{j : 1 \leq j < i, x_i = x_j, a_j > a_i\}|$$

So ψ_i is the number of crossings of edge $(x_i, x_i + a_i)$ *from the inside*—the term on the left gives the external crossings, and the term on the right is the internal crossings. Now let $M_j = \{\psi_i : x_i + a_i \equiv j \bmod d\}$, and then the graph of $(d, \boldsymbol{x}, \boldsymbol{a})$ is constructed as the sequence of cards $C^{M_0}, C^{M_1}, ..., C^{M_{d-1}}$. So what we have done is to establish the number of crossings from the inside of each edge for one period of the juggling pattern, and then, noting that all crossings from the inside occur on the card where the ball is caught, we can pick the cards to construct the graph. It is clear from this construction that each juggling triple corresponds to a sequence of juggling cards, and from any sequence of juggling cards we can read off the juggling triple.

□

Note that if $\alpha_i = 1$, then $\begin{bmatrix} n \\ \alpha_i \end{bmatrix} = [n]$, and so Theorem 3.31 reduces to Theorem 3.26 when $\alpha_i = 1$ for all i.

3.9 q-Stirling numbers

These results can be used to prove some interesting identities related to the q-analogue of Stirling numbers of the second kind. Recall that the Stirling number of the second kind $S(n, k)$ counts the number of ways of putting n objects into k boxes.

Definition 3.32. The *q-Stirling numbers of the second kind* are defined by the recurrence

$$S[n, k] = q^{k-1} \cdot S[n-1, k-1] + [k] \cdot S[n-1, k]$$

where $n \geq 2$, $k \geq 1$, with initial conditions $S[1, 1] = 1$ and $S[1, k] = 0$ for $k \neq 1$.

We can develop a combinatorial interpretation for q-Stirling numbers of the second kind by generalising that for Stirling numbers of the second kind.

Definition 3.33. Let $\Pi_k n$ be the set of all partitions of $\boldsymbol{\delta}_{n-1}$ into k blocks. Let $int(i,j)$ be the (exclusive) *interval between i and j*, that is

$$int(i,j) = \{n \in \mathbb{Z} : \min(i,j) < n < \max(i,j)\}$$

Of course $int(i,j) = int(j,i)$.

Definition 3.34. Let B and C be disjoint, non-empty subsets of $\{1, 2, ..., n\}$. Define the *intertwining number*, $\iota(B,C)$, by

$$\iota(B,C) = |\{(b,c) \in B \times C : int(b,c) \cap (B \cup C) = \emptyset\}|$$

For a partition $\pi \in \Pi_k n$ where $\pi = (B_1, B_2, ..., B_k)$ define the intertwining number as

$$\iota(\pi) = \sum_{1 \leq i < j \leq k} \iota(B_i, B_k)$$

Note that $\iota(B,C) = \iota(C,B)$ so $\iota(\pi)$ is independent of the ordering of the sets in the partition. It happens that the intertwining number of a partition corresponds to the number of crossings of a finite juggling diagram if each set in the partition defines one ball in the diagram. For example consider Figure 22. We have the partition $\pi = \{\{1,3,6\},\{2,4\},\{5\}\}$. The intertwining number is calculated to be: $\iota(\pi) = \iota(\{1,3,6\},\{2,4\}) + \iota(\{1,3,6\},\{5\}) + \iota(\{2,4\},\{5\}) = 4+2+1 = 7$. It can be clearly seen in the diagram that there are seven crossings.

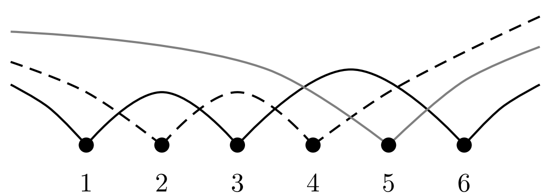

Figure 22: The intertwining number of the partition $\pi = \{\{1,3,6\},\{2,4\},\{5\}\}$ can be calculated by the crossings in this graph. Source: [12].

We can show that $\iota(\pi)$ is the same as the number of crossings from the definition: $int(b,c)$ is the set of all numbers between b and c (exclusive). So then if $int(b,c) \cap (B \cup C) = \emptyset$ then there must be a crossing between b and c. That is, if $int(b,c) = \emptyset$ then b is adjacent to c and there must be a crossing since, by construction, b is in one path and c is in another path. So the edge leaving b must cross the edge entering c. Crossings occur between any adjacent pairs that are in different paths. On the other hand, if $int(b,c) \neq \emptyset$ and yet $int(b,c) \cap (B \cup C) = \emptyset$, then all the numbers $int(b,c)$ are in paths different from the paths containing b and c. Given that we are only looking at the crossings between these paths, and not crossings with the paths containing the values in the interval, this is effectively the same as $int(b,c) = \emptyset$ for the purposes of counting crossings. We essentially remove the intervening values and place b next to c (or vice-versa), and so we have a crossing.

Lemma 3.35.

$$S[n,k] = \sum_{\pi \in \Pi_k n} q^{\iota(\pi)}$$

Proof: We can show this by demonstrating that the right hand side satisfies the recurrence in Definition 3.32. Call the right hand side $G[n, k]$.

Let us first isolate those partitions where the element n is in a subset by itself, which corresponds to a path P_n that is only caught once, at time n. So we have $n-1$ elements left to distribute among the $k-1$ remaining subsets, which correspond to $k-1$ paths. From Figure 23 it can be seen that each of these $k-1$ paths must cross the path P_n from the inside exactly once. So we have a factor of q^{k-1}. This gives us the term $q^{k-1}G[n-1, k-1]$.

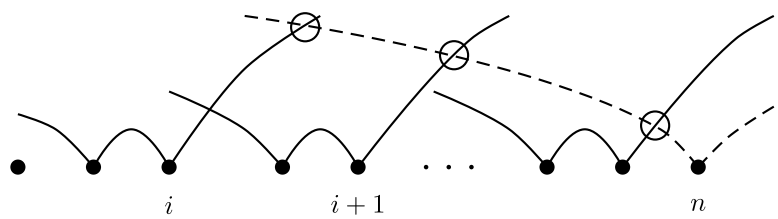

Figure 23: Isolating partitions where the element n is in a subset by itself.

Remaining are those partitions where n does not appear in its own subset. We partition the $n-1$ other elements into k subsets and then 'add in' the last element. Arrange the subsets in ascending order, according to the largest element in each subset. Then if we put the element n into the kth subset, we don't create any extra crossings, which can be seen in Figure 24. So we have a factor of $q^0 = 1$.

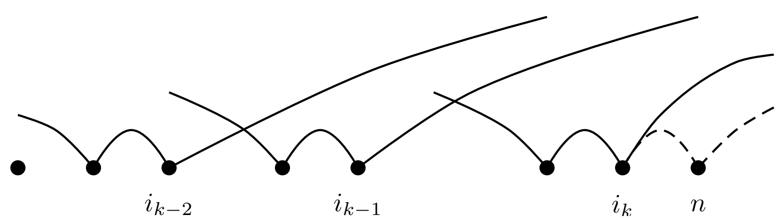

Figure 24: Placing the element n into the kth subset.

If we put it into the $(k-1)$th subset then we create one crossing (see Figure 25) and we pick up a factor of q.

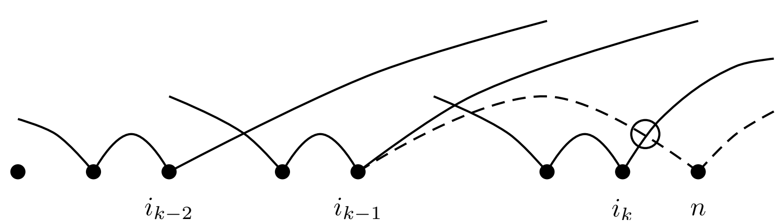

Figure 25: Placing the element n into the $(k-1)$th subset.

We continue in this way, and if we add the element n into the first subset, then we create $k-1$ new crossings, picking up a factor of q^{k-1}. Thus we create $k-j$ more crossings when we add the largest element n into the jth subset of the partition, and we gain a factor of q^{k-j}. Hence we also have the term $(1+q+...+q^{k-1})G[n-1,k] = [k]G[n-1,k]$.

It only remains to check the initial conditions. If $n = k = 1$ then we note that $\iota(\pi) = 0$ since there are no terms in the sum that defines $\iota(\pi)$. So $G[1,1] = q^0 = 1$. If $n = 1$ and $k > 1$ then there are no partitions, and so there are no terms in the sum of $G[1,k]$ and so $G[1,k] = 0$. This fulfills the initial conditions of Definition 3.32 and so we are done.

□

We can now prove the following theorem which was discussed by Leonard Carlitz in [8] long before juggling became interesting to mathematicians.

Theorem 3.36.

$$[n]^d = \sum_{m=0}^{d} S[d,m] \cdot [m]! \cdot \begin{bmatrix} n \\ m \end{bmatrix}$$

Proof: The left hand side of the identity is, by Theorem 3.26, the sum of the weights of simple juggling triples with period d and at most n balls. Recall that simple juggling triples have only external crossings. Now we contract d consecutive vertices, $k\cdot d, k\cdot d+1, ..., k\cdot d+(d-1)$, and examine what becomes of the crossings under contraction. Note that some crossings may disappear since if an edge lies completely within the set of contracted vertices, then it will not exist in the contracted graph. The contraction of a simple juggling triple $(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a})$ is carried out explicitly by forming a new sequence $\boldsymbol{b} = (b_0, b_1, ..., b_{d-1})$ where $b_i = \lfloor \frac{a_i+i}{d} \rfloor$. $a_i + i$ is where the ball from beat i ends up, so if $a_i + i < d$ then the throw is contracted to no throw at all and $\frac{a_i+i}{d} < 1$, so $b_i = 0$. Once we have the sequence $\boldsymbol{b}$ then we delete all the zero entries to obtain the sequence $\boldsymbol{c} = (c_0, c_1, ..., c_m)$. We then form the juggling triple $(1, \mathbf{0}_m, \boldsymbol{c})$. This sequence is a multiplex sequence of period 1, with all m throws happening on the same beat.

Note that m corresponds to the number of distinct paths (that is, balls) that were caught during the contracted interval, and so these m paths partition the interval into m disjoint subsets. Call this partition π. $\iota(\pi)$, the intertwining number of π is the number of crossings of the graph that occur in the interval.

Note that $ball(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a}) = ball(1, \mathbf{0}_m, \boldsymbol{c})$, since all the paths in $(d, \boldsymbol{\delta}_{d-1}, \boldsymbol{a})$ are infinitely long, they must also appear in the contracted graph. For example, the juggling triple $(3,(0,1,2),(4,4,1))$ of Figure 4 becomes $(1,(0,0,0),(1,1,1))$ under contraction, the graph of this triple is contained in Figure 26.

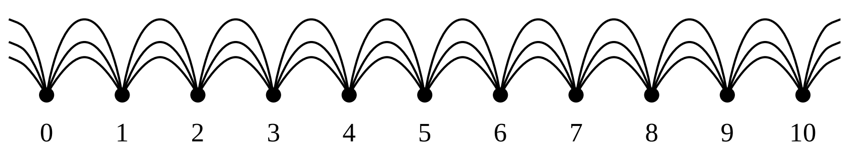

Figure 26: The graph of $(3,(0,1,2),(4,4,1))$ once it has been contracted to $(1,(0,0,0),(1,1,1))$.

We can see from the construction of $\boldsymbol{c}$ that $1 \leq m \leq d$. We can also see how this is true in terms of the graph. Recall that we start with a simple juggling pattern, that is the in/outdegree of every vertex is 1. So that means that a maximum of d edges terminate/originate inside the interval $(0,1,...,d-1)$ since there are only d vertices being contracted, so there can only be a maximum of d balls being caught—in other words, we can only partition a set of d elements into a maximum of d subsets. m is the in/outdegree of the edges after contraction, which must be bounded by the number of vertices that terminated at/originated from the d vertices before contraction. So the sum must terminate at m.

Now we examine what happens to a typical (external) crossing $((x, y), (u, v))$ under contraction.

We have four possibilities. If the vertices y and u are contracted together (Figure 27) then they must have both been inside the contracted interval, which implies that the crossing also occurred inside the interval, and so it disappears. It will be counted by $S[d, m]$ according to Lemma 3.35.

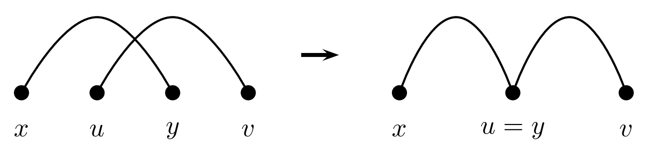

Figure 27: Effect on the crossing $((x, y), (u, v))$ when y and u are contracted together.

Now we assume that y and u are not contracted together. If y and v are contracted together (Figure 28), then this can be seen an inversion of a permutation. That is, for every pair $((x, y), (u, v))$, if $y < v$ we have a crossing, but if $v < y$ then we do not have a crossing. Since y and v belong to different paths, they must be in different subsets of the partition. If s_y and s_v are the subsets containing y and v respectively then we will have a crossing if s_y occurs before s_v, and not otherwise. So the number of crossings will be the same as the number of inversions in a permutation of the subsets of the partition. Recall that there are m subsets, and so these inversions will be counted by $[m]!$ ([10], pg 541).

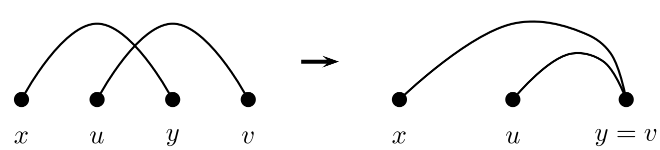

Figure 28: Effect on the crossing $((x, y), (u, v))$ when y and v are contracted together.

The third case is that x and u are contracted together (Figure 29), then the crossing becomes an internal crossing of $(1, \mathbf{0}_m, \boldsymbol{c})$.

The last possibility is that none of the vertices are contracted together, then the crossing remains an external crossing of $(1, \mathbf{0}_m, \boldsymbol{c})$. These last two cases together are counted by $\begin{bmatrix} n \\ m \end{bmatrix}$ since by Theorem 3.31 this counts both internal and external crossings of $(1, \mathbf{0}_m, \boldsymbol{c})$.

And so we have the result.

$\square$

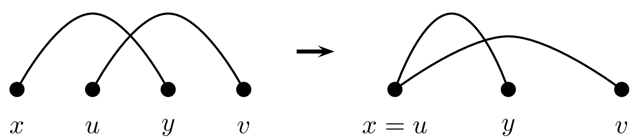

Figure 29: Effect on the crossing $((x,y),(u,v))$ when x and u are contracted together.

3.10 Unitary vector compositions

The method of proof used in Theorem 3.36 turns out to be quite useful for proving other identities involving Gaussian coefficients. We find some interesting results if we turn our attention to a generalisation of set partitions, called vector compositions.

Definition 3.37. A *composition* of a vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{N}^d$ is a an ordered set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_k\}$ where $\boldsymbol{v}_i \in \mathbb{N}^d$ such that $\boldsymbol{v}_1 + \boldsymbol{v}_2 + ... + \boldsymbol{v}_k = \boldsymbol{\alpha}$. Let $f_k(\boldsymbol{\alpha})$ be the number of compositions of $\boldsymbol{\alpha}$ into k parts.

Call a composition *unitary* if every entry of $\boldsymbol{v}_i$ is 1 or 0 for all i. Let $g_k(\boldsymbol{\alpha})$ be the number of unitary compositions of $\boldsymbol{\alpha}$ into k parts.

Note that the symbol $\boldsymbol{\alpha}$ has been used deliberately since this vector will relate to the α_i used earlier to denote the in/outdegree of vertex i. Also note that the index set of $\boldsymbol{\alpha}$ begins with 1 and ends with d, whereas earlier index sets have begun at 0 and ended at $d-1$.

For example let $\boldsymbol{\alpha} = (2,1)$, then there are two unitary compositions of $\boldsymbol{\alpha}$ into two parts, $(1,1)+(1,0)$ and $(1,0)+(1,1)$. So $g_2((2,1)) = 2$. For $k=3$ we have $(1,0)+(1,0)+(0,1)$, $(1,0)+(0,1)+(1,0)$ and $(0,1)+(1,0)+(1,0)$, and so $g_3((2,1)) = 3$. From these examples, we can see that we are dealing with ordered sums of vectors, whose entries are 1 or 0, and whose sum is $\boldsymbol{\alpha}$.

Note that

$$g_k(\underbrace{(1,1,...,1)}_{n}) = k! \cdot S(n,k)$$

The $S(n,k)$ factor specifies the number of ways that the n entries of $(1,1,...,1)$ can be arranged into k groups, and the $k!$ factor counts the number of ways of arranging the groups. The q-analogue of this identity also holds, which will be explained later.

Before we can apply this concept to juggling sequences we need some preliminary definitions.

Definition 3.38. For a vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{N}^d$, let $\|\boldsymbol{\alpha}\| = \sum_{i=1}^d \alpha_i$ and $\boldsymbol{z}(\boldsymbol{\alpha}) = (\underbrace{1,1,...,1}_{\alpha_1}, \underbrace{2,2,...,2}_{\alpha_2}, ..., \underbrace{d,d,...,d}_{\alpha_d})$.

Also let $(\boldsymbol{d}+\boldsymbol{1})_k = (\underbrace{d+1, d+1, ..., d+1}_{k})$

Definition 3.39. Let $\boldsymbol{\alpha} \in \mathbb{N}^d$. A *unitary compositional triple* of $\boldsymbol{\alpha}$ into k parts is an ordered triple $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ where

1. $\boldsymbol{x} = (\boldsymbol{0}_k, \boldsymbol{z}(\boldsymbol{\alpha}))$ and $\boldsymbol{y} = (\boldsymbol{z}(\boldsymbol{\alpha}), (\boldsymbol{d}+\boldsymbol{1})_k)$, where the comma in $\boldsymbol{x}$ and $\boldsymbol{y}$ indicates the concatenation of the two relevant vectors.

2. The elements of $\boldsymbol{a}$ satisfy $1 \leq a_i \leq d$.

3. $\boldsymbol{a} + \boldsymbol{x} \sim \boldsymbol{y}$

Note that conditions 1 and 3 imply that each of the vectors that comprise $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ must have length $m = \|\boldsymbol{\alpha}\| + k$.

(Be sure to note the nomenclature: a **unitary composition** is defined in Definition 3.37 while a **unitary compositional triple** is from Definition 3.39.)

Using the vector $\boldsymbol{\alpha} = (2, 1)$ from the earlier example, we see that there are two unitary compositional triples into two parts both with $\boldsymbol{x} = (0, 0, 1, 1, 2)$ and $\boldsymbol{y} = (1, 1, 2, 3, 3)$. The two possible $\boldsymbol{a}$ vectors are $(1, 1, 1, 2, 1)$ and $(1, 1, 2, 1, 1)$.

There are three unitary compositional triples of $\boldsymbol{\alpha} = (2, 1)$ into three parts: $\boldsymbol{x} = (0, 0, 0, 1, 1, 2)$, $\boldsymbol{y} = (1, 1, 2, 3, 3, 3)$ and $\boldsymbol{a} = (1, 1, 2, 2, 2, 1)$ or $(1, 2, 1, 2, 2, 1)$ or $(2, 1, 1, 2, 2, 1)$.

We would like to show that a unitary compositional triple is equivalent to a unitary composition. We do this by defining a graph for the unitary compositional triple, in a similar way to that defined for juggling triples $(d, \boldsymbol{x}, \boldsymbol{a})$ earlier.

For the unitary compositional triple $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ we define the graph G. The vertex set of G is $\{0, 1, ..., d, d+1\}$ and the edge set is given by $E(G) = \{(x_j, x_j + a_j) : 1 \leq j \leq m\}$. Note that for $1 \leq i \leq d$, vertex i has in/outdegree of α_i, which is guaranteed by the construction of $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$. Vertex 0 has indegree 0 and outdegree k, while vertex $d + 1$ has indegree k and outdegree 0. So this graph can be viewed as a finite juggling graph on $d+2$ vertices, using k balls, all of which must be caught at least once before the edges all terminate at vertex $d+1$ as we cannot have any edge $(0, d+1)$ (this follows from the second criterion in Definition 3.39).

This graph may have multiple edges originating and terminating at any vertex, so several unitary compositional triples may correspond to the same graph. To specify a graph uniquely for each unitary compositional triple, we assign an ordering to the edges if there is more than one originating or terminating at a vertex.

When $x_j = i = x_k$, that is when two edges both leave vertex i, then the order is given by $(x_j, x_j + a_j) < (x_k, x_k + a_k)$ if $j < k$, otherwise $(x_k, x_k + a_k) < (x_j, x_j + a_j)$.

When $x_j + a_j = i = x_k + a_k$, that is when two edges both enter vertex i, then $(x_j, x_j + a_j) < (x_k, x_k + a_k)$ if $x_j > x_k$, or if $x_j = x_k$ and $j < k$.

By matching up the ordering of the edges leaving and entering the $d + 2$ vertices, we can arrive at a unique graph for each unitary juggling triple. This results in a unique decomposition of the graph into k paths $P_1, P_2, ..., P_k$. Now define the characteristic vector $\boldsymbol{\chi}_i = (\chi_{i,1}, \chi_{i,2}, ..., \chi_{i,d})$ for path P_i. Define $\boldsymbol{\chi}_i$ only on the vertex set $(1, 2, ..., d)$, that is we are ignoring the path P_i at vertices 0 and $d + 1$, by

$$\chi_{i,j} = \begin{cases} 1 & \text{if path } P_i \text{ is incident with vertex } j \\ 0 & \text{if path } P_i \text{ is not incident with vertex } j \end{cases}$$

So we can think of $\boldsymbol{\chi}_i$ as recording the times that ball i is caught. Clearly $\boldsymbol{\chi}_1 + \boldsymbol{\chi}_2 + ... + \boldsymbol{\chi}_k = \boldsymbol{\alpha}$ since this sum gives us the total number of balls being caught on each beat between 1 and d (inclusive), which is exactly $\boldsymbol{\alpha}$ (recall that the index set of $\boldsymbol{\alpha}$ begins at 1 and ends at d).

So now we can see that the vectors $\boldsymbol{\chi}_i$ comprise a composition of $\boldsymbol{\alpha}$, in fact, they are a unitary composition.

Referring back to the earlier example, for the unitary compositional triple of $\boldsymbol{\alpha} = (2, 1)$ into two parts, $((0, 0, 1, 1, 2), (1, 1, 2, 3, 3), (1, 1, 1, 2, 1))$ has two paths:

$$\begin{array}{ll} P_1 & 0 \to 1 \to 2 \to 3 \\ P_2 & 0 \to 1 \to 3 \end{array}$$

See Figure 30 (a). For this unitary compositional triple we have $\boldsymbol{\chi}_1 = (1, 1)$ and $\boldsymbol{\chi}_2 = (1, 0)$, and so the corresponding unitary composition is $(1, 1) + (1, 0)$.

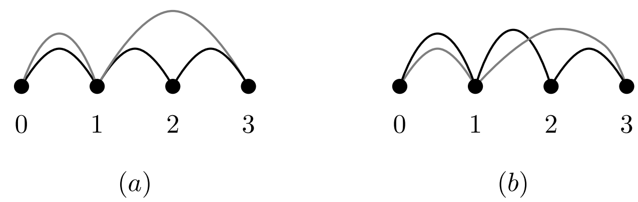

Figure 30: Graphs showing the relationship between unitary compositions and unitary compositional triples. Path P_i will land on vertex k if $\boldsymbol{\chi}_{i,k} = 1$. (a) $((0, 0, 1, 1, 2), (1, 1, 2, 3, 3), (1, 1, 1, 2, 1))$ and $(1, 1) + (1, 0)$, (b) $((0, 0, 1, 1, 2), (1, 1, 2, 3, 3), (1, 1, 2, 1, 1))$ and $(1, 0) + (1, 1)$.

For the other unitary compositional triple of $\boldsymbol{\alpha} = (2, 1)$ into two parts, $((0, 0, 1, 1, 2), (1, 1, 2, 3, 3), (1, 1, 2, 1, 1))$, we have $(1, 0) + (1, 1)$ (see Figure 30 (b)).

For equivalence, we also need to show that there is a unique unitary compositional triple for each unitary composition. Let $(r_1, r_2, ..., r_d) + (s_1, s_2, ..., s_d) + ... + (t_1, t_2, ..., t_d)$ be the unitary composition of $\boldsymbol{\alpha}$. Recall from Definition 3.39 that $\boldsymbol{x}$ and $\boldsymbol{y}$ only depend on $\boldsymbol{\alpha}$ and k, and so they are already uniquely determined. Explicitly: there must be k edges leaving from vertex 0, so we must have $\mathbf{0}_k$ as the first element in $\boldsymbol{x}$ and there must be k edges terminating at vertex $d + 1$ so $\boldsymbol{y}$ must have $(\boldsymbol{d} + \mathbf{1})_k$ in the second entry, and we can find $\boldsymbol{z}(\boldsymbol{\alpha})$ from Definition 3.38. So as long as $\boldsymbol{a}$ is unique then we have finished.

To show this, I will show that the composition uniquely determines a graph, from which $\boldsymbol{a}$ can be constructed. Draw the $d + 2$ vertices, $(0, 1, ..., d + 1)$ in a horizontal line. Now look at $(r_1, r_2, ..., r_d)$, the first term in the composition, and, if any entry r_i is 1, then draw a line, in colour R, from vertex i to vertex j, where $i < j$ and r_j is the next non-zero entry. This defines the path for all vertices $1 \leq i \leq d$. To complete the path, draw lines, still in colour R, from vertex 0 to vertex k, where r_k is the first non-zero entry, and from vertex m to vertex $d + 1$ where r_m is the last non-zero entry.

Repeat with each of the terms in the composition in a different colour, being careful to start each subsequent edge above the previous one as the unitary composition is an ordered sum. So the horizontal order in the sum corresponds to the vertical order in the graph; if term S is to the right of term T, then the path corresponding to term T will be above that corresponding to term S. Using the graph and $\boldsymbol{x}$ it is a simple matter to construct $\boldsymbol{a}$, which is unique. That is, we treat $\boldsymbol{x}$ as the base vector of a juggling triple. Then, we work from the left hand side of vector $\boldsymbol{x}$ and assign the entries of $\boldsymbol{a}$ according to the horizontal order described above.

The remarks above are summarised in the following lemma.

Lemma 3.40. *Let $\boldsymbol{\alpha}$ be a vector of non-negative integers. There is a bijection between the unitary compositional triples of $\boldsymbol{\alpha}$ and the unitary compositions of $\boldsymbol{\alpha}$.*

We can define crossings on the graph of the unitary compositional triple in an identical way to that of the juggling triples.

Definition 3.41. Let Γ be the graph of a unitary compositional triple. An *external crossing* of Γ is a pair of edges $((x,y),(u,v)) \in E(\Gamma)$ where $x < u < y < v$.

An *internal crossing* of Γ is a pair $(i,j) \in \mathbb{Z}^d$ such that $1 \leq i < j \leq m$, $x_i = x_j$ and $a_i > a_j$.

Denote the total number of crossings of a unitary compositional triple $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ by $cross(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$.

Definition 3.42. Let the *weight* of a unitary compositional $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ triple be $q^{cross(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})}$

This allows the definition of the q-analogue of $g_k(\boldsymbol{\alpha})$.

Definition 3.43.

$$g_k[\boldsymbol{\alpha}] = \sum_{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})} q^{cross(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})}$$

where the sum is over all unitary compositional triples of $\boldsymbol{\alpha}$ into k parts.

By looking at the diagrams above, we see that there are no crossings in the graph of $((0,0,1,1,2),(1,1,2,3,3),(1,1,1,2,1))$, but there is one internal crossing in the graph of $((0,0,1,1,2),(1,1,2,3,3),(1,1,2,1,1))$. So $g_2[(2,1)] = q+1$. Similarly, we can find that $g_3[(2,1)] = q^4+q^3+q^2$.

The q-analogue of the identity stated below Definition 3.37 can now be established.

Lemma 3.44.

$$g_k[\underbrace{(1,1,...,1)}_{n}] = [k]! \cdot S[n,k]$$

Proof: By Definition 3.43 the left hand side counts the number of crossings in the graphs of $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$, where the $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ are the unitary compositional triples of $\boldsymbol{\alpha} = \underbrace{(1,1,...,1)}_{n}$ into k parts. Since $\alpha_i = 1$ for all i, this means that $\boldsymbol{\alpha}$ partitions the set $(1,2,...,n)$ into k subsets. So by Lemma 3.35 the crossings in the graphs of the $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ are counted by $S[n,k]$. Since all the paths in all the graphs terminate at vertex $d+1$, it can be seen from the definition that there can be no crossings between vertices d and $d+1$. However, there can be some internal crossings between vertices 0 and 1. As in the proof of Theorem 3.36, the number of these crossings is the same as the number of inversions of a partition of k elements, so we have an extra factor of $[k]!$.

□

Definition 3.43 allows us to prove the following theorem, which was first proved by Haglund in his doctoral thesis [14].

Theorem 3.45.

$$\sum_{k=1}^{\|\boldsymbol{\alpha}\|} g_k[\boldsymbol{\alpha}] \cdot \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ \alpha_0 \end{bmatrix} \cdot \begin{bmatrix} n \\ \alpha_1 \end{bmatrix} \cdots \begin{bmatrix} n \\ \alpha_{d-1} \end{bmatrix}$$

Proof: The proof of this theorem is very similar to that of Lemma 3.35, using the same idea of contracting vertices of a juggling graph. We see from Theorem 3.31 that the right hand side counts the weights of all juggling triples of period d with at most n balls, having in/outdegree of α_i at time point i.

Now for the left hand side, contract every set of d consecutive vertices in the graphs corresponding to these juggling triples. We are left with a graph of a juggling pattern, of period d, where k balls are being caught on each beat. The crossings in this graph are counted by $\left[{n \atop k}\right]$. The crossings that vanish due to contraction will be counted by $g_k[\boldsymbol{\alpha}]$. This is true since, by the remarks following Definition 3.39, we can view the graphs of unitary compositional triples as finite juggling patterns on the vertex set $(0, 1, ..., d+1)$. On vertices $(1, 2, ..., d)$ these patterns correspond to the contracted vertices. There are however some extra crossings that may occur by edges that begin outside the interval but terminate inside it. These will show up as the internal crossings between vertices 0 and 1 in the graph of the unitary compositional triple.

The sum must terminate at $k = \|\boldsymbol{\alpha}\|$, since otherwise we violate condition 2 of Definition 3.39, that is we would have $a_i > d$ for all i such that $d+1 < i < k + \|\boldsymbol{\alpha}\| - (d+1)$.

$\square$

There is an immediate, obvious consequence of Theorem 3.45 which we state as a corollary.

Corollary 3.46. *The value of $g_k[\boldsymbol{\alpha}]$ is independent of the order of the entries of $\boldsymbol{\alpha}$.*

3.11 Juggling quadruples

We can generalise the juggling triples by removing the restriction that the indegree is equal to the outdegree of each vertex. In a juggling sense, this means that balls can be caught and not thrown again immediately.

Definition 3.47. Let d be a positive integer, $\boldsymbol{x} = (x_1, x_2, ..., x_m)$, $\boldsymbol{y} = (y_1, y_2, ..., y_m)$ be vectors of integers, and $\boldsymbol{a} = (a_1, a_2, ..., a_m)$ be a vector of positive integers. A *juggling quadruple* $(d, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ satisfies the following conditions:

- $0 \leq x_i, y_i \leq d-1$ for all $i = 1, 2, ..., m$
- $(\boldsymbol{x} + \boldsymbol{a}) \bmod d \sim \boldsymbol{y}$

Call d the *period*, $\boldsymbol{x}$ the *throw vector*, $\boldsymbol{y}$ the *catch vector*, and $\boldsymbol{a}$ the *juggling sequence.*

It can be seen that when $\boldsymbol{x} = \boldsymbol{y}$ a juggling quadruple is equivalent to a juggling triple as in Definition 3.21. We note that the number of balls is not well-defined for a juggling quadruple. However $\sum_{i=1}^{m} a_i$ is well-defined and will serve instead. Note that the following identity holds

$$\sum_{i=1}^{m} a_i \equiv \sum_{i=1}^{m} (y_i - x_i) \bmod d$$

To see that the number of balls is not defined, we need to look at the graph G of a juggling quadruple. The vertex set of G is $\mathbb{Z}$, and the edge set is given by

$$E(G) = \{(x_i + k \cdot d, x_i + a_i + k \cdot d) : 1 \leq i \leq m, k \in \mathbb{Z}\}$$

It can be seen in the graph contained in Figure 31 that, according to Definition 3.23, the number of balls is not defined; there are no clearly defined infinite, edge disjoint paths. If we do try to complete the graph such that the paths become infinite, then we run into a problem of non-uniqueness. For instance, in Figure 31, the obvious 'fix' to the problem is to draw two edges from vertices $2 + n \cdot d$ to $3 + n \cdot d$, we will then have a definite number of infinite, edge disjoint paths. There would be three balls/paths in this case. However, we could have equally drawn edges from vertices $2 + n \cdot d$ to $6 + n \cdot d$, in which case we would have five balls/paths, and so on. So the number of balls is clearly not well-defined.

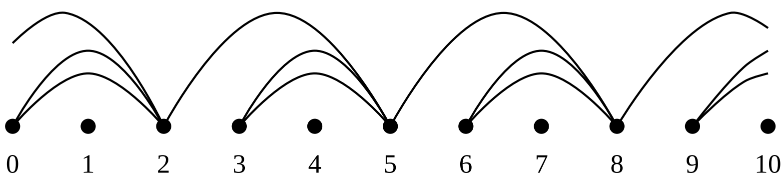

Figure 31: The graph of the juggling quadruple $(3,(0,0,2),(2,2,2),(2,2,3))$. Source [12].

Recall that earlier $\boldsymbol{\alpha}$ was the vector of in/outdegrees for a juggling triple. We need to be more specific for a juggling quadruple.

Definition 3.48. For a juggling quadruple, $(d, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ let $\boldsymbol{\alpha}$ be the *outdegree vector* and $\boldsymbol{\beta}$ be the *indegree vector.*

We define crossings in the same way as for juggling triples and unitary compositional triples.

Definition 3.49. Let G be the graph of a juggling quadruple $(d, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$. An *external crossing* of G is a pair of edges $((x,y),(u,v)) \in E(G)$ where $x < u < y < v$.

An *internal crossing* of G is a pair $(i,j) \in \mathbb{Z}^d$ such that $1 \leq i < j \leq m$, $x_i = x_j$ and $a_i > a_j$.

Denote the total number of crossings of a juggling quadruple $(d, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ by $cross(d, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$.

Let the *weight* of a juggling quadruple be defined by $q^{cross(d, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})}$.

This allows the following theorem, which is a generalisation of Theorem 3.31.

Theorem 3.50. *Let $(d, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ be a juggling triple such that $\sum_{i=1}^{m} a_i \leq N \equiv \sum_{i=1}^{m}(y_i - x_i) \bmod d$. Then*

$$\sum_{(d, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})} q^{cross(d, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})} = \begin{bmatrix} n_0 \\ \beta_0 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ \beta_1 \end{bmatrix} \cdots \begin{bmatrix} n_{d-1} \\ \beta_{d-1} \end{bmatrix}$$

where the sum is over all juggling quadruples and $n_0, n_1, ..., n_{d-1}$ is the (unique) solution to the system of equations

$$N = n_0 + n_1 + ... + n_{d-1}$$
$$n_{i+1} = n_i - \beta_i + \alpha_i, \quad i = 0, 1, ..., d-1$$

where the indices are taken mod d.

Proof: The method of the proof is the same as that for Theorem 3.31. First note that we can interpret n_{i+1} as the number of balls in the air between time points i and $i+1$, that is n_{i+1} is the total number of edges that exist during the time between i and $i+1$. For instance, in the diagram above, $n_6 = 3$ and $n_8 = 1$. As in Theorem 3.31 we use juggling cards, except now, for each time point i, we want a deck that will show us all possible ways to catch β_i balls out of the n_i balls that were in the air immediately before the time point i. Call this deck ${}^{n_i}D_{\beta_i}$ Note that there will be no crossings created by throwing the α_i balls; all the crossings will come from catching the β_i balls, and so it makes no difference which values of α_i we have in our deck. As in Theorem 3.31 we think of the balls being caught as 'x's and the balls not being caught as 'y's. On the right hand side of each card, the 'x's are grouped at the bottom and the 'y's are grouped at the top, while on the left they are in no particular order. If we rearrange the order on the left hand side to match that on the right side, using the relation $yx = qxy$, then from the combinatorial interpretation of the Gaussian coefficient, the sum of the weights is $\begin{bmatrix} n_i \\ \beta_i \end{bmatrix}$ for the ith card. So for a juggling quadruple whose graph has indegree vector β, with n_i balls in the air immediately before time point i, we use deck ${}^{n_i}D_{\beta_i}$, and we have the result.

□

As we generalised the concept of a juggling triple in Definition 3.47, we can also generalise the idea of a unitary compositional triple.

Definition 3.51. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Z}$, such that $\|\boldsymbol{\alpha}\| = \|\boldsymbol{\beta}\|$. A *generalised unitary compositional triple* of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ where

1. $\boldsymbol{x} = (\boldsymbol{0}_k, \boldsymbol{z}(\boldsymbol{\alpha}))$ and $\boldsymbol{y} = (\boldsymbol{z}(\boldsymbol{\beta}), (\boldsymbol{d}+\boldsymbol{1})_k)$, where the comma in $\boldsymbol{x}$ and $\boldsymbol{y}$ indicates the concatenation of the two relevant vectors.

2. The elements of $\boldsymbol{a}$ satisfy $0 < a_i \leq d$.

3. $\boldsymbol{a} + \boldsymbol{x} \sim \boldsymbol{y}$

Let $c_k(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be the number of generalised unitary compositional triples of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ into k parts.

Note that $\|\boldsymbol{\alpha}\| = \|\boldsymbol{\beta}\|$ must be true for this definition to make sense; it will be impossible to satisfy Condition 3 if it were not true. In a juggling context, it means that the total number of balls caught must equal the total number of balls thrown during the time for one period of the pattern.

A graph for these generalised unitary compositional triples can be defined in a similar way to the graphs of unitary compositional triples. Although there is, of course, no way to match the arcs through the whole pattern, as was done before, since now the arcs need not be continuous. That is, an arc may terminate at a beat and then appear to restart some beats later. So there is no need to bother with defining an order for the incoming and outgoing edges.

Definition 3.52. Let Γ be the graph of a generalised unitary compositional triple. An *external crossing* of Γ is a pair of edges $((x,y),(u,v)) \in E(\Gamma)$ where $x < u < y < v$.

An *internal crossing* of Γ is a pair $(i,j) \in \mathbb{Z}^d$ such that $1 \leq i < j \leq m$, $x_i = x_j$ and $a_i > a_j$.

Denote the total number of crossings of a generalised unitary compositional triple $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ by $cross(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$.

This allows the definition of the q-analogue of $c_k(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Definition 3.53. Define $c_k[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ by

$$c_k[\boldsymbol{\alpha}, \boldsymbol{\beta}] = \sum_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{a})} q^{cross(\boldsymbol{x},\boldsymbol{y},\boldsymbol{a})}$$

where the sum is over all generalised unitary compositional triples of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ into k parts.

So we can generalise Theorem 3.45.

Theorem 3.54.

$$\sum_{k=1}^{\|\alpha\|} c_k[\boldsymbol{\alpha}, \boldsymbol{\beta}] \cdot \begin{bmatrix} n_0 \\ k \end{bmatrix} = \begin{bmatrix} n_0 \\ \beta_0 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ \beta_1 \end{bmatrix} \cdots \begin{bmatrix} n_{d-1} \\ \beta_{d-1} \end{bmatrix}$$

Proof: The proof is the same as for Theorem 3.45. On the right hand side we see that we have counted the sum of the weights of all juggling quadruples that obey the conditions in Theorem 3.50. For the left hand side we contract d consecutive vertices and look at the crossings. We are left with a period 1 juggling pattern, of n_0 balls (recall that n_0 is the number of balls in the air immediately before time period 0). The crossings in this pattern are counted by $\left[\begin{smallmatrix} n_0 \\ \beta_i \end{smallmatrix}\right]$. The crossings that disappear are counted by $c_k[\boldsymbol{\alpha}, \boldsymbol{\beta}]$

□

3.12 Juggling multicards and $\boldsymbol{\alpha}$-non-increasing juggling triples

In his paper of 2002, Jonathan Stadler [25] develops these ideas to prove another of Haglund's identities involving vector compositions, and then further generalises the result. While Stadler's work is based on Ehrenborg and Readdy's ideas of juggling triples and juggling cards, he needs slightly different concepts for his proofs.

Definition 3.55. A *juggling multicard of length k* is a sequence of k juggling cards $\boldsymbol{C} = (C_1, C_2, ..., C_k)$. Let c_i be the number of crossings on card C_i, and $cross(\boldsymbol{C}) = \sum_{i=1}^{k} c_i$

Denote by $D_{n,(k)}$ the set of all multicards of length k with n balls such that $c_1 \geq c_2 \geq ... \geq c_k$.

Note that this notation varies slightly from that used earlier to discuss juggling cards. Previously, the index of each card counted the number of crossings on that card; now the index is a record of the card's position in the multicard.

Definition 3.56. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_t)$ be a sequence of positive integers. The *lth interval in $\boldsymbol{\alpha}$* is

$$int_{\boldsymbol{\alpha}} l = \{q : \alpha_1 + \alpha_2 + ... + \alpha_{l-1} \leq q \leq \alpha_1 + \alpha_2 + ... + \alpha_l - 1\}$$

For the case $l = 1$, let $\alpha_0 = 0$.

Note that this definition is effectively a sequential partition of $\{0, 1, 2, ..., (\sum_{i=1}^{t} \alpha_i) - 1\}$. The elements of the partition are $int_{\boldsymbol{\alpha}} l$ and $|int_{\boldsymbol{\alpha}} l| = \alpha_l$. For example, if $\boldsymbol{\alpha} = (2, 1, 4, 2)$ then

$$\begin{aligned}
&int_{\boldsymbol{\alpha}} 1 = \{q : 0 \leq q \leq \alpha_1 - 1\} = \{0, 1\} \\
&int_{\boldsymbol{\alpha}} 2 = \{q : \alpha_1 \leq q \leq \alpha_1 + \alpha_2 - 1\} = \{2\} \\
&int_{\boldsymbol{\alpha}} 3 = \{q : \alpha_1 + \alpha_2 \leq q \leq \alpha_1 + \alpha_2 + \alpha_3 - 1\} = \{3, 4, 5, 6\} \\
&int_{\boldsymbol{\alpha}} 4 = \{q : \alpha_1 + \alpha_2 + \alpha_3 \leq q \leq \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 1\} = \{7, 8\}
\end{aligned}$$

To continue, we use a specific type of juggling triple, which is effectively a simple juggling triple, as defined in Definitions 3.21 and 3.22, with the special property that the ball being caught must be 'below' the level of the ball caught on the previous beat. For example, if a ball which is three edges from the bottom is caught at time point i, then for all time points j such that $i < j \leq d-1$ no ball which is more than three edges from the bottom can be caught.

Definition 3.57. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_t)$ be a sequence of positive integers. An *$\boldsymbol{\alpha}$-non-increasing juggling triple* (or $\boldsymbol{\alpha}$-nijt, to borrow Stadler's convention) is a juggling triple $(d, \boldsymbol{x}, \boldsymbol{a})$ where $d = \sum_i \alpha_i := \|\boldsymbol{\alpha}\|$ and

1. $\boldsymbol{x} = \boldsymbol{\delta}_{d-1} = \boldsymbol{\delta}_{\|\boldsymbol{\alpha}\|-1}$
2. If both $(x_i + a_i) \bmod d$ and $(x_j + a_j) \bmod d$ belong to $int_{\boldsymbol{\alpha}} l$ and $(x_i + a_i) \bmod d < (x_j + a_j) \bmod d$, then $a_i \geq a_j$.

Note that Condition (1) is just the statement that we are only looking at simple patterns, that is only one ball is thrown on every beat. Condition (2) encapsulates the idea mentioned before Definition 3.57 that successive catches cannot be 'higher' than previous catches. The importance of this condition is that it implies that if the α_i vertices in the interval $int_{\boldsymbol{\alpha}} i$ are contracted together, then no internal crossings will appear in the resulting pattern. This is important for the proofs later. Condition (2) is easy to understand in the context of a juggling graph, however we need another equivalent condition. Let Condition $(2')$ be

$(2')$ If both $(x_i + a_i) \bmod d$ and $(x_j + a_j) \bmod d$ belong to $int_\alpha l$ and $(x_i + a_i) \bmod d < (x_j + a_j) \bmod d$, then $cross(x_i + a_i) \geq cross(x_j + a_j)$

where $cross(\epsilon_k)$ is the number of crossings *from the inside* of the edge $\epsilon_k = (x_k, x_k + a_k)$.

Theorem 3.58. *Condition* (2) *in* Definition 3.56 *is equivalent to condition* $(2')$.

Proof: $(2) \Rightarrow (2')$

First we note that there are only external crossings since the patterns are all simple. Take $(x_i + a_i) \bmod d$ and $(x_j + a_j) \bmod d$ both belonging to $int_{\boldsymbol{\alpha}} l$. Assume that $(x_i + a_i + 1) \bmod d = (x_j + a_j) \bmod d$, and so by Condition (2) $a_i \geq a_j$. If $a_j = 1$ there cannot be any other balls caught between the times that this ball was thrown and then caught again so $cross(x_j + a_j) = 0$ and clearly $cross(x_i + a_i) \geq cross(x_j + a_j)$. So assume that $a_j > 1$. Then the edge ϵ_j must cross the edge ϵ_i *from the inside*. If this were not true then the ball being caught at $(x_j + a_j)$ must have been thrown before the ball being caught at $(x_i + a_i)$ (see Figure 32). However, since $(x_i + a_i + 1) \bmod d = (x_j + a_j) \bmod d$, this implies that $a_i < a_j$ which violates Condition (2). Now we can claim that $cross(x_i + a_i) \geq cross(x_j + a_j)$. Every edge that crosses ϵ_j *from the inside* also crosses ϵ_i *from the inside*, except for the edge that originates at $(x_i + a_i)$. This edge will cross ϵ_j, but since ϵ_i is crossed by ϵ_j so far we have an equal number of crossings *from the inside* (see Figure 33). However, there may be more edges crossing ϵ_i *from the inside* than ϵ_j; this may be true if there is some ball that is thrown after ball i and before ball j (see Figure 34).

Finally we just need to show that the assumption $(x_i + a_i + 1) \bmod d = (x_j + a_j) \bmod d$ is merely a computational tool and makes no difference to the conclusion. Assume instead that we have $(x_i + a_i + s) \bmod d = (x_j + a_j) \bmod d$ where $s > 1$. Since we are dealing with simple juggling triples, for every time point between $(x_i + a_i) \bmod d$ and $(x_j + a_j) \bmod d$ we must have exactly one

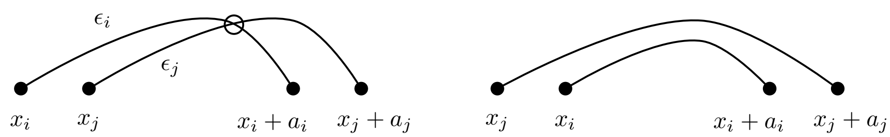

Figure 32: Graph showing that if $a_j > 1$ and ϵ_j does not cross ϵ_i *from the inside* then $a_i < a_j$.

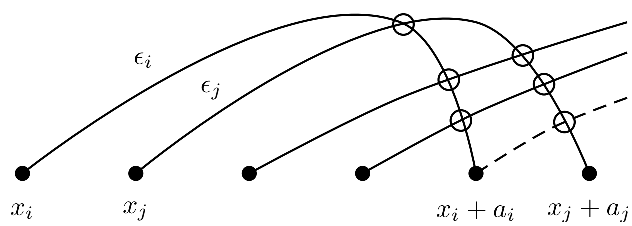

Figure 33: If ball j is thrown one beat after ball i then there are an equal number of crossings *from the inside*.

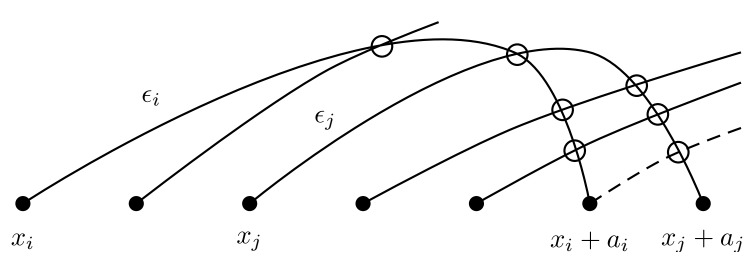

Figure 34: If there is more than one beat between the times that ball i and ball j were thrown then there may be more crossings of ϵ_i *from the inside* than ϵ_j.

edge terminating and another originating. However, any edge that we introduce may create a new crossing *from the inside* with ϵ_j. We have two options for the new edges: if the new edges cross ϵ_i *from the inside*, then we are done, since each of these edges (or a following one) must also cross ϵ_j *from the inside.* If a new edge does not cross ϵ_i *from the inside*, then that means that it did not originate at any of the vertices between x_i and $x_i + a_i$. But, since this is a simple juggling triple, there must have been another edge that did originate in this interval that does not terminate somewhere between $(x_i + a_i) \bmod d$ and $(x_j + a_j) \bmod d$, and so it will cross ϵ_i *from the inside* but not ϵ_j. Therefore, if we allow $s > 1$, there is no change to the relative number of crossings *from the inside* from the case $s = 1$.

$(2) \Leftarrow (2')$

For the converse take $cross(x_i + a_i) \geq cross(x_j + a_j)$. Again we can assume that $(x_i + a_i + 1) \bmod d = (x_j + a_j)$ for the same reasons as above. Now if $a_j = 1$ then we are finished since $a_i \in \mathbb{N}$. So take $a_j > 1$ in which case we know, from above, that ϵ_j must cross ϵ_i *from the inside* so $x_i < x_j$ and $a_i \geq a_j$.

□

We would like to prove a theorem using $\boldsymbol{\alpha}$-nijts that is analogous to Theorem 3.31, however we need an identity first.

Lemma 3.59.

$$\sum_{0 \leq c_k \leq c_{k-1} \leq \ldots \leq c_1 \leq m} q^{c_1 + c_2 + \ldots + c_k} = \begin{bmatrix} m+k \\ k \end{bmatrix}$$

Proof: First we want to change the inequalities of the sum. Let $c_l = d_l - (k - l)$. So

$$\begin{array}{lcl} c_k & = & d_k \\ c_{k-1} & = & d_{k-1} - 1 \\ & \vdots & \\ c_1 & = & d_1 - (k-1) \end{array}$$

Note that now we have the strict inequality $d_l < d_{l-1}$ for all l. Let $n = m + k - 1$ and the sum becomes

$$\begin{aligned} &\sum_{0 \leq d_k < d_{k-1} < \ldots < d_1 \leq n} q^{d_1 - (k-1) + d_2 - (k-2) + \ldots + d_{k-1} - 1 + d_k} \\ &= q^{-\frac{1}{2}k(k-1)} \sum_{0 \leq d_k < d_{k-1} < \ldots < d_1 \leq n} q^{d_1 + d_2 + \ldots + d_{k-1} + d_k} \\ &= q^{-\frac{1}{2}k(k-1)} F(k,n) \end{aligned}$$

where the sum in denoted by the function $F(k, n)$.

Let $G(k, n) = \left[\begin{smallmatrix} m+k \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} n+1 \\ k \end{smallmatrix}\right]$. Now we want to show that $G(k, n) = q^{-\frac{1}{2}k(k-1)} F(k, n)$. We do this by showing that $G(k, n)$ satisfies the recurrence from the proof of Theorem 3.19:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} q^k + \begin{bmatrix} n \\ k-1 \end{bmatrix}$$

Using a method commonly used for deriving a recurrence for the binomial function, we develop a recurrence for $F(k, n)$ by examining two possible cases.

Case 1: Assume $d_k = 0$, and recall that $d_{k-1} > d_k = 0$ in which case we have

$$\sum_{1<d_{k-1}<...<d_1\leq n} q^{d_1+d_2+...+d_{k-1}}$$

Now shift the sum by letting $d_l \mapsto d_l + 1$ and we have

$$q^{k-1} \sum_{0\leq d_{k-1}<...<d_1\leq n-1} q^{d_1+d_2+...+d_{k-1}} = q^{k-1}F(k-1, n-1)$$

Case 2: Assume $d_k \geq 1$ and we have

$$\sum_{1\leq d_k<d_{k-1}<...<d_1\leq n} q^{d_1+d_2+...+d_{k-1}+d_k}$$

Again let $d_l \mapsto d_l + 1$.

$$q^k \sum_{0\leq d_k<d_{k-1}<...<d_1\leq n-1} q^{d_1+d_2+...+d_{k-1}+d_k} = q^k F(k, n-1)$$

So $F(k, n) = q^{k-1}F(k-1, n-1) + q^k F(k, n-1)$

$$\begin{aligned}
\Rightarrow G(k,n) &= q^{-\frac{1}{2}k(k-1)}F(k,n) \\
&= q^{(k-1)-\frac{1}{2}k(k-1)}F(k-1,n-1) + q^{k-\frac{1}{2}k(k-1)}F(k,n-1) \\
&= q^{\frac{1}{2}k(k-1)(k-2)}F(k-1,n-1) + q^k q^{-\frac{1}{2}k(k-1)}F(k,n-1) \\
&= G(k-1,n-1) + q^k G(k,n-1)
\end{aligned}$$

and we are done.

□

Theorem 3.60. *Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_t)$ be a sequence of positive integers. Then*

$$\sum_{(d,\boldsymbol{x},\boldsymbol{a})} q^{cross(d,\boldsymbol{x},\boldsymbol{a})} = \begin{bmatrix} n+\alpha_1-1 \\ \alpha_1 \end{bmatrix} \begin{bmatrix} n+\alpha_2-1 \\ \alpha_1 \end{bmatrix} \cdots \begin{bmatrix} n+\alpha_t-1 \\ \alpha_t \end{bmatrix}$$

where the sum is over all $\boldsymbol{\alpha}$-nijts of at most n balls.

Proof: The method of this proof is the same as that for Theorem 3.31: We calculate the weight of multicards in a deck $D_{n,(k)}$ and then show a bijection between sequences of multicards and $\boldsymbol{\alpha}$-nijts (note that this theorem requires only the map from $\boldsymbol{\alpha}$-nijts to multicards to be surjective, however a bijection will be useful later).

For an arbitrary multicard $\boldsymbol{C}$ we know that $cross(\boldsymbol{C}) = c_1+c_2+...+c_k$, since the crossings of the multicard are just the crossings on the individual juggling cards that make up the multicard. Then, by Lemma 3.59

$$\sum_{C\in D_{n,(k)}} q^{cross(C)} = \sum_{n-1\geq c_1\geq c_2\geq ...\geq c_k\geq 0} q^{c_1+c_2+...+c_k} = \begin{bmatrix} n-1+k \\ k \end{bmatrix}$$

So if the graph of an $\boldsymbol{\alpha}$-nijt can be constructed from a sequence of t multicards, then the sum of the weights of these $\boldsymbol{\alpha}$-nijts is

$$\begin{bmatrix} n+\alpha_1-1 \\ \alpha_1 \end{bmatrix} \begin{bmatrix} n+\alpha_2-1 \\ \alpha_1 \end{bmatrix} \cdots \begin{bmatrix} n+\alpha_t-1 \\ \alpha_t \end{bmatrix}$$

All that remains is to show the bijection. Take an $\boldsymbol{\alpha}$-nijt $(d, \boldsymbol{x}, \boldsymbol{a})$. From Theorem 3.58 all $\boldsymbol{\alpha}$-nijts have property $(2')$ which is equivalent to the $c_1 \geq c_2 \geq ... \geq c_k$ condition for the multicards. So, on the interval $int_{\boldsymbol{\alpha}} l$, we can construct the graph of $(d, \boldsymbol{x}, \boldsymbol{a})$ from the deck $D_{n,(\alpha_l)}$.

For the converse suppose we have a sequence $\boldsymbol{C}_i$ of juggling cards on $int_{\boldsymbol{\alpha}} i$, $i = 1, 2, ..., t$, where $\boldsymbol{C}_i$ comes from the deck $D_{n,(\boldsymbol{\alpha}_i)}$, then we have an $\boldsymbol{\alpha}$-nijt since $c_1 \geq c_2 \geq ... \geq c_k$ and Condition $(2')$ is equivalent to Condition (2). The vector $\boldsymbol{x} = \boldsymbol{\delta}_{d-1}$ and $d = \|\boldsymbol{\alpha}\|$ are determined by Definition 3.57.

□

3.13 Vector compositions

Theorem 3.60 is useful for proving an identity involving vector compositions. Earlier we restricted ourselves to *unitary* compositions, now we investigate general vector compositions (see Definition 3.37 for the definition of vector compositions). Following the earlier method, we define a *compositional triple* (cf. the *unitary* compositional triple), and then show that this is equivalent to a vector composition. The compositional triple is then used to prove the identity contained in Theorem 3.65, which involves vector compositions.

Definition 3.61. Let $\boldsymbol{w}(\boldsymbol{\alpha}) = (1, 2, ..., \|\boldsymbol{\alpha}\|)$. A *compositional triple* of a vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_d)$ into k parts is an ordered triple $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ where

1. $\boldsymbol{x} = (\boldsymbol{0}_k, \boldsymbol{w}(\boldsymbol{\alpha}))$ and $\boldsymbol{y} = (\boldsymbol{w}(\boldsymbol{\alpha}), (\|\boldsymbol{\alpha}\| + \boldsymbol{1})_k)$, where the comma in $\boldsymbol{x}$ and $\boldsymbol{y}$ indicates the concatenation of the two relevant vectors.
2. The elements of $\boldsymbol{a}$ satisfy $1 \leq a_i \leq \|\boldsymbol{\alpha}\|$.
3. $\boldsymbol{a} + \boldsymbol{x} \sim \boldsymbol{y}$
4. If $\alpha_1 + \alpha_2 + ... + \alpha_l + 1 \leq a_i + x_i < a_j + x_j \leq \alpha_1 + \alpha_2 + ... + \alpha_{l+1}$ then $a_i \geq a_j$.

By constructing a graph of a vector composition we can show the equivalence to compositional triples. If $\boldsymbol{v}_1 + \boldsymbol{v}_2 + ... + \boldsymbol{v}_k = \boldsymbol{\alpha}$ is a composition of $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_t)$ into k parts, then we can construct a juggling graph with k balls on the vertex set $(0, 1, ..., \|\boldsymbol{\alpha}\|)$. This graph is best explained through the use of an example, contained in Figure 35. For our example, let $\boldsymbol{\alpha} = (7, 8)$, $k = 4$ and $\boldsymbol{v}_1 = (3, 1), \boldsymbol{v}_2 = (1, 4), \boldsymbol{v}_3 = (2, 3), \boldsymbol{v} = (1, 0)$. We use paths P_1,P_2,...,P_k to represent the k balls, with P_1 corresponding to $\boldsymbol{v}_1$, P_2 corresponding to $\boldsymbol{v}_2$ and so on. All the balls are thrown on beat 0 and they are all caught on beat $\|\boldsymbol{\alpha}\| + 1$, but for all other vertices we have exactly one path incident with them. The paths leave vertex 0 in ascending order, that is P_1 is the lowest path, P_2 is the second lowest path and so on, with P_k being the highest path, in our example this will be P_4 (Figure 35 (i)).

Now over the next α_1 vertices, we want path P_i to be incident with $v_{i,1}$ vertices where $v_{i,1}$ is the first entry in vector $\boldsymbol{v}_1$. Note that the paths enter the first block of α_1 vertices in the order $P_k, P_{k-1}, ..., P_2, P_1$ from the top, which corresponds to the permutation $(k, k-1, ..., 2, 1)$ of $(1, 2, ..., k)$. Call this permutation σ^0 since this was the order established by the way the paths left vertex 0, which we defined. We take all vectors such that the first entry is non-zero, and we let them intersect with vertices in the order σ^0, ie. in descending order. So if vectors $\boldsymbol{v}_i$ and $\boldsymbol{v}_{i+k}$ $(k > 0)$ both have non-zero first entries, then path P_{i+k} will intersect with a vertex before path P_i does (Figure 35 (ii)).

Then we take all vectors that have first entry ≥ 2 and we let their corresponding paths intersect with vertices in descending order (Figure 35 (iii)).

We continue in this way until we have filled the α_1 vertices following vertex 0 (Figure 35 (iv)).

Now define a new permutation σ^1 by arranging the vectors $\boldsymbol{v}_i$ so that their first entries form a non-decreasing sequence. If two vectors have the same first entry then order them according to σ^0. In our example, the non-decreasing sequence of vectors is $\boldsymbol{v}_4, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_1$, giving $\sigma^1 = (4, 2, 3, 1)$. Both $\boldsymbol{v}_2$ and $\boldsymbol{v}_4$ have first entry 1, but 4 came before 2 in σ^0 so 4 comes before 2 in σ^1 as well. Note that this vertical ordering matches the order of the paths after the first α_1 vertices.

To fill the next α_2 vertices, we repeat the same procedure as above, this time using the second entry of the vectors $\boldsymbol{v}_i$, and the order that the paths intersect with the vertices is given by σ^1 (Figure 35 (v)). We determine the permutation σ^2 by ordering the vectors $\boldsymbol{v}_i$ according to their second entries. In our example, the sequence of vectors is $\boldsymbol{v}_4, \boldsymbol{v}_1, \boldsymbol{v}_3, \boldsymbol{v}_2$ and so $\sigma^2 = (4, 1, 3, 2)$. In our example, we have only two entries of $\boldsymbol{\alpha}$ and so to complete the graph, we just connect all the paths to vertex $\|\boldsymbol{\alpha}\| + 1$ with the order, from highest to lowest, given by σ^2 (Figure 35 (vi)).

In the general case, we continue this process, filling up the vertex set $V = \{(\alpha_1 + \alpha_2 + ... + \alpha_{j-1} + 1), ..., (\alpha_1 + \alpha_2 + ... + \alpha_j)\}$ by letting paths P_i intersect with $v_{i,j}$ vertices according to the permutation σ^{j-1}. As in our example, once vertices $0, 1, ..., \|\alpha\|$ have been filled (that is when we have repeated the process t times, where t is the number of entries of $\boldsymbol{\alpha}$), then let all the paths intersect with vertex $\|\alpha\| + 1$ where, from highest to lowest, we order the paths according to σ^t

From this graph we can read off the unique $\boldsymbol{a}$ vector of the compositional triple. We do this by treating the graph as a finite juggling graph. The first ball thrown will land some number of beats later, say b_1. So the first entry of $\boldsymbol{a}$ is b_1. We continue this to identify the $\|\boldsymbol{\alpha}\|$ entries of $\boldsymbol{a}$ and so there is a unique compositional triple for each vector composition. In the example we have been using $\boldsymbol{a} = (4, 3, 2, 1, 15, 3, 5, 2, 4, 1, 3, 3, 3, 6, 2, 2, 2, 1)$.

Conversely, we could construct a graph of the compositional triple according to the method for graphing unitary compositional triples described after Definition 3.39, and then identify the corresponding vector composition. We identify the k paths in the graph and assign the vectors $\boldsymbol{v}_i$ to paths P_i by letting the first (lowest) path originating at vertex 0 correspond to vector $\boldsymbol{v}_1$, the second path correspond to vector $\boldsymbol{v}_2$ and so on until the topmost path is assigned to vector $\boldsymbol{v}_k$. To establish the first coordinate in each of the vectors, we count the number of times that each path intersects with a vertex during the first α_1 vertices. The second coordinate is the number of times a path is incident with a vertex during the next α_2 beats, and so on. In this way we will find a unique vector composition of $\boldsymbol{\alpha}$ for each compositional triple.

The remarks above are summarised by the following lemma.

Lemma 3.62. *Let $\boldsymbol{\alpha}$ be a vector of positive integers. Every vector composition of $\boldsymbol{\alpha}$ into k parts is equivalent to a compositional triple of $\boldsymbol{\alpha}$ into k parts.*

Definition 3.63. Let $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ be a compositional triple. Define the number of crossings, $cross(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$, as the number of crossings of the graph of this compositional triple.

Recall from Definition 3.37 that the number of compositions of $\boldsymbol{\alpha}$ into k parts is $f_k(\boldsymbol{\alpha})$. By the equivalence in Lemma 3.62, $f_k(\boldsymbol{\alpha})$ is also the number of

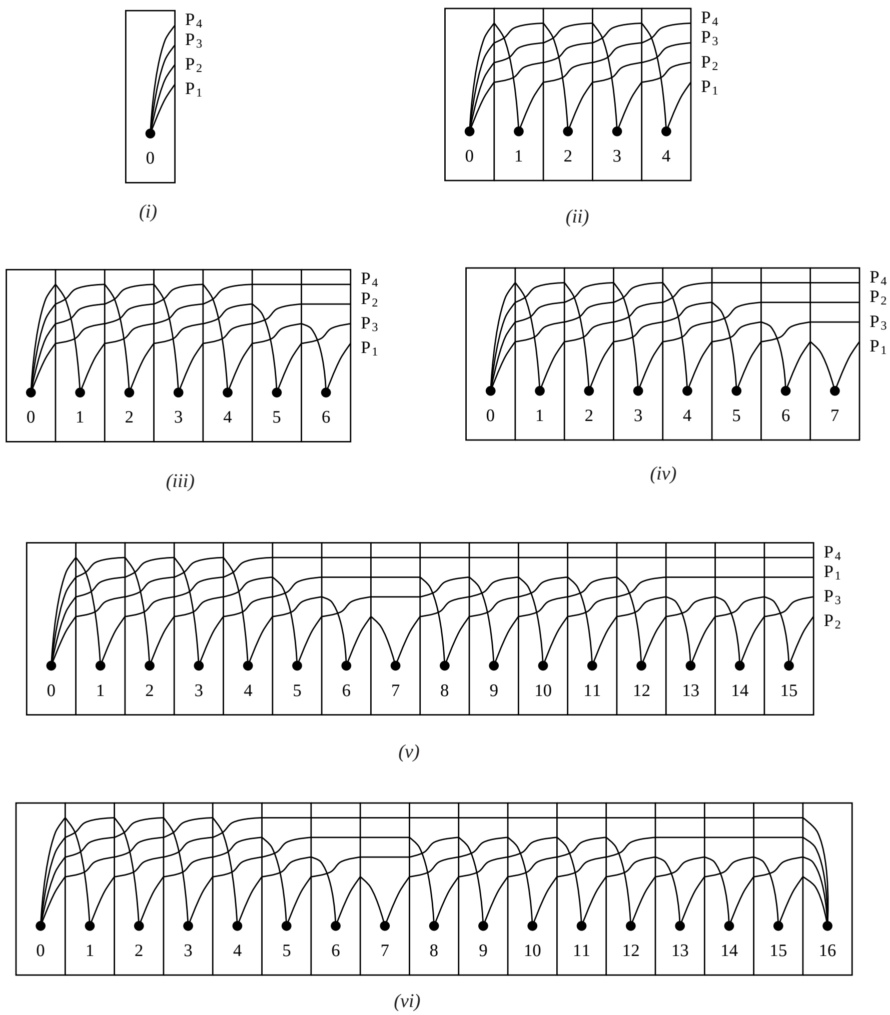

Figure 35: Construction of the graph of the vector composition
$\boldsymbol{\alpha} = (7, 8) = (3, 1) + (1, 4) + (2, 3) + (1, 0)$, expressed as juggling multicards.

compositional triples of $\boldsymbol{\alpha}$ into k parts. We can now define the q-analogue of $f_k(\boldsymbol{\alpha})$.

Definition 3.64. Define $f_k[\boldsymbol{\alpha}]$ by

$$f_k[\boldsymbol{\alpha}] = \sum_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{a})} q^{cross(\boldsymbol{x},\boldsymbol{y},\boldsymbol{a})}$$

where the sum is over all compositional triples of $\boldsymbol{\alpha}$ into k parts.

We can now provide a proof of another of Haglund's identities.

Theorem 3.65. *Let $\boldsymbol{\alpha}$ be a sequence of positive integers. Then*

$$\sum_{k=0}^{\|\boldsymbol{\alpha}\|} f_k[\boldsymbol{\alpha}] \cdot \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n+\alpha_1-1 \\ \alpha_1 \end{bmatrix} \cdot \begin{bmatrix} n+\alpha_2-1 \\ \alpha_2 \end{bmatrix} \cdots \begin{bmatrix} n+\alpha_d-1 \\ \alpha_d \end{bmatrix}$$

Proof: From Theorem 3.60 the right hand side counts the crossings in all $\boldsymbol{\alpha}$-nijts with at most n balls. For the left hand side, we follow the same method as used earlier: we contract $\|\boldsymbol{\alpha}\|$ vertices together leaving a multiplex pattern of period 1 and k balls. The crossings in these patterns are counted by $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ by Theorem 3.31. Since we have a bijection between $\boldsymbol{\alpha}$-nijts and compositional triples, and between compositional triples and vector compositions, the crossings that disappear on contraction are counted by $f_k[\boldsymbol{\alpha}]$.

□

3.14 General vector compositions

We now generalise these results to more general vector compositions. First we need to generalise the idea of an $\boldsymbol{\alpha}$-nijt.

Definition 3.66. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_t)$ be a sequence of positive integers, and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_t)$ a sequence of non-negative integers. An $(\boldsymbol{\alpha};\boldsymbol{\gamma})$-*non-increasing juggling triple* is a juggling triple $(d, \boldsymbol{x}, \boldsymbol{a})$ where

1. $(d, \boldsymbol{x}, \boldsymbol{a})$ is an $\boldsymbol{\alpha}$-nijt.
2. During the interval $int_{\boldsymbol{\alpha}} l$ the γ_l balls that have been in the air the longest at the beginning of the interval (that is, at time $\alpha_1 + \alpha_2 + ... + \alpha_{l-1}$), are not incident with any vertices.

As before, we call this new juggling triple an $(\boldsymbol{\alpha};\boldsymbol{\gamma})$-nijt for convenience. If $\gamma_i = 0$ for all $i = 1, 2, ..., t$ then the $(\boldsymbol{\alpha};\boldsymbol{\gamma})$-nijt reduces to an $\boldsymbol{\alpha}$-nijt. Note that, in terms of juggling multicards, Condition 2 says that the γ_l highest paths at the start of a multicard are not caught anywhere inside that multicard. So during $int_{\boldsymbol{\alpha}} l$ only the lower $n - \gamma_l$ balls may be caught, the γ_l balls above this appear as straight lines across the multicard. This means that we are using a set of multicards $C \in D_{n,(k)}$ such that $n - \gamma_l - 1 \geq c_1 \geq c_2 \geq ... \geq c_k$. Call this subdeck $D^{\gamma_l}_{n,(k)}$.

Theorem 3.67. *Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_t)$ be a sequence of positive integers and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_t)$ a sequence of non-negative integers. Then*

$$\sum_{(d,\boldsymbol{x},\boldsymbol{a})} q^{cross(d,\boldsymbol{x},\boldsymbol{a})} = \begin{bmatrix} n-\gamma_1+\alpha_1-1 \\ \alpha_1 \end{bmatrix} \begin{bmatrix} n-\gamma_2+\alpha_2-1 \\ \alpha_1 \end{bmatrix} \cdots \begin{bmatrix} n-\gamma_t+\alpha_t-1 \\ \alpha_t \end{bmatrix}$$

where the sum is over all $(\boldsymbol{\alpha};\boldsymbol{\gamma})$-nijts of at most n balls.

Proof: The proof is almost identical to that used for Theorem 3.60, except that this time we use the deck $D^{\gamma_l}_{n,(\alpha_l)}$ at time l. By combining the comments preceeding the statement of this theorem with Lemma 3.59 we see that

$$\sum_{C \in D^{\gamma_l}_{n,(k)}} q^{cross(C_l)} = \sum_{n-\gamma_l-1 \geq c_1 \geq c_2 \geq \ldots \geq c_k \geq 0} q^{c_1+c_2+\ldots+c_k} = \begin{bmatrix} n-\gamma_l+k-1 \\ k \end{bmatrix}$$

Since these $(\boldsymbol{\alpha};\boldsymbol{\gamma})$-nijts correspond to sequences of multicards drawn from the decks $D^{\gamma_i}_{n,(k)}$, we have the result.

□

So with this new juggling triple we can generalise the vector compositions.

Definition 3.68. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_t)$ be a sequence of positive integers and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_t)$ a sequence of non-negative integers. A *composition of* $\boldsymbol{\alpha}$ *mod* $\boldsymbol{\gamma}$ (written $\boldsymbol{\alpha}/\boldsymbol{\gamma}$) *into* k *parts* is a sequence of vectors $(\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_3)$ such that

1. $\boldsymbol{v}_1 + \boldsymbol{v}_2 + ... + \boldsymbol{v}_3 = \boldsymbol{\alpha}$ is a composition of $\boldsymbol{\alpha}$.
2. $v_{\sigma^{i-1}_j, i} = 0$ for $1 \leq j \leq \gamma_i$ and $1 \leq i \leq t$.

Let $f_k(\boldsymbol{\alpha}/\boldsymbol{\gamma})$ be the number of compositions of $\boldsymbol{\alpha}/\boldsymbol{\gamma}$ into k parts.

Recall that σ^{i-1} gives the order of the paths as they enter multicard $\boldsymbol{C}_i$, so Condition (2) will mean that the highest γ_i paths are not caught on $\boldsymbol{C}_i$, that is, they have no intersections with vertices over that multicard.

Since these compositions $\boldsymbol{\alpha}/\boldsymbol{\gamma}$ are still vector compositions, they have a corresponding compositional triple. We can graph them by the method described immediately before Lemma 3.62. So we can make the following definition.

Definition 3.69. Define the q-analogue of $f_k(\boldsymbol{\alpha}/\boldsymbol{\gamma})$ by

$$f_k[\boldsymbol{\alpha}/\boldsymbol{\gamma}] = \sum_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{a})} q^{cross(\boldsymbol{x},\boldsymbol{y},\boldsymbol{a})}$$

where the sum is over all compositional triples $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ which are equivalent to the compositions of $\boldsymbol{\alpha}/\boldsymbol{\gamma}$ into k parts.

Theorem 3.70. *Let* $\boldsymbol{\alpha}$ *be a sequence of positive integers, and* $\boldsymbol{\gamma}$ *a sequence of non-negative integers. Then*

$$\sum_{k=0}^{\|\boldsymbol{\alpha}\|} f_k[\boldsymbol{\alpha}/\boldsymbol{\gamma}] \cdot \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-\gamma_1+\alpha_1-1 \\ \alpha_1 \end{bmatrix} \begin{bmatrix} n-\gamma_2+\alpha_2-1 \\ \alpha_1 \end{bmatrix} \cdots \begin{bmatrix} n-\gamma_t+\alpha_t-1 \\ \alpha_t \end{bmatrix}$$

Proof: By following the same method of proof as Theorem 3.65 (only now with $(\boldsymbol{\alpha};\boldsymbol{\gamma})$-nijts instead of $\boldsymbol{\alpha}$-nijts) and using Theorem 3.67 we arrive at the result.

□

Using this $\boldsymbol{\alpha}/\boldsymbol{\gamma}$ structure we can also generalise Theorem 3.45. Though we first need to generalise Theorem 3.31.

Theorem 3.71.

$$\sum_{(d,\boldsymbol{x},\boldsymbol{a})} q^{(d,\boldsymbol{x},\boldsymbol{a})} = \begin{bmatrix} n-\gamma_1 \\ \alpha_1 \end{bmatrix} \begin{bmatrix} n-\gamma_2 \\ \alpha_2 \end{bmatrix} \cdots \begin{bmatrix} n-\gamma_t \\ \alpha_t \end{bmatrix}$$

where α_i *is the in/out-degree of time point* i*, and where the sum is over all juggling triples with period* t *and at most* n *balls, such that the* γ_i *highest balls in the associated graph are not caught at time* i mod t*.*

Proof: The proof is an easy adaptation of the proof of Theorem 3.31, only this time using the subset of deck D_{α_i} where the highest γ_i balls are not caught.

□

Definition 3.72. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_t)$ be a sequence of positive integers and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_t)$ a sequence of non-negative integers. A *unitary composition of* $\boldsymbol{\alpha}$ *mod* $\boldsymbol{\gamma}$ *into* k *parts* is a sequence of vectors $(\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_3)$ such that

1. $\boldsymbol{v}_1 + \boldsymbol{v}_2 + ... + \boldsymbol{v}_3 = \boldsymbol{\alpha}$ is a unitary composition of $\boldsymbol{\alpha}$ from Definition 3.37.
2. $v_{\sigma_j^{i-1},i} = 0$ for $1 \leq j \leq \gamma_i$ and $1 \leq i \leq t$.

Let $g_k(\boldsymbol{\alpha}/\boldsymbol{\gamma})$ be the number of unitary compositions of $\boldsymbol{\alpha}/\boldsymbol{\gamma}$ into k parts.

Note that while this is just a special case of Definition 3.68, we have kept it separate to highlight that it will be applied in a quite different manner. Also note that this definition is just a generalisation of Definition 3.39, and so these unitary compositions of $\boldsymbol{\alpha}/\boldsymbol{\gamma}$ into k parts, are also unitary compositions of $\boldsymbol{\alpha}$ into k parts. Recall that, by the remarks following Definition 3.39, we know a unitary compositions of $\boldsymbol{\alpha}$ into k parts is equivalent to a unitary compositional triple of $\boldsymbol{\alpha}$ into k parts, for which we have a method for creating a unique graph, with a defined number of crossings. This graph is related to juggling cards, and does not use multicards. So we can now see the difference between the use of Definitions 3.68 and 3.72: the graph of Definition 3.68 has the highest γ_i balls not being caught on *multicard* i (ie. during the period $int_{\boldsymbol{\alpha}} i$), whereas the graph of Definition 3.72 has the highest γ_i balls not being caught on *card* i (ie. at time point i).

Definition 3.73. Define the q-analogue of $g_k(\boldsymbol{\alpha}/\boldsymbol{\gamma})$ by

$$g_k[\boldsymbol{\alpha}/\boldsymbol{\gamma}] = \sum_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{a})} q^{cross(\boldsymbol{x},\boldsymbol{y},\boldsymbol{a})}$$

where the sum is over all unitary compositional triples $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ which are equivalent to the unitary compositions of $\boldsymbol{\alpha}/\boldsymbol{\gamma}$ into k parts.

Now we can state the generalisation of Theorem 3.45.

Theorem 3.74. *Let* $\boldsymbol{\alpha}$ *be a sequence of positive integers, and* $\boldsymbol{\gamma}$ *a sequence of non-negative integers. Then*

$$\sum_{k=0}^{\|\boldsymbol{\alpha}\|} g_k[\boldsymbol{\alpha}/\boldsymbol{\gamma}] \cdot \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-\gamma_1 \\ \alpha_1 \end{bmatrix} \begin{bmatrix} n-\gamma_2 \\ \alpha_1 \end{bmatrix} \cdots \begin{bmatrix} n-\gamma_t \\ \alpha_t \end{bmatrix}$$

Proof: The proof is identical to that of Theorem 3.45: the right hand side is given by Theorem 3.71; for the left hand side we contract every set of t consecutive vertices, the crossings of the resulting graph are counted by $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$, while the crossings that disappear are counted by $g_k[\boldsymbol{\alpha}/\boldsymbol{\gamma}]$.

□

4 Concluding remarks

Mathematics and juggling have both been practiced throughout recorded history, and the origins of each are lost in the past. However, it was not until recently that the former was applied to the to the study of the latter, and even more recently that the latter has found application in solving the problems of the former.

Physical juggling can be described by well-chosen mathematical function and their resulting sequences. These descriptions have opened up new possibilities for jugglers, not least of which is the easy communication of juggling patterns. Another area where this abstraction has been applied is change ringing, a type of bell ringing that dates from the seventeenth century, where bells (often church bells) are rung in particular sequences.[4] It is not difficult to imagine that other sequential procedures could also be described by a similar mathematical treatment.

More importantly, however, is that juggling sequences have been shown to possess remarkable utility in the elucidation and solution of previously unrelated mathematical problems, leading to some straightforward and concise proofs. It was shown that there is a close connection between vector compositions and juggling patterns, expressed as juggling triples, quadruples or $\boldsymbol{\alpha}$-nijts. An interesting question remaining is to ask if there is a overall bijection that will relate juggling patterns and vector compositions.

The juggling concept may also be extended to examine affine Weyl groups other than $\tilde{A}_{d-1}$ (which was examined in this paper). It may also be possible to develop a completely group theoretic generalisation of juggling in the following way: given a group G and a normal subgroup $H \lhd G$, consider all permutations π of G such that $\pi(g \cdot h) = \pi(g) \cdot h$ for all $h \in H$. This may yield some interesting insights into group theory from a juggling viewpoint.

The juggling context presents us with a philosophy of applied mathematics that develops both the application and the mathematics itself, resulting in an enrichment of both worlds.

[4] This hobby still has some devoted practitioners; see [21].

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