University of Melbourne
Department of Mathematics and Statistics

High precision computation of some probability distributions in random matrix theory

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I would also like to thank all my Honours classmates and the staff members in the Mathematics department for their help throughout the year, especially Anne. She has been a moral support for me throughout the year in Honours and is always willing to help me with the coursework.

To my partner for fetching me home and being so understanding. Lastly, I would like to thank my Mum, sister (Ah Bebe) and my Dad for paying my university fees, without it I would not have such a wonderful experience.

Introduction: Summary of the research topic

Random matrix ensembles are typically defined by specifying a probability density function (p.d.f.) in terms of the matrices. A typical example is the ensemble of real symmetric matrices where a matrix, $M = M^T$ with p.d.f. proportional to $\exp(-\frac{1}{2}\mathrm{Tr}M^2)$. In applications, one is interested in the p.d.f. not of the matrices but rather their eigenvalues. For these, there are a number of known evaluations in terms of Painlevé transcendents.

Painlevé transcendents are specified as the solution of particular non-linear differential equations. To compute the p.d.f.'s and their moments to high precision, it is necessary to compute the Painlevé transcendents to high precision. Such high precision calculations are the topic of this thesis. In particular, the high precision evaluations of the Painlevé transcendents is obtained by performing a power series solution of the corresponding differential equation and these are used to give power series expansions of the corresponding p.d.f.'s, which also yield high precision evaluations.

The details of the high precision calculations are given in this thesis, as are the corresponding Mathematica programs. The theoretical results so obtained are compared against the empirical computation of eigenvalue p.d.f.'s for certain random matrix ensembles.

Chapter 1

1.1 Some different types of matrices

Real orthogonal matrices

An *orthogonal matrix* is a $N \times N$ square matrix R whose transpose is its inverse and so $R^T R = I_N$. An *orthogonal matrix* with real elements is known as a *real orthogonal matrix*. Taking the determinant of both sides of the defining equation and using the general properties of a determinant $\det R = \det R^T$, $\det AB = \det A \det B$ we see that $(\det R)^2 = 1$ and thus $\det R = 1$ or $\det R = -1$.

Real orthogonal matrices are special cases of *unitary matrices*, to be discussed subsequently. One general feature of *unitary matrices* is that the modulus of the eigenvalues equals 1, and so this must be true of *real orthogonal matrices*.

For an $N \times N$ *real orthogonal matrix* R with $\det R = 1$ and simple (i.e. non-degenerate) eigenvalues, it must be that the eigenvalues come in complex conjugate pairs $\exp(\pm i\theta_j)$ ($j = 1, \ldots, \frac{N}{2}$ and $0 < \theta_j < \pi$) for N even, while for N odd there will be an eigenvalue $\lambda = 1$, with the remaining eigenvalues coming in complex conjugate pairs. For this reason, *real orthogonal matrices* with odd rank ($(2N+1) \times (2N+1)$ matrices) should be considered separately from those with even rank ($(2N) \times (2N)$ matrices). In the case that $\det R = 1$, these matrices are denoted $O^+(2N+1)$ and $O^+(2N)$ respectively..

The set of all $(2N+1) \times (2N+1)$ *real orthogonal matrices* with determinant -1 is denoted $O^-(2N+1)$. For such matrices, there must exist one eigenvalue of $\lambda = -1$ and the remaining $2N$ eigenvalues in complex conjugate pair. The set of all $(2N) \times (2N)$ *real orthogonal matrices* with determinant -1 is denoted $O^-(2N)$ and such matrices have two eigenvalues $\lambda = \pm 1$ and the remaining $2N - 2$ eigenvalues in complex conjugate pairs

We remark that multiplication by (-1) takes a matrix from $O^+(2N+1)$ to a matrix $O^-(2N+1)$ and vice versa.

Hermitian matrix

A *Hermitian matrix* is a square matrix H whose conjugate transpose (also called the Hermitian adjoint) $(H^*)^T = H^t$ is equal to the matrix H, which means $(H^*)^T = H^t = H$.

There are two distinct classes of *Hermitian matrices*, depending on whether the elements are real or complex. According to the finite-dimensional spectral theorem, any *Hermitian matrix* can be diagonalised by a *unitary matrix* and the resulting diagonal matrix, which gives the eigenvalues, has only real elements. In general, all eigenvalues of *Hermitian matrices* are real and eigenvectors with distinct eigenvalues are orthogonal (see Appendix A).

Unitary matrices

A *unitary matrix* is a square matrix U with the property that the conjugate transpose $(U^*)^T U = U^t U = I_N$. As with *Hermitian matrices*, there are two distinct classes of unitary matrices of interest, depending on whether the elements are real or complex

Our interest is in the eigenvalues of these unitary matrices. Let $\mathbf{v}$ be a normalized eigenvector of U with corresponding eigenvalue λ, $U\mathbf{v} = \lambda\mathbf{v}$. Let (,) denote the inner product. We see that

$$\begin{aligned}(U\mathbf{v}, U\mathbf{v}) &= (\lambda\mathbf{v}, \lambda\mathbf{v}) \\ &= |\lambda|^2(\mathbf{v}, \mathbf{v}) \\ &= |\lambda|^2\end{aligned}$$

On the other hand

$$\begin{aligned}(U\mathbf{v}, U\mathbf{v}) &= (U^t U\mathbf{v}, \mathbf{v}) \\ &= (\mathbf{v}, \mathbf{v}) \\ &= 1\end{aligned}$$

So, in general, for a unitary matrix $|\lambda| = 1$. A general reference for the above theory is [6].

1.2 Definition of random matrix ensembles

Gaussian random matrices

Hermitian matrices with real entries are referred to as real symmetric matrices. They have the property that $X = X^T$.

Definition 1 *The Gaussian orthogonal ensemble (GOE) refers to random real symmetric $N \times N$ matrices X such that the diagonal and upper diagonal elements are independently chosen with probability density functions (p.d.f.'s) $\frac{1}{\sqrt{2\pi}} \exp \frac{-x_{jj}^2}{2}$ and $\frac{1}{\sqrt{\pi}} \exp -x_{jk}^2$ respectively.*

The p.d.f.'s in the GOE's definition are examples of the normal (Gaussian) distribution $\frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{-(x-\mu)^2}{2\sigma^2}$ which is to be denoted $\mathrm{N}[\mu, \sigma]$. In terms of this notation, the diagonal elements have p.d.f. $\mathrm{N}[0, 1]$ and the off diagonal elements p.d.f. $\mathrm{N}[0, \frac{1}{\sqrt{2}}]$. We note that the joint p.d.f. of all the independent elements is given by

$$\begin{aligned}P(X) &= \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi}} \exp \frac{-x_{jj}^2}{2} \prod_{1 \le j < k \le N} \frac{1}{\sqrt{\pi}} \exp -x_{jk}^2 \\ &= A_N \exp(-\sum_{j,k=1}^{N} \frac{x_{jk}^2}{2}) \\ &= A_N \exp(-\frac{1}{2}\mathrm{Tr}(X^2)) \end{aligned} \tag{1.1}$$

where A_N is the normalization and Tr denotes the trace. Let R be a *real orthogonal matrix* ($RR^T = I$). The above formula and the property of the trace that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ gives

$$\begin{aligned}P(R^T X R) &= A_N \exp(-\frac{1}{2}\mathrm{Tr}(R^T X R)^2) \\ &= A_N \exp(-\frac{1}{2}\mathrm{Tr}(R^T R X)^2) \\ &= A_N \exp(-\frac{1}{2}\mathrm{Tr}(X^2)) \\ &= P(X) \end{aligned} \tag{1.2}$$

Thus the GOE joint p.d.f. has the property that it is unchanged by similarity transformations with orthogonal matrices. By similarity transformations, it means to transform an $N \times N$ matrix A into an $N \times N$ matrix B such that $P^{-1}AP = B$, where A and B share the eigenvalues (but not necessarily the same eigenvectors). This is the reason for the word 'orthogonal' in Definition 1.

Next, the case of *Hermitian matrices* with complex entries is considered.

Definition 2 *The Gaussian unitary ensemble (GUE) refers to random Hermitian $N \times N$ matrices X such that the diagonal elements (which must be real) and the upper triangular elements $x_{jk} = u_{jk} + iv_{jk}$ are independently chosen with p.d.f. $\frac{1}{\sqrt{\pi}} \exp -x_{jj}^2$ and $\frac{2}{\pi} \exp -2(u_{jk}^2 + v_{jk}^2) = \frac{2}{\pi} \exp -2|x_{jk}|^2$ respectively. Equivalently, the diagonal entries have distribution $\mathrm{N}[0, \frac{1}{\sqrt{2}}]$, while the upper triangular elements have distribution $\mathrm{N}[0, \frac{1}{2}] + i\mathrm{N}[0, \frac{1}{2}]$*

The joint p.d.f. of all the independent elements is

$$\begin{aligned} P(X) &= \prod_{j=1}^{N} \frac{1}{\sqrt{\pi}} \exp -x_{jj}^2 \prod_{1 \leq j < k \leq N} \frac{2}{\pi} \exp -2|x_{jk}|^2 \\ &= A_N \prod_{j,k=1}^{N} \exp -|x_{jk}|^2 \\ &= A_N \exp -\mathrm{Tr}(X^2) \end{aligned} \tag{1.3}$$

where A_N is the normalization. The invariance $P(U^{-1}XU) = P(X)$, which is obtained similarly through the steps shown in Eq. (1.2), for any unitary matrix U follows immediately.

Unitary matrices

At a technical level, there is a unique meaning to choosing a unitary matrix uniformly at random. At a practical level, such matrices can be generated by diagonalising Gaussian random matrices.

Generally, a *Hermitian matrix* X is diagonalised by a unitary matrix U according to the formula $U^{-1}XU = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. The j^{th} column of U is the normalised eigenvector corresponding to the eigenvalue λ_j. If the elements of X are of a particular number system (real or complex) then so will be the elements of U. These unitary matrices have the sought property that they are generated uniformly at random provided the *Hermitian matrix* is from the GOE (real case) or from the GUE (complex case).

Real orthogonal matrices

As with *unitary matrices*, there is a technical meaning to choosing a real symmetric matrix uniformly at random. *Real orthogonal matrices* are the restrictions of unitary matrices to have real elements. As remarked above, random matrices of this type can therefore be generated as the matrix of eigenvectors which results from diagonalising matrices from the GOE. An expanded discussion on the definition of random matrix ensembles can be found in [7].

Chapter 2

2.1 Empirical eigenvalue distributions

2.1.1 Gaussian ensembles

In application of Gaussian random matrices to physics (see e.g. [3]), the p.d.f. for the spacing between consecutive eigenvalues near the middle of the spectrum can be compared against experimental data. A very accurate approximate analytic form is known for this p.d.f. [4]. To state this:
Let $\beta = 1$ represent GOE and $\beta = 2$ represent GUE.
Let $p_\beta^w(s)$ denote the corresponding p.d.f. where the superscript stands for Wigner who gave this approximation in the case $\beta = 1$. Then

$$\begin{aligned} p_1^w(s) &= \frac{\pi s}{2} \exp(-\frac{\pi s^2}{4}) \\ p_2^w(s) &= \frac{32 s^2}{\pi^2} \exp(-\frac{4 s^2}{\pi}) \end{aligned}$$

In these formulae, we have $\int_0^\infty p_\beta^w(s) ds = 1$ and the mean spacing between the eigenvalues has been normalized to unity as well, where $\int_0^\infty s p_\beta^w(s) ds = 1$. Furthermore, it is assumed that the rank of the matrix is large (i.e. size of each matrix is $N \times N$, for N large). These formulae reveal the phenomenum of eigenvalue repulsion, as they all have the property $p_\beta^w(0) = 0$. For small s, $p_\beta^w(s) \propto s^\beta$ so the strength of the repulsion depends on the ensemble under consideration.

The first task to be undertaken is the empirical determination of $p_\beta^w(s)$ for the GOE and GUE using matrices of rank $N = 16$. This will be presented in the form of bar graphs and superimposed will be the corresponding approximate analytic form $p_\beta^w(s)$. The p.d.f.'s are determined empirically by the following procedure:

- An $N \times N$ matrix is generated from the appropriate ensemble (GOE or GUE). It's eigenvalues and the spacing between the $\left[\frac{N}{2}\right]$th and $(\left[\frac{N}{2}\right] + 1)$th eigenvalues are computed. In this project, N is chosen to be 16 and, thus, the spacing is between the 8th and 9th eigenvalues. This spacing is then denoted by y_1. The above procedure is then repeated M times, for large M, leading to an array of spacings $\{y_1, y_2, y_3, \ldots, y_M\}$.

- From the array above, calculate the average spacing, $\triangle$ as :

$$\triangle = \frac{1}{M} \sum_{j=1}^{M} y_j$$

The array of y_j values is then scaled :

$$\overline{y}_j = \frac{y_j}{\triangle} \text{ for each } j = 1, 2, \ldots, M.$$

- For each interval $(\frac{j}{20}, \frac{j+1}{20})$ with $j = 0, \ldots, 50$, one computes

$$\text{number of } \overline{y}_j \in (\frac{j}{20}, \frac{j+1}{20})$$

We remark that we want to normalise the above expression, given length of each subinterval $= \frac{1}{20}$, such that

$$\sum_{\text{subintervals}} \text{number of } \textbf{scaled } \overline{y}_j \times \text{length of each subinterval} = 1 \tag{2.1}$$

As a result, we need to divide number of $\overline{y}_j \in (\frac{j}{20}, \frac{j+1}{20})$ by $\frac{M}{20}$ to give the height of the bar in the interval $(\frac{j}{20}, \frac{j+1}{20})$ of the corresponding bar graph.

2.1.2 Unitary random matrices

The spectrum of a *unitary random matrix* U has the special property that it is unchanged by rotation on the unit circle in the complex plane since $U \exp i\phi$ is also a *unitary random matrix*. This means the spacings between all pairs of consecutive eigenvalues are statistically equivalent.

In practice, with U as 16×16, we choose to record the (angular) spacings $\theta_2 - \theta_1$, $\theta_6 - \theta_5$, $\theta_{10} - \theta_9$, $\theta_{14} - \theta_{13}$ from the list of eigenvalues $\exp i\theta_j$ for $j = 1, \ldots, 16$ and $0 < \theta_1 < \theta_2 < \ldots < 2\pi$. This enlarges the list of data, while avoiding the strong correlation between spacings that are in immediate succession.

Theoretically, it is known that in the large M limit, the p.d.f. for the spacings between consecutive eigenvalues of *unitary random matrices* is the same as the p.d.f. for the spacings between consecutive eigenvalues near the middle of the spectrum in matrices from the GUE, and hence, well approximated by $p_2^w(s)$. See, for example, [7].

To illustrate this, the empirical spacing distribution is computed according to the following procedure (similar to the one used for the Gaussian ensemble) :

- A 16×16 matrix X is generated from the GUE and diagonalised according to the formula of diagonalising a *unitary matrix* mentioned previously. We know that the matrix U in this formula is a random unitary matrix. The eigenvalues of U are computed and the angular spacings denoted by the following $\varphi_1^{(1)} = \theta_2 - \theta_1$, $\varphi_1^{(2)} = \theta_6 - \theta_5$, $\varphi_1^{(3)} = \theta_{10} - \theta_9$, $\varphi_1^{(4)} = \theta_{14} - \theta_{13}$ are stored. The above procedure is then repeated M times, for large M, leading to an array of $\varphi_k^{(i)}$ values $\{\varphi_1^{(1)}, \varphi_1^{(2)}, \varphi_1^{(3)}, \varphi_1^{(4)}, \varphi_2^{(1)}, \varphi_2^{(2)}, \varphi_2^{(3)}, \varphi_2^{(4)}, \ldots, \varphi_M^{(1)}, \varphi_M^{(2)}, \varphi_M^{(3)}, \varphi_M^{(4)}\}$.
- From the array above, calculate the average spacing $\triangle$,

$$\triangle = \frac{1}{4M} \sum_{k=1}^{M} (\varphi_k^{(1)} + \varphi_k^{(2)} + \varphi_k^{(3)} + \varphi_k^{(4)})$$

The array of $\varphi_k^{(i)}$ values is then scaled

$$\overline{\varphi}_k^{(i)} = \frac{\varphi_k^{(i)}}{\triangle} \text{ for each } k = 1, 2, \ldots, M \text{and } i = 1, \ldots, 4.$$

- For each interval $(\frac{j}{20}, \frac{j+1}{20})$ for $j = 0, \ldots, 50$, one computes

$$(\text{number of } \varphi_k^{(1)} \in (\frac{j}{20}, \frac{j+1}{20})) + (\text{number of } \varphi_k^{(2)} \in (\frac{j}{20}, \frac{j+1}{20}))$$
$$+(\text{number of } \varphi_k^{(3)} \in (\frac{j}{20}, \frac{j+1}{20})) + (\text{number of } \varphi_k^{(4)} \in (\frac{j}{20}, \frac{j+1}{20}))$$

Using Eq. (2.1), this number is divided by $\frac{4M}{20}$ to give the height of the bar in the interval $(\frac{j}{20}, \frac{j+1}{20})$ of the corresponding bar graph.

2.1.3 Real orthogonal random matrices

As mentioned previously, the spectrum is dependent on the rank being even or odd. For definiteness, we only consider matrices of odd rank. For matrices in $O^+(2N+1)$, there is an eigenvalue at $\theta = 0$. Labelling the angles of the remaining N independent eigenvalues, $0 < \theta_1 < \theta_2 < \ldots < \theta_N < \pi$, our interest is in the distribution of θ_1 and the distribution of θ_N. We are interested in the distribution of the same angles in the case of matrices from $O^-(2N+1)$. However, here the fixed eigenvalue is at $\theta = \pi$, since multiplying a matrix from $O^-(2N+1)$ by (-1) transforms the matrix to $O^+(2N+1)$. This property also implies the distribution of θ_1 and θ_N for $O^-(2N+1)$ must coincide with the distribution of θ_N and θ_1 for $O^+(2N+1)$ respectively. It therefore suffices to computes the p.d.f. $p^+(s)$ of θ_1 in $O^+(2N+1)$ and the p.d.f. $p^-(s)$ of θ_1 in $O^-(2N+1)$. Approximate formula analogous to these noted in the Gaussian case are

$$\begin{aligned} p^{+w}(s) &= p_2^w(s) \\ &= \frac{32s^2}{\pi^2}\exp(-\frac{4s^2}{\pi}) \\ p^{-w}(s) &= \frac{2}{\pi}\exp(-\frac{s^2}{\pi}) \end{aligned}$$

The empirical distribution of θ_1 in $O^+(2N+1)$ and $O^-(2N+1)$ can be computed according to the following procedure:

- A 15×15 matrix X is generated from the GOE and diagonalised. As already noted, the matrix of eigenvectors, U say, is a random real orthogonal matrix. The matrix will belong to $O^+(15)$ if the $\det X = 1$ or $O^-(15)$ if the $\det U = -1$. In both cases, θ_1 and θ_7 are computed. If the matrix belongs to $O^+(15)$ then θ_1 is set equal to φ_1^+ while θ_7 is set equal to φ_1^-. If the matrix belongs to $O^-(15)$ then θ_1 is set equal to φ_1^-, while θ_7 is set equal to φ_1^+. The above procedure is then repeated M times, for large M, leading to two array of φ_j^+ and φ_j^- values: $\{\varphi_1^+, \varphi_2^+, \ldots, \varphi_M^+\}, \{\varphi_1^-, \varphi_2^-, \ldots, \varphi_M^-\}$.
- Using the procedure detailed above, we form two histograms which give the empirical distribution of θ_1 and θ_7 approporiately scaled.

2.2 Results

The procedures stated in sections 2.1.1, 2.1.2 and 2.1.3 have been coded into Matlab programs, 'GAUENSEM.m', 'URM.m' and 'ORM.m' respectively, and their outputs displayed as bar graphs in Figs. 2.1 to 2.5.

In Fig. 2.1, the bar graph in blue is generated using random matrices belonging to the GOE according to the procedure mentioned in section 2.1.1, while the bar graph in red is plotted using $p_1^w(s)$. In Fig. 2.2, the bar graph in blue is generated using random matrices belonging to the GUE according to the procedure mentioned in section 2.1.1, while the bar graph in red is plotted using $p_2^w(s)$.

In Fig. 2.3, the bar graph in blue is generated using *unitary matrices* obtained by diagonalising random matrices belonging to the GUE according to the procedure mentioned in section 2.1.2., while the bar graph in red is plotted using $p_2^w(s)$.

For the 2 figures, Figs. 2.4 and 2.5, both the blue bar graphs are generated using *real orthogonal matrices* obtained by diagonalising random matrices belonging to the GOE according to the procedure mentioned in section 2.1.3, with φ_j^+ and φ_j^- respectively. The 2 bar graphs in red are plotted using $p^{+w}(s)$ and $p^{-w}(s)$ respectively.

So as we can see, in all 3 cases, Wigner's approximations give accurate approximations to the empirical distributions.

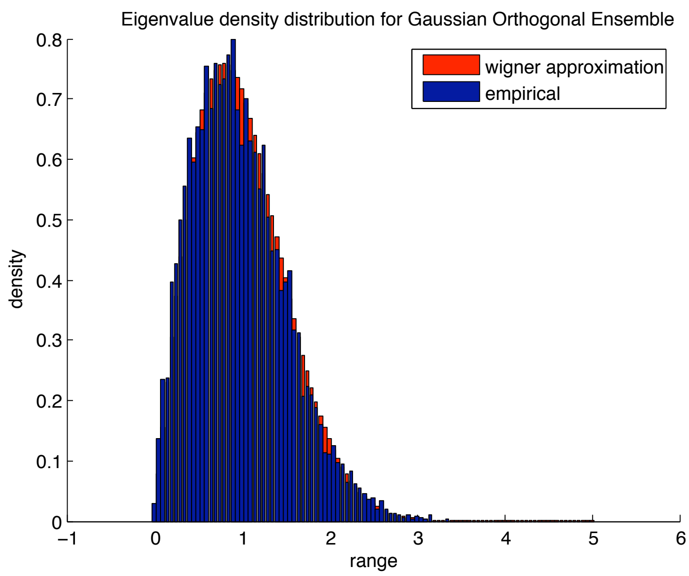
Eigenvalue density distribution for Gaussian Orthogonal Ensemble
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Figure 2.1:

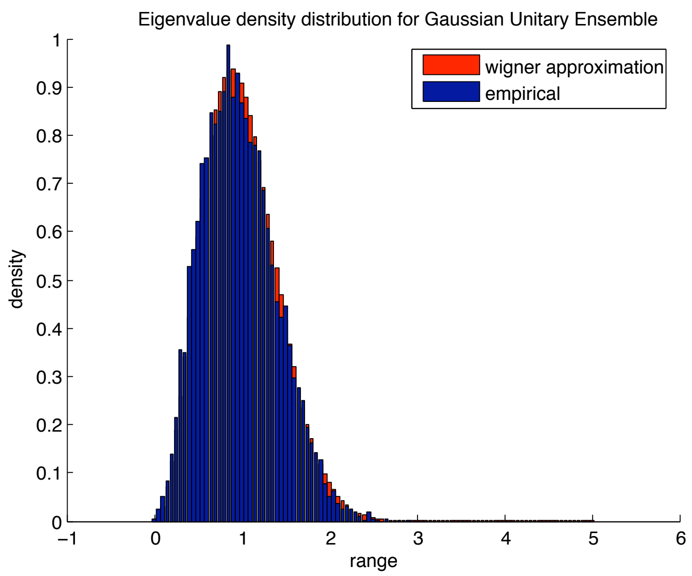
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Figure 2.2:

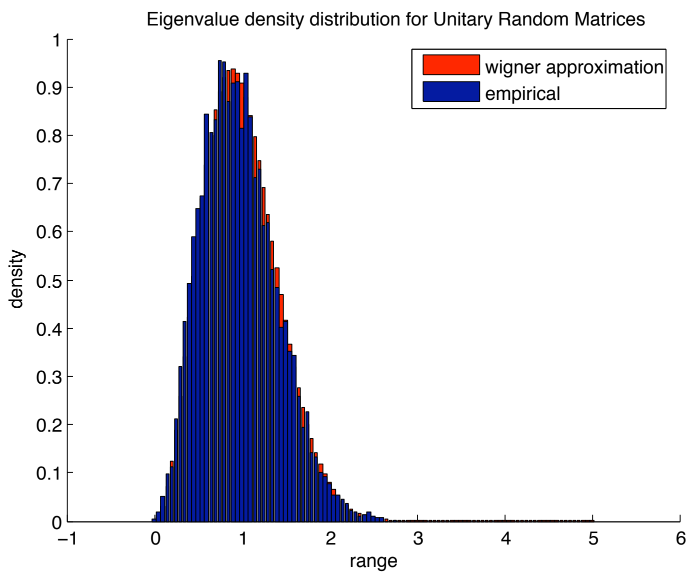
Eigenvalue density distribution for Unitary Random Matrices
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Figure 2.3:

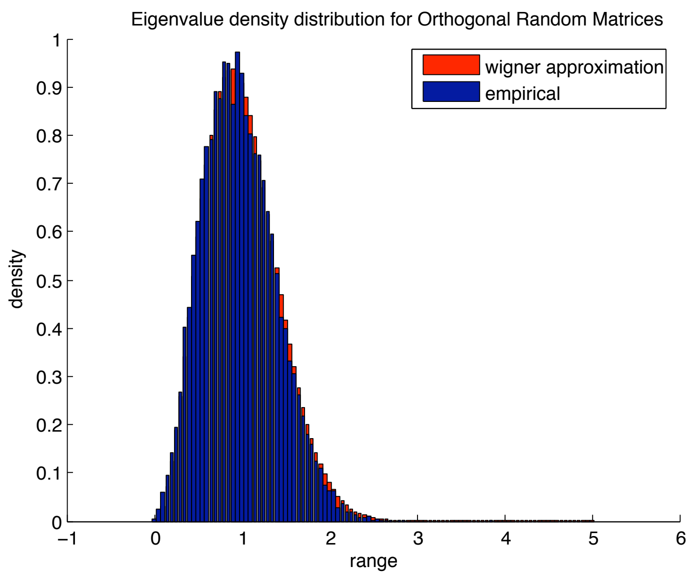
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Figure 2.4:

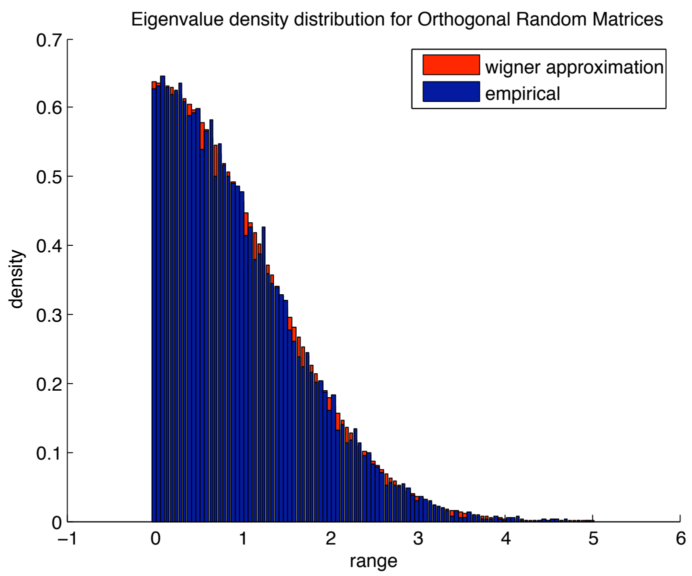
Eigenvalue density distribution for Orthogonal Random Matrices
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Figure 2.5:

Chapter 3

3.1 Spacing distribution in terms of Painlevé transcendents

In the limit of $N \to \infty$, all the eigenvalue distributions discussed in the previous sections can be expressed in terms of particular solutions of some non-linear equations [7]. These equations are examples of the σ-form of the *Painlevé* equations [2] and [5].

The *Painlevé* equations are certain non-linear second order equations with a special property: the only movable singularities are poles. On this point, one recalls in general, non-linear differential equations have singularities which depend on the initial conditions. One would like to distinguish non-linear equations with movable poles from those with moveable essential singularities. For example $\frac{dy}{dt} = y^2 + 1$ has the general solution $y = \tan(t + c)$ where c is determined by the initial condition . Hence, in this case, all the singularities are movable first order poles.

On the other hand, the non-linear equation $\frac{dy}{dt} = \frac{1}{\alpha y^{\alpha-1}}$ for $\alpha = 2, 3, \dots$ has the general solution $y = (t - c)^{\frac{1}{\alpha}}$. Here the singularity is a movable branch point, and hence, an essential singularity.

The non-linear equation relevant to the eigenvalue spacing distributions is

$$(t\frac{d^2u}{dt^2})^2 - a^2(\frac{du}{dt})^2 + \frac{du}{dt}(4\frac{du}{dt} - 1)(u - t\frac{du}{dt}) = 0 \tag{3.1}$$

for values of the parameter $a = +\frac{1}{2}$. This equation is an example of the σ-form of the third Painlevé equation [1]. The following evaluations are known, where $u(t, a)$ is the solution to Eq. (3.1)

$$p^-(s) = -\frac{d}{ds}\exp\Big(-\int_0^{(\pi s)^2} u(t; -\frac{1}{2})\frac{dt}{t}\Big) \tag{3.2}$$

$$p^+(s) = -\frac{d}{ds}\exp\Big(-\int_0^{(\pi s)^2} u(t; \frac{1}{2})\frac{dt}{t}\Big) \tag{3.3}$$

$$p_1(s) = \frac{d^2}{ds^2}\exp\Big(-\int_0^{(\frac{\pi s}{2})^2} u(t; -\frac{1}{2})\frac{dt}{t}\Big) \tag{3.4}$$

$$p_2(s) = \frac{d^2}{ds^2}\exp\Big(-\int_0^{\frac{\pi s}{2})^2}\Big[(u(t; -\frac{1}{2}) + u(t; \frac{1}{2})\Big]\frac{dt}{t}\Big) \tag{3.5}$$

The function $u(t; a)$ satisfies Eq. (3.1) subject to the boundary conditions:

$$u(t; -\frac{1}{2}) \underset{t\to 0}{\sim} \frac{t^{\frac{1}{2}}}{\pi} \tag{3.6}$$

$$u(t; \frac{1}{2}) \underset{t\to 0}{\sim} \frac{t^{\frac{3}{2}}}{3\pi} \tag{3.7}$$

An eigenvalue distribution more general than $p^-(s)$ and $p^+(s)$ for $O^-(2N+1)$ and $O^+(2N+1)$ respectively, can also be evaluated in terms of solutions of Eq. (3.1). We denote the angles of the N independent complex eigenvalues by $0 < \theta_1 < \theta_2 < \ldots < \theta_M < \pi$ and let $p^-(s;k)$ be the distribution of θ_k in $O^-(2N+1)$ and $p^+(s;k)$ be the distribution of θ_k in $O^+(2N+1)$. It is known that from [7].

$$\begin{aligned} p^-(s;k) &= p^-(s;k-1) - \frac{(-1)^k}{k!}\frac{d}{ds}\frac{\partial^k}{\partial\xi^k}\Big\{\exp\Big(-\int_0^{(\pi s)^2} u(t;-\frac{1}{2};\xi)\frac{dt}{t}\Big)\Big\}\Big|_{\xi=1} \\ p^+(s;k) &= p^+(s;k-1) - \frac{(-1)^k}{k!}\frac{d}{ds}\frac{\partial^k}{\partial\xi^k}\Big\{\exp\Big(-\int_0^{(\pi s)^2} u(t;\frac{1}{2};\xi)\frac{dt}{t}\Big)\Big\}\Big|_{\xi=1} \end{aligned}$$

Here the function $u(t;a;\xi)$ satisfies Eq. (3.1) subject to the boundary conditions

$$u(t;-\frac{1}{2};\xi) \underset{t\to 0}{\sim} \frac{\xi t^{\frac{1}{2}}}{\pi} \qquad (3.8)$$

$$u(t;\frac{1}{2};\xi) \underset{t\to 0}{\sim} \frac{\xi t^{\frac{3}{2}}}{3\pi} \qquad (3.9)$$

3.2 A brief introduction: Solving the third Painlevé differential equation

Given the third Painlevé differential equation (D.E.) Eq. (3.1) subject to boundary conditions (B.C.) Eqs. (3.6) and (3.7), or more generally Eqs. (3.8) and (3.9), we seek power series solutions

$$u(t) = \begin{cases} ct^{\frac{1}{2}} + \sum_{p=0}^{\infty} c_p t^{1+\frac{p}{2}}, a = -\frac{1}{2} & , \\ dt^{\frac{3}{2}} + \sum_{p=0}^{\infty} d_p t^{2+\frac{p}{2}}, a = \frac{1}{2} & , \end{cases}$$

where c and d are non-zero constants. For definiteness, we will only be dealing with the second power series solution involving d throughout this article.

To begin, the power series solution $u(t)$ is substituted into the D.E. (3.1) in order to derive some important recurrence formulae. It turns out that $u(t)$ expanded about $t = 0$ only has a finite radius of convergence. Let t_0 be a point inside this radius and compute $u(t_0)$ and $\frac{du(t_0)}{dt}$. A new power series $u(t) = \sum_{n=0}^{\infty} d_n (t - t_0)^n$ where $d_0 = u(t_0)$ and $d_1 = \frac{du(t_0)}{dt}$ is then sought. Again, the power series is substituted into the D.E. (3.1) and the recurrence relations for the coefficients d_n are obtained. This procedure is repeated many times and so $u(t)$ is represented by many overlapping power series. These are then computed into a Mathematica file *'Solutions of the Third Painlevé'*.

The Mathematica file *'Solutions of the Third Painlevé'* is divided into 4 main segments. The first three segments depict the usage of Mathematica in calculating the power series solutions of the D.E. (3.1) numerically, while the last segment depicts two graphs and their relevant moments for comparison and will be explained thoroughly in sections later on.

3.3 Solving the third Painlevé D.E. using power series solution expanded about $t = 0$

3.3.1 Power series solution expanded about $t = 0$

Using the power series expansions about the point $t = 0$, we determine the coefficients of d_p, $p = 0, \ldots$ in terms of d using recurrence relations.

To see how this works, keep only the first 3 terms. Then,

$$\begin{aligned} u(t) &= dt^{\frac{3}{2}} + d_0 t^2 + d_1 t^{\frac{5}{2}} \\ \frac{du}{dt} &= \frac{3}{2} dt^{\frac{1}{2}} + 2d_0 t + \frac{5}{2} d_1 t^{\frac{3}{2}} \\ \frac{d^2u}{dt^2} &= \frac{3}{4} dt^{-\frac{1}{2}} + 2d_0 + \frac{15}{4} d_1 t^{\frac{1}{2}} \end{aligned}$$

Subsituting into the D.E. (3.1) gives

$$\frac{3}{2} dd_0 t^{\frac{3}{2}} + (3d_0^2 + \frac{7}{2} dd_1 + \frac{3}{2} d^2) t^2 = 0$$

Equating the coefficients of t to zero then shows

$$\begin{aligned} \frac{3}{2} dd_0 = 0 \quad &\Rightarrow \quad d_0 = 0 \text{ since } d \neq 0 \text{ is a non-zero constant.} \\ 3d_0^2 + \frac{7}{2} dd_1 + \frac{3}{2} d^2 = 0 \quad &\Rightarrow \quad d_1 = -\frac{3}{7} d \end{aligned}$$

Proceeding similarly, by extending $u(t)$ to include more terms, we can determine the recurrence relation of d_p in terms of d. First note

$$\begin{aligned} u(t) &= dt^{\frac{3}{2}} + \sum_{p=0}^{\infty} d_p t^{2+\frac{p}{2}} \\ \frac{du}{dt} &= \frac{3}{2} dt^{\frac{1}{2}} + \sum_{p=0}^{\infty} d_p (2 + \frac{p}{2}) t^{1+\frac{p}{2}} \\ \frac{d^2u}{dt^2} &= \frac{3}{4} dt^{-\frac{1}{2}} + \sum_{p=0}^{\infty} d_p (2 + \frac{p}{2})(1 + \frac{p}{2}) t^{1+\frac{p}{2}} \end{aligned}$$

These power series must be substituted into the D.E. (3.1) and the power series of each of the individual terms computed. The main tool is the formula for the multiplication of power series

$$\begin{aligned} (\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n) &= \sum_{n=0}^{\infty} c_n x^n \\ \text{with } c_n &= \sum_{l=0}^{n} a_l b_{n-l} \end{aligned}$$

We'll work out each term before subsitituting into the D.E. (3.1).

First term:

$$(t \frac{d^2u}{dt^2})^2 = t \Big(\frac{3}{4} d + \sum_{p=0}^{\infty} d_p (2 + \frac{p}{2})(1 + \frac{p}{2}) t^{\frac{p+1}{2}} \Big)^2$$

$$= t\sum_{p=0}^{\infty}\overline{\gamma}_p t^{\frac{p}{2}} \tag{3.10}$$

where

$$\overline{\gamma}_p = \sum_{l=0}^{p}\gamma_l\gamma_{p-l}$$

$$\gamma_0 = \frac{3}{4}d$$

$$\gamma_l = d_{l-1}(2+\frac{l-1}{2})(1+\frac{l-1}{2}),\ l=1,2,\ldots,p$$

Second term:

$$(\frac{du}{dt})^2 = t\Big(\frac{3}{2}d+\sum_{p=0}^{\infty}d_p(2+\frac{p}{2})t^{\frac{p+1}{2}}\Big)^2$$

$$= t\sum_{p=0}^{\infty}\overline{\beta}_p t^{\frac{p}{2}} \tag{3.11}$$

where

$$\overline{\beta}_p = \sum_{l=0}^{p}\beta_l\beta_{p-l}$$

$$\beta_0 = \frac{3}{2}d$$

$$\beta_l = d_{l-1}(2+\frac{l-1}{2})$$

Third term:

$$\frac{du}{dt}(4\frac{du}{dt}-1)(u-t\frac{du}{dt}) = \Big(\frac{3}{2}t^{\frac{1}{2}}d+\sum_{p=0}^{\infty}d_p(2+\frac{p}{2})t^{1+\frac{p}{2}}\Big)$$

$$\times\Big(t^{\frac{3}{2}}[-\frac{1}{2}d+\sum_{p=0}^{\infty}d_p(-\frac{p}{2}-1)t^{\frac{p+1}{2}}]\Big)$$

$$\times\Big(t^{\frac{1}{2}}[-t^{-\frac{1}{2}}+6d+\sum_{p=0}^{\infty}d_p(8+2p)t^{\frac{p+1}{2}}]\Big)$$

$$= \Big(t^2\sum_{p=0}^{\infty}\delta_p t^{\frac{p}{2}}\Big)\Big(t^{\frac{1}{2}}[-t^{-\frac{1}{2}}+\sum_{p=0}^{\infty}A_p t^{\frac{p}{2}}]\Big)$$

where

$$\delta_p = \sum_{l=0}^{p}b_l B_{p-l}$$

$$b_0 = \frac{3}{2}d$$

$$B_0 = -\frac{1}{2}d$$

$$A_0 = 6d$$

$$b_l = d_{l-1}(2+\frac{l-1}{2})$$

$$B_l = d_{l-1}(\frac{1-l}{2}-1)$$

$$A_p = d_{p-1}[8+2(p-1)]$$

The equation can be further simplified to

$$\frac{du}{dt}(4\frac{du}{dt}-1)(u-t\frac{du}{dt}) \quad = \quad t^{\frac{5}{2}}[\sum_{p=0}^{\infty} t^{\frac{p}{2}} X_p] - t^2 \sum_{p=0}^{\infty} t^{\frac{p}{2}} \delta_p \tag{3.12}$$

where

$$X_p \quad = \quad \sum_{l=0}^{p} \delta_l A_{p-l}$$

Substitute Eqs. (3.10), (3.11), (3.12) into D.E. (3.1):

$$t\sum_{p=0}^{\infty}\overline{\gamma}_p - \frac{1}{4}\left(t\sum_{p=0}^{\infty}\overline{\beta}_p t^{\frac{p}{2}}\right) + t^{\frac{5}{2}}\sum_{p=0}^{\infty} X_p t^{\frac{p}{2}} - t^2\sum_{p=0}^{\infty}\delta_p t^{\frac{p}{2}} \quad = \quad 0$$

Equating like powers

It remains to equate like powers of t

t**:**

$$\overline{\gamma}_0 - \frac{1}{4}\overline{\beta}_0 \quad = \quad 0$$

But $\overline{\gamma}_0 = {\gamma_0}^2$, $\overline{\beta}_0 = {\beta_0}^2$, so this reads

$${\gamma_0}^2 - \frac{1}{4}{\beta_0}^2 \quad = \quad 0$$

Further $\gamma_0 = \frac{3}{4}d$ and $\beta_0 = \frac{3}{2}d$, so this equation is satisfied identically.

$t^{\frac{3}{2}}$**:**

$$\overline{\gamma}_1 - \frac{1}{4}\overline{\beta}_1 \quad = \quad 0$$

But $\overline{\gamma}_1 = 3dd_0$, $\overline{\beta}_1 = 6dd_0$ and so

$$3dd_0 - \frac{6}{4}dd_0 \quad = \quad 0$$

which in turn implies $d_0 = 0$.

t^2**:**

$$\overline{\gamma}_2 - \frac{1}{4}\overline{\beta}_2 - \delta_0 \quad = \quad 0$$

But $\overline{\gamma}_2 = 2\gamma_0\gamma_2 + {\gamma_1}^2$ and $\overline{\beta}_2 = 2\beta_0 + {\beta_1}^2$, further $\gamma_2 = \frac{15}{4}d_2$ and so,

$$\frac{30}{8}dd_1 + 3d_0^2 + \frac{3}{4}d^2 \quad = \quad 0$$

which in turn implies

$$d_1 \quad = \quad -\frac{1}{5}d$$

$t^{\frac{5}{2}}$**:**

$$\overline{\gamma}_3 - \frac{1}{4}\overline{\beta}_3 - \delta_1 + X_0 \quad = \quad 0$$

Using similar procedure for the previous three powers of t, we obtained

$$9dd_2 - \frac{9}{4}dd_2 + 15d_0d_1 - \frac{5}{2}d_0d_1 - \frac{9}{2}d^3 + \frac{5}{2}dd_0 \quad = \quad 0$$

which implies

$$d_2 \quad = \quad \frac{2}{3}d^2$$

t^n**,** $n \geq \frac{5}{2}$**:**

$$\overline{\gamma}_{p+3} - \frac{1}{4}\overline{\beta}_{p+3} - \delta_{p+1} + X_p \quad = \quad 0 \text{ for } p = 0, 1, \ldots \tag{3.13}$$

Recurrence formula for d_{p+2}

It will turn out that the highest subscripted d_j in Eq. (3.13) is d_{p+2}. It appears linearly and can be expressed in terms of lower subscripted d_j which are regarded as having already been computed. To make this explicit, we first recall the definitions of $\overline{\gamma}$, $\overline{\beta}$, δ and X and compute the terms $\overline{\gamma}_{p+3}$, $\overline{\beta}_{p+3}$, δ_{p+1} and X_p that appeared in (3.13):

$$\begin{aligned}
\overline{\gamma}_{p+3} &= \sum_{l=0}^{p+3} \gamma_l \gamma_{p+3-l} \\
&= 2\gamma_0\gamma_{p+3} + \sum_{l=1}^{p+2} \gamma_l \gamma_{p+3-l} \\
&= 2(\frac{3}{4}d)(d_{p+2})(2 + \frac{p+2}{2})(1 + \frac{p+2}{2}) + \sum_{l=1}^{p+2} \gamma_l \gamma_{p+3-l} && (3.14) \\
\overline{\beta}_{p+3} &= \sum_{l=0}^{p+3} \beta_l \beta_{p+3-l} \\
&= 2\beta_0\beta_{p+3} + \sum_{l=1}^{p+2} \beta_l \beta_{p+3-l} \\
&= 2(\frac{3}{2}d)(d_{p+2})(2 + \frac{p+2}{2}) + \sum_{l=1}^{p+2} \beta_l \beta_{p+3-l} \\
&= \frac{3}{2}d(6+p)(d_{p+2}) + \sum_{l=1}^{p+2} \beta_l \beta_{p+3-l} && (3.15) \\
\delta_{p+1} &= \sum_{l=0}^{p+1} b_l B_{p+1-l} && (3.16) \\
X_p &= \sum_{l=0}^{p} \delta_l A_{p-l} && (3.17)
\end{aligned}$$

Subsitute Eqs. (3.14), (3.15), (3.16), (3.17) into Eq. (3.13):

$$\frac{3}{8}d(6+p)(4+p)d_{p+2} + \sum_{l=1}^{p+2} \gamma_l \gamma_{p+3-l} - \frac{1}{4}\Big\{\frac{3}{2}d(6+p)d_{p+2} + \sum_{l=1}^{p+2} \beta_l \beta_{p+3-l}\Big\}$$
$$- \sum_{l=0}^{p+1} b_l B_{p+1-l} + \sum_{l=0}^{p} \delta_l A_{p-l} = 0$$

Next, we group the terms with d_{p+2} to arrive at the recurrence formula for d_{p+2}:

$$\begin{aligned}\frac{3}{8}(6+p)(4+p)-\frac{3}{8}(6+p)\}(d)(d_{p+2}) &= -\sum_{l=1}^{p+2}\gamma_l\gamma_{p+3-l}+\frac{1}{4}\sum_{l=1}^{p+2}\beta_l\beta_{p+3-l} \\ &\quad -\sum_{l=0}^{p}\delta_l A_{p-l}+\sum_{l=0}^{p+1}b_l B_{p+1-l}\end{aligned}$$

and hence

$$\begin{aligned}d_{p+2} &= \frac{8}{3(6+p)(3+p)d}\Bigg\{-\sum_{l=1}^{p+2}\gamma_l\gamma_{p+3-l}+\frac{1}{4}\sum_{l=1}^{p+2}\beta_l\beta_{p+3-l} \\ &\quad -\sum_{l=0}^{p}\delta_l A_{p-l}+\sum_{l=0}^{p+1}b_l B_{p+1-l}\Bigg\}\end{aligned} \tag{3.18}$$

At this point of the study, the coefficients in the power series solution in the case of the parameter $a=\frac{1}{2}$ have been fully determined by a recurrence. This recurrence cannot be solved analytically. However, it is well suited to computation via a symbollic computer algebra package. For this, the package Mathematica is used. Its implementation forms Segment 1 of *'Solutions of the Third Painlevé'* .

3.3.2 *'Solutions of the Third Painlevé'*: Segment 1

Recursive solution expanded about $t = 0$

Segment 1 is produced according to the equations derive in section 3.3.1. Below are the definitions of the variables used in the code with references to the variables used in the equations previously:

mygamma $\leftrightarrow \gamma$, *mybeta* $\leftrightarrow \beta$, *smallb* $\leftrightarrow b$, *bigB* $\leftrightarrow B$, *mydelta* $\leftrightarrow \delta$, *myA* $\leftrightarrow A$, *myverybigX* $\leftrightarrow X$, $d \leftrightarrow d$ (remains the coefficient d)

Initial conditions

Initialise the variables by the base cases :

$\gamma_0 = \frac{3}{4}d$, $\beta_0 = \frac{3}{2}d$, $b = \frac{3}{2}d$, $B = -\frac{1}{2}d$, $\delta_0 = b_0 B_0 = -\frac{3}{4}d^2$

Recursive definitions

Definitions:

mygati $\leftrightarrow \overline{\gamma}_p$, *mybeet* $\leftrightarrow \overline{\beta}_p$, *myverybigX* $\leftrightarrow X_p$

This part merely transforms the Eqs. (3.10) to (3.12) and (3.18) to its recursive definitions. The last definition is for d_{p+2} which calls upon the required definitions of each variable. This is used to compute each d_{p+2} for $p = 0, 1, \ldots$ in terms of d only.

The first 150 such coefficients are calculated, $d[n]$ for $n = 0, 1, \ldots, 150$. The variable d is assigned with the special value $\frac{1}{3\pi}$ before storing as a table into the file *'dcoe.dat'*.

Analysing the radius of convergence

For a power series of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ the root test, sometimes known as Cauchy's radical test, gives for the radius of convergence R, the formula

$$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$$

where a_n refers to the coefficients of the power series.

Using Mathematica, we retrieve the coefficients of our power series from the *'dcoe.dat'*. However, $u(t) = d(t^{\frac{1}{2}})^3 + \sum_{n=0}^{\infty} d_n (t^{\frac{1}{2}})^{(n+4)}$ so we need to let $t^{\frac{1}{2}} = z$ and apply the root test. The list of values found for $\frac{1}{|d[n]|^{\frac{2}{n}}}$, $n = 130, \ldots, 150$, is displayed in the relevant segment in *'Solutions of the Third Painlevé'*.

Power series expansion of the τ function

The following segment corresponds to Eqs. (3.2) to (3.6), in particular Eq. (3.3). We replaced $u(t, a)$ with the power series solution of $u(t)$ for the coefficient d. We define:

$$\begin{aligned} iu(s) &= \int_0^s \frac{u(x)}{x} dx \\ &= \int_0^s \frac{dx^{\frac{3}{2}} + \sum_{p=0}^{\infty} d_p x^{2+\frac{p}{2}}}{x} dx \\ &= \frac{2}{3} d s^{\frac{3}{2}} + \sum_{p=0}^{\infty} \frac{d_p s^{\frac{p}{2}+2}}{\frac{p}{2}+2} \end{aligned}$$

$f(x)$ is then defined to be the power series expansion of $\exp(-iu(x))$ and truncated to a normal series $ff(t)$ such that

$$ff(t) = \text{Series}[\exp -\Big\{\frac{2}{3}dt^{\frac{3}{2}} + \sum_{p=0}^{\infty} d_p t^{\frac{p}{2}+2}\}] \tag{3.19}$$

Then replace $ff(t)$ with $ff(x^2)$, differentiate it w.r.t. x and assign $dvff(x)$ to be the Mathematica definition equivalent to Eq. (3.3),

$$dvff(x) \quad = \quad \text{Series}[-\frac{d}{dx}ff(x^2)] \tag{3.20}$$

The logical flow of thought is to insert this subpart in Segment 1. Howver, due to technical computation limitations, the moments have to be computed numerically, which require the variables to be initialised first. Hence, this subpart is brought back to Segment 2 in *'Solutions of the Third Painlevé*.

3.4 Solving the third Painlevé D.E. using power series solution expanded about a general point

3.4.1 Power series solution expanded about a general point

The power series will only converge for t values smaller than R^2 because of the finite radius of convergence R. Let t_0 be a point inside the radius of convergence. We then seek a power series expansion solution of the D.E. (3.1) with the knowledge of $u(t_0)$ and $\frac{du(t_0)}{dt}$ as initial data. Equivalently, introducing a new variable $s = t - t_0$ into the D.E. (3.1), which then reads

$$((s+t_0)\frac{d^2u}{dt^2})^2 - \frac{1}{4}(\frac{du}{dt})^2 + \frac{du}{dt}(4\frac{du}{dt} - 1)(u - (s+t_0)\frac{du}{dt}) = 0 \tag{3.21}$$

the task is to find the power series solution

$$u(s) = \sum_{p=0}^{\infty} a_p s^p \tag{3.22}$$

where $a_0 = u(t_0)$, $a_1 = \frac{du(t_0)}{dt}$ and are thus known.

Following the similar procedure as section 3.3.1, we find the derivatives of the power series and subsitute into Eq. (3.21).

First term:

$$\begin{aligned}
(s+t_0)\frac{d^2u}{dt^2} &= \sum_{p=0}^{\infty} a_p p(p-1)s^{p-1} + t_0 \sum_{p=0}^{\infty} a_p p(p-1)s^{p-2} \\
&= \sum_{p=0}^{\infty} a_{p+1}(p+1)(p)s^p + t_0 \sum_{p=0}^{\infty} a_{p+2}(p+2)(p+1)s^p \\
((s+t_0)\frac{d^2u}{dt^2})^2 &= \sum_{p=0}^{\infty} \overline{A}_p s^p \qquad (3.23) \\
\text{where} & \\
\overline{A}_p &= \sum_{l=0}^{p} A_l A_{p-l} \\
A_l &= l(l+1)a_{l+1} + t_0(l+1)(l+2)a_{l+2}
\end{aligned}$$

Second term:

$$\begin{aligned}
(\frac{du}{dt})^2 &= (\sum_{p=0}^{\infty} a_p p s^{p-1})^2 \\
&= (\sum_{p=0}^{\infty} a_{p+1}(p+1)s^p)^2 \\
&= \sum_{p=0}^{\infty} \overline{B}_p s^p \qquad (3.24) \\
\text{where} & \\
\overline{B}_p &= \sum_{l=0}^{p} B_l B_{p-l} \\
B_l &= a_{l+1}(l+1)
\end{aligned}$$

Third term:

$$
\begin{aligned}
u-(s+t_0)\frac{du}{dt} &= \sum_{p=0}^{\infty} a_p s^p - (s+t_0)\sum_{p=0}^{\infty} a_p p s^{p-1} \\
&= \sum_{p=0}^{\infty} a_p s^p - \sum_{p=0}^{\infty} a_p p s^p - t_0 \sum_{p=0}^{\infty} a_{p+1}(p+1)s^p \\
&= \sum_{p=0}^{\infty} \Big(a_p(1-p) - t_0 a_{p+1}(p+1)\Big)s^p
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{du}{dt}(u-(s+t_0)\frac{du}{dt}) &= \sum_{p=0}^{\infty} a_{p+1}(p+1)s^p \sum_{p=0}^{\infty} \Big(a_p(1-p) - t_0 a_{p+1}(p+1)\Big)s^p \\
&= \sum_{p=0}^{\infty} \overline{R}_p s^p \qquad (3.25)
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{R}_p &= \sum_{l=0}^{p} r_l R_{p-l} \\
r_l &= a_{l+1}(l+1) \\
R_l &= a_l(1-l) - t_0 a_{l+1}(l+1)
\end{aligned}
$$

Also

$$
\begin{aligned}
4\frac{du}{dt}-1 &= \sum_{p=0}^{\infty} 4a_{p+1}(p+1)s^p - 1 \\
&= (4a_1-1) + \sum_{p=1}^{\infty} 4a_{p+1}(p+1)s^p \qquad (3.26)
\end{aligned}
$$

Combining the Eqs. (3.25) and (3.26), we obtained the third term

$$
\frac{du}{dt}(u-(s+t_0)\frac{du}{dt})(4\frac{du}{dt}-1) = \sum_{p=0}^{\infty} T_p s^p \qquad (3.27)
$$

where

$$
\begin{aligned}
T_p &= \sum_{l=0}^{p} X_l \overline{R}_{p-l} \\
X_0 &= 4a_1 - 1 \\
X_p &= 4a_{p+1}(p+1)
\end{aligned}
$$

Substitute the Eqs. (3.23), (3.24) and (3.27) into Eq. (3.19)

$$
\sum_{p=0}^{\infty} \overline{A}_p - \frac{1}{4}\sum_{p=0}^{\infty} \overline{B}_p s^p + \sum_{p=0}^{\infty} T_p s^p = 0 \qquad (3.28)
$$

Equating like powers

Following the similar procedure of equating like powers as seen in section 3.3.1, we equate like powers of s in Eq. (3.28),

$$\overline{A}_p - \frac{1}{4}\overline{B}_p + T_p = 0 \tag{3.29}$$

Rewrite (3.29) in terms of the definitions of $\overline{A}_p$, $\overline{B}_p$, T_p:

$$2A_0A_p + \sum_{l=1}^{p-1} A_lA_{p-l} - \frac{1}{4}\sum_{l=0}^{p} B_lB_{p-l} + \sum_{l=0}^{p} X_l\overline{R}_{p-l} = 0, \text{ for } p = 1, 2, \ldots \tag{3.30}$$

Recurrence formula for a_{p+2}

After further substitution and simplification similar to those performed in section 3.2.2, we arrive at the recurrence formula for a_{p+2}:

$$a_{p+2} = \frac{1}{4(t_0)^2a_2(p+1)(p+2)}\Big\{ - 4t_0a_2p(p+1)a_{p+1} \\ - \sum_{l=1}^{p-1} A_lA_{p-l} + \frac{1}{4}\sum_{l=0}^{p} B_lB_{p-l} - \sum_{l=0}^{p} X_l\overline{R}_{p-l}\} \tag{3.31}$$

The right hand side involves smaller subscripted coefficients a_0, a_1, $\ldots$, a_{p+1} for all p except $p = 0$. In the latter case, a quadratic results for a_2. Using Eq. (3.29) and definitions of $\overline{A}_0$, $\overline{B}_0$, $\overline{T}_0$, we find that

$$a_2 = \pm\frac{\sqrt{(\frac{a_1^2}{4} - (4{a_1}^2 - a_1)(a_0 - a_1t_0))\frac{1}{t_0^2}}}{2} \tag{3.32}$$

where the sign is chosen to equal the sign of $\frac{d^2u(t_0)}{dt^2}$.

3.4.2 *'Solutions of the Third Painlevé'*: Segment 2

Recursive solution expanded about a general point

This segment makes reference to section 3.4.1 where we now introduce a new parameter, n, for generalising the point of expansion. Then n refers to the nth expansion point. The stratgey here is to follow up from the code in Segment 1 so that $u(t)$ is to be expanded about a general point $t_0 = s_0 + is[n]$, where s_0 is initialised to 1 as the starting point of expansion and $is[n]$ is initialised to $\frac{n}{2}$ as an increment for the point of expansion. As n increments, the point of expansion increases. The essential feature of this procedure is that the point $s_0 + is[n]$ is always inside the radius of convergence of successive power series. It allows for the function to be accurately computed over a large range of t values.

Initial condition

The initial values are $a_0 = u(s_0 + is[n])$, $a_1 = \frac{du(s_0+is[n])}{dt}$, while a_2 is calculated from the formula given in Eq. (3.32)

$$a_2 = \text{Sign}\left[\frac{d^2u(s_0 + is[n])}{dt^2}\right]\frac{\sqrt{(\frac{a_1^2}{4} - (4{a_1}^2 - a_1)(a_0 - a_1t_0))\frac{1}{t_0^2}}}{2}$$

Initialise the variables by the base case:

$$a_0 = u(s_0),\ a_1 = \frac{du(s_0)}{dt}$$

as well as the formula for a_2 given above. $zxX[0,0]$ and $zaAt[1,0]$ give the base case of X_p and $\overline{A}_p$ used in Eq. (3.27) and Eq. (3.23) respectively, where the first expansion point is set to $t_0 = 1$.

Recursive definitions

Definitions:

$$zaA \leftrightarrow A_l,\ zaAt \leftrightarrow \overline{A}_p,\ zbB \leftrightarrow B_l,\ zbBt \leftrightarrow \overline{B}_p$$

$$zr \leftrightarrow r_l,\ zrR \leftrightarrow R_l,\ zrRt \leftrightarrow \overline{R}_p,\ zxX \leftrightarrow X_p,\ ztT \leftrightarrow T_p,\ ,\ a \leftrightarrow a(remainsthecoefficienta)$$

As in section 3.3.2, the following segment is the recursive definitions of Eqs. (3.23) to (3.27) and (3.31) using Mathematica. In particular, the last formula, a_{p+2}, calls upon the required definitions of each variable and uses the Eq. (3.31).

Plotting the power series solution of D.E. (3.1)

The solution of the D.E. (3.1) is now given by a total of $nn + 2$ power series. These power series have finite radius of convergence and so only represent the function inside certain intervals. Denoting the function by $f(t)$, we choose the specific intervals implied by setting

$$f(t) = gg(t) + \sum_{jj=0}^{nn} g[t, jj] \tag{3.33}$$

where

$$gg(t) = \begin{cases} u(t),\ 0 \le t \le s_0 & , \\ 0,\ \text{otherwise} & , \end{cases}$$

$$g(t) = \begin{cases} zu[t, jj],\ s_0 + is[jj] \le t \le s_0 + is[jj+1] & , \\ 0,\ \text{otherwise} & , \end{cases}$$

Using this, we plot the graph of $f(t)$ for $0 \le t \le 100$ as shown in Mathematica file.

Power series expansion of the τ function about a general point

Here, we want to extend the recursive definitions used in computing the τ function in Segment 1, using the strategy mentioned at the beginning of this section so that the point of expansion is about $s_0+is[n]$. From the power series solution $u(s)$, Eq. (3.22), we substitute s with $t-(s_0+is[n])$ (since $s = t - t_0 = t - (s_0 + is[n])$) and replace a_p with $a[k, n]$ (where $p \leftrightarrow k$) to come up with a new power series solution $zu[t, n]$ expanded about $s_0 + is[n]$:

$$zu[t, n] = \sum_{k=0}^{\infty} a[k, n](t - (s_0 + is[n]))^k \tag{3.34}$$

and its first and second derivative w.r.t. t. To compute the τ function, we require Eq. (3.3) to be evaluated using the power series solution Eq. (3.34) in place of $u(t;\frac{1}{2})$ about $t_0 = s_0 + is[n]$. But first, rewrite $\frac{1}{t}$ in terms of a power series.

$$\begin{aligned}
\frac{1}{t} &= \frac{1}{t - t_0 + t_0} \\
&= \frac{1}{t_0}\left(\frac{1}{1 + \frac{t-t_0}{t_0}}\right) \\
&= \frac{1}{t_0}\sum_{p=0}^{\infty}(-1)^p \frac{(t-t_0)^p}{(t_0)^p} \\
&= \sum_{p=0}^{\infty} A_p (t-t_0)^p \qquad (3.35)
\end{aligned}$$

where

$$A_p = \frac{1}{t_0}\left(-\frac{1}{t_0}\right)^p$$

And we define $bigA = A_p$ in *'Solutions of the Third Painlevé'*. Using Eqs. (3.34) and (3.35) we can write $\frac{zu}{t}$ in its power series form.

$$\begin{aligned}
\frac{zu}{t} &= \sum_{p=0}^{\infty} bigA[p,n](t-t_0)^p \sum_{k=0}^{\infty} a[k,n](t-t_0)^k \\
&= \sum_{k=0}^{\infty} at[k,n](t-t_0)^k \qquad (3.36)
\end{aligned}$$

where

$$at[k,n] = \sum_{p=0}^{k} a[p,n]bigA[k-p,n]$$

Next, define

$$\begin{aligned}
iy[t,n] &= \int_{t_0}^{t} \frac{zu[x,n]}{x}dx \\
&= \sum_{k=0}^{\infty} \frac{at[k,n](t-t_0)^{k+1}}{k+1} \qquad (3.37)
\end{aligned}$$

$g(x)$ is then defined to be the power series expansion of $\exp(-iy[x,n])$ about $t = t_0$ and truncated to a normal series $sj[t,n]$ such that

$$sj[t,n] = \text{Series}[\exp - \left\{ \sum_{k=0}^{\infty} \frac{at[k,n](t-t_0)^{k+1}}{k+1} \right\}] \qquad (3.38)$$

Finally, replacing t with x^2, we differentiate $sj[x^2,n]$ w.r.t. x and assign $dvsj[x,n]$ to be the Mathematica definition of Eq. (3.3) for the calculations of τ function in this case, so that

$$dvsj[x,n] = \text{Series}[-\frac{d}{dx} sj[x^2,n]] \qquad (3.39)$$

The procedure must be iterated. We reinitialise the base cases a_0, a_1 and a_2 with numeric evaluations using definitions of Eqs. (3.34) to (3.36) to extend the τ function calculations about any general point.

$$\begin{aligned}
a_0 &= \mathrm{N}[zu[t_0, j], ac] \\
a_1 &= \mathrm{N}[zud[t_0, j], ac] \\
a_2 &= \mathrm{Sign}[zsd[t_0, j]] \times \frac{\sqrt{(\frac{a_1^2}{4} - (4{a_1}^2 - a_1)(a_0 - a_1 t_0))\frac{1}{t_0^2}}}{2}
\end{aligned}$$

Recall that $a[k, n]$ refers to the coefficients of the power series solution Eq. (3.34). We then set $a[0, n]$, $a[1, n]$, $a[2, n]$ equal to a_0, a_1, a_2. Next, we set $zxX[0, 0]$ equal to the first term of X_p in Eq. (3.27) and $zaAt[1, 0]$ equal to the second term of $\overline{A}_p$ in Eq. (3.23), where the expansion point is arbitrary.

Calculating moments of the τ function

In general, we are interested in evaluating the different n moments of the τ function as seen in Eq. (3.20) and seek to extend the calculations of the nth moment about a general point $s_0 + is[n]$.

Given that the general formula for the nth moment of a positive real-valued function $p(t)$ of a real variable is $m(n) = \int_0^\infty t^n p(t)dt$, we are able to evaluate $m(n)$ for the τ function by substituting $p(t)$ by the power series of τ function.

However, we note that the power series of the τ function, first calculated in Eq. (3.20), has a finite radius of convergence. So, in order to extend the integration of $t^n p(t)$ from 0 to ∞, we need to split the calculation of $m(n)$ into 2 distinct parts, the first corresponds to expanding about the starting point $t = 0$ and the second about a general point $s_0 + is[j]$. Denoting the split function by $tq(n)$, we choose the specific intervals implied by setting

$$tq(n) = jaq(n) + \sum_{j=0}^{\infty} jol[j, n]$$

where

$$jaq(n) = \int_0^{\frac{\sqrt{s_0}}{\pi}} s^n \Big\{ -\frac{d}{ds} \exp\Big(-\int_0^{(\pi s)^2} u(t; -\frac{1}{2})\frac{dt}{t} \Big) \Big\} ds \tag{3.40}$$

$$jol[j, n] = \int_{\frac{\sqrt{s_0+is[j]}}{\pi}}^{\frac{\sqrt{s_0+is[j+1]}}{\pi}} s^n \Big\{ -\frac{d}{ds} \exp\Big(-\int_0^{(\pi s)^2} u(t; -\frac{1}{2})\frac{dt}{t} \Big) \Big\} ds \tag{3.41}$$

About point of expansion $t = 0$

The τ function in Eq. (3.40) has already been computed in section 3.3.2, and evaluated to as being $dvff(x)$ in Eq. (3.20). Hence, after replacing s with $\frac{s}{\pi}$, Eq. (3.40) simplifies to

$$\begin{aligned}
jaq(n) &= \int_0^{\sqrt{s_0}} (\frac{s}{\pi})^n \Big\{ -\frac{d}{ds} \exp\Big(-\int_0^{(s)^2} u(t; -\frac{1}{2})\frac{dt}{t} \Big) \Big\} ds \\
&= (\frac{1}{\pi})^n \int_0^{\sqrt{s_0}} s^n dvff(s) ds
\end{aligned} \tag{3.42}$$

About point of expansion $s_0 + is[j]$

Define a new power series

$$s[x^2, j] = \sum_{k=0}^{\infty} gia[k, j]\Big(x^2 - (s_0 + is[j])\Big)^k \tag{3.43}$$

where $j = 0, \ldots$ such that $s[x^2, j] = \exp(-\int_0^{x^2} u(t; -\frac{1}{2})\frac{dt}{t})$. After differentiating $s[x^2, j]$ in Eq. (3.43) w.r.t. x, we substitute it into Eq. (3.3) to obtain the equivalent Mathematica definition for the τ function

$$\begin{aligned} ds[x, j] &= -\frac{d}{dx} s[x^2, j] \\ &= \sum_{k=0}^{\infty} gia[k, j] k(2x) \Big(x^2 - (s_0 + is[j]) \Big)^{k-1} \end{aligned} \tag{3.44}$$

Since the Mathematica function $ds[s, j]$, when replaced x by s, is defined as having the upper limit of the integration to be s^2 instead of $(\pi s)^2$, we need to replace s by $\frac{s}{\pi}$ in Eq. (3.41). This allows us to use Eqs. (3.43) and (3.44) in expressing Eq. (3.41) in terms of the power series $s[x^2, j]$ and $ds[x, j]$. By doing so, the contribution for the nth moment of the τ function becomes

$$\begin{aligned} jol[j, n] &= (\frac{1}{\pi})^n \int_{\sqrt{s_0+is[j]}}^{\sqrt{s_0+is[j+1]}} s^n \Big[-\frac{d}{ds} \Big(\exp\Big(-\int_0^{s^2} u(t; -\frac{1}{2}) \frac{dt}{t} \Big) \Big) \Big] ds \\ &= (\frac{1}{\pi})^n \int_{\sqrt{s_0+is[j]}}^{\sqrt{s_0+is[j+1]}} x^n \Big(-\frac{d}{dx} s[x^2, j] \Big) dx \\ &= (\frac{1}{\pi})^n \int_{\sqrt{s_0+is[j]}}^{\sqrt{s_0+is[j+1]}} x^n ds[x, j] dx \end{aligned} \tag{3.45}$$

Recall the definitions of $s[x^2, j]$ and $ds[x, j]$, we rewrite Eq. (3.45)

$$\begin{aligned} &(\frac{1}{\pi})^n \int_{\sqrt{s_0+is[j]}}^{\sqrt{s_0+is[j+1]}} x^n \Big\{ -\frac{d}{dx} \Big(\sum_{k=0}^{\infty} gia[k, j] \Big(x^2 - (s_0 + is[j]) \Big)^k \Big) \Big\} dx \\ &= (\frac{1}{\pi})^n \sum_{k=0}^{\infty} k \times gia[k, j] \int_{\sqrt{s_0+is[j]}}^{\sqrt{s_0+is[j+1]}} x^n \Big\{ (2x) \Big(x^2 - (s_0 + is[j]) \Big)^{k-1} \Big\} dx \end{aligned}$$

To remove the square roots on the limit, let $y = x^2$, $dy = 2xdx$

$$= (\frac{1}{\pi})^n \sum_{k=0}^{\infty} k \times gia[k, j] \int_{s_0+is[j]}^{s_0+is[j+1]} y^{\frac{n}{2}} \Big\{ \Big(y - (s_0 + is[j]) \Big)^{k-1} \Big\} dy$$

To change the lower limit of the integration to 0, replace y with $y + (s_0 + is[j])$

$$= (\frac{1}{\pi})^n \sum_{k=0}^{\infty} k \times gia[k, j] \int_0^{(s_0+is[j+1])-(s_0+is[j])} \Big(y + (s_0 + is[j]) \Big)^{\frac{n}{2}} y^{k-1} dy$$

Replace y with $(s_0 + is[j])y$

$$= (\frac{1}{\pi})^n \sum_{k=0}^{\infty} k \times gia[k, j] (s_0 + is[j])^{\frac{n}{2}+k} \int_0^{\frac{s_0+is[j+1]}{s_0+is[j]}-1} y^{k-1} (y+1)^{\frac{n}{2}} dy$$

Let $er = \frac{s_0+is[j+1]}{s_0+is[j]} - 1$

$$= (\frac{1}{\pi})^n \sum_{k=0}^{\infty} k \times gia[k, j] (s_0 + is[j])^{\frac{n}{2}+k} \int_0^{er} y^{k-1} (y+1)^{\frac{n}{2}} dy$$

We note that $\int_0^{er} y^{k-1}(1+y)^{\frac{n}{2}} dy$ can be easily evaluated using the special Mathematica function, incomplete beta function Beta$[z, a, b] = \int_0^z y^{a-1}(1-y)^{b-1} dy$. And so to change $\int_0^{er} y^{k-1}(1+y)^{\frac{n}{2}} dy$ into the form of the 'Beta function', we replace y with $-y$

$$\begin{aligned} \int_0^{er} y^{k-1}(1+y)^{\frac{n}{2}} dy &= \int_0^{-er} (-y)^{k-1}(1-y)^{\frac{n}{2}} d(-y) \\ &= \int_0^{-er} (-1)^k y^{k-1}(1-y)^{\frac{n}{2}} dy \end{aligned} \tag{3.46}$$

Hence $jol[k, n]$ becomes

$$jol[j, n] = (\frac{1}{\pi})^n \sum_{k=0}^{\infty} (-1)^k k \times gia[k, j](s_0 + is[j])^{\frac{n}{2}+k} \text{Beta}[-er, k, 1 + \frac{n}{2}] \tag{3.47}$$

Finally, substituing Eqs. (3.42) and (3.47) into the equation of $tq[n]$, we compute the corresponding nth moment and tabulate the results.

Computation of the nth moment	
n	$\mathbf{p^+(s)}$
0	0.9999999999999999999999999999999997731767405
1	0.78271575829254364686477177982432925449821388120
2	0.7192336365093331884273890064852259171308740817
3	0.745693053967521055814661677407593453848340621
4	0.85156873600003934808214859905582137675863047
5	1.0540205779357301438902439126011597302270349

3.5 Solving the third Painlevé D.E. using special power series involving ξ expanded about general point of expansion

3.5.1 Power series solution involving ξ

We now introduce a variable ξ, first seen in section 3.1, as a parameter of the power series solution u to the D.E. (3.1) so that $u = u(t;\xi)$. Let $u(t;1) = u(t)$ and $\frac{\partial u(t;1)}{\partial \xi} = w$. The power series solution for the B.C. (3.9) has already been obtained from the theory of section 3.2. In particular, the parameter d in that section is equal to $\frac{\xi}{3\pi}$. Hence

$$w = d\frac{\partial u(t)}{\partial d}\bigg|_{t\to 0} = \frac{1}{3\pi} \tag{3.48}$$

where $u(t)$ is the power series solution of section 3.2, while u itself is the power series obtained from the coefficients computed in section 3.3.

Taking partial derivative of the new D.E. w.r.t. ξ and setting $\xi = 1$, gives the following equation

$$\begin{aligned} &2t^2\frac{d^2u}{dt^2}\frac{d^2w}{dt^2} - \frac{1}{2}\frac{du}{dt}\frac{dw}{dt} + \frac{dw}{dt}(4\frac{du}{dt}-1)(u-t\frac{du}{dt}) \\ &+4\frac{du}{dt}\frac{dw}{dt}(u-t\frac{du}{dt}) + \frac{du}{dt}(4\frac{du}{dt}-1)(w-t\frac{dw}{dt}) = 0 \end{aligned} \tag{3.49}$$

We seek a power series solution for $w(t)$ about a general point t_0 , $w(t) = \sum_{k=0}^{\infty} c_k(t-t_0)^k$, with the power series soluton of $u(t) = \sum_{k=0}^{\infty} a_k(t-t_0)^k$, regarded as known. For this, a similar procedure to that of section 3.4.1 is used. However, here we consider the terms in Eq. (3.49) according to factors of $\frac{d^2w}{dt^2}$, $\frac{dw}{dt}$ and w. Let $s = t - t_0$, where t_0 is the point of expansion.

$\frac{d^2w}{dt^2}$:

$$\begin{aligned} 2t^2\frac{d^2u}{dt^2} &= 2(s^2+2st_0+t_0^2)\sum_{k=0}^{\infty} k(k-1)a_k s^{k-2} \\ &= \sum_{k=0}^{\infty}\Big(2k(k-1)a_k + 4t_0(k+1)ka_{k+1} + 2t_0^2(k+2)(k+1)a_{k+2}\Big)s^k \end{aligned}$$

Hence

$$2t^2\frac{d^2u}{dt^2}\frac{d^2w}{dt^2} = \sum_{p=0}^{\infty}\gamma_p s^p \tag{3.50}$$

where

$$\begin{aligned} \gamma_p &= \sum_{k=0}^{p}\mu_k\nu_{p-k} \\ \mu_k &= 2k(k-1)a_k + 4t_0(k+1)ka_{k+1} + 2t_0^2(k+2)(k+1)a_{k+2} \\ \nu_k &= (k+2)(k+1)c_{k+2} \end{aligned}$$

$\frac{dw}{dt}$:

$$\begin{aligned} &\Big(-\frac{1}{2}\frac{du}{dt} + (4\frac{du}{dt}-1)(u-t\frac{du}{dt}) + 4\frac{du}{dt}(u-t\frac{du}{dt}) - t\frac{du}{dt}(4\frac{du}{dt}-1)\Big) \\ &\quad = \Big(-\frac{1}{2}\frac{du}{dt} + (8\frac{du}{dt}-1)(u-(s+t_0)\frac{du}{dt}) \\ &\qquad +(s+t_0)\frac{du}{dt}(1-4\frac{du}{dt})\Big) \end{aligned} \tag{3.51}$$

Rewrite each term in Eq. (3.51) in terms of the power series solution $u(t)$.

First term:

$$-\frac{1}{2}\frac{du}{dt} = -\frac{1}{2}\sum_{k=0}^{\infty}(k+1)a_{k+1}s^k \tag{3.52}$$

Second term:

$$\begin{aligned} 8\frac{du}{dt} - 1 &= 8\sum_{k=0}^{\infty}(k+1)a_{k+1}s^k - 1 \\ &= (8a_1 - 1) + 8\sum_{k=1}^{\infty}(k+1)a_{k+1}s^k \end{aligned}$$

and

$$u - (s+t_0)\frac{du}{dt} = \sum_{k=0}^{\infty}\Big(a_k(1-k) - t_0(k+1)a_{k+1}\Big)s^k$$

Combining the 2 equations

$$(8\frac{du}{dt} - 1)(u - (s+t_0)\frac{du}{dt}) = \sum_{p=0}^{\infty}\tau_p s^p \tag{3.53}$$

where

$$\begin{aligned} \tau_p &= \sum_{k=0}^{p}\overline{\mu}_k\overline{\nu}_{p-k} \\ \overline{\mu}_k &= a_k(1-k) - t_0(k+1)a_{k+1} \\ \overline{\nu}_k &= 8(k+1)a_{k+1} \ (k \neq 0) \\ \overline{\nu}_0 &= 8a_1 - 1 \end{aligned}$$

Third term:

$$(s+t_0)\frac{du}{dt} = \sum_{k=0}^{\infty}\Big(ka_k + t_0(k+1)a_{k+1}\Big)s^k$$

and

$$1 - 4\frac{du}{dt} = (1 - 4a_1) - 4\sum_{k=1}^{\infty}(k+1)a_{k+1}s^k$$

Combining the 2 equations

$$(s+t_0)\frac{du}{dt}(1 - 4\frac{du}{dt}) = \sum_{p=0}^{\infty}\epsilon_p s^p \tag{3.54}$$

where

$$\begin{aligned} \epsilon_p &= \sum_{k=0}^{p}f_k g_{p-k} \\ f_k &= ka_k + t_0(k+1)a_{k+1} \\ g_k &= -4(k+1)a_{k+1} \ (k \neq 0) \\ g_0 &= 1 - 4a_1 \end{aligned}$$

Substitute Eqs. (3.52) to (3.54) into Eq. (3.51) and equate only the factors of $\frac{dw}{dt}$ in Eq. (3.51) to $\sum_{p=0}^{\infty}\chi_p s^p$, where

$$\chi_p = -\frac{1}{2}(p+1)a_{p+1} + \tau_p + \epsilon_p \tag{3.55}$$

Hence

$$\frac{dw}{dt}\Big(-\frac{1}{2}\frac{du}{dt} + (8\frac{du}{dt}-1)\Big(u-(s+t_0)\frac{du}{dt}\Big) - (s+t_0)\frac{du}{dt}(4\frac{du}{dt}-1)\Big)$$

$$= \sum_{p=0}^{\infty} J_p s^p \tag{3.56}$$

where

$$\begin{aligned} J_p &= \sum_{k=0}^{p}\chi_k w_{p-k} \\ w_k &= (k+1)c_{k+1} \end{aligned}$$

w:

$$\begin{aligned} \frac{du}{dt}(4\frac{du}{dt}-1) &= \Big(\sum_{k=0}^{\infty}(k+1)a_{k+1}s^k\Big)\Big(4a_1 - 1 + 4\sum_{k=1}^{\infty}(k+1)a_{k+1}s^k\Big) \\ &= \sum_{p=0}^{\infty} W_p s^p \end{aligned} \tag{3.57}$$

where

$$\begin{aligned} W_p &= \sum_{k=0}^{p} h_k H_{p-k} \\ h_k &= (k+1)a_{k+1} \\ H_k &= 4(k+1)a_{k+1}\ (k \neq 0) \\ H_0 &= 4a_1 - 1 \end{aligned}$$

Hence

$$\frac{du}{dt}(4\frac{du}{dt}-1)w = \sum_{p=0}^{\infty} K_p s^p \tag{3.58}$$

where

$$K_p = \sum_{k=0}^{p} W_k c_{p-k}$$

Finally, substituting Eqs. (3.50),(3.56) and (3.58) into Eq. (3.49) gives

$$\sum_{p=0}^{\infty}\Big(\gamma_p + J_p + K_p\Big)s^p = 0$$

and thus

$$\gamma_p + J_p + K_p = 0$$

Recalling that $\gamma_p = \sum_{k=0}^{p}\mu_k\nu_{p-k}$ shows

$$\mu_0\nu_p + \sum_{k=1}^{p}\mu_k\nu_{p-k} + J_p + K_p = 0$$

while the definitions of μ_0 and ν_0 then give

$$4t_0^2 a_2(p+2)(p+1)c_{p+2} + \sum_{k=1}^{p} \mu_k \nu_{p-k} + J_p + K_p \quad = \quad 0$$

Simplifying the above expression, we arrive at the recurrence formula for c_p:

$$c_p \quad = \quad -\Big(\frac{1}{4t_0^2 a_2(p)(p-1)}\Big)\Big(\gamma_{p-2} + J_{p-2} + K_{p-2}\Big) \tag{3.59}$$

We note that setting $p = 2$ gives an explicit expression for c_2

$$8t_0^2 a_2 c_2 + J_0 + K_0 = 0$$

Recall the definitions of J_0 and K_0

$$\begin{aligned}
J_0 &= \chi_0 w_0 \\
&= (-\frac{a_1}{2} + \tau_0 + \epsilon_0)c_1 \\
&= (-\frac{a_1}{2} + \overline{\mu}_0\overline{\nu}_0 + f_0 g_0)c_1 \\
&= (-\frac{a_1}{2} + (a_0 - t_0 a_1)(8a_1 - 1) + t_0 a_1(1 - 4a_1))c_1
\end{aligned}$$

$$\begin{aligned}
K_0 &= W_0 c_0 \\
&= h_0 H_0 c_0 \\
&= a_1(4a_0 - 1)c_0
\end{aligned} \tag{3.60}$$

And so

$$c_2 = \frac{1}{16a_2 t_0^2}\Big(2c_1(a_0 - 2t_0 a_1) + a_1((2 - 8a_0)c_0 + c_1 - 16a_0 c_1 + 24c_1 t_0 a_1)\Big) \tag{3.61}$$

3.5.2 *'Solutions of the Third Painlevé'*: Segment 3

As before, this segment corresponds to section 3.5.1. However, there are 2 distinct cases to be considered here. The first case corresponds to expanding $w(t)$ about $t = 0$ as depicted in Segment 3.1, while the second case corresponds to expanding $w(t)$ about a general point $t_0 = s_0 + is[n]$ as depicted in Segment 3.2.

Segment 3.1: Recursive solution involving ξ expanded about t=0

Below are the definitions of the variables used with references to the variables seen in section 3.5.1.
Definitions:

$$ud(s) \leftrightarrow w(t),\ a[k,n] \leftrightarrow a_k,\ derivd[p] \leftrightarrow c_p,\ xvt \leftrightarrow \nu_k,\ xbh \leftrightarrow H_k,\ xg \leftrightarrow g_0$$

Recursive definition for the coefficient of $ud(= w(t))$ expanded about $t = 0$

From Eq. (3.48), we note that the coefficients c_p in $w(t)$ can in fact be computed using the recurrence relation of d_{p+2} in Eq. (3.18). This is done by differentiating Eq. (3.18) w.r.t. d and multiplying d to it. Hence, the recursive definitions for c_p is in terms of d, where d is the coefficient belonging to the power series $u(t)$, have been computed previously in Segment 1. We named this definition as *derivd*[*p*] and use it to compute the first 150 coefficients belonging to the power series solution, $w(t)$ and the variable d is again assigned to $\frac{1}{3\pi}$ before storing as a table into the file *derivdcoe.dat*.

Initial condition

$ud(s)$ is defined to be equal to the power series $w(t)$. d_0 and d_1 are initialised using the power series $ud(s)$, computed numerically . Initialise the variables by the base case:

$$\nu_0 = 8a_1 - 1,\ H_0 = 8a_1 - 1,\ g_0 = 1 - 4a_1,\ d_0 = w(s_0),\ d_1 = \frac{dw(s_0)}{ds}$$

Power series expansion of the τ function for $p^+(s;1)$ about $t = 0$

The τ function we are interested in calculating here corresponds to the special case of $w(t)$ when $t = 0$. First define

$$\begin{aligned} xdiu(s) &= \int_0^s \frac{w(t)}{t} dt \\ &= \int_0^s \frac{ct^{\frac{3}{2}} + \sum_{k=0}^{\infty} c_p t^{2+\frac{p}{2}}}{t} dt \\ &= \frac{2}{3} cs^{\frac{2}{3}} + \sum_{p=0}^{\infty} \frac{c_p s^{\frac{p}{2}+2}}{\frac{p}{2}+2} \end{aligned} \tag{3.62}$$

In order to derive the recursive definition for the τ function in Mathematica, we recall from section 3.1, the equation $p^+(s;k)$ which is the τ function we seek. Thus, letting $k = 1$,

$$p^+(s;1) = p^+(s;0) + \frac{d}{ds}\frac{\partial}{\partial \xi}\Big\{ \exp\Big(- \int_0^{(\pi s)^2} u(t; \frac{1}{2}; \xi) \frac{dt}{t} \Big) \Big\}\Big|_{\xi=1}$$

Since the integration of u is w.r.t. t only, we can first differentiate u w.r.t. ξ and replace it by w, given that $w = \frac{\partial u(t,\xi)}{d\xi}\Big|_{\xi=1}$. Also, noting that $p^+(s;0)$ is in fact equal to $p^+(s)$, we substitute Eq. (3.3) into the above equation. And so, $p^+(s;1)$ equates to

$$-\frac{d}{ds}\Big\{\exp\Big(-\int_0^{(\pi s)^2} u(t)\frac{dt}{t}\Big)\Big\} - \frac{d}{ds}\Big\{\Big(\exp\Big(-\int_0^{(\pi s)^2} u(t)\frac{dt}{t}\Big)\Big)\Big(\int_0^{(\pi s)^2} w(t)\frac{dt}{t}\Big)\Big\}$$
$$= -\frac{d}{ds}\Big\{\exp\Big(-\int_0^{(\pi s)^2} u(t)\frac{dt}{t}\Big) + \Big(\exp\Big(-\int_0^{(\pi s)^2} u(t)\frac{dt}{t}\Big)\Big)\Big(\int_0^{(\pi s)^2} w(t)\frac{dt}{t}\Big)\Big\} \quad (3.63)$$

Recalling from Segment 1, Series[exp($-\int_0^{(s)^2} u(t)\frac{dt}{t}$)] is defined to be $ff(s)$, whereas Series[$\int_0^{(s)^2} w(t)\frac{dt}{t}$] has been defined to be $xdiu(s)$. We sought to simplify the above τ function in terms of the corresponding Mathematica defintitions $ff(s)$ and $xdiu(s)$,

$$vvb(s^2) = \text{Series}[ff(s^2)(1 + xdiu(s^2))]$$

Hence, the Mathematica function equivalent of the τ function $p^+(s;1)$ is

$$dv(s) = \text{Series}[-\frac{d}{ds}vvb(s^2)] \quad (3.64)$$

Segment 3.2: Recursive solution involving ξ expanded about a general point

Recursive definitions

Definitions:

$$xu \leftrightarrow \mu_k,\ xv \leftrightarrow \nu_k,\ xga \leftrightarrow \gamma_p,\ xut \leftrightarrow \overline{\mu}_k,\ xvt \leftrightarrow \overline{\nu}_k,\ xt \leftrightarrow \tau_p$$

$$xf \leftrightarrow f_k,\ xg \leftrightarrow g_k,\ xe \leftrightarrow \epsilon_p,\ xxi \leftrightarrow \chi_p,\ xw \leftrightarrow w_k,\ xj \leftrightarrow J_p$$

$$xh \leftrightarrow h_k,\ xbh \leftrightarrow H_k,\ xbw \leftrightarrow W_p,\ xbk \leftrightarrow K_p,\ d \leftrightarrow c \text{ (coefficients belonging to } w(t))$$

The following segment is the recursive Mathematica definitions for Eqs. (3.50) to (3.59). In particular, the last recursive definition corresponds to the recurrence formula d_p as seen in Eq. (3.59) and calls upon the previously defined variables.

Power series expansion of the τ function for $p^+(s;1)$ about a general point

After computing the τ function for the first point of expansion, the next step is to extend the recursive definitions in this segment so as to compute the τ function for $p^+(s;1)$, similar to those carried out in section 3.4.2. To do so, we define a power series, similar to that of Eq. (3.34), about $s_0 + is[n]$:

$$xuz[t,n] = \sum_{k=0}^{\infty} d[k,n](t - (s_0 + is[n]))^k \quad (3.65)$$

and compute its first derivative w.r.t. t.
After substituting the power series solution xuz, Eq. (3.65), and the power series expansion of $\frac{1}{t}$,

Eq. (3.35), into Eq. (3.3) , the integrand of Eq. (3.3) becomes

$$\begin{aligned}\frac{xuz[t,n]}{t} &= \sum_{p=0}^{\infty} bigA[p,n](t-t_0)^p \sum_{k=0}^{\infty} d[k,n](t-t_0)^k \\ &= \sum_{k=0}^{\infty} xdat[k,n](t-t_0)^k \\ \text{where} & \\ xdat[k,n] &= \sum_{p=0}^{k} d[p,n] bigA[k-p,n]\end{aligned}$$

Next, we define $xdiy[t,n] = \int_{t_0}^{t} \frac{xuz[x,n]}{x} dx$ so that

$$xdiy[t,n] = \sum_{k=0}^{\infty} \frac{xdat[k,n](t-t_0)^{k+1}}{k+1} \tag{3.66}$$

Again, the procedure must be iterated and the base cases d_0 and d_1 are reintialised with the numeric evaluations as below

$$\begin{aligned} d_0 &= \mathrm{N}[xuz[t_0, j], ac] \\ d_1 &= \mathrm{N}[xdz[t_0, j], ac]\end{aligned}$$

Calculating moments of the τ function

This section follows from section 3.4.2 under the same name of 'Calculating moments of the τ function'. However, in that section, we are interested in calculating the moments of the τ function for $u(t)$. Whereas, in this section, we are interested in calculating the moments of the τ function for $p^+(s;1)$ that is $m(n) = \int_0^\infty s^n p^+(s;1) ds$. We split the calculation of $m(n)$ into two distinct parts due to the same reasoning given in section 3.4.2. Hence, using Eq. (3.63), we denote the split function $xdtq(n)$ as

$$xdtq(n) = mj(n) + \sum_{0}^{\infty} xdjol[j,n]$$

where

$$mj(n) = \int_0^{\frac{\sqrt{s_0}}{\pi}} s^n p^+(s;1) ds \tag{3.67}$$

$$xdjol[j,n] = \int_{\frac{\sqrt{s_0+is[j]}}{\pi}}^{\frac{\sqrt{s_0+is[j+1]}}{\pi}} s^n p^+(s;1) ds \tag{3.68}$$

and

$$\begin{aligned} p^+(s;1) &= -\frac{d}{ds}\Big\{ \exp\Big(- \int_0^{(\pi s)^2} u(t)\frac{dt}{t}\Big) \\ &\quad + \Big(\exp\Big[- \int_0^{(\pi s)^2} u(t)\frac{dt}{t}\Big]\Big)\Big(\int_0^{(\pi s)^2} w(t)\frac{dt}{t}\Big)\Big\}\end{aligned}$$

About point of expansion s_0

The τ function in Eq. (3.67) has been computed in Segment 3.1 as being $dv(s)$. Hence, after replacing s with $\frac{s}{\pi}$, Eq. (3.67) becomes

$$mj(n) = -(\frac{1}{\pi})^n \int_0^{\sqrt{s_0}} s^n dv(t) dt \tag{3.69}$$

About point of expansion $s_0 + is[j]$

To be able to extend the calculations of the moment for the τ function, define a new power series

$$app[x^2, j] = \sum_{k=0}^{\infty} xdgia[k, j]\Big(x^2 - (s_0 + is[j])\Big)^k \tag{3.70}$$

where $j = 0, \ldots$ such that $app[x^2, j]$ equals to the terms inside the curly brackets of $p^+(s; 1)$

$$\begin{aligned} app[x^2, j] &= \exp\Big(-\int_0^{x^2} u(t)\frac{dt}{t}\Big) + \Big(\exp\Big[-\int_0^{x^2} u(t)\frac{dt}{t}\Big]\Big)\Big(\int_0^{x^2} w(t)\frac{dt}{t}\Big) \\ &= \exp\Big(-\int_0^{x^2} u(t)\frac{dt}{t}\Big) \times \Big\{1 + \int_0^{x^2} w(t)\frac{dt}{t}\Big\} \end{aligned}$$

Next, we try to define $app[x^2, j]$ in terms of previously defined Mathematica definitions by replacing the right hand side of the above equation.

Using Eq. (3.43), $s[x^2, j] = \exp(-\int_0^{x^2} u(t)\frac{dt}{t})$, expanded about $s_0 + is[j]$ and noting from the definition of $xdiy[t, n]$, Eq. (3.66), that

$$1 + \int_0^{x^2} w(t)\frac{dt}{t} = 1 + xdiu(s_0) + \sum_{n=0}^{j-1} xdiy[s_+ is[n+1], s_0 + is[n]] + xdiy[x^2, j] \tag{3.71}$$

which is to be denoted by $xds[x^2, j]$ gives

$$app[x^2, j] = xds[x^2, j] \times s[x^2, j]$$

Substitute $app[x^2, j]$ into $p^+(s; 1)$, we have

$$\begin{aligned} dpp[x, j] &= -\frac{d}{dx} app[x^2, j] \\ &= \sum_{k=0}^{\infty} xdgia[k, j]k(2x)\Big(x^2 - (s_0 + is[j])\Big)^{k-1} \end{aligned} \tag{3.72}$$

We note that in $app[x^2, j]$, replacing x with s, the upper limit is s^2, instead of $(\pi s)^2$ as seen in Eq. (3.68). And so, we need to replace s with $\frac{s}{\pi}$ in Eq. (3.68) to be able to use those Mathematica definitions. Hence, the contribution for the nth moment of the τ function for $p^+(s; 1)$ becomes

$$\begin{aligned} xdjol[k, n] &= (\frac{1}{\pi})^n \Big\{\int_{\sqrt{s_0 + is[k]}}^{\sqrt{s_0 + is[k+1]}} s^n \Big[-\frac{d}{ds}\Big\{\exp\Big(-\int_0^{s^2} u(t)\frac{dt}{t}\Big) \times \Big\{1 + \Big(\int_0^{s^2} w(t)\frac{dt}{t}\Big)\Big\}\Big] ds\Big\} \\ &= (\frac{1}{\pi})^n \Big\{\int_{\sqrt{s_0 + is[k]}}^{\sqrt{s_0 + is[k+1]}} s^n dpp[s, j] ds\Big\} \end{aligned} \tag{3.73}$$

Similar algebraic manipulations as seen from Eq. (3.45) to Eq. (3.46) are performed on Eq. (3.73), where the upper limits of the integration is now set as $xder(j) = \frac{s0 + is[j+1]}{s0 + is[j]} - 1$. So using the incomplete beta function, Beta$[z, a, b]$, in the same manner as before, $xdjol[k, n]$ becomes

$$\begin{aligned} xdjol[k, n] &= (\frac{1}{\pi})^n \sum_{k=0}^{\infty} (-1)^k k \times xdgia[k, j](s0 + is[j])^{\frac{n}{2} + k} \\ &\quad \times \mathrm{Beta}[-xder[j], k, 1 + \frac{n}{2}] \end{aligned} \tag{3.74}$$

Finally, substitute Eqs. (3.68) and (3.74) into the equation of $xdtq(n)$, we compute the corresponding nth moment for $p^+(s;1)$ and tabulate the results.

Computation of the nth moment	
n	$\mathbf{p^+(s;1)}$
0	0.9999999999999999999999999219306
1	1.76454678940017569780582656626
2	3.26498044185987864525719869820
3	6.3064271456737087320662405114
4	12.668147588875690318749478761

3.6 Comparing Wigner approximation with exact results '*Solutions of the Third Painlevé*': Segment 4

In this section, we seek to compare the Wigner approximation $p^{+w}(s)$ and the exact form $p^{+}(s)$ using two different methods, the first is by comparing the graphs and the second by comparing their respective moments.

In section 2.1.3, the Wigner approximation $p^{+w}(s)$ is derived under the assumption that the first moment $\int_0^\infty sp^{+w}(s)ds = 1$. Whereas, in Segment 2 'Computing moments', the first moment of the exact form, $p^{+}(s)$ is calculated to be $\int_0^\infty sp^{+}(s)ds \approx 0.7827$.
Therefore, in order to compare the 2 resultant graphs of $p^{+w}(s)$ and $p^{+}(s)$, we need to introduce a scale factor α into $p^{+w}(s)$:

$$p^{+w}(s)ds = \frac{32}{\pi^2}(\frac{s}{\alpha})^2 \exp\Big(-\frac{4}{\pi}(\frac{s}{\alpha})^2\Big)\frac{ds}{\alpha}$$

such that $\int_0^\infty p^{+w}(s)ds$ remains equal to 1, while its first moment becomes

$$\int_0^\infty sp^{+w}(s)ds = \alpha = \int_0^\infty sp^{+}(s)ds$$

Hence, choosing $\alpha = 0.7827$, $p^{+w}(s)$ becomes

$$\frac{1}{0.7827}\frac{32}{\pi^2}(\frac{s}{0.7827})^2 \exp\Big(-\frac{4}{\pi}(\frac{s}{0.7827})^2\Big)$$

By replacing the scaled equation $p^{+w}(s)$ into the procedure of section 2.1.3 and superimpose it onto the graph of $p^{+}(s)$, we are now able to compare the two graphs as shown in '*Solutions of the Third Painlevé*'. A colour printed version of the graphs is provided on the page after '*Solutions of the Third Painlevé*', where the graph in red corresponds to the Wigner approximation $p^{+w}(s)$ and the one in black $p^{+}(s)$.

The moments of the scaled $p^{+w}(s)$ can be evaluated exactly. Given the nth moment is $\int_0^\infty s^n p^{+w}(s)ds$, substitute the above formula for $p^{+w}(s)$,

$$\int_0^\infty s^n\Big\{\frac{1}{\alpha}(\frac{32}{\pi^2})(\frac{s}{\alpha})^2 \exp\Big(-\frac{4}{\pi}(\frac{s}{\alpha})^2\Big)\Big\}ds = \alpha^n \int_0^\infty s^n \frac{32}{\pi^2}s^2 \exp\Big(-\frac{4}{\pi}s^2\Big)ds$$

Let $\frac{4}{\pi}s^2 = t$, $\frac{8}{\pi}sds = dt$, then

$$\begin{aligned}\int_0^\infty s^n p^{+w}(s)ds &= \alpha^n(\frac{\pi}{8})(\frac{32}{\pi^2})\int_0^\infty (\frac{\pi t}{4})^{\frac{n+1}{2}}\exp(-t)dt \\ &= \alpha^n(\frac{4}{\pi})(\frac{\pi}{4})^{\frac{n+1}{2}}\int_0^\infty t^{\frac{n+1}{2}}\exp(-t)dt\end{aligned}$$

where $\int_0^\infty t^{\frac{n+1}{2}}\exp(-t)dt$ is in the form of a known Γ function, $\Gamma(z) = \int_0^\infty t^{z-1}\exp(-t)dt$, with z replaced by $\frac{n+3}{2}$ giving

$$\int_0^\infty s^n p^{+w}(s)ds = \alpha^n(\frac{\pi}{4})^{\frac{n-1}{2}}\Gamma(\frac{n+3}{2}) \tag{3.75}$$

Computation of the nth moment		
n	$\mathbf{p^{+w}(s)}$	$\mathbf{p^{+}(s)}$
1	0.7827	0.7827
2	0.7217	0.7192
3	0.7532	0.7457
4	0.8681	0.8516
5	1.0872	1.0540

Concluding remarks: Further computations

In this thesis, a high precision computation of the p.d.f. $p^+(s)$ as defined by Eq. (3.3) has been performed. Similarly, an analogous computation of $p^+(s;1)$ has been performed. The method of computation has relied on high precision power series solution of the particular *Painlevé* equation, Eq. (3.1), subject to the boundary conditions, (3.7) and (3.9).

By retaining 150 coefficients in the power series and expanding the solution about 802 points of expansion with overlapping radii of convergence, moments of these distributions have been obtained to an accuracy of around 48 decimals for $p^+(s)$ and 31 decimals for $p^+(s;1)$.

This same program could be carried out for the p.d.f.'s $p^-(s)$, $p_1(s)$ and $p_2(s)$ as specified by Eqs. (3.2), (3.4) and (3.5). Similarly, $p^-(s;1)$ could be similarly computed. With more effort $p^+(s;k)$ and $p^-(s;k)$ could be computed for $k \geq 2$. However, this involves coupling $k+1$ dependent variables (recall Eq. (3.49) in the case $k=1$) which makes the programming more complex. Also, for each higher value of k, our experience in going from computing $p^+(s)$ to $p^+(s;1)$ is that the accuracy will decrease.

Appendix A

A.1 Some theorems for ***Hermitian*** matrices

The follwing results are well known (see e.g.[8]).

Theorem 1 ***Let*** H ***be a*** $N \times N$ **Hermitian matrix.** ***Then all eigenvalues of*** H ***are real numbers.***

Proof: Let $H\mathbf{v} = \lambda\mathbf{v}$ *where* λ *is the eigenvalue and* $\mathbf{v}$ *is a column eigenvector* $\begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$ *(assume normalized). Then, using* $\mathbf{v}^t = (\mathbf{v}^*)^T = (v_1^*, \ldots, v_M^*)$*,*

$$\begin{aligned} \mathbf{v}^t H\mathbf{v} &= (\mathbf{v}^*)^T \lambda\mathbf{v} \\ &= \lambda(\mathbf{v}^*)^T\mathbf{v} \\ &= \lambda \end{aligned} \tag{A.1}$$

The final line follows because $(\mathbf{v}^*)^T\mathbf{v}$ *is the complex dot product,*

$$\begin{aligned} (\mathbf{v}^*)^T\mathbf{v} &= |v_1|^2 + \ldots + |v_N|^2 \\ &= 1 \end{aligned}$$

for $\mathbf{v}$ *normalised. But from* $H\mathbf{v} = \lambda\mathbf{v}$ *and the fact that* H *is* Hermitian*, we have*

$$\begin{aligned} \mathbf{v}^t H &= (H\mathbf{v})^t \\ &= (\lambda\mathbf{v})^t \\ &= \lambda^*(\mathbf{v}^*)^T \\ &= \lambda^*\mathbf{v}^t \end{aligned}$$

Substitute that into LHS of Eq. (1):

$$\begin{aligned} \mathbf{v}^t H\mathbf{v} &= \lambda^*\mathbf{v}^t\mathbf{v} \\ &= \lambda^* \end{aligned}$$

and consequently $\lambda = \lambda^*$ *which says the eigenvalues are real.*

Theorem 2 ***Let*** λ_1 ***and*** λ_2 ***be distinct eigenvalues of the*** $N \times N$ **Hermitian matrix** H ***with corresponding eigenvectors*** $\mathbf{v}$ ***and*** $\mathbf{w}$***. Then*** $\mathbf{v}$ ***and*** $\mathbf{w}$ ***are orthogonal.***

Proof:
To show they are orthogonal, we must show that the complex dot product $\mathbf{v}^t\mathbf{w} = 0$*.*
Using $H\mathbf{v} = \lambda_1\mathbf{v}$*, we have*

$$\mathbf{w}^t H\mathbf{v} = \lambda_1\mathbf{w}^t\mathbf{v}$$

But, since H *is* Hermitian

$$\begin{aligned} \mathbf{w}^t H\mathbf{v} &= (H\mathbf{w})^t\mathbf{v} \\ &= \lambda_2\mathbf{w}^t\mathbf{v} \end{aligned}$$

Since λ_1 *and* λ_2 *are distinct, comparing the two equations, we require* $\mathbf{v}^t\mathbf{w} = 0$*, as required.*

Appendix B

B.1 *'Solutions of the Third Painlevé'*

Appendix C

C.1 GAUENSEM.m

C.2 URM.m

C.3 ORM.m

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Fig. (2.1)

Fig. (2.2)

Fig. (2.3)

Fig. (2.4)

Fig. (2.5)

Fig. (2.1)

$$
\begin{aligned}
p^-(s;k) &= p^-(s;k-1) - \frac{(-1)^k}{k!}\frac{d}{ds}\frac{\partial^k}{\partial\xi^k}\Big\{\exp\Big(-\int_0^{(\pi s)^2} u(t;-\frac{1}{2};\xi)\frac{dt}{t}\Big)\Big\}\Big|_{\xi=1} \\
p^+(s;k) &= p^+(s;k-1) - \frac{(-1)^k}{k!}\frac{d}{ds}\frac{\partial^k}{\partial\xi^k}\Big\{\exp\Big(-\int_0^{(\pi s)^2} u(t;\frac{1}{2};\xi)\frac{dt}{t}\Big)\Big\}\Big|_{\xi=1}
\end{aligned}
$$

$$
u(t) = \sum_{p=0}^{\infty} a_p (t - s_0)^p
$$

$$
d_{p-1}, d_{p-2}, \ldots, d_0
$$

$$
a_{p-1}, a_{p-2}, \ldots, a_0
$$