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Equivalence of Construction Methods for Sturmian Words

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Introduction

In his book Liber Abaci, Leonardo Pisano, or Fibonacci, described rabbit breeding according to the rule that a baby rabbit matures in one month and produces a new baby rabbit every month thereafter. This can be represented using the substitution $a \mapsto A, A \mapsto Aa$, where a represents a baby rabbit, and A represents a mature rabbit. This rabbit breeding problem led to the famed Fibonacci sequence $1, 1, 2, 3, 5, \ldots$, where the n^{th} Fibonacci number, F_n is given by the recurrence $F_{n+1} = F_n + F_{n-1}$, with $F_0 = 1$, and is the total number of rabbits in successive months starting with one baby rabbit. This in turn led to the related Fibonacci substitution sequence $AaAAaAaAAa\ldots$ specifying the evolution of the rabbit population. The sequence has the feature that the ratio of A's to a's in a sequence of length n converges to $\phi = \frac{1+\sqrt{5}}{2}$, the golden ratio, as $n \to \infty$ [15].

The Fibonacci substitution sequence turns out to have many interesting properties that can be generalized to a wider class of sequences. Fibonacci words are two-distance, meaning that for each subword of length n they have at most two different weights. Also, for each of these lengths n, they have $n + 1$ distinct subwords, which is a property that defines Sturmian words. It is possible to produce the Fibonacci substitution sequence geometrically by looking at the sequence of intersections between a straight line of slope ϕ and the unit grid, which is known as a cutting sequence. Substitution sequences such as the Fibonacci substitution sequence may also be reduced using an inverse substitution, and these sequences are commonly called characteristic words, studied by Series [16] in relation to cutting sequences created from tesselations of the hyperbolic plane. That the Fibonacci substitution sequence has these properties implies that there is a relationship between them, and it is their equivalence that we seek to display in the first chapter, following Lunnon and Pleasants [11].

The study of doubly infinite binary sequences is important in the field of quasicrystals, and it has been shown that the one-dimensional analogue of Penrose tilings, a model for these quasicrystals, is the Fibonacci substitution sequence. If we wish to build this sequence up through substitution from an initial seed, it is necessary to define two words f^{lower} and f^{upper}, which are doubly infinite analogues of the Fibonacci substitution sequence and they differ at only two positions. The difference between these two words is displayed in the second chapter when we deal with mechanical words.

The second chapter of this thesis aims to answer a question posed by de Bruijn, "which sequences have infinitely many predecessors?", or equivalently, which sequences can be infinitely reduced with inverse transforms [3]. We look at classifying mechanical words into different equivalence classes, which we accomplish by studying different algebraic transformations which are analogous to the substitution rules examined in the first chapter. Our approach is different to that of Litvin and Litvin [9], as we show that the sets of subwords for different words with the same slope are the same.

The transformation θ_m also leads us to study continued fractions, which have in turn been studied since antiquity, and are associated with the work of Euclid. In particular, Euler's great memoir, De Fractionibus Continius, laid the groundwork for the modern theory in 1737 [13]. Allouche and Shallit investigate transformations of mechanical words s, and their relationship to the continued fraction expansions of the slope α of s.

$$\alpha = [m_0, m_1, m_2, m_3, \ldots] = m_0 + \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cfrac{1}{\cdots}}}}$$

This is accomplished by investigating the fixed points of sequences of θ_m transformations acting on mechanical words that intersect one grid point, but focus only on the right infinite word that begins at the first element to the right of the grid point [1]. We also extend this theory by deriving a formula for the fixed point of a sequence transformations acting on a word that doesn't intersect a grid point, and consequently the continued fraction expansions of α and ρ in the resulting word $s_{\alpha,\rho}$.

Chapter 1

Equivalence of models of doubly-infinite sequences

A sequence in which each member is chosen from a finite number of elements is often called a *word*, and it's elements *letters*. Such sequences have been studied by mathematicians for centuries, most famously (at least implicitly) by Leonardo of Pisa (c. 1170 -1250), or Fibonacci. His simple model of rabbit breeding, characterised by the growth of baby rabbits, a to adult rabbits A, and the birth of a new baby rabbit at each time step can be specified at the substitution rules $a \mapsto A, A \mapsto Aa$. Starting with one baby rabbit a, successive words can be generated. These have the property that the total number of elements after n steps is equal to the n^{th} Fibonacci number. After an infinite number of time steps, a semi-infinite word, known as the *Fibonacci word*, is obtained and it is invariant under the substitutions $a \mapsto A, A \mapsto Aa$.

More recently mathematicians have examined the properties of several different types of words and sequences, such as *characteristic words, cutting sequences, two-distance words*, and *linear sequences*. The aim of this chapter, following Lunnon and Pleasants [11], is to prove the equivalence of these words and to then show that they are *Sturmian* by showing that non-periodic two-distance words are in fact Sturmian words.

1.1 Doubly-infinite words

We define a *doubly-infinite word* to be a sequence that takes it's letters from a finite alphabet $A \subset \mathbb{Z}^+ \cup \{0\}$, say $\{0, 1\}$, and is infinite to the left and right of a fixed origin. If $|A| = 2$, we call the words created from A *binary*and we shall only consider binary words below. Similarly, a semi-infinite word is a sequence indexed by the positive integers. Hence it takes on values to the right of the origin only.

Probably the most well known semi-infinite word is the Fibonacci word, $f = 1011010\ldots$, generated by repeated application of the substitution rule $0 \mapsto 1, 1 \mapsto 10$ applied to the seed 1. To generate doubly infinite Fibonacci words, we can repeatedly apply the substitution $0 \mapsto 1, 1 \mapsto 10$ to an initial seed $\underline{1}1$, with the 0^{th} element underlined, to give

$$\begin{array}{c} \underline{1}1 \\ 1\underline{0}10 \\ 10\underline{1}101 \\ 1011\underline{0}10110 \\ 1011010\underline{1}10110101 \\ 1011010110110\underline{1}011010110110 \\ \vdots \end{array}$$

Here the rule for the location of the origin in successive substitutions is that it is equal to the rightmost symbol of the substitution mapping of the previously underlined element. The resulting word appears to be invariant under the substitution, except at the positions $n = -1, 0$, where it alternates between 10 and 01. With this in mind, we define the *lower doubly infinite Fibonacci word*, f^{lower} to be the word with 10 in positions $n = -1, 0$, and the *upper doubly infinite Fibonacci word* f^{upper} to be the word with 01 in positions $n = -1, 0$. We shall see in Chapter 2 that these words are so named due to their relationship with mechanical words.

The Fibonacci word exhibits a number of special features which in fact apply to a wider class of words. One feature relates to its complexity. To make this notion precise, denote by $p(s, n)$ the number of distinct subwords of length n of a word s. For the Fibonacci word, f, the possible subwords of length n, for n small, are as given in the following table.

n	allowed subwords
2	10, 01, 11
3	101, 011, 010, 110
4	1010, 1011, 0110, 0101, 1101
5	10101, 10110, 01101, 01011, 11010, 11011

Thus it appears that $p(f,n) = n+1$ and in particular that $p(f,n)$ is strictly increasing. In fact, this latter feature is necessary for the Fibonacci word to be non-periodic.

Theorem 1.1.1. *Suppose there exists an m such that*

$$p(s,m) = p(s,m+1).$$

Then the doubly infinite word, s, must be periodic.

Proof. Suppose $p(s,m) = p(s,m+1)$, and let $p(s,m) = r$.

Consider a fixed subword of length *m*, say

$$x = a_1 a_2 \dots a_m$$

Then,

$$\text{'every occurrence of } x \text{ must be followed by the same letter'}, \tag{1.1.1}$$

as otherwise the equality $p(s,m) = p(s,m+1)$ would not hold.

Consider now the $r+1$ strings of letters

$$\begin{array}{ccc} b_0 b_1 & \dots & b_{m-1} \\ b_1 b_2 & \dots & b_m \\ & \vdots & \\ b_r b_{r+1} & \dots & b_{r+m-1} \end{array}$$

Since $p(s, m) = r$, these $r + 1$ subwords cannot all be distinct. Hence there must exist indices *i, j* such that for $0 \leq i < j \leq r$ we have

$$b_i \dots b_{i+m-1} = b_j \dots b_{j+m-1}$$

But according to (1.1.1), $b_{i+m} = b_{j+m}$, and so

$$b_{i+1} \dots b_{i+m} = b_{j+1} \dots b_{j+m}$$

Repeating the argument shows that

$$b_{i+l} = b_{j+l}, \qquad \text{for all } l \geq m$$

or in other words

$$b_l = b_{(j-i)+l}, \qquad \text{for all } l \geq m + i,$$

telling us that the word is, for $l \geq m + i$, periodic to the right of period $(j - i)$.

We see that it is also true that

‘every occurrence of *x* must be preceded by the same letter’ (1.1.2)

Repeating the above argument, but now using (1.1.2), we see that

$$b_{i-l} = b_{j-l}, \qquad l = 1, 2, \dots$$

and so

$$b_{-l} = b_{j-i-l}, \qquad l = 1, 2, \dots$$

telling us that, for $l = 1, 2, \dots$, the word is periodic to the left of period $(j - i)$.

Since the choice of origin is completely arbitrary, ‘left’ and ‘right’ overlap. We conclude that the word is periodic. ■

Corollary 1.1.2. *For a non-periodic binary word s,*

$$p(s, n) \geq n + 1.$$

Proof. Let s be a non-periodic binary word. Therefore from Theorem 1.1.1 we require that $p(s, n-1) < p(s, n)$, or equivalently that

$$p(s, n-1) + 1 \leq p(s, n), \tag{1.1.3}$$

as otherwise s would be periodic. Repeatedly making use of (1.1.3) tells us that

$$p(s, 1) + n - 1 \leq p(s, n),$$

and as the word is binary, $p(s, 1) = 2$, so we have the required result. ■

Words that satisfy Corollary 1.1.2, but with strict equality, are a special type of word which we shall call *Sturmian words*. These words have studied extensively due to their connection to quasicrystals, and we shall look at special properties of these words later.

1.2 Two-distance words

Define the *weight* of a subword to be the number of 1's it contains. A sequence is said to be *two-distance* if for every length l there are at most two weights for subwords of length l.

In relation to the Fibonacci subwords listed in the previous table, the weights are read off as

l	weights
2	1, 1, 2
3	2, 2, 1, 2
4	2, 3, 2, 2, 3
5	3, 3, 3, 3, 3, 4

This is consistent with the Fibonacci word being two-distance. A basic property relating to the notion of weights is the following.

Lemma 1.2.1. *Suppose that for a given length l, there are subwords of weights w_1 and w_2 where $w_1 < w_2$. Then there are subwords of all weights $w_1, w_1 + 1, \ldots, w_2$.*

Proof. The weights of two subwords with starting points which differ by 1 can differ by at most 1. Hence, by moving from the subword of weight w_1 by a shift of 1 at a time, all subword weights between w_1 and w_2 must be encountered. ■

Corollary 1.2.2. *For a doubly-infinite word s,*

(i) If s is two-distance, then for every length l there is a number $w(l)$ such that every subword of length l has weight either $w(l)$ or $w(l) + 1$.

(ii) If s is not two-distance then there exists an l such that there are subwords of length l with weights $w(l)$ and $w(l) + 2$.

1.3 Characteristic words

Substitution such as that used to define the Fibonacci word is a well-used method for the creation of doubly-infinite two-distance words. de Bruijn [3] called this *deflation* for it's relation to Penrose tilings. We can also look at the opposite of this, *inflation*, which involves applying this substitution in reverse. Obviously, not all substitutions produce two-distance sequences.

A reverse of a substitution is to remove the element directly before isolated elements. For example, $\ldots 000100100 \ldots$ would become $\ldots 00\emptyset 10\emptyset 100 \ldots = \ldots 0010100 \ldots$ by removing the 0 before every isolated 1.

With this in mind, *characteristic words* can be defined in terms of the concept of certain reductions.

(i) For a word without consecutive 1's, define the *0-reduction* to be the word obtained by removing the 0 immediately before each 1.

(ii) For a word without consecutive 0's, define the *1-reduction* to be the word obtained by removing the 1 immediately before each 0.

(iii) A word is said to be *characteristic* if there is an infinite chain of reductions.

This is closely related to the two-distance property. In fact the following result holds.

Theorem 1.3.1. *Every two-distance word is characteristic.*

Proof. A two-distance word, s, either has no consecutive 0's or no consecutive 1's for if it did there would be weights 0 and 2 for words of length 2. Suppose it has no consecutive 1's, so that it can be 0-reduced. We wish to show that the resulting word s' is two-distance.

Proceeding by contradiction, we suppose that s' is not two-distance. Then there are subwords w_1 and w_2 of lengths l in s' with weights w and $w + 2$, from Corollary 1.2.2(ii). In w_2, we insert an extra 0 between each pair of 1's. This gives us a word in s of length $l + w + 1$ of weight $w + 2$. In w_1, insert an extra 0 between every pair of 1's and also to the left of the leftmost 1, and to the right of the rightmost 1. This gives a word in s of length $l + w + 1$ and of weight w. There cannot be words in s of the same length of weights w and $w + 2$, so we have a contradiction. ■

1.4 Cutting sequences

Similar to the billiards problems presented by Kinsley and Moore [7] where a billiard ball is struck with an initial direction on a frictionless billiard table, *cutting sequences* can be thought of as the sequence created by labelling the horizontal and vertical sides of a square billiard table with two distinct labels, and studying the sequence of collisions between the ball and the sides of the table.

Consider the square grid of vertical and horizontal lines through integer points in $\mathbb{R}^2$. On an arbitrary line L of positive slope mark the points where it crosses the grid lines and label them 0 for a vertical grid line, and 1 for a horizontal grid line. For lines that pass through the intersection of two grid lines mark them either 01 or 10. It is not important which convention you choose, as long as it is consistent, and the significance of this statement will be shown in Chapter 2. The sequence created by these intersections is our *cutting sequence*.

We observe that this sequence will always have one of the elements 0 or 1 only appearing in isolation. Also, there are either $\lfloor \alpha \rfloor$ or $\lfloor \alpha \rfloor + 1$ of the non-isolated element appearing between consecutive isolated elements, where α is the slope of the line L, and $\lfloor x \rfloor$ is the floor function.

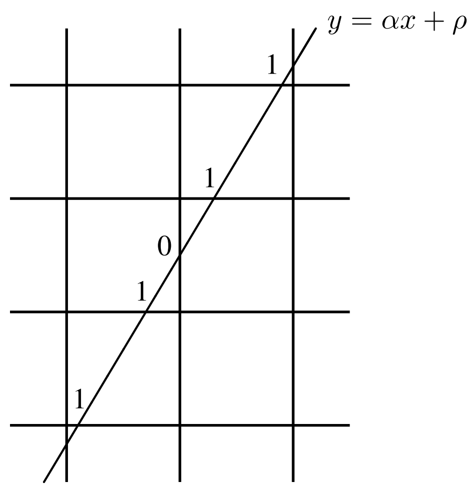

Figure 1.1: A cutting sequence $\ldots 11011 \ldots$

This is equivalent to saying that a cutting sequence created by the line L can be uniquely factored into parts that are either 10^n or 10^{n-1} if $\alpha < 1$, and 01^n and 01^{n-1} if $\alpha > 1$, where $n = \lfloor \alpha \rfloor + 1$.

We see that this is analogous to folding out infinitely many copies of our square billiard table into a square grid in $\mathbb{R}^2$ and striking the ball again, where the horizontal sides of the billiard table were labelled with a 1, and the vertical sides with a 0. As for characteristic sequences, this idea is closely related to the two-distance property, which we see in the following theorem.

Theorem 1.4.1. *Every cutting sequence is two distance*

Proof. Let S be the cutting sequence associated with the line L in the plane. Let W be a subword in S of length l and weight w. Let M be the segment of the line L whose end-points are the points corresponding to the first and last symbols of W.

Let u be the length of the projection M onto the x-axis. Then

$$(\#\text{of 0's in } W) - 1 \leq u < (\#\text{0's in } W) + 1.$$

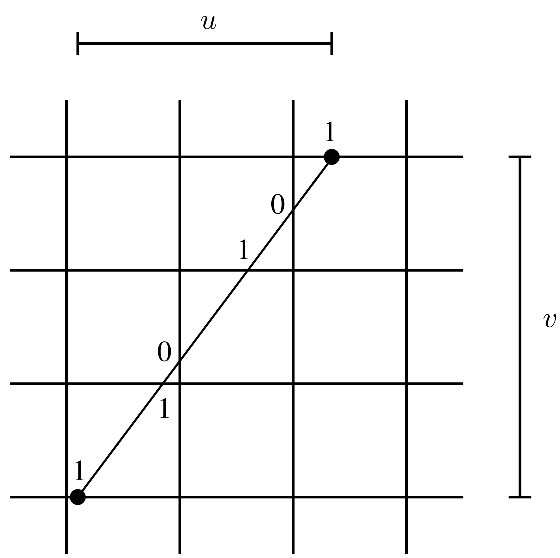

Figure 1.2: A line segment M

but # 0's in $W = l - w$, so

$$l - w - 1 \leq u < l - w + 1.$$

Since the set up is such that $u > 0$, this inequality can be improved for $w = l$ and $w = l - 1$ to

$$\begin{aligned} 0 < u < 1, &\qquad w = l, \\ 0 < u < 2, &\qquad w = l - 1. \end{aligned}$$

Let v be the length of the projection of M onto the y-axis. Then

$$(\text{\#of 1's in } W) - 1 \leq v < (\text{\#1's in } W) + 1.$$

But, we know that there are w 1's in W by definition, so

$$w - 1 \leq v < w + 1$$

Since $v > 0$, we can see that for $w = 0$ and $w = 1$ this can be improved to

$$0 < v < 1, \qquad w = 0,$$
$$0 < v < 2, \qquad w = 1.$$

Now for $w \neq l, l-1$ the inequalies

$$l - w - 1 \leq u \text{ and } v < w + 1$$

together imply

$$\frac{v}{u} < \frac{w+1}{l-w-1} \tag{1.4.1}$$

Similarly the inequalities

$$w - 1 \leq v \qquad \text{and} \qquad u < l - w + 1$$

together imply

$$\frac{w-1}{l-w+1} < \frac{v}{u} \tag{1.4.2}$$

Now, with l fixed let w_1 and w_2 be two different weights, $w_1 < w_2$. Substituting w_1 in (1.4.1) and w_2 in (1.4.2) it follows that

$$\frac{w_2 - 1}{l - w_2 + 1} < \frac{w_1 + 1}{l - w_1 - 1}, \qquad w_1 \neq l - 1.$$

Multiplying both sides by the positive number $(l - w_1 - 1)(l - w_2 + 1)$ and cancelling terms gives

$$w_2 - 1 < w_1 + 1 \tag{1.4.3}$$

For $w_2 = w_1 + 1$ (1.4.3) obviously holds true. Conversely, for $w_2 > w_1 + 1$ the inequality in (1.4.3) implies that all integers $\geq w_1 + 1$ are less than $w_1 + 1$, which is a clear contradiction, so we must have that $w_2 = w_1 + 1$, and so the cutting sequence is two-distance. ■

1.4.1 Linear sequences

If a subword can be obtained from a finite portion of a cutting sequence, we shall call it *linear*. Therefore if we concatenate an infinite number of linear sequences that belong to the same cutting sequence we can recreate that cutting sequence, or a translation of it, provided it is still two-distance. We shall now prove that the subwords of a characteristic sequence are linear, and therefore the corresponding infinite sequence must also be linear.

Lemma 1.4.2. *If a subword W is linear, then so are $W|_{0\mapsto 10}$ and $W|_{1\mapsto 01}$.*

Proof. Let L be a line segment corresponding to W. If W begins with a 0, let us adopt the convention that L begins just above a horizontal grid line. If W begins with a 1, we adopt the convention that L begins just to the right of a vertical grid line. If W ends with a 0, let it stop on a vertical line, and if W ends with a 1, let it stop on a horizontal grid line. With these conventions, we can now consistently join line segments.

Now suppose that L undergoes the linear transformation $(x, y) \mapsto (x, x + y)$. We consider separately portions of L between successive vertical lines, extended downwards to just below the nearest horizontal line if it starts with a 0, and extended upwards to just on the closest horizontal line if it ends with a 1.

Due to the invariance of the lattice under translations by integer multiples, we can consider the transformation as leaving the intercept with the first vertical line it crosses unchanged, and shifting up by one unit the intercept with the second vertical line it crosses, as we are considering separate portions of L that are one unit wide in the x direction. From this, we see that in all cases the effect of the transformation is the substitution $0 \mapsto 10$. In particular, this tells us that $W|_{0\mapsto 10}$ is linear.

To study the substitution $W|_{1\mapsto 01}$, we suppose that L undergoes the linear transformation $(x, y) \mapsto (x + y, y)$. We consider separately portions of L between successive horizontal lines, extended downwards to just before the closest vertical line if it starts with a 1, and extended upwards to the closest vertical line if it starts with a 0.

Once again, the invariance of the lattice under transformations by integer multiples shows that the transformation can be considered as leaving the intercept with the first horizontal line it crosses unchanged, and shifting the intercept with the second horizontal line it crosses by one unit. We see that the effect of the transformation is the substitution $1 \mapsto 01$, showing that $W|_{1\mapsto 01}$ is linear. ∎

Now that we have proven that the transformations $W|_{0\mapsto 10}$ and $W|_{1\mapsto 01}$ produce linear sequences, we are able to use them as inverse reductions to prove the following Theorem.

Theorem 1.4.3. *Every subword of a characteristic sequence is linear, and consequently every characteristic sequence is linear.*

Proof. By definition, a characteristic sequence has an infinite descending chain of reductions $r_1, r_2, \ldots$ where each r_j corresponds to $10 \mapsto 0$ or $01 \mapsto 1$.

Focus attention now on a particular subword and apply this chain of reductions, but now according to the interpretation $10 \mapsto \not{1}0$ or $01 \mapsto \not{0}1$, where we are removing the elements before the 'isolated' 0's or 1's. Therefore, we are thinking of the elements in a subword retaining their position, unless (or until) they are deleted by the reduction procedure. This must result in either

(i) deleting all the elements in a subword;

(ii) the only remaining elements are all 0's or all 1's, as are all the remaining elements in the word itself.

In situation (ii) the original word must have been $(0^k1)^\infty$ or $(1^k0)^\infty$ which are both linear.

In situation (i), locate the closest remaining element on the left (for definiteness). This element is itself linear. The subword can be reconstructed by applying the operations $r_p^{-1}, r_{p-1}^{-1}, \ldots, r_1^{-1}$, where p is the number of reductions it took to delete the subword. By Lemma 1.4.2 we see that this sequence of operations produces a linear sequence.

Suppose now that the subword W_i is chosen to have i elements on either side of the element at position 0, so that $|W_i| = 2i + 1$. Let L_i be the corresponding linear sequence which reproduces W_i. We then must have that the subword W_i is contained about position 0 in each $L_n, n \geq i$. Let L_∞ be the line constructed out of the point of accumulation (y_0, λ) of the y-intercepts, $0 \leq y_0 < 1$ and the slopes of the L_n, λ. Then the elements about the origin are those of L_n for L_n large enough, and thus those of the W_i, telling us that L_∞ corresponds to a word formed out of the subwords W_i. As every word formed purely out of characteristic subwords is characteristic itself, we see that every characteristic word must be linear. ■

Corollary 1.4.4. *Every characteristic sequence is a cutting sequence.*

1.5 Sturmian words

We define a *Sturmian word* to be an infinite word s, with the property that $p(s,n) = n+1$. We seek to show, following Lothaire [10], that non-periodic, binary two-distance words are Sturmian, but first we need an upper bound on the number of subwords in a set of two-distance binary subwords of length n.

If $u, v \in X_n$ where X_n is the set of subwords of some infinite word x of length n , we shall make use of the quantity $|w(u) - w(v)|$, where $|w(u) - w(v)| = 1$ if x is two-distance and $|w(u) - w(v)| \geq 2$ if x is not two-distance (from Corollary 1.2.2(ii)).

We see that for a doubly-infinite binary word s to be two-distance, we require that for any length n, it has subwords that are of weights differing by at most 1. This concept is clearly displayed in the following.

Lemma 1.5.1. *Let x be an infinite binary word. If the word x is not two-distance then there exists a word t such that $0t0$ and $1t1$ are both subwords of x.*

Proof. Let X_n be the set of subwords of x of length n. Assume x is not two-distance, and hence X_n is not two-distance for $n \geq n^*$, where n^* is the minimal length for the loss of the two-distance property. Let $u, v \in X_{n^*}$ be subwords of x that are not two-distance. As u and v are of finite length and n^* is minimal, their first and last letters are distinct. Therefore, we can assume, without loss of generality, that the first letter of u is 0, and the first letter of v is 1. Therefore, we can factorize u and v into $u = 0tau'$ and $v = 1tbv'$ for some words t, u' and v' and letters a and b, where $a \neq b$, again without loss of generality. We have

$$\begin{aligned} |w(u) - w(v)| &= |w(0tau') - w(1tbv')| \\ &= |w(0ta) - w(1tb)| + |w(u') - w(v')| \end{aligned}$$

If $b = 0$ and $a = 1$, $|w(0t1) - w(1t0)| = 0$, and this implies $|w(u) - w(v)| = |w(u') - w(v')|$, which contradicts the minimality of n^*. Therefore we must have that $a = 0$ and $b = 1$, and so

$$|w(u) - w(v)| = |w(0t0u') - w(1t1v')|.$$

Again enforcing minimality, we have that $u = 0t0$ and $v = 1t1$. ■

This now allows us to provide an upper bound on the number of subwords of length n of any infinite two-distance binary word.

Lemma 1.5.2. *Let x be an infinite two-distance binary word. Then,*

$$p(x, n) \leq n + 1.$$

Proof. Let X_n be the set of subwords of x of length n. The statement obviously holds for $n = 0, 1$ and it holds for $n = 2$ as X_2 cannot contain both 00 and 11 if x is two-distance.

We proceed by contradiction. Let $n \geq 3$ be the smallest integer such that the statement is false. Therefore, for our statement to be true, $p(x, n-1) \leq n$ and $p(x, n) \geq n + 2$.

The suffix of each x_n in X_n of length $n - 1$ must be in X_{n-1}. Using the pigeonhole principle, we see that there must exist two distinct words $u, u^* \in X_{n-1}$ such that the words $0u$, $1u$, $0u^*$ and $1u^*$ are elements of X_n.

Since $u \neq u^*$, there exists a word v such that $v0$ and $v1$ are prefixes of u and u^*. But, this means that $0v0$ and $1v1$ are both words in X_n, so by Lemma 1.5.1 X_n is not two-distance and we have a contradiction. ■

With an upper bound on the number of subwords of length n for any non-periodic two-distance word, we can now prove the following theorem.

Theorem 1.5.3. *Every non-periodic two-distance word, s, is Sturmian.*

Proof. If s is non-periodic, then $p(s, n) \geq n+1, \forall n$ by Corollary 1.1.2. If s is two-distance, then by Lemma 1.5.2 $p(s, n) \leq n + 1,\ \forall n$. Therefore s is Sturmian. ■

We now see that we have proven

Theorem 1.5.4. *For a non-periodic binary word s, the following statements are equivalent*

(i) *s is two-distance,*

(ii) *s is a cutting sequence,*

(iii) *s is linear,*

(iv) *s is characteristic,*

(v) *s is Sturmian.*

Proof. From Theorem 1.3.1, Theorem 1.4.1, Theorem 1.4.3, Corollary 1.4.4 and Theorem 1.5.3 the result follows. ■

Chapter 2

Mechanical words and the θ_m transformation

In this chapter transformation properties of mechanical words are studied. We seek to answer a problem proposed by de Bruijn [3], as a prelude to his study of Penrose tilings. For some specified substitution transformation, the problem is to classify characteristic sequences with the property that remain in the same equivalence class.

A *local equivalence class (LE-class)*, is a collection of sequences such that if S_1 and S_2 are in the LE-class, then all the subwords of S_1 occur in S_2 and all the subwords of S_2 occur in S_1. To appreciate the significance of this notion, we begin by making note of the following result.

Theorem 2.0.5. *Let $x_1 \neq x_2 \in (0, 1)$ and suppose α is irrational. Then the lines L_1 and L_2 with slope α passing through the points (x_1, y) and (x_2, y) define different sequences.*

In preparation for the proof of this result, a number of lemmas are required.

Lemma 2.0.6. *For all positive integers K there exists an integer $k \neq 0$ such that for any irrational number α, the fractional part of $k\alpha$, denoted $\{k\alpha\}$, is such that $\{k\alpha\} < \frac{1}{K}$.*

Proof. Divide $[0,1]$ into K subintervals of length $\frac{1}{K}$. Consider the numbers

$$0, \{\alpha\}, \{2\alpha\}, \dots, \{K\alpha\}.$$

Since there are $K+1$ numbers, and K subintervals, by the pigeonhole principle there is at least one subinterval containing two numbers. Hence there exists distinct positive integers q_1, q_2 such that

$$|\{q_1\alpha\} - \{q_2\alpha\}| \leq \frac{1}{K}.$$

Noting that

$$|\{q_1\alpha\} - \{q_2\alpha\}| = |\{(q_1 - q_2)\alpha\}|$$

the result follows with $|q_1 - q_2| = k$. ■

Lemma 2.0.7. *Let α be irrational. The sequence of fractional parts*

$$\{\alpha + \beta\}, \{2\alpha + \beta\}, \dots, \{n\alpha + \beta\}, \dots$$

is dense in $[0,1]$.

Proof. We must show that for any $x \in (0,1)$ and for any $\delta > 0$ such that $[x - \frac{\delta}{2}, x + \frac{\delta}{2}] \subset [0,1]$, there exists a positive integer k such that $\{k\alpha + \beta\} \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]$. By Lemma 2.0.6, there exists an integer m such that $\{m\alpha\} = \delta' < \delta$. Now, let j be an integer such that

$$j\delta' + \beta \in [x - \frac{\delta}{2}, x + \frac{\delta}{2}]$$

(note that the interval $[x - \frac{\delta}{2}, x + \frac{\delta}{2}]$ is of length δ, so such an interval can always be found). But $j\delta' + \beta = \{j\delta' + \beta\} = \{j\{m\alpha\} + \beta\} = \{jm\alpha + \beta\}$, so taking $k = jm$ gives the required result. ■

With the lemmas established, the proof of Theorem 2.0.5 can now be given.

Proof of Theorem 2.0.5. It is sufficient to show that there is always an grid point between the lines L_1 and L_2.

Let $x_2 > x_1$, and let X_1, X_2 be points on L_1, L_2 with the same y coordinates. If there is no grid point in between L_1 and L_2 then we must always have $\{X_2\} > \{X_1\}$. But

$X_2 = \frac{Y-\rho_2}{\alpha_2}$ for some Y. Lemma 2.0.7 tells us that Y can be chosen to be an integer such that $0 < X_2 < \delta$ for any δ. Choosing δ small enough implies $\{X_1\} > \{X_2\}$ and thus establishes the result. ■

Given two lines L_1 and L_2 with the same slope, by Theorem 2.0.5 we know that they must produce cutting sequences S_1 and S_2 respectively such that $S_1 \neq S_2$. Nonetheless, we can show that the sequences S_1 and S_2 belong to the same equivalence class.

Theorem 2.0.8. *Cutting sequences S_1 and S_2 defined by lines L_1 and L_2 are in the same LE-class if and only if they have the same slope.*

Proof. Suppose S_1 and S_2 are in the same LE-class. Let the proportion of $\#1's : \#0's$ be μ_1 and μ_2 respectively (note that these proportions are the slope of the lines). Suppose $\mu_1 \neq \mu_2$. Recalling that all cutting sequences are two-distance, and so have two possible weights for a word of length n, it must be that for some $n = n^*$ the two possible weights differ in S_1 and S_2. Let these weights be w_{11}, w_{12} for S_1 and w_{21}, w_{22} for S_2, with $w_{11} \neq w_{21}, w_{22}$. This means that the subword of length n^* with weight w_{11} is not in S_2, which contradicts S_1 and S_2 being in the same LE-class.

Suppose next that L_1 and L_2 both have slope μ, so that $\#1's : \#0's = \mu$ in both S_1 and S_2. If S_1 and S_2 are not in the same LE-class, then there is a subword of length n in S_1 which is not in S_2. Without loss of generality, let 1 be the symbol in S_1 which may occur consecutively. Let the number of times it can occur consecutively be k or $k+1$ in subwords of length n in L_1, and be j or $j+1$ in subwords of length n in L_2. By the assumption that there is a subword of length n in S_1 which is not in S_2, it must be that $j \neq k$. But this would mean that 1 occurs consecutively k or $k+1$ times in all subwords of length $n+m$ in $L_1, (m \geq 0)$, while it occurs consecutively j or $j+1$ times in all subwords of length $n+m$ in L_2. This would imply that $\#1's : \#0's$ is different for L_1 and L_2, giving a contradiction. ■

2.1 Mechanical words

The geometrical picture of a cutting sequence suggests a class of $\{0, 1\}$ sequences known as *mechanical words*. These words can be expressed as an arithmetic function of two real numbers, a form which is conducive for algebraic manipulation. In particular, it allows for the precise determination of the action of certain substitutions. In the following sections, we are mainly following Lothaire [10] and Allouche and Shallit [1].

Consider a straight line $y = \alpha x + \rho$ where $\alpha > 0$, and ρ is unrestricted. If we look at the intersections of this line and the lines of the grid $x = j, y = k$ where $(j, k \in \mathbb{Z})$, we get a doubly infinite sequence of intersection points $\dots P_{-1}P_0P_1\dots$ where P_n is the n^{th} intersection point from a point which we choose to be an origin, denoted $n = 0$. We see that if $P_n = (x_n, y_n)$ has $y_n \in \mathbb{Z}$, then $y = \alpha x + \rho$ intersects a horizontal line in the grid, and similarly if $x_n \in \mathbb{Z}$, then $y = \alpha x + \rho$ must intersect a vertical line.

There are two natural classes of sequences of points with integer coordinates which can be associated with $\{P_n\}$. One is the sequence $\{A_n\}$, where $A_n = (x_n, \lfloor y_n \rfloor)$ which is the closest lattice point at or below each vertical intersection. The other is the sequence $\{A'_n\}$, where $A'_n = (x_n, \lceil y_n \rceil)$ which is the closest lattice point at or above each vertical intersection. The two sequences of integer points are in turn associated with two binary sequences $\{s_{\alpha,\rho}(n)\}$ and $\{s'_{\alpha,\rho}(n)\}$ referred to as the *lower* and *upper mechanical words* respectively. These are defined, up to a constant, as the difference between successive y-coordinates in the sequences $\{A_n\}, \{A'_n\}$. Thus

$$\begin{aligned} s_{\alpha,\rho}(n) &= \lfloor y_{n+1} \rfloor - \lfloor y_n \rfloor - \lfloor \alpha \rfloor \\ &= \lfloor (n+1)\alpha + \rho \rfloor - \lfloor n\alpha + \rho \rfloor - \lfloor \alpha \rfloor \end{aligned} \tag{2.1.1}$$

and

$$\begin{aligned} s'_{\alpha,\rho}(n) &= \lceil y_{n+1} \rceil - \lceil y_n \rceil - \lfloor \alpha \rfloor \\ &= \lceil (n+1)\alpha + \rho \rceil - \lceil n\alpha + \rho \rceil - \lfloor \alpha \rfloor \end{aligned} \tag{2.1.2}$$

The cutting sequence introduced in Section$\{1.4\}$ can be expressed as a mechanical word. For definiteness, the convention that intersections with lattice points are denoted 10 will be adopted. We begin by introducing a further sequence of lattice points associated with

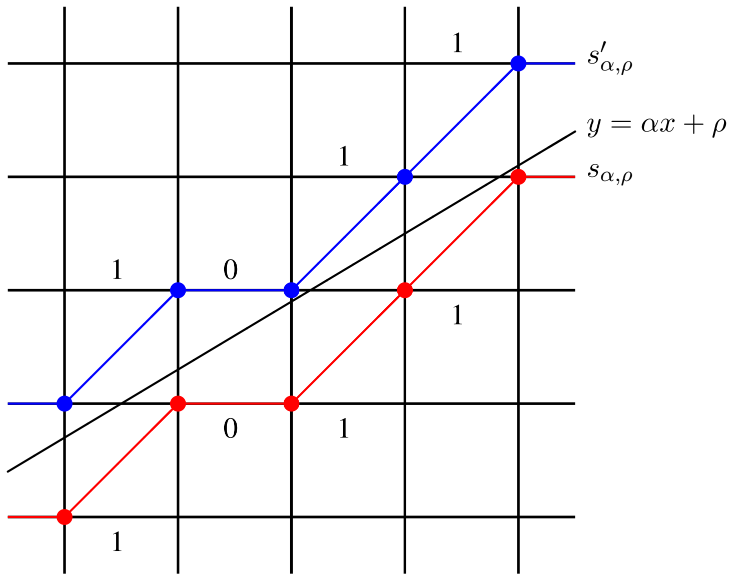

Figure 2.1: The mechanical words $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ when $y = \alpha x + \rho$ doesn't intersect a grid point

$\{P_n\}$. These are the points $\{B_n\}$, with $B_n = (u_n, v_n)$ such that B_n is the point below P_n if P_n corresponds to a vertical intersection, and is the point below and to the right if P_n is horizontal. We can write this as

$$B_n = (u_n, v_n) = \begin{cases} (x_n, \lfloor y_n \rfloor) & \text{if } P_n \text{ is a vertical cut,} \\ (\lceil x_n \rceil, y_n - 1) & \text{if } P_n \text{ is a horizontal cut} \end{cases}$$

Noting that B_{n+1} is either one unit above, or one unit to the right of each B_n, it follows

$$u_n + v_n = n, \ \forall n.$$

The motivation for introducing $\{B_n\}$ is the fact that the cutting sequence $\{K_{\alpha,\rho}(n)\}$ for the line $y = \alpha x + \rho$ is given by

$$K_{\alpha,\rho}(n) = v_{n+1} - v_n = 1 + u_n - u_{n+1} \tag{2.1.3}$$

and with the result of the following Lemma, we can display the equivalence between cutting sequences and mechanical words.

Lemma 2.1.1. *For $\alpha > 0$ and general ρ,*

$$v_n = \left\lfloor \frac{\alpha}{1+\alpha}\, n + \frac{\rho}{1+\alpha} \right\rfloor .$$

Proof. As the equation of the line is $y = \alpha x + \rho$, it follows that P_n is vertical if and only if

$$v_n \le u_n\alpha + \rho < 1 + v_n$$

which using $u_n + v_n = n$ can be rewritten as

$$v_n \le (n - v_n)\alpha + \rho < 1 + v_n.$$

This in turn is equivalent to

$$v_n(1+\alpha) \le n\alpha + \rho < 1 + (1+\alpha)v_n$$

which gives

$$v_n \le n\frac{\alpha}{1+\alpha} + \frac{\rho}{1+\alpha} < \frac{1}{1+\alpha} + v_n.$$

Hence the formula holds for vertical P_n.

We observe that P_n is horizontal if and only if

$$1 + v_n \le u_n\alpha + \rho < 1 + v_n + \alpha.$$

Since $u_n + v_n = n$ this can be rewritten as

$$1 + v_n \le (n - v_n)\alpha + \rho < 1 + v_n + \alpha$$

or equivalently

$$1 + (1+\alpha)v_n \le n\alpha + \rho < (1+\alpha)v_n + (1+\alpha)$$

which gives

$$v_n + \frac{1}{1+\alpha} \le \frac{\alpha}{1+\alpha}\, n + \frac{\rho}{1+\alpha} < v_n + 1.$$

Hence, the formula also holds for horizontal P_n, and so holds for all P_n. ∎

Proposition 2.1.2. *For $\alpha > 0$ and general ρ,*

$$K_{\alpha,\rho}(n) = s_{\frac{\alpha}{1+\alpha},\frac{\rho}{1+\alpha}}(n).$$

Proof. From (2.1.1), (2.1.3) and Lemma 2.1.1 the result follows. ∎

Another convention for cutting sequences passing through lattice points is to denote this by the two letters 01 instead of 10. In this case, with the cutting sequence denoted $\{K'_{\alpha,\rho}(n)\}$,

$$K'_{\alpha,\rho}(n) = v'_{n+1} - v_n$$

while the analogue of Lemma 2.1.1 gives

$$v'_n = \left\lceil \frac{\alpha}{1+\alpha}n + \frac{\rho}{1+\alpha} \right\rceil.$$

Consequently

$$K'_{\alpha,\rho}(n) = s'_{\frac{\alpha}{1+\alpha},\frac{\rho}{1+\alpha}}. \tag{2.1.4}$$

This formula is consistent with the fact that $s_{\alpha,\rho} = s'_{\alpha,\rho}$ for $\alpha n + \rho \notin \mathbb{Z}$, which in turn is immediate from the geometrical definition, and further that if $\alpha n + \rho = k \in \mathbb{Z}$, then $(s_{\alpha,\rho}(n-1), s_{\alpha,\rho}(n)) = (1,0)$, while $(s'_{\alpha,\rho}(n-1), s'_{\alpha,\rho}(n)) = (0,1)$.

Let's consider the particular cutting sequence $\{K_{\phi,\phi}(n)\}$ where ϕ is the golden ratio. It follows from Proposition 2.1.2 that

$$K_{\phi,\phi}(n) = \left\lfloor \frac{1}{\phi}(n+2) \right\rfloor - \left\lfloor \frac{1}{\phi}(n+1) \right\rfloor.$$

The first few members are

$$\begin{aligned} K_{\phi,\phi} &= (\ldots,0,0,\underline{1},1,2,3,3,4,\ldots) - (\ldots,-1,0,\underline{0},1,1,2,3,3,\ldots) \\ &= \ldots,1,0,\underline{1},0,1,1,0,1,\ldots \end{aligned}$$

where we have underlined the element corresponding to $n = 0$.

This seems to be the word f^{lower}, a fact which can be confirmed by studying the transformation properties of mechanical words with respect to substitution. Before this study can be undertaken, a variant of an upper mechanical word needs to be introduced.

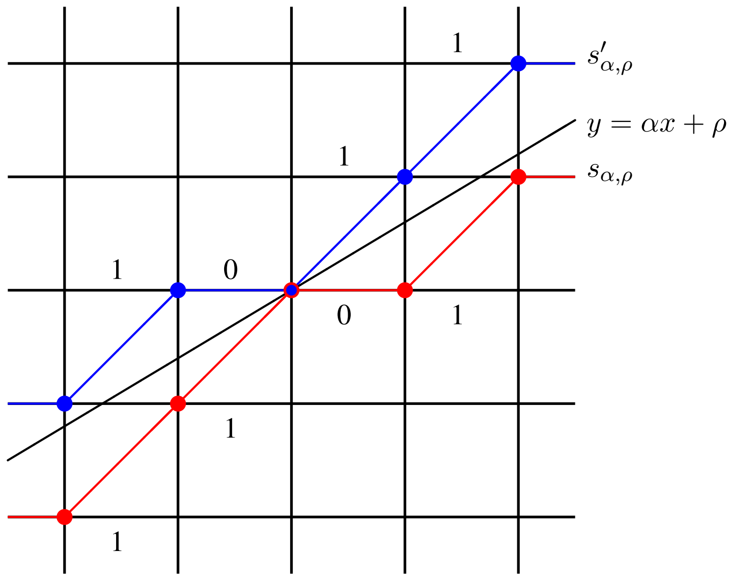

Figure 2.2: The mechanical words $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ when $y = \alpha x + \rho$ intersects a grid point

2.2 The reciprocal mechanical word, $g_{\beta,\gamma}$

We define the reciprocal mechanical word $g_{\beta,\gamma}$ by $g_{\beta,\gamma} = \{g_{\beta,\gamma}(n)\}_{n=-\infty}^{\infty}$, where

$$g_{\beta,\gamma}(n) = \begin{cases} 1 & \text{if } n = \lfloor k\beta + \gamma \rfloor \text{ for some integer } k, \\ 0 & \text{otherwise.} \end{cases}$$

with $\beta > 1$.

This word is useful for its mathematical equivalence to certain mechanical words. We shall make use of this relationship when we look at the E and G transforms, which we shall introduce later.

As an example, if we take $\beta = \phi$ and γ = 0, we can generate the first few members of the reciprocal mechanical word by calculating $\lfloor k\phi \rfloor$ for consecutive integer k

k	$\ldots$	-1	0	1	2	3	4	5	$\ldots$
$k\phi$	$\ldots$	-1.62..	0	1.62..	3.24..	4.85..	6.47..	8.09..	$\ldots$
$\lfloor k\phi \rfloor = n$	$\ldots$	-2	0	1	3	4	6	8	$\ldots$

We see here that $g_{\beta,\gamma}$ = 1 for n = -2, 0, 1, 3, 4, 6, 8, ... and 0 for n = -1, 2, 5, 7, ... The reciprocal mechanical word $g_{\phi,0}$ is therefore $g_{\phi,0} = \ldots 1, 0, \underline{1}, 1, 0, 1, 1, 0, 1, 0, 1 \ldots$, with the 0^{th} element underlined, which appears to be the doubly infinite upper Fibonacci word, f^{upper}.

Theorem 2.2.1. *If* $\beta > 1$*, then*

$$g_{\beta,\gamma}(n) = s'_{\frac{1}{\beta},-\frac{\gamma}{\beta}}(n).$$

Proof. Suppose first that n is such that $g_{\beta,\gamma}(n)$ = 1. Then $n = \lfloor k\beta + \gamma \rfloor$ for some integer k. It follows that

$$n \leq k\beta + \gamma < n + 1.$$

Writing $k\beta + \gamma$ = $n + \epsilon$ where n and ϵ are the integer and fractional parts respectively of $k\beta + \gamma$, where obviously $0 \leq \epsilon < 1$, shows that

$$\begin{aligned} (k-1)\beta + \gamma &= k\beta + \gamma - \beta \\ &= n + \epsilon - \beta \\ &< n \end{aligned} \qquad \text{and} \qquad \begin{aligned} (k+1)\beta + \gamma &= k\beta + \gamma + \beta \\ &= n + \epsilon + \beta \\ &> n + 1 \end{aligned}$$

Hence

$$(k-1)\beta + \gamma < n \leq k\beta + \gamma < n + 1 < (k+1)\beta + \gamma$$

which is equivalent to

$$k - 1 < \frac{n-\gamma}{\beta} \leq k < \frac{n+1-\gamma}{\beta} < k + 1.$$

It follows from this that

$$\left\lceil \frac{n+1-\gamma}{\beta} \right\rceil = k + 1, \text{ and } \left\lceil \frac{n-\gamma}{\beta} \right\rceil = k.$$

Consequently,

$$s'_{\frac{1}{\beta},-\frac{\gamma}{\beta}}(n) = (k+1) - k = 1.$$

Now we consider n values such that $g_{\beta,\gamma}(n) = 0$. Then, $\forall k \in \mathbb{Z}$, $n \neq \lfloor k\beta + \gamma \rfloor$. The condition $\beta > 1$ implies that there must exist, for each $n \neq \lfloor k\beta + \gamma \rfloor$, an integer b such that

$$\lfloor (b-1)\beta + \gamma \rfloor < n < n+1 < \ldots < \lfloor b\beta + \gamma \rfloor.$$

Thus

$$(b-1)\beta + \gamma < n < n+1 < \ldots < b\beta + \gamma$$

or equivalently

$$(b-1) < \frac{n-\gamma}{\beta} < \frac{n+1-\gamma}{\beta} < \ldots < b.$$

It follows that

$$\left\lceil \frac{n-\gamma}{\beta} \right\rceil = \left\lceil \frac{n+1-\gamma}{\beta} \right\rceil = b$$

and so in this case

$$s'_{\frac{1}{\beta},-\frac{\gamma}{\beta}}(n) = b - b = 0.$$

■

2.3 The E and G transforms

We define the transforms

$$E := \begin{cases} 0 & \mapsto 1 \\ 1 & \mapsto 0 \end{cases}$$

and

$$G := \begin{cases} 0 & \mapsto 0 \\ 1 & \mapsto 01 \end{cases}$$

These transforms operate on mechanical words to create new mechanical words, and we shall use them later to define a new transform, θ_m. Lothaire [10] studied these transforms in detail.

Lemma 2.3.1. *For any irrational number α and real number ρ, the following relations hold*

$$E(s_{\alpha,\rho}) = s'_{1-\alpha,1-\rho}$$

and

$$E(s'_{\alpha,\rho}) = s_{1-\alpha,1-\rho}$$

Proof. For $n \geq 0$,

$$\begin{aligned} s'_{1-\alpha,1-\rho}(n) &= \lceil (1-\alpha)(n+1) + 1 - \rho \rceil - \lceil n(1-\alpha) + 1 - \rho \rceil \\ &= 1 - (\lceil -\alpha n - \rho \rceil - \lceil -\alpha(n+1) - \rho \rceil) \end{aligned}$$

and as $-\lceil -r \rceil = \lfloor r \rfloor$ for every real number r,

$$\begin{aligned} &= 1 - (\lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor) \\ &= 1 - s_{\alpha,\rho}(n) \end{aligned}$$

Hence, the first equality is proven, and the second is analogous. ■

Lemma 2.3.2. *Let $0 < \alpha < 1, 0 < \rho \leq 1$. Then*

$$G(s'_{\alpha,\rho}) = s'_{\frac{\alpha}{\alpha+1},\frac{\rho}{\alpha+1}}$$

and if $0 \leq \rho < 1$, then

$$G(s_{\alpha,\rho}) = s_{\frac{\alpha}{\alpha+1},\frac{\rho}{\alpha+1}}$$

Proof. As $s'_{\beta,\gamma}(n) = g_{\frac{1}{\beta},-\frac{\gamma}{\beta}}$ (rewriting Theorem 2.2.1), we wish to prove

$$G(s'_{\alpha,\rho}) = g_{1+\frac{1}{\alpha},-\frac{\rho}{\alpha}} \tag{2.3.1}$$

This result is established by showing that the position N of the m^{th} digit 1 in the sequence $G(s'_{\alpha,\rho})$ satisfies the condition for the occurence of a digit 1 in the reciprocal mechanical word $g_{1+\frac{1}{\alpha},-\frac{\rho}{\alpha}}$. From the definition of $g_{1+\frac{1}{\alpha},-\frac{\rho}{\alpha}}$,

$$N+1=\left\lfloor(1+\frac{1}{\alpha})m-\frac{\rho}{\alpha}\right\rfloor$$

or equivalently

$$N-m+1=\left\lfloor\frac{m}{\alpha}-\frac{\rho}{\alpha}\right\rfloor. \tag{2.3.2}$$

In obtaining the left hand side of the first equation, the fact that the positions are counted from the origin at position 0 accounts for the +1.

On the left hand side of (2.3.1) , from the 0^{th} sequence member and to the right we have

$$G(s'_{\alpha,\rho})=G(s'_{\alpha,\rho}(0))G(s'_{\alpha,\rho}(1))\ldots G(s'_{\alpha,\rho}(n))\ldots$$

As G(0) = 0 and G(1) = 01, the m^{th} digit 1 to the right of and including the origin occurs as the output of $G(s'_{\alpha,\rho}(m^*))$ where $s'_{\alpha,\rho}(m^*), m^*\geq 0$ is the m^{th} digit 1 in the sequence $s'_{\alpha,\rho}(0)s'_{\alpha,\rho}(1)\ldots$ It follows that

$\#$ 0's up to the m^{th} 1 in output $= m^*$

$\#$ 1's up to the m^{th} 1 in output $= m-1$

and so

$$\begin{aligned} N &= \text{position of the } m^{th}\ 1 \\ &= \#\ \text{0's} + \#\ \text{1's up to the } m^{th}\ 1 \\ &= m^*+m-1 \end{aligned} \tag{2.3.3}$$

There is another consequence of $s'_{\alpha,\rho}(m^*)$ being the m^{th} digit 1 in $s'_{\alpha,\rho}(0)\ s'_{\alpha,\rho}(1)\ldots$ This is that

$$\begin{aligned} m &= s'_{\alpha,\rho}(0)+s'_{\alpha,\rho}(1)+\ldots+s'_{\alpha,\rho}(m^*) \\ &= \lceil\alpha(m^*+1)+\rho\rceil-\lceil\rho\rceil \\ &= \lceil\alpha(m^*+1)+\rho\rceil-1 \end{aligned} \tag{2.3.4}$$

where to obtain the second equality, the definition (2.1.2) of $s'_{\alpha,\rho}$ has been used, together with it's telescoping property under summation, and that $0 < \alpha < 1$. In the final equality the fact that $0 < \rho \leq 1$ has been used. Using this equation, the fact that $s'_{\alpha,\rho}(m^*) = 1$, and the definition (2.1.2), it follows

$$\lceil \alpha m^* + \rho \rceil = m \tag{2.3.5}$$

Considering (2.3.4) and (2.3.5) together shows

$$\begin{aligned} \alpha m^* + \rho &\leq m < \alpha(m^* + 1) + \rho \\ \Rightarrow m^* &\leq \frac{m}{\alpha} - \frac{\rho}{\alpha} < m^* + 1 \\ \Rightarrow m^* &= \left\lfloor \frac{m}{\alpha} - \frac{\rho}{\alpha} \right\rfloor \end{aligned}$$

Substituting (2.3.6) into (2.3.3), and comparing with (2.3.2), we see that N, the position of the $m^{th}1$ is the same for both sides of (2.3.1).

It remains to consider the equality (2.3.1) for positions to the left of the origin. For such positions $G(s'_{\alpha,\rho}) = \ldots G(s'_{\alpha,\rho}(-n)) \ldots G(s'_{\alpha,\rho}(-2))G(s'_{\alpha,\rho}(-1))$.

Again, as $G(0) = 0$ and $G(1) = 01$, we have that the m^{th} digit 1 to the left of the origin occurs at the output of $G(s_{\alpha,\rho}(-m^*))$, where $s_{\alpha,\rho}(-m^*)$ is the m^{th} digit 1 reading from right to left in the sequence $\ldots s'_{\alpha,\rho}(-2)s'_{\alpha,\rho}(-1)$. We see from this that the position $-N$ of the m^{th} digit 1 to the left of the origin satisfies (2.3.3). The analogue of (2.3.4) is that

$$\begin{aligned} m &= s'_{\alpha,\rho}(-m^*) + s'_{\alpha,\rho}(-m^* + 1) + \ldots + s'_{\alpha,\rho}(-1) \\ &= \lceil \rho \rceil - \lceil \alpha(-m^*) + \rho \rceil \\ &= 1 + \lfloor \alpha(-m^*) - \rho \rfloor \end{aligned} \tag{2.3.6}$$

Since $s'_{\alpha,\rho}(-m^*) = 1$, it follows from this that

$$\lceil \alpha(-m^* + 1) + \rho \rceil = -m + 2$$

or equivalently

$$\lfloor \alpha(m^* - 1) - \rho \rfloor = m - 2. \tag{2.3.7}$$

Together (2.3.6) and (2.3.7) imply

$$\alpha(m^* - 1) - \rho < m - 1 \le \alpha m^* - \rho$$

which is equivalent to

$$m^* = \left\lceil \frac{m-1}{\alpha} + \frac{\rho}{\alpha} \right\rceil \tag{2.3.8}$$

On the right hand side of (2.3.1), from the definition of $g_{1+\frac{1}{\alpha},-\frac{\rho}{\alpha}}$

$$\begin{aligned} -N &= \left\lfloor (1+\frac{1}{\alpha})(-m+1) - \frac{\rho}{\alpha} \right\rfloor \\ &= -\left\lceil (1+\frac{1}{\alpha})(m-1) + \frac{\rho}{\alpha} \right\rceil \\ &= -m+1-\left\lceil \frac{m-1}{\alpha} + \frac{\rho}{\alpha} \right\rceil \end{aligned} \tag{2.3.9}$$

where the term $+1$ in $(-m+1)$ accounts for the positions being counted from -1. This agrees with the equation for N obtained from (2.3.8) and (2.3.3) taken together.

The proof for $G(s_{\alpha,\rho}) = s_{\frac{\alpha}{\alpha+1},\frac{\rho}{\alpha+1}}$ is analogous. ■

2.4 The θ_m transform

We can now use the E and G transforms to define a new transform, θ_m, specified by

$$\theta_m = G^{m-1} \circ E \circ G := \begin{cases} 0 & \mapsto 0^{m-1}1 \\ 1 & \mapsto 0^{m-1}10 \end{cases} \tag{2.4.1}$$

This transform and a slight variation were studied by de Bruijn [3]. We see that the special case $m = 1$ is the substitution rule that uniquely defines the Fibonacci substitution sequence.

Theorem 2.4.1. *For m* ≥ 1 *and* $0 < \alpha, \rho < 1$*, one has*

$$\theta_m(s_{\alpha,\rho}) = s'_{\frac{1}{\alpha+m},\frac{1+\alpha-\rho}{\alpha+m}}. \tag{2.4.2}$$

Proof. Using Lemma 2.3.1, Lemma 2.3.2 and (2.4.1), the result follows, by successive application of the corresponding transformation rules. ■

Corollary 2.4.2. *For* $m \geq 1$ *and* $\alpha > 0$*,*

$$\theta_m(s_{\alpha,\alpha}) = s'_{\frac{1}{m+\alpha},\frac{1}{m+\alpha}}.$$

Corollary 2.4.3. *For* $m \geq 1$ *and* $\alpha = \frac{-m+\sqrt{m^2+4}}{2}$*, one has*

$$\theta_m(s_{\alpha,\rho}) = s'_{\alpha,\alpha(1-\rho-m)+1}.$$

From Corollary 2.4.3, we see that taking $m = 1$, we indeed have that $s_{\frac{1}{\phi},0} = f^{lower}$ and $s'_{\frac{1}{\phi},0} = f^{upper}$, confirming the statements made in Section$\{2.1\}$ and Section$\{2.2\}$.

As mechanical words are equivalent to cutting sequences, we see from Theorem 2.0.8 and Corollary 2.4.3 that words created from a line with slope $\alpha = \frac{-m+\sqrt{m^2+4}}{2}, m \geq 1$ are invariant under transformation by θ_m. Hence these words remain in the same LE-class after transformation by θ_m, which answers our original question posed at the beginning of this chapter.

2.5 Continued fractions and the θ_m transform

It is also interesting to study the effect of consecutive θ_m transformations acting on a mechanical word $s_{\beta,\gamma}$, where m can vary after each transformation, as α and ρ in the resulting word $s_{\alpha,\rho}$ have special properties.

Restrict attention now to irrational α. Consider the line $y = \alpha x + \rho$. Let (x_0, y_0) be a point on this line. Then all other points (x, y) on the line satisfy

$$\alpha = \frac{y - y_0}{x - x_0}. \tag{2.5.1}$$

Suppose furthermore that both x_0 and y_0 are integer. Because α is irrational, it follows immediately from (2.5.1) that there are no other points (x, y) on the line which are both rational. A consequence is that there is at most one grid point that the line $y = \alpha x + \rho$ passes through for α irrational.

Now, as $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ only differ at the points where our line intersects a grid point, we can define a binary sequence $\{c_{\alpha,\rho}(n)\}$ which is formed from the elements of either $s_{\alpha,\rho}$ or $s'_{\alpha,\rho}$ to the right of this intersection. We shall call this sequence the *coequal word*, and it is defined as

$$c_{\alpha,\rho}(n) = s_{\alpha,\alpha}(n), \qquad n > n^* \tag{2.5.2}$$

where n^* is the sequence member corresponding to a grid point intersection. The simplest case is when $\rho = \alpha$ and thus $n^* = -1$. Let us write $c_{\alpha,\alpha} = c_\alpha$.

From Corollary 2.4.2 we see that

$$\theta_m(c_\alpha) = c_{\frac{1}{m+\alpha}}. \tag{2.5.3}$$

This can be used to connect continued fractions to coequal - and hence mechanical - words. Every irrational number $\gamma > 0$ has a unique simple continued fraction expansion

$$\gamma = m_0 + \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cfrac{1}{\cdots}}}} \tag{2.5.4}$$

where $m_0, m_1, \ldots$ are integers and $m_0 \geq 0$, $m_i > 0$ for $i \geq 1$. This information can be written in a more compact form

$$\gamma = [m_0, m_1, m_2, \ldots],$$

where the integers m_i are called the *partial quotients* of γ. If a number α has a continued fraction expansion which is eventually periodic, we overline the part of the continued fraction expansion which is repeated periodically, ie.

$$\alpha = [0, a_1, a_2, \ldots, a_n, \overline{a_{n+1}, a_{n+2}, \ldots, a_{n+i}}]$$

for some integers $i, n \geq 0$. If $\alpha = [0, m_1, m_2, \ldots]$ is the continued fraction expansion of an irrational number α, where $0 < \alpha < 1$ and if for some irrational β, with $0 < \beta < 1$,

$$\beta = [0, m_{i+1}, m_{i+2}, \ldots]$$

we shall write

$$\alpha = [0, m_1, m_2, \ldots, m_i + \beta].$$

With these notations established, the following result is a consequence of Corollary 2.4.3 and (2.5.3).

Corollary 2.5.1. *If* $\alpha = [0, m_1, m_2, \ldots, m_i + \beta]$ *for some irrational* α *and* $0 < \alpha, \beta < 1$*, then*

$$c_\alpha = \theta_{m_1} \circ \theta_{m_2} \circ \ldots \circ \theta_{m_i}(c_\beta)$$

From this result in turn the substitution mapping leaving invariant a class of coequal words can be determined.

Corollary 2.5.2. *If* $0 < \alpha < 1$ *is an irrational real number with the purely periodic continued fraction expansion*

$$\alpha = [0, \overline{m_1, m_2, \ldots, m_n}].$$

then the coequal word c_α *is a fixed point of the sequence of transforms*

$$\theta_{m_1} \circ \theta_{m_2} \circ \ldots \circ \theta_{m_n}$$

Define ρ_k to be the value of ρ and α_k to be the value of α after k transformations θ_{m_j} to $s_{\beta,\gamma}$, so that

$$s_{\alpha_k,\rho_k} = \theta_{m_{i-k+1}} \circ \theta_{m_{i-k+2}} \circ \ldots \circ \theta_{m_i}(s_{\beta,\gamma}), \qquad 1 \le k \le i.$$

Applying the transforms $\theta_{m_i}, \theta_{m_{i-1}}, \ldots$ consecutively, we find that

$$\{\alpha_1, \alpha_2, \ldots\} = \left\{ \frac{1}{m_i + \beta}, \frac{1}{m_{i-1} + \dfrac{1}{m_i + \beta}}, \ldots \right\}. \tag{2.5.5}$$

and

$$\{\rho_1, \rho_2, \ldots\} = \left\{ \alpha_1(1 + (\beta - \gamma)), \alpha_2(1 - \frac{\beta - \gamma}{m_i + \beta}), \alpha_3(1 + \frac{\frac{\beta-\gamma}{m_i+\beta}}{m_{i-1} + \frac{1}{m_i+\beta}}), \ldots \right\} \tag{2.5.6}$$

From (2.5.6) we see that there seems to be a pattern to the consecutive values of $\rho_k' s,\ 1 \leq k \leq i$. In fact if we define $\alpha_0 = 1$, then the following holds true.

Proposition 2.5.3. *If the mechanical word $s_{\beta,\gamma}$ does not pass through a grid point and $0 < \beta, \gamma < 1$, then*

$$\alpha = \alpha_k = [0, m_{i-k+1}, m_{i-k+2}, \ldots, m_i + \beta]$$

and

$$\rho_k = \alpha_k(1 - (-1)^k \alpha_{k-1}\alpha_{k-2} \ldots \alpha_1 \alpha_0(\beta - \gamma))$$

are such that

$$s_{\alpha_k,\rho_k} = \theta_{m_{i-k+1}} \circ \theta_{m_{i-k+2}} \circ \ldots \circ \theta_{m_i}(s_{\beta,\gamma}), \qquad 1 \leq k \leq i. \tag{2.5.7}$$

Proof. We proceed by induction. As s does not pass through a grid point, (2.4.1) can be rewritten as

$$\theta_m(s_{\beta,\gamma}) = s_{\frac{1}{\beta+m}, \frac{1+\beta-\gamma}{\beta+m}}. \tag{2.5.8}$$

This shows (2.5.7) is true for $k = 1$. We suppose now that it is true for general $k < i$, and seek to show that it is true for $k + 1$ substituted in (2.5.7). Now

$$\begin{aligned} s_{\alpha_{k+1},\rho_{k+1}} &= \theta_{m_{i-k}}(s_{\alpha_k,\rho_k}) \\ &= s_{\frac{1}{\alpha_k+m_{i-k}}, \frac{1+\alpha_k-\rho_k}{\alpha_k+m_{i-k}}} \end{aligned}$$

Hence

$$\begin{aligned} \alpha_{k+1} &= \frac{1}{\alpha_k + m_{i-k}}, \\ \rho_{k+1} &= \frac{1 + \alpha_k - \rho_k}{\alpha_k + m_{i-k}} \\ &= a_{k+1}(1 + \alpha_k - \rho_k) \end{aligned}$$

Substituting in the formulas for α_k in the first formula and ρ_k in the second, as assumed in the induction step, indeed gives the same formulas back with $k \mapsto k + 1$. ■

In the case $k = i$, Proposition 2.5.3 states that

$$s_{\alpha_i,\rho_i} = \theta_{m_1} \circ \theta_{m_2} \circ \ldots \circ \theta_{m_i}(s_{\beta,\gamma})$$

with

$$\begin{aligned} \alpha_i &= [0, m_1, m_2, \ldots, m_i + \beta], \\ \rho_i &= \alpha_i(1 - (-1)^i \alpha_{i-1}\alpha_{i-2} \ldots \alpha_1\alpha_0(\beta - \gamma)). \end{aligned}$$

Note that this reduces to the statement of Corollary 2.5.1 in the case $\gamma = \beta$.

Chapter 3

Further Research

We can consider doubly infinite substitution sequences to be tilings of one-dimensional spaces, where the alphabet A is the set of tiles. This perspective has higher dimensional analogues such as the Penrose kite and dart tiling of the plane, introduced by Penrose in 1978 [14]. Penrose tilings can contain infinite chains of long and short 'bow ties' which display similar properties to those of the one-dimensional Fibonacci substitution sequence. We can grow Penrose tilings by repeated substitution of a starting patch of tiles using the kite-dart substitution rule (See Figure 3.1[1]) $D \mapsto D + K$, $K \mapsto 2K + D$, which can be written as a substitution matrix

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Long and short bow ties become a copy of themselves with added tiles around the extremities after substitution, and this copy is aligned along the same axis. The ratio of long to short bow ties in these infinite chains (and also the ratio of kites to darts) is $\phi : 1$. To see this note that the substitution matrix for the Fibonacci substitution sequence is

$$M_f = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

[1]Figure 3.1 provided by E. O. Harriss,
The Tiling Encyclopedia
http://tilings.math.uni-bielefeld.de/tilings/substitution_rules/penrose_kite_dart

and so $P = M_f^2$. If we label long bow ties L and short bow ties S then the substitution rule for the long and short bow ties is $S \mapsto L, L \mapsto L + S$, which is the Fibonacci substitution sequence.

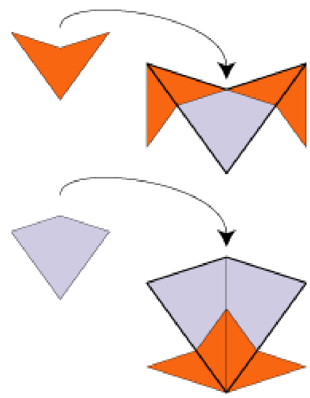

Figure 3.1: The deflation of Penrose's kite and dart

In addition to the notion of a cutting sequence, there is a further geometrical perspective on mechanical words, which is of particular interest for its generalization to higher dimensions and corresponding application to aperiodic (non-periodic) tilings of the plane. For this, let α be irrational and consider the reciprocal mechanical word $g_{\alpha,0}$. Draw the lines $y = \alpha x$ and $y = \alpha x - 1$ on the square lattice. Because $n\alpha \geq \lfloor n\alpha \rfloor \geq n\alpha - 1$, we see that $g_{\alpha,0}(n) = 1$ if $(k, \lfloor k\alpha \rfloor)$ with $\lfloor k\alpha \rfloor = n$ is inside the strip for some integer k. Hence the $y-$coordinates of the set of all integer points inside the strip gives the values of n such that $g_{\alpha,0}(n) = 1$. Translating the strip by drawing the lines $y = \alpha x + \rho$ and $y = \alpha x - 1 + \rho$, the same statement now applies to the reciprocal mechanical word $g_{\alpha,\rho}$. Further, the word can be represented geometrically by projecting these integer points onto the line $y = \alpha x + \rho$. Only two segment lengths are possible, with the short length corresponding to a 0 and the long length corresponding to a 1.

This so called projection method gives further insight into the invariance of certain cutting sequences under substitutions. Consider in particular the θ_1 transform. The substitution $0 \mapsto 1$ with 0 relating to crossing the x-axis and 1 to crossing the y-axis can be viewed as the linear transformation $x \mapsto y$, while the substitution $1 \mapsto 10$ can be viewed as the linear transformation $y \mapsto x + y$. These two linear transformations are specified by the matrix M_f defined above.

The eigenvalues and corresponding eigenvectors for this transformation are

$$\lambda_\pm = \frac{1 \pm \sqrt{5}}{2}, \qquad \vec{u}_\pm = \begin{pmatrix} 1 \\ \lambda_\pm \end{pmatrix}$$

Note that $\vec{u}_+ \cdot \vec{u}_- = 0$, which can be anticipated from the transformation matrix being symmetric. The directions parallel to $\vec{u}_+$ are expanded by the factor λ_+ (note $|\lambda_+| > 1$) while the directions parallel to $\vec{u}_-$ are contracted by the factor λ_- (note $|\lambda_-| < 1$). The strip between $y = \lambda_+ x$ and $y = \lambda_+ x - 1$ is mapped to the strip between $y = \lambda_+ x - \lambda_-$ and $\lambda_+ x$. The same sequence of long and short segments (now multiplied by λ_+) is seen in the transformed strip.

Penrose rhomb tilings, which are equivalent to Penrose kite and dart tilings, can be constructed by choosing a plane of irrational slope through the 5-dimensional integer lattice, and projecting all points within a certain perpendicular distance onto the plane [3]. The associated theory could more than fill a further Honours year thesis...

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