Particle system realisations of determinantal processes

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Abstract

Special classes of non-intersecting or interlacing particle systems, inspired by various statistical models, and their description in terms of determinantal correlation functions are the main themes of this thesis. Another unifying aspect of our subject matter is that the particle systems permit scaling limits in which their joint law becomes identical to the joint law of the eigenvalues for certain ensembles of random matrices.

Particle systems resulting from a queueing model, as well as particle systems relating to tilings of hexagons, are reviewed from the viewpoint of the joint PDF and their scaling to random matrix forms. An interlacing particle system relating to a limit of an $a \times b \times c$ hexagon with a large is introduced, and forms the majority of the subject matter for the thesis. The particle system is analyzed by the computation of single line PDFs and correlation functions, as well as density profiles and scaled correlation functions in certain scaling limits.

This in turn is possible due to there being an underlying determinantal structure to the joint PDFs which carries to the correlations themselves. The functional forms obtained involve classical orthogonal polynomials, and their asymptotic properties allow scaled limits to be calculated. The scaled functional forms exhibit a universality property, being common (mostly) to a class of models which exhibit fluctuations known from random matrix theory.

Finally, a particle system relating to tilings of the Aztec diamond is studied from the viewpoint of the joint PDF, allowing seemingly original methods to be used to show known results such as the total number of possible tilings of an Aztec diamond, and the so-called arctic circle boundary.

Declaration

This is to certify that:

- (i) the thesis comprises only my original work towards the PhD except where indicated in the Preface,
- (ii) due acknowledgement has been made in the text to all other material used,
- (iii) the thesis is fewer than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Benjamin John Fleming

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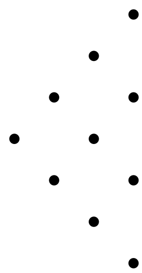
Contents

1 Introduction

The setting of non-intersecting particle systems is very natural and indeed very familiar. Thus for example a rule for the dynamics of a hard core lattice gas may be that at each tick of the clock one particle is chosen at random and attempts to jump to one of its neighbouring lattice sites. However, if this lattice site is occupied the move is rejected. It is clear that the space-time trajectories of this process give rise to non-intersecting trajectories. Our interest overlaps with the one-dimensional version of this setting, which in fact was introduced into statistical mechanics by Fisher in 1984 [22] under the name of the lock-step model of vicious walkers.

Interlaced particles are perhaps less familiar. What we have in mind are configurations such as those in the following diagram



so that the interlacing takes place between particles on successive vertical lines. In fact there are many statistical mechanical systems in which such configurations show themselves, however this is typically with respect to some auxiliary variables and so is typically not literal. One of the first examples of an interlaced particle system to appear within the analysis of a statistical model occurred in the work of Baryshnikov on queues [3]. The details are covered in §1.1 below. It is also true that non-intersecting and and interlaced particle systems can be present as auxiliary variables in the one statistical model. This is the case for certain tiling models, in particular the tiling of a hexagon by three species of rhombus (see Figure 1 and §1.3). It is the aim of this chapter to give a detailed account of some of the prominent known statistical models which relate to non-intersecting and interlacing particle systems.

Additionally, interlacing is a natural feature of the eigenvalues of successive sub-matrices of symmetric/Hermitian matrices. In many of the interlaced particle systems that appear in statistical models in which the particles are confined to certain lattice positions, under certain continuum scaling limits the particle systems converge to the eigenvalues of particular random matrices. This was first noted by Baryshnikov in the case of queues [3].

After reviewing Baryshnikov's work in §1.1, the joint PDF for eigenvalues of nested submatrices from the random matrix ensemble GUE* is computed in §1.2 for purposes of demonstrating the result from [3] that a scaling limit of the joint PDF for variables from the queueing model coincides with this random matrix form.

Tilings of the hexagon, half-hexagon and Aztec diamond motivate the subsequent sections of the chapter, from viewpoints of both interlacing particles and non-intersecting paths, where again joint PDFs and scaling limits occupy our attention.

1.1 Queueing models and the RSK correspondence

In [3], Baryshnikov investigates the process D_k, $k = 1, 2, \ldots$, given by

$$D_k = \sup_{\substack{0=t_0<t_1<\ldots \\ \cdots<t_{k-1}<t_k=1}} \sum_{i=0}^{k-1} [B_i(t_{i+1}) - B_i(t_i)] \tag{1.1}$$

with B_i being independent Brownian motions. It is shown that this process, which describes the limiting behaviour of queuing times for k jobs through a large number of queues, has the law of the process of the largest eigenvalues of the successive sub-blocks of an infinite random matrix drawn from the GUE* (Gaussian Unitary Ensemble of complex Hermitian matrices, standard real normals $\mathrm{N}[0, 1]$ on the diagonal, and standard complex normals $\mathrm{N}[0, 1/\sqrt{2}] + i\mathrm{N}[0, 1/\sqrt{2}]$ as the independent off diagonal entries; see the standard texts [52], [27]). In particular, the distribution of D_k is the distribution of the largest eigenvalue of the $k \times k$ submatrix of a Hermitian matrix drawn from the GUE*.

Consider the 'queueing process' represented by an $M \times N$ non-negative integer matrix X, where each entry $x_{i,j}$ of X represents the time it takes *queue* j to process *job* i using a labeling convention that $x_{i,j}$ is the i-th value from the *bottom* and the j-th value from the left. At $t = 0$ all the jobs are queued behind Q1, and generally a job i at the head of a queue j moves to queue $j + 1$ after waiting the time $x_{i,j}$. For example the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 2 \\ 0 & 2 & 1 \end{bmatrix} \tag{1.2}$$

corresponds to the queueing process

$t=0$	Q1	Q2	Q3	$t=1$	Q1	Q2	Q3	$t=2$	Q1	Q2	Q3
	•	•				•				•	•
	•					•				•	
						•					

$t=3$	Q1	Q2	Q3	$t=4$	Q1	Q2	Q3	$t=5$	Q1	Q2	Q3
		•				•				•	•
		•				•					

$t=6$	Q1	Q2	Q3	$t=7$	Q1	Q2	Q3	$t=8$	Q1	Q2	Q3
			•				•				
			•								

We notice that the first job appears to begin in Q2 because $x_{1,1} = 0$, meaning it takes no time for Q1 to process job 1.

If we define $T_{i,j}$ to be the time at which *job* i leaves *queue* j, then $T_{i,j}$ obeys the recurrence

$$T_{i,j} = \max(T_{i-1,j}, T_{i,j-1}) + x_{i,j} \tag{1.3}$$

Here it is required that $T_{0,j} = T_{i,0} = 0$. The first term comes from the fact that *queue* j can't begin processing *job* i until both: *queue* j is ready to process *job* i (*queue* j has already processed *job* $i-1$, which occurs at $t = T_{i-1,j}$); and *job* i is ready to be processed by *queue* j (*job* i has already been processed by *queue* $j-1$, which occurs at $t = T_{i,j-1}$). Note that $T_{i,N}$ is the exit time of job i from the final queue. In the above examples we see that the exit times are $T_{1,3} = 3$, $T_{2,3} = 7$, $T_{3,3} = 8$.

Baryshnikov's paper concentrates on the case where each $x_{i,j}$ is an *iid* random value (with finite variance), and the number of queues is taken to infinity, with the number of jobs kept finite (the regime "near the edge"). Set

$$D_k^{(N)} = \frac{T_{k,N} - EN}{\sqrt{vN}} \tag{1.4}$$

where E is the mean value of the $x_{i,j}$ and v is the variance. A result of Glynn and Whitt [36], nearly 10 years before [3], gives that the processes $D^{(N)} = (D_k^{(N)},\, k = 1, 2, \dots)$ converge in law as $N \to \infty$ to the stochastic process $D = (D_k,\, k = 1, 2, \dots)$ defined in (1.1). That this large N limit can be expressed as a sum of Brownian motions as shown in (1.1) involves mathematics outside our theme, and more can be found in [36]. Rather, it is (1.4) that will be of use to us in showing links between D_k and eigenvalues of $k \times k$ random matrices.

To show the main result of his paper, Baryshnikov takes advantage of the fact that (1.1) holds true independent of the details of the distribution specifying $x_{i,j}$. It turns out that if the geometric distribution is chosen to be

$$\Pr(x_{i,j} = k) = (1-q)q^k \qquad (k = 0, 1, \dots) \tag{1.5}$$

the queueing process becomes exactly solvable. We note that for this distribution

$$E = \frac{q}{1-q}, \qquad v = \frac{q}{(1-q)^2} \tag{1.6}$$

in (1.4). Use is now made of the *Robinson-Schensted-Knuth (RSK) correspondence*, a bijection between non-negative integer matrices and pairs of semistandard Young tableaux. A *Young diagram* λ is a set of rows of boxes, all left justified, such that each row has at least as many boxes as the row below it. We identify a Young diagram with k boxes and r rows with the partition:

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r), \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0, \quad \sum_{i=1}^{r} \lambda_i = k$$

of non-negative integers. Here, λ_i represents the number of boxes in row i of λ. A (semistandard) Young tableau is a filling of the boxes of a Young diagram λ by natural numbers so that the numbers are weakly increasing rightwards in rows and are strictly increasing downwards in columns.

Given a tableau P, we denote the underlying Young diagram $\lambda = \mathrm{sh}(P)$. Let P be filled by numbers from $\{1, 2, \dots, M\}$. Note that then we must have the number of rows in λ, $r \leq M$. If the tableau P has the non-negative integer j repeated f_j times ($j = 1, \dots, M$) (f_j is referred to as the frequency of j), and $\{a_j\}_{j=1,\dots,M}$ are a set of weights, we associate with P the weight

$$W(P) = a_1^{f_1} a_2^{f_2} \dots a_M^{f_M} \tag{1.7}$$

For two sets of weights $\{a_i\}_{i=1,\dots,M}$ and $\{b_j\}_{j=1,\dots,N}$, and an $M \times N$ non-negative integer matrix $X = [x_{i,j}]_{\substack{i=1,\dots,M \\ j=1,\dots N}}$, the weight of X is defined as

$$\prod_{i=1}^{M} \prod_{j=1}^{N} (a_i b_j)^{x_{i,j}}$$

If we denote the set of all such $M \times N$ matrices whose entries sum up to k as $W_{M,N,k}$, then the RSK correspondence (see e.g. [33]) is a bijection between $W_{M,N,k}$ and the set of pairs of weighted semistandard Young tableaux (P, Q) with $\mathrm{sh}(P) = \mathrm{sh}(Q) = \lambda$ having k boxes. Then P has entries from $\{1, \ldots, M\}$ and weights $\{a_i\}_{i=1,\ldots,M}$ while Q has entries from $\{1, \ldots, N\}$ and weights $\{b_j\}_{j=1,\ldots,N}$, and we denote the pair of tableaux corresponding to the matrix $w \in W_{M,N,k}$ as $(P(w), Q(w))$.

For a non-negative integer matrix $X = [x_{i,j}]_{\substack{i=1,\ldots,M \\ j=1,\ldots N}}$ define

$$l_{m,n} = \max \sum_{(1,1)\,\mathrm{to}\,(m,n)} x_{i,j}$$

where the sum is over indices of the matrix, connected as "up" or "right" neighbours starting at $x_{1,1}$ and finishing at $x_{m,n}$. From this definition, it is intuitive to see that $l_{m,n}$ obeys the recurrence

$$l_{m,n} = \max\left(l_{m-1,n}, l_{m,n-1}\right) + x_{m,n} \tag{1.8}$$

where $l_{0,n} = l_{m,0} = 0$, and so comparing with (1.3) we have that $l_{i,j} = T_{i,j}$, the time that *job* i leaves *queue* j in the queuing process define by X.

We now make use of an important feature of the RSK correspondence [33]. To state this, given a Young tableau P filled with elements $\{1, \ldots, M\}$, we define the Young diagram $\lambda^K(P)$, $1 \leq K \leq M$ as the Young diagram left after 'deleting' all the boxes that are filled with an element from $\{K+1, \ldots, M\}$ in P.

Lemma 1.1. *Fix a matrix $w \in W_{M,N,k}$. The sequence $l_{1,N}, l_{2,N} \ldots, l_{M,N}$ coincides with the sequence of the lengths of the first rows of the Young diagrams $\lambda^1(P), \ldots, \lambda^M(P)$ associated with the Young tableau $P(w)$:*

$$l_{K,N} = \lambda_1^K(P),\ K = 1, \ldots, M$$

Similarly, the sequence $l_{M,1}, l_{M,2}, \ldots, l_{M,N}$ coincides with the sequence of the lengths of the first rows of the Young diagrams $\lambda^1(Q), \ldots, \lambda^M(Q)$ associated with the Young tableau $Q(w)$:

$$l_{M,K} = \lambda_1^K(Q),\ K = 1, \ldots, N$$

We now introduce the *Schur polynomials* $s_\lambda(a_1, \ldots, a_M)$ defined to be the sum of the weight $W(P)$ over all Young tableaux P of shape λ and fillings from 1 to M. Hence, recalling (1.7),

$$s_\lambda(a_1, \ldots, a_M) = \sum_{P:\mathrm{sh}(P)=\lambda} W(P) = \sum_{P:\mathrm{sh}(P)=\lambda} a_1^{f_1} a_2^{f_2} \ldots a_M^{f_M}$$

In terms of the Schur polynomials, the probability $P(\lambda)$ that the RSK correspondence applied to a random $M \times N$ matrix w with entries

$$\Pr(x_{i,j} = k) = (1 - a_i b_j)(a_i b_j)^k \tag{1.9}$$

yields the Young diagram of shape $\lambda = (\lambda_1, \ldots, \lambda_M)$ is given by

$$P(\lambda) = s_\lambda(a_1, \ldots, a_M) s_\lambda(b_1, \ldots, b_N) \prod_{i=1}^{M} \prod_{j=1}^{N} (1 - a_i b_j) \tag{1.10}$$

To make use of Lemma 1.1, we ask the question of the joint probability that the RSK correspondence applied to a random $(M+1) \times N$ matrix with entries chosen according to (1.9) yields the Young diagram of shape μ and that the RSK correspondence applied to the $M \times N$ bottom left sub-block yields the Young diagram of shape κ. It is shown in [32] that this is equal to

$$P(\mu,\kappa) = \chi(\mu \prec \kappa) s_\kappa(a_1,\ldots,a_M) s_\mu(b_1,\ldots,b_N) a_{M+1}^{\sum_{l=1}^{n}(\mu_l-\kappa_l)} \prod_{i=1}^{M+1}\prod_{j=1}^{N}(1-a_i b_j) \tag{1.11}$$

where $n = \min(M+1, N)$ and $\chi(\mu \prec \kappa)$ is the interlacing condition defined in §6.1.

This joint probability allows us to find the conditional probability $P(\mu|\kappa)$, that given an $M \times N$ matrix corresponds to a pair of Young tableaux of shape κ, the probability that the $(M+1)\times N$ matrix obtained by adding a row corresponds to a pair of Young tableaux of shape μ. Thus, using $P(\mu|\kappa) = \dfrac{P(\mu,\kappa)}{P(\kappa)}$ it follows from (1.10) and (1.11) that

$$P(\mu|\kappa) = \chi(\mu \prec \kappa)\frac{s_\mu(b_1,\ldots,b_N)}{s_\kappa(b_1,\ldots,b_N)} a_{M+1}^{\sum_{l=1}^{n}(\mu_l-\kappa_l)} \prod_{j=1}^{N}(1-a_{M+1}b_j) \tag{1.12}$$

Proposition 1.2. *Consider a random $M \times N$ ($M \le N$) matrix with iid geometric entries with parameter q. The joint probability that under the RSK correspondence this matrix is such that the principal $K \times N$ sub-matrices ($K = 0,\ldots,M$) corresponds to pairs of semi-standard tableaux with shape $\lambda^K = (\lambda_1^K,\ldots,\lambda_K^K)$ is equal to*

$$(1-q)^{MN} q^{\frac{1}{2}\sum_{i=1}^{M}\lambda_i^M} s_{\lambda^M}(a_1,\ldots,a_N)\big|_{a_1=\cdots=a_N=q^{\frac{1}{2}}} \prod_{K=1}^{M}\chi(\lambda^{K-1}\prec\lambda^K) \tag{1.13}$$

Proof. Since the $K=0$ case can be taken as unity, the sought probability is just $\prod_{K=0}^{M-1} P(\lambda^{K+1}|\lambda^K)$. Substituting (1.12) and setting $a_i = b_j = \sqrt{q}$ for all $i,j = 1,\ldots,N$ gives (1.13) □

Recalling Lemma 1.1 and the theory below (1.8), we can restate Proposition 1.2 in terms of exit times from a queueing process.

Corollary 1.3. *Consider a queueing system of the type illustrated below (1.2). Suppose that the service times $x_{i,j}$ are iid geometric random variables with parameter q. The joint probability that the exit times from the final queue N of jobs i, $\{T_{i,N}\}_{i=1,\ldots,M}$ is the same as the joint probability of $\{\lambda_1^i\}_{i=1,\ldots,M}$ as implied by (1.13).*

Remember that our aim is to specify the joint distribution of the scaled exit times (1.4). With this in mind, using the fact that [33]

$$s_\lambda(a_1,\ldots,a_n)\big|_{a_1=\cdots=a_n=q^{\frac{1}{2}}} = q^{\frac{k}{2}} \prod_{1\le i<j\le n}\frac{\lambda_i-\lambda_j+j-i}{j-i}$$

where $k = |\lambda|$, the number of boxes in λ, (1.13) simplifies to

$$(1-q)^{MN} q^k \prod_{1\le i<j\le N}\frac{\lambda_i^M-\lambda_j^M+j-i}{j-i}\prod_{K=1}^{M}\chi(\lambda^{K-1}\prec\lambda^K) \tag{1.14}$$

where $k = |\lambda^M|$.

Since λ^M is the shape of a pair of Young tableaux, one filled with elements $\{1,\ldots,M\}$, and $M \le N$, we must have that $\lambda_i = 0$ for $M < i \le N$. Then

$$\prod_{1\le i<j\le N}\frac{\lambda_i^M-\lambda_j^M+j-i}{j-i} = \prod_{1\le i<j\le M}\frac{\lambda_i^M-\lambda_j^M+j-i}{j-i}\prod_{i=1}^{M}\prod_{j=M+1}^{N}\frac{\lambda_i^M+j-i}{j-i}$$

and (1.14) becomes

$$(1-q)^{MN}q^k \prod_{1\le i<j\le M} (\lambda_i^M - \lambda_j^M + j - i) \prod_{i=1}^{M} \frac{(\lambda_i^M + N - i)!}{(\lambda_i^M + M - i)!(N-i)!} \prod_{K=1}^{M} \chi(\lambda^{K-1} \prec \lambda^K) \quad (1.15)$$

In order to obtain a form in keeping with (1.4), we make the substitution $h_i^K = \dfrac{\lambda_i^K - EN}{\sqrt{vN}}$, where E and v are given by (1.6), and take the limit $N \to \infty$. Because of our change in variables, we must also multiply by $(vN)^{\frac{M(M+1)}{2}}$ since

$$\prod_{K=1}^{M} \bigwedge_{i=1}^{K} d\lambda_i^K = (vN)^{\frac{M(M+1)}{2}} \prod_{K=1}^{M} \bigwedge_{i=1}^{K} dh_i^K$$

Then, recalling (1.5), applying Stirling's formula (valid for $a \neq 0$)

$$\begin{aligned}(aN + b\sqrt{N} + c)! = \sqrt{2\pi}(aN)^{aN+b\sqrt{N}+c+1/2} \\ \times \exp\left(-aN + \frac{b^2}{2a} + \frac{b}{a}\left(c + \frac{1}{2} - \frac{b^2}{6a}\right)\frac{1}{\sqrt{N}} + \mathrm{O}(1/N)\right)\end{aligned} \quad (1.16)$$

to (1.15) and multiplying by $(vN)^{\frac{M(M+1)}{2}}$ gives us the PDF for $\{h_i^K\}_{\substack{i=1,\ldots,K \\ K=1,\ldots,M}}$.

Proposition 1.4. *The joint PDF for the scaled exit times* (1.4) *in the* $N \to \infty$ *limit is equal to the joint PDF of the variables* $\{h_1^K\}_{K=1,\ldots,M}$*. The latter in turn is implied by the joint PDF of* $\{h_i^K\}_{\substack{i=1,\ldots,K \\ K=1,\ldots,M}}$*, which is equal to*

$$\frac{1}{(2\pi)^{\frac{M}{2}}} \Delta(h_1^M, \ldots, h_M^M) \prod_{i=1}^{M} e^{\frac{-(h_i^M)^2}{2}} \prod_{K=2}^{M} \chi(h^{K-1} \prec h^K) \quad (1.17)$$

where Δ *is the Vandermonde determinant*

$$\Delta(x_1, \ldots, x_n) = \prod_{1\le i<j\le n} (x_i - x_j) = \det[x_{n+1-i}^{j-1}]_{i,j=1,\ldots,n} \quad (1.18)$$

The same specialisation of parameters and limiting procedure applied to (1.13) to obtain (1.17) can be applied to (1.10) to obtain the PDF of $\{h_i^M\}_{i=1,\ldots,M}$. Thus we obtain

$$\frac{1}{(2\pi)^{\frac{M}{2}} \prod_{l=1}^{M-1} l!} \left(\Delta(h_1^M, \ldots, h_M^M)\right)^2 \prod_{i=1}^{M} e^{-(h_i^M)^2/2} \quad (1.19)$$

This same expression results, as it must, by integrating (1.17) over the variables $\{h_i^K\}_{\substack{i=1,\ldots,K \\ K=1,\ldots,M-1}}$. In fact a simple formula can be given for the joint PDF of $\{h_i^K\}_{\substack{i=1,\ldots,K \\ K=m,\ldots,M}}$ for any $1 \le m \le M$.

Lemma 1.5. *Integrating out the variables* $\{h_i^K\}_{\substack{i=1,\ldots,K \\ K=1,\ldots,m-1}}$ *in* (1.17) *gives*

$$\frac{1}{(2\pi)^{\frac{m}{2}} \prod_{l=1}^{m-1} l!} \Delta(h_1^m, \ldots, h_m^m) \Delta(h_1^M, \ldots, h_M^M) \prod_{K=m+1}^{M} \chi(h^{K-1} \prec h^K) \quad (1.20)$$

Proof. We proceed by induction. In the base case $m = 1$ there is nothing to prove. Suppose now that the result is true for $m = l$. Making use of (1.18) shows we want to integrate

$$\det\left[(h_{l+1-i}^l)^{j-1}\right]_{i,j=1,\ldots,l} \chi(h^l \prec h^{l+1})$$

over $\{h_i^l\}_{i=1,\dots,l}$. The integration can be done row by row to give

$$\det\left[\int_{h_{l+2-i}^{l+1}}^{h_{l+1-i}^{l+1}} u^{j-1}du\right]_{i,j=1}^{l} = \det\left[\frac{1}{j}\left((h_{l+1-i}^{l+1})^j - (h_{l+2-i}^{l+1})^j\right)\right]_{i,j=1}^{l} = \frac{1}{l!}\Delta(h_1^{l+1},\dots,h_{l+1}^{l+1}) \quad (1.21)$$

where the final equality follows by first extracting the factor of $1/j$ from each column j, then observing that the remaining determinant is anti-symmetric in $\{h_i^{l+1}\}_{i=1,\dots,l+1}$, is homogeneous of degree $l(l+1)/2$ and has coefficient of $\prod_{j=1}^{l}(h_j^{l+1})^{l+1-j}$ unity. Substituting (1.21) back in the remaining terms of (1.20) establishes the case $m = l+1$. □

After the study of Baryshnikov [3], Johansson [39] introduced a number of statistical models based on the RSK correspondence. One of the most prominent was a particular example of the much studied totally asymmetric simple exclusion process (TASEP). This consists of particles on the integer lattice, conditioned so that no two particles can occupy the one site. At each time step, only particles with their right neighbouring site vacant can move, and they must stay where they are with probability $1-q$, or move to the vacant site to their right with probability q. In [39], the particular initial condition that all sites at the negative integers are occupied is shown to result from an analysis based on the RSK correspondence, and thus on certain interlaced variables.

We remark that interlacing variables have provided the key to the subsequent analysis of other instances of the TASEP model. Thus suppose that the particles always jump one step to the right provided the site is empty, but that they do this not at each tick of the clock but after a random exponential weight time with mean one beginning the moment the right neighbouring site is vacant. Sasamoto [61] used interlacing variables to give a decomposition of the transition probability, from a general initial condition to a general fixed position [63], and this theme has been further developed beginning with [6].

1.2 The GUE* eigenvalue process

We now go about finding a PDF for the eigenvalues of sub-matrices of the GUE* process, with the aim being to show that the joint PDF (1.17) occurs in that setting. We inductively define a sequence of matrices $\{X_n\}_{n=1,2,\dots}$ by $X_1 = c$ and

$$X_{n+1} = \begin{bmatrix} X_n & \vec{v} \\ \vec{v}^\dagger & c \end{bmatrix}$$

where $c \sim \mathrm{N}[0,1]$, $\vec{v}$ is a column vector of v_i's where $v_i \sim \mathrm{N}[0,\frac{1}{\sqrt{2}}] + i\mathrm{N}[0,\frac{1}{\sqrt{2}}]$, and $\vec{v}^\dagger$ denotes the conjugate transpose of $\vec{v}$. Then X_n represents top $n \times n$ sub-block of a matrix taken from the GUE*. GUE* matrices have a distribution which is unchanged by mappings $X \to UXU^\dagger$ or $X \to U^\dagger XU$ with U unitary. To see this, we use the fact that the distribition of a GUE* matrix is proportional to

$$e^{-\frac{1}{2}\mathrm{Tr}X^2} = e^{-\frac{1}{2}\mathrm{Tr}(U^\dagger XU)^2}$$

It is further true that the product of differentials of the independent entries of X, (dX), is unchanged by the same mapping (see e.g. [27]). Hence X_{n+1} has the same distribution as

$$X_{n+1}^{'} = \begin{bmatrix} U_n^\dagger & \vec{0}_n \\ \vec{0}_n^T & 1 \end{bmatrix}\begin{bmatrix} X_n & \vec{v} \\ \vec{v}^\dagger & c \end{bmatrix}\begin{bmatrix} U_n & \vec{0}_n \\ \vec{0}_n^T & 1 \end{bmatrix}$$

where $U_n^\dagger X_n U_n = D_n := \mathrm{diag}\, X_n$, the diagonal matrix of the eigenvalues $\{\lambda_i^{(n)}\}$ of X_n. So

$$X'_{n+1} = \begin{bmatrix} D_n & U_n^\dagger \vec{v} \\ \vec{v}^\dagger U_n & c \end{bmatrix} \sim \begin{bmatrix} D_n & \vec{v} \\ \vec{v}^\dagger & c \end{bmatrix}$$

since for U unitary and $\vec{v}$ complex Gaussian, $U\vec{v}$ has the same distribution as $\vec{v}$. Thus, to find the distribution of the eigenvalues of the X_n, it suffices to find the eigenvalues of the matrices A_n, where

$$A_{n+1} = \begin{bmatrix} D_n & \vec{v} \\ \vec{v}^\dagger & c \end{bmatrix}$$

Noting that

$$\det(\mathbf{1}_{n+1}x - A_{n+1}) = \det(\mathbf{1}_n x - A_n)\left(x - c - \sum_{j=1}^{n} \frac{q_j}{x - \lambda_j^{(n)}}\right)$$

where $q_j = v_j \bar{v}_j \sim \Gamma[1,1]$, $\Gamma[s,\sigma]$ being the gamma distribution with PDF proportional to $t^{s-1}e^{-t/\sigma}$, it follows that, for $p_n(x)$ the characteristic polynomial for A_n, we have

$$\frac{p_{n+1}(x)}{p_n(x)} = x - c - \sum_{j=1}^{n} \frac{q_j}{x - \mu_j} \tag{1.22}$$

where $\mu_j = \lambda_j^{(n)}$ is the j-th largest eigenvalue of A_n

Proposition 1.6. *[32] The zeroes* $\{\lambda_i\}$ *of the random rational function* (1.22) *with* $c \sim \mathrm{N}[0,1]$, $q_j \sim \Gamma[1,1]$ *and* μ_j *given, have the PDF*

$$\frac{1}{\sqrt{2\pi}} \frac{\Delta(\lambda_1, \dots, \lambda_{n+1})}{\Delta(\mu_1, \dots, \mu_n)} \exp\left(-\frac{1}{2}(\sum_{j=1}^{n+1} \lambda_j^2 - \sum_{j=1}^{n} \mu_j^2)\right) \chi(\mu \prec \lambda) \tag{1.23}$$

where $\chi(\mu \prec \lambda)$ *is the interlacing condition defined in §6.1*

Proof. Because the q_j are positive, graphical considerations imply the interlacing condition $\chi(\mu \prec \lambda)$. Assume $c = a$ for some given constant a. From (1.22) we have

$$x - a - \sum_{i=1}^{n} \frac{q_i}{x - \mu_i} = \frac{\prod_{j=1}^{n+1}(x - \lambda_j)}{\prod_{l=1}^{n}(x - \mu_l)} \tag{1.24}$$

Taking the large x expansion and equating coefficients of x^0 gives

$$\sum_{i=1}^{n+1} \lambda_i = a + \sum_{j=1}^{n} \mu_j \tag{1.25}$$

showing that we only need to find the first n of the λ_i (since the RHS is given), while equating coefficients of x^{-1} gives

$$-\sum_{i=1}^{n} q_i = \frac{1}{2}a^2 - \frac{1}{2}\left(\sum_{j=1}^{n+1} \lambda_j^2 - \sum_{l=1}^{n} \mu_l^2\right) \tag{1.26}$$

From the residue at $x = \mu_i$ in (1.24) it follows that

$$-q_i = \frac{\prod_{j=1}^{n+1}(\mu_i - \lambda_j)}{\prod_{\substack{l=1 \\ l \neq i}}^{n}(\mu_i - \mu_l)} \tag{1.27}$$

Since we know the distribution of the $\{q_i\}$, we want to change variables from $\{q_i\}_{i=1,\dots,n}$ to $\{\lambda_i\}_{i=1,\dots,n}$. From (1.25) and (1.27) we see that

$$\frac{\partial q_i}{\partial \lambda_j} = q_i \left(\frac{\lambda_j - \lambda_{n+1}}{(\mu_i - \lambda_j)(\mu_i - \lambda_{n+1})} \right)$$

and so, up to a possible sign,

$$\bigwedge_{l=1}^{n} dq_l = \det \left[\frac{1}{\mu_i - \lambda_j} \right]_{i,j=1,\dots,n} \prod_{k=1}^{n} q_k \left(\frac{\lambda_k - \lambda_{n+1}}{\mu_k - \lambda_{n+1}} \right) \bigwedge_{k=1}^{n} d\lambda_k$$

Using the Cauchy double alternant identity [17, pg 57]

$$\det \left[\frac{1}{\mu_i - \lambda_j} \right]_{i,j=1,\dots,n} = \frac{\prod_{1 \le i < j \le n} (\mu_i - \mu_j)(\lambda_i - \lambda_j)}{\prod_{i,j=1}^{n} (\mu_i - \lambda_j)} \tag{1.28}$$

along with (1.26), (1.27), and the fact that the $\{q_i\}$ have PDF $e^{-\sum_{i=1}^{n} q_i}$, we have the PDF of $\{\lambda_i\}_{i=1,\dots,n}$, for $c = a$

$$e^{\frac{a^2}{2}} \frac{\prod_{1 \le i < j \le n+1} (\lambda_i - \lambda_j)}{\prod_{1 \le i < j \le n} (\mu_i - \mu_j)} \exp \left(-\frac{1}{2} (\sum_{j=1}^{n+1} \lambda_j^2 - \sum_{j=1}^{n} \mu_j^2) \right)$$

where (1.25) holds. Multiplying by the PDF of $c = a$, $\frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}}$, and integrating over a, consistent with the restriction (1.25) on a, gives (1.23) $\square$

Taking the product of (1.23) for $n = 0, \dots, M-1$ gives the joint PDF of the eigenvalues of the A_n, $\{\lambda_i^{(n)}\}_{\substack{i=1\dots,n \\ n=1,\dots,M}}$

$$p_{GUE^*,M}(\{\lambda_i^{(n)}\}) = \frac{1}{(2\pi)^{\frac{M}{2}}} \prod_{k=1}^{M} e^{\frac{-(\lambda_k^{(M)})^2}{2}} \prod_{1 \le i < j \le M} \left(\lambda_i^{(M)} - \lambda_j^{(M)} \right) \prod_{n=0}^{M-1} \chi(\lambda^{(n)} \prec \lambda^{(n+1)}) \tag{1.29}$$

We see that this is equal to (1.17) with $\lambda_i^{(j)} = h_i^j$. (1.4) tells us that $h_1^k = D_k$, so the result of Baryshnikov that the D_k have the same law as the nested sequence of submatrices from GUE* has been shown. Also, since (1.29) and (1.17) are equal, we have that the PDF for the eigenvalues of an $M \times M$ GUE* matrix are given by (1.19)

1.3 The hexagon

A theme of this thesis is to be statistical systems which relate to eigenvalues of nested sub-blocks of random matrices. The first result of this type, showing that scaled exit times in a queueing process have this interpretation, has been reviewed above. Historically, the next result of this type, due to Johansson and Nordenstam [43], related to a certain random tilings of a hexagon.

In favourable circumstances, when looking at random tilings of certain shapes, one can define 'particles', and evaluate the PDF of the positions of said particles. Here, we follow the work of [43] and [55] and show that the PDF of the eigenvalues of GUE* minors, as found above, can be realised as the scaling limit of a random tiling of a hexagon by rhombi. It is easy to see that a rhombus tiling of an $a \times b \times c$ hexagon is in bijection with a set of a simple symmetric random walks conditioned never to intersect, starting at $(0, 2i)$ and ending at $(b+c, -b+c+2i)$ for $i = 0, \dots, a-1$. These walks will be addressed later in §1.7. For now, we are interested in the holes left between

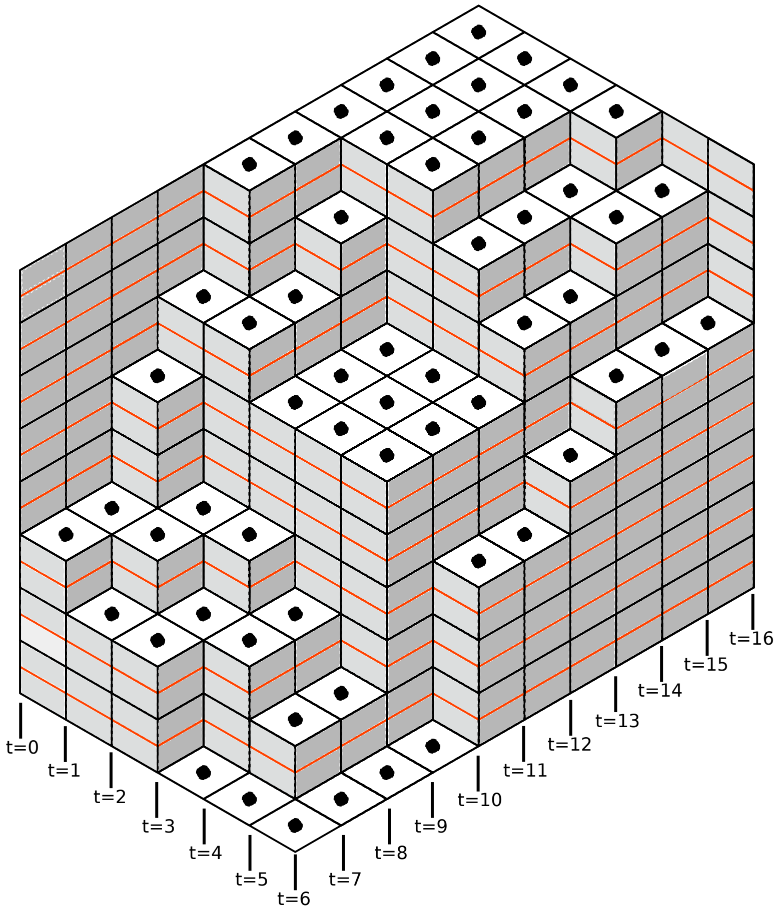


Figure 1: An example of a random tiling of an $8 \times 6 \times 10$ hexagon by rhombi, with the corresponding interlaced particles and non-intersecting walks shown. The shading is to highlight the interpretation of the particles as heights, as described at the end of §1.3

the walkers, the particles in the horizontal rhombi in Figure 1. For simplicity, we will restrict ourselves to the $b \leq c$ case, the other case being equivalent by reflection. On line t there will be $r(t)$ particles, where

$$r(t) = \begin{cases} t & t \leq b \\ b & b \leq t \leq c \\ b + c - t & c \leq t \end{cases} \tag{1.30}$$

Let $x^{(t)} = (x_1^{(t)}, \dots x_{r(t)}^{(t)})$ be the positions of the particles on line t, where by labelling convention, $x_i^{(t)} > x_j^{(t)}$ for $i < j$. Since no particle may be outside the hexagon, the highest possible position for $x_1^{(t)}$, $g(t)$ say, and the lowest possible position for $x_{r(t)}^{(t)}$, $h(t)$ say, are given by

$$g(t) = \begin{cases} 2(a-1) + t & t \leq c \\ 2(a+c-1) - t & t \geq c \end{cases} \qquad h(t) = \begin{cases} -t & t \leq b \\ -2b + t & t \geq b \end{cases} \tag{1.31}$$

For combinatorial reasons, which are clear to see from the picture, the particles fulfill the interlacing requirement $\chi(x^{(t)} \prec x^{(t+1)})$. We also have the restriction that

$$x_i^{(t)} - t \ \text{ even } \ \forall\, i, t. \tag{1.32}$$

Given that every possible tiling is equally likely by definition, the probability of some configuration $\bar{x} = (x^{(0)}, \dots, x^{(b+c)})$ can be written

$$p(\bar{x}) = \frac{1}{C_{a,b,c}} \prod_{t=0}^{b+c-1} \chi(x^{(t)} \prec x^{(t+1)})$$

for $\chi(x^{(t)} \prec x^{(t+1)})$ as defined in §6.1, where the virtual particles $x_{t+1}^{(t)} = -t-2$ for $t = 0, \dots, b-1$ and $x_0^{(t)} = 2(a+c) - t$ for $t = c+1, \dots, b+c$ have been included in the appropriate $\{x^{(t)}\}$, and furthermore the condition (1.32) is required. $C_{a,b,c}$ is some normalization constant, which is equal to the number of possible configurations for an $a \times b \times c$ hexagon.

From [54] we have the following Lemma and Theorem.

Lemma 1.7. *Let $t \leq b$. Given some configuration $x^{(t)}$, the number of configurations to the left of line t, i.e.*

$$G_t(x^{(t)}) = \sum_{x^{(1)}, \dots, x^{(t-1)}} \prod_{n=1}^{t-1} \chi(x^{(n)} \prec x^{(n+1)})$$

with $x_{n+1}^{(n)}$ defined as above, is

$$G_t(x_1, \dots, x_t) = c_t \Delta(x_1, \dots, x_t) \tag{1.33}$$

where

$$c_t = \frac{1}{2^{\frac{t(t-1)}{2}} \prod_{k=1}^{t-1} k!}.$$

Proof. Induction on t. For $t = 1$, it is true that $G_1(x) \equiv 1$. Assuming the statement is true for case t, consider case $t+1$.

$$G_{t+1}(x_1, \dots, x_{t+1}) = \sum_{i=1}^{t} \sum_{\substack{x_i > y_i > x_{i+1} \\ y_i - x_i \text{ odd}}} c_t \begin{vmatrix} 1 & y_1 & \dots & y_1^{t-1} \\ \vdots & & & \vdots \\ 1 & y_t & \dots & y_t^{t-1} \end{vmatrix}$$

Now, for a fixed i and with $x_i \geq x_{i+1} + 2$,

$$\begin{aligned} \sum_{\substack{x_i > y_i > x_{i+1} \\ y_i - x_i \text{ odd}}} y_i^j &= (x_i - 1)^j + (x_i - 3)^j + \dots + (x_{i+1} + 1)^j \\ &= q_j(x_i) - q_j(x_{i+1}) \end{aligned} \tag{1.34}$$

where

$$q_j(x) = \begin{cases} \displaystyle\sum_{k=-M}^{\frac{x-2}{2}} (2k+1)^j, & x \text{ even} \\ \displaystyle\sum_{k=-M}^{\frac{x-1}{2}} (2k)^j, & x \text{ odd} \end{cases} \tag{1.35}$$

for large enough positive integer M. By looking at the behaviour of $q_j(x)$ as $x \to \infty$, we see that $q_j(x) = \dfrac{x^{j+1}}{2(j+1)} + O(x^j)$, so inputting (1.34) and performing column operations to remove dependence on lower order terms gives

$$G_{t+1}(x_1, \dots, x_{t+1}) = c_t \begin{vmatrix} \frac{1}{2}(x_1 - x_2) & \frac{1}{4}(x_1^2 - x_2^2) & \dots & \frac{1}{2t}(x_1^t - x_2^t) \\ \vdots & & & \vdots \\ \frac{1}{2}(x_t - x_{t+1}) & \frac{1}{4}(x_t^2 - x_{t+1}^2) & \dots & \frac{1}{2t}(x_t^t - x_{t+1}^t) \end{vmatrix}$$

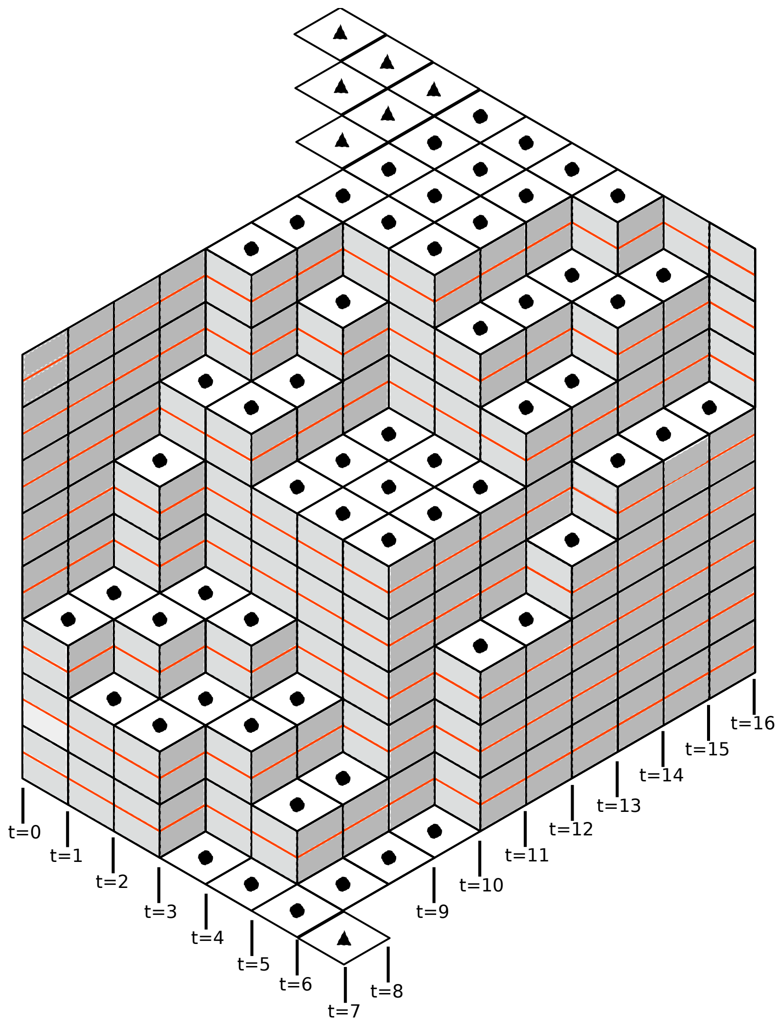


Figure 2: The hexagon from Figure 1, with virtual rhombi and particles (represented by triangles) introduced as described in the $b < t < c$ case in the proof of Theorem 1.8 (here $t = 7$).

Finally, we pull out constants and convert to final form

$$G_{t+1}(x_1, \ldots, x_{t+1}) = \frac{c_t}{2^t t!} \Delta(x_1, \ldots, x_{t+1})$$

□

Theorem 1.8. *The probability that the particles are at positions* $(x_1, \ldots, x_{r(t)})$ *on line* t *is*

$$p_t(x_1, \ldots, x_{r(t)}) = Z_{t,a,b,c}^{-1} \left(\Delta(x_1, \ldots, x_{r(t)})\right)^2 \prod_{i=1}^{r(t)} f_t(x_i) \tag{1.36}$$

where $Z_{t,a,b,c}$ *is some normalizing constant and*

$$f_t(x) = \prod_{k=1}^{|c-t|} (g(t) + 2k - x) \prod_{k=1}^{|b-t|} (x - h(t) + 2k) \tag{1.37}$$

where $g(t)$*,* $h(t)$ *are as in* (1.31)

Proof. The idea of the proof is illustrated in Figure 2. We introduce virtual particles above and below the hexagon so that the number of possible tilings to each side of the line t is represented by a (1.33) for an appropriate line number.

Case $t \leq b$: The area to the left of t can be tiled in $G_t(x_1, \ldots, x_t)$ ways and the area to the right of t can be tiled in $G_{b+c-t}(x_{-c+t+1}, \ldots, x_1, \ldots, x_t, \ldots, x_b)$ ways, where $x_{t+i} = h(t) - 2i$ for $i = 1, \ldots, b-t$ and $x_{-i+1} = g(t) + 2i$ for $i = 1, \ldots, c-t$. Then

$$p_t(x_1, \ldots, x_t) = \frac{1}{C_{a,b,c}} G_t(x_1, \ldots, x_t) G_{b+c-t}(x_{-c+t+1}, \ldots, x_1, \ldots, x_t, \ldots, x_b) \tag{1.38}$$

The theorem follows since the part of the Vandemonde that has to do with the virtual particles is, up to a constant, $\prod_{i=1}^{r(t)} f_t(x_i)$

Case $b < t < c$: The area to the left of t can be tiled in $G_t(x_1, \ldots, x_b, \ldots, x_t)$ ways where $x_{b+i} = h(t) - 2i$ for $i = 1, \ldots, t-b$, and the area to the right of t can be tiled in $G_{b+c-t}(x_{-c+t+1}, \ldots, x_1, \ldots, x_t)$ ways, where $x_{-i+1} = g(t) + 2i$ for $i = 1, \ldots, c-t$ and $x_{t+i} = h(t) - 2i$, for $i = 1, \ldots, b-t$. The theorem follows as before.

Case $c \leq t$: The area to the left of t can be tiled in $G_t(x_{c-t+1}, \ldots, x_1, \ldots, x_{b+c-t}, \ldots, x_c)$ ways where $x_{b+c-t+i} = h(t) - 2i$ for $i = 1, \ldots, t-b$, and $x_{-i+1} = g(t) + 2i$ for $i = 1, \ldots, t-c$, and the area to the right of t can be tiled in $G_{b+c-t}(x_1, \ldots, x_t)$ ways. Again the theorem follows.

□

We now fix the line number t such that $0 < t < b$ and define the random variable $X_t^{\text{Hex}} = (x^{(1)}, \ldots, x^{(t)})$. From Lemma 1.7, Theorem 1.8 and the requirement that every possible tiling of the hexagon is equally likely, we have that the PDF of X_t^{Hex} is

$$\begin{aligned} P_{(t)}^{\text{Hex}}(x^{(1)}, \ldots, x^{(t)}) &= \frac{p_t(x^{(t)})}{G_t(x^{(t)})} \prod_{n=1}^{t-1} \chi(x^{(n)} \prec x^{(n+1)}) \\ &= \frac{1}{c_t Z_{t,a,b,c}} \Delta(x_1^{(t)}, \ldots, x_t^{(t)}) \prod_{i=1}^{t} f_t(x_i^{(t)}) \prod_{n=1}^{t-1} \chi(x^{(n)} \prec x^{(n+1)}) \end{aligned} \tag{1.39}$$

Proposition 1.9. *Let the points $y_i^{(j)} := (x_i^{(j)} - N)/\sqrt{3N/2}$ be a rescaling of the points $x_i^{(j)}$, where $N = a = b = c$ is the length of all sides of the hexagon. Given that the $x_i^{(j)}$ have PDF $P_{(t)}^{\text{Hex}}$ as described in* (1.39)*, one has*

$$P_{(t)}^{\text{Hex}}(x^{(1)}, \ldots, x^{(t)}) \to P_{GUE,t}(y^{(1)}, \ldots, y^{(t)})$$

where $P_{GUE,M}$ is as in (1.29)*, as $N \to \infty$, where the convergence is uniform on compact sets with respect to the $x_i^{(j)}$.*

Proof. From (1.31) and (1.37) we have that, for $a = b = c = N$,

$$f_t(x) = \prod_{k=1}^{N-t} (2N - 2 + t - x + 2k)(x + t + 2k) \tag{1.40}$$

If we make the substitution $x = \sqrt{\alpha N} y + N$ then

$$\begin{aligned} f_t(x) &= \prod_{k=1}^{N-t} (N - \sqrt{\alpha N} y + t + 2(k-1))(N + \sqrt{\alpha N} y + t + 2k) \\ &= 2^{2(N-t)} \frac{(\frac{3}{2}N - \frac{1}{2}\sqrt{\alpha N} y - \frac{1}{2}t - 1)!}{(\frac{1}{2}N - \frac{1}{2}\sqrt{\alpha N} y + \frac{1}{2}t - 1)!} \frac{(\frac{3}{2}N + \frac{1}{2}\sqrt{\alpha N} y - \frac{1}{2}t)!}{(\frac{1}{2}N + \frac{1}{2}\sqrt{\alpha N} y + \frac{1}{2}t)!} \end{aligned}$$

Setting $\alpha = \dfrac{3}{2}$ and applying Stirling's formula (1.16) gives

$$f_t\left(\sqrt{\frac{3N}{2}} y + N\right) = N^{2N-2t} 3^{3N-t} \exp\left(-2N - \frac{y^2}{2} - y\sqrt{\frac{2}{3N}} + \mathrm{O}(1/N)\right) \tag{1.41}$$

For ease of notation, we define $F_{t,N}$ so that the leading order term of the RHS of (1.41) can be written $F_{t,N} e^{-\frac{1}{2}y^2}$. Then, noting that

$$\prod_{k=1}^{t} \bigwedge_{i=1}^{k} dx_i^{(k)} = \frac{1}{2^{\frac{t(t+1)}{2}}} \left(\frac{3N}{2}\right)^{\frac{t(t+1)}{4}} \prod_{k=1}^{t} \bigwedge_{i=1}^{k} dy_i^{(k)}$$

since the original $x_i^{(k)}$'s possible positions were confined to every *second* integer, we have

$$\begin{aligned} P_{(t)}^{\mathrm{Hex}}(x^{(1)}, \dots, x^{(t)}) = \frac{(F_{t,N})^t}{c_t Z_{t,N,N,N}} \frac{1}{2^{\frac{t(t+1)}{2}}} \left(\frac{3N}{2}\right)^{\frac{t^2}{2}} \prod_{k=1}^{t} e^{-(y_i^{(t)})^2/2} \\ \times \prod_{1 \le i < j \le n} (y_i^{(t)} - y_j^{(t)}) \prod_{n=1}^{t-1} \chi(y^{(n)} \prec y^{(n+1)}) \end{aligned} \tag{1.42}$$

Applying Stirling's formula (1.16) to $Z_{t,N,N,N}$, which we find in Proposition 2.2 in §2.1, gives

$$Z_{t,N,N,N} = \pi^{\frac{t}{2}} \frac{3^{\frac{t}{2}(6N-t)}}{2^{\frac{t}{2}(t+1)}} \left(\frac{N}{e}\right)^{2Nt} N^{\frac{-3t^2}{2}} \prod_{i=0}^{t-1} i! + \mathrm{O}(1/N)$$

and this along with (1.41) gives

$$\frac{(F_{t,N})^t}{c_t Z_{t,N,N,N}} \frac{1}{2^{\frac{t(t+1)}{2}}} \left(\frac{3N}{2}\right)^{\frac{t^2}{2}} = \frac{1}{(2\pi)^{\frac{t}{2}}} + \mathrm{O}(N^{-1/2})$$

and so the PDF in (1.42) converges to the GUE* PDF (1.29). □

Thus we have in fact revised two known examples of probabilistic systems which, in appropriate scaling limits, have a PDF the same as that for the GUE* minor process. This tells us that the latter is a fixed point for a certain universality class. Moreover, the members belonging to the class can be extended by the consideration of domino tilings of the Aztec diamond described in the next section.

Another viewpoint of the hexagon tiling by rhombi is that of the so called solid-on-solid model. Consider the $b \times c$ integer grid $\{(i,j) : 1 \le i \le b,\ 1 \le j \le c\}$. At each site (i,j), associate a height variable $x_{i,j}$ with some distribution $h(x)$. Furthermore, require that for fixed i

$$x_{i,1} \le x_{i,2} \le \cdots \le x_{i,c} \tag{1.43}$$

and that for fixed j

$$x_{1,j} \le x_{2,j} \le \cdots \le x_{b,j} \tag{1.44}$$

Thus a directed up/right lattice path must encounter successively non-decreasing heights.

Rotating the rectangular grid by 45^o the lattice points form lines parallel to the y-axis. For a $b \times c$ grid there are $b + c - 1$ lines, with the number of grid points on line t given by $r(t)$ as in (1.30). Let $y_i^{(t)}$ denote the height of the i-th lattice point counting from the top on line t. Then

$$y_i^{(t)} = \begin{cases} x_{t-i+1,b-i+1} & t \leq c \\ x_{c-i+1,b+c-t-i+1} & t > c \end{cases} \tag{1.45}$$

Looking at Figure 1, it is not hard to see that a rhombus tiling of an $a \times b \times c$ hexagon can also be interpreted as a representation of $b \times c$ grid of heights obeying the same restrictions on the heights (1.43) and (1.44). Specifically, we imagine the edges of the hexagon are parallel to the x, y and z axes, and have that the a-th non-intersecting path is always at height a in the z direction, with the particles being on surfaces parallel to the x, y plane. Then in this viewpoint, with $x_i^{(t)}$ given as in the earlier parts of this section (i.e the position of the i-th particle on the t-th line in the two dimensional viewpoint), we have the relationship

$$y_i^{(t)} = \begin{cases} i + \dfrac{x_i^{(t)} - t}{2} & t \leq c \\ i - c + \dfrac{x_i^{(t)} + t}{2} & t > c \end{cases} \tag{1.46}$$

Thus, if we define $h(x)$, the distribution of the heights $x_{i,j}$, to be the discrete uniform distribution on heights 0 to a, then the definitions of y (1.45) and (1.46) are equivalent.

1.4 The Aztec diamond

The Aztec diamond of order N is the union of all lattice squares within the diamond shaped region $\{(x, y) : |x| + |y| \leq N + 1\}$. It was shown in [19] that there are $2^{N(N+1)/2}$ possible tilings of an Aztec diamond of order N by 2×1 dominoes, and since then a number of derivations distinct from those given in [19] have been found, for example [40, 20]. As with the hexagon tiling of the previous section, a tiling of the Aztec diamond by these dominoes is in bijection with a family of lattice paths. To see this, with the top left lattice square specified as white, introduce a checkerboard colouring of all the lattice squares making up the Aztec diamond. For a horizontal domino which covers a white-black (black-white) pair of squares when reading left to right, no segment (a horizontal segment) of path is marked. For a vertical domino which covers a white-black (black-white) pair of squares when reading top to bottom, a right-up (right-down) segment of path is marked. This results in a family of N non-intersecting lattice paths, with segments up sloping, down sloping or horizontal, starting at equally spaced points on the bottom down sloping edge, and finishing at the corresponding points on the bottom up sloping edge. See Figure 3 for an example.

Our interest is in the interlaced particle system implied by a domino tiling of the Aztec diamond. For its specification, with the Aztec diamond checkerboard coloured as already described, let the horizontal dominoes such that the left square is colour black (white) be called of S (N) type. Similarly, let the vertical dominoes such that the top square covered is black (white) be called of E (W) type [19]. Suppose now that the E and S type dominoes are shaded and we add a number of labelled diagonal lines equal to the order of the Aztec diamond, as seen in Figure 4. The k-th line passes through the interior of k shaded tiles, and these intersections are considered as specifying the positions of k particles [41, 43]. In an appropriate co-ordinate system, these particles

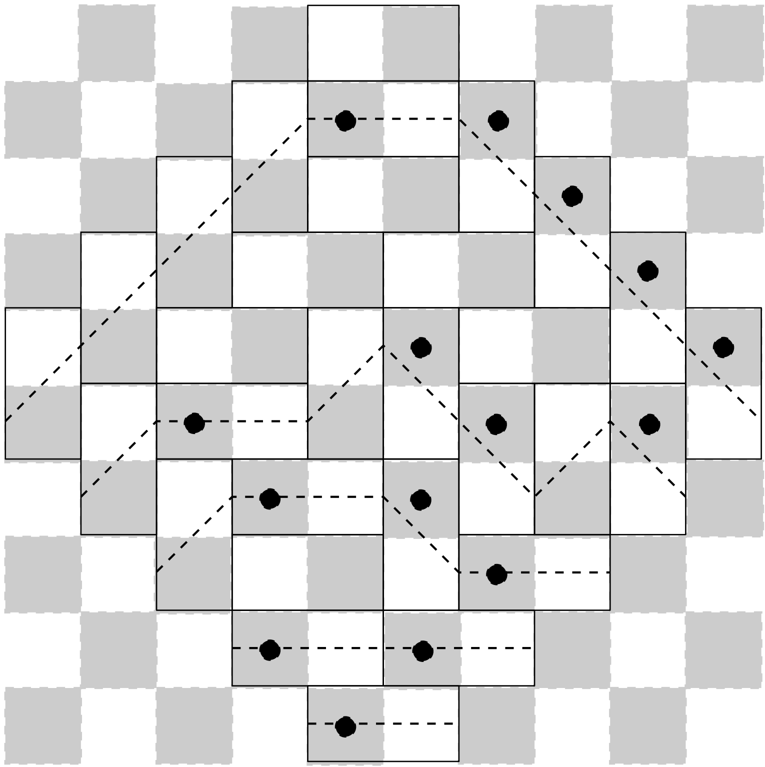


Figure 3: An example of the checkerboard colouring of an Aztec diamond of order 5, with lattice paths and particles inserted as described at the beginning of §1.4 and below (1.47) respectively. Note that the k-th path from the bottom crosses k squares containing particles.

occupy distinct positions $x_1^{(k)} > \cdots > x_k^{(k)}$ restricted to the lattice points $0, 1, 2, \ldots, N$ on line k $(k = 1, \ldots, N)$. Most importantly, the particles must satisfy the interlacing condition

$$x_{i+1}^{(k+1)} \leq x_i^{(k)} \leq x_i^{(k+1)} \quad \text{for} \quad i = 1, \ldots, k-1 \tag{1.47}$$

The positions of the particles can also be determined by the non-intersecting paths. Any black square (in the checkerboard coloring) in which a path crosses from the top left half of the square to the bottom right half contains a particle (see Figure 3). It is then true that the k-th path from the bottom passes through k particles.

In [43] this weighted particle process corresponding to the Aztec diamond tiling, defined through its correlations and restricted to the first n lines, was shown in a certain scaling limit to coincide with the minor process of the GUE*.

1.5 The antisymmetric GUE eigenvalue process

Just as the scaled particles of a rhombus tiling of a hexagon have the same probabilistic law as the eigenvalues of GUE* sub-matrices, it was shown in [31] that the scaled particles of a rhombus tiling of a half-hexagon have the same probabilistic law as the eigenvalues of antisymmetric GUE

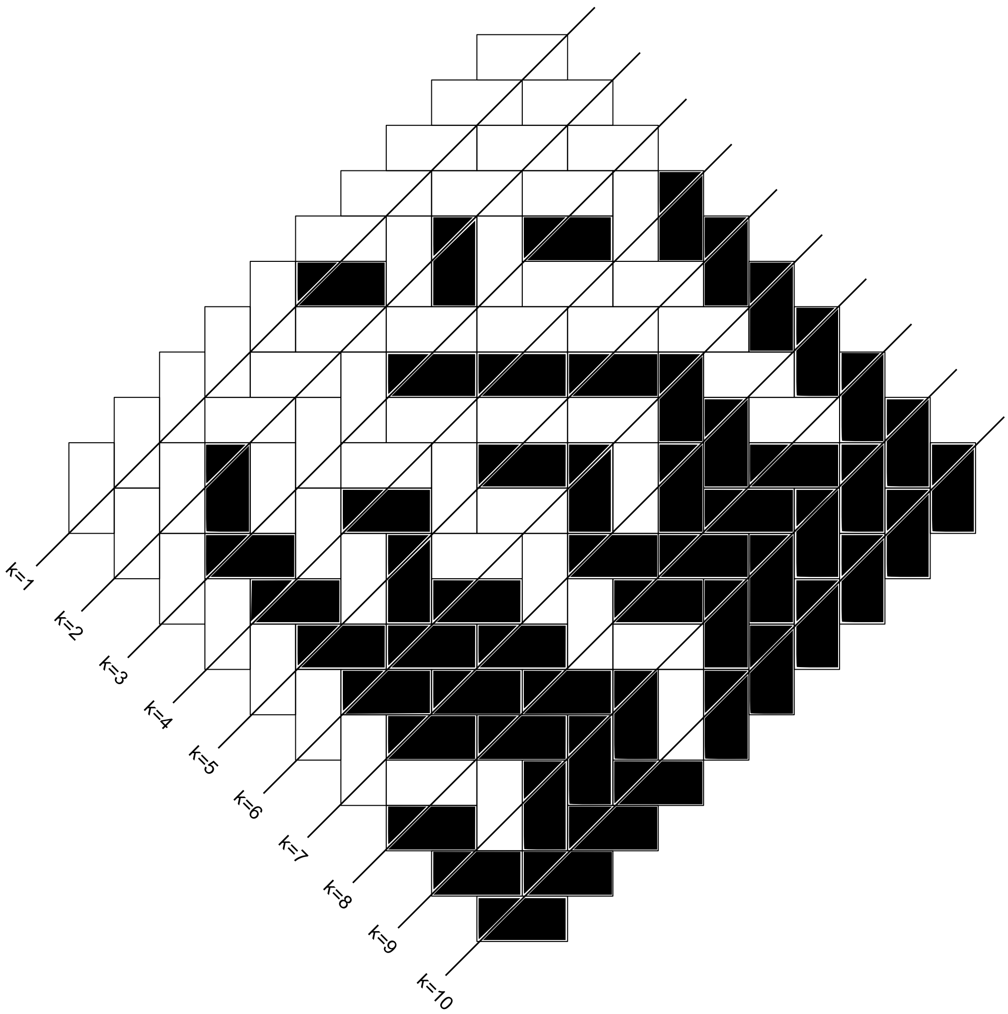


Figure 4: An example of a tiling of an Aztec diamond of order 10 by dominoes, with the E and S type dominoes shaded, with 10 labelled lines added. Note that the north, south, east and west points of the Aztec diamond are predominantly tiled by N type, S type, E type and W type dominoes respectively, and that the k-th line passes through k shaded tiles.

sub-matrices. This will be discussed in the next section, while in this section the topic for consideration is the eigenvalue problem. The anti-symmetric GUE ensemble is the probability measure on purely imaginary Hermitian matrices with density $Z^{-1}e^{-\mathrm{Tr}H^2/2}$, Z a normalization constant. Equivalently, form a real Gaussian matrix X with entries chosen independently from $\mathrm{N}[0, \frac{1}{\sqrt{2}}]$ and set $H = \frac{1}{2}(X - X^T)$. Following the work in [31], we would like to compute the joint PDF of $(\lambda^{(1)}, \dots, \lambda^{(n)})$, where $\lambda^{(k)} = (\lambda_1^{(k)}, \dots, \lambda_{\lfloor \frac{k}{2} \rfloor}^{(k)})$ are the positive eigenvalues of the $k \times k$ principal sub-block of H. First, two lemmas are required

Lemma 1.10. *[31] Let $0 < a_1 < \cdots < a_n$ be fixed real numbers. Let $q_1, \dots, q_n$ be i.i.d.* $\exp(1)$ *random variables, specified by the PDF e^{-x} $(x > 0)$. Consider the random rational function*

$$p(\lambda) = \lambda - \sum_{i=1}^{n} \frac{\lambda q_i}{\lambda^2 - a_i^2} \tag{1.48}$$

$p(\lambda)$ has n positive zeroes denoted $0 < b_1 < \cdots < b_n$, and their PDF is

$$2^n \frac{\Delta(b^2)}{\Delta(a^2)} \prod_{i=1}^{n} b_i e^{-b_i^2 + a_i^2} \chi(a \prec b) \tag{1.49}$$

where $\chi(a \prec b)$ is the interlacing condition defined in §6.1

Proof. As with the rational function (1.22), graphical considerations imply that there must be exactly n positive zeros $b_1, \dots, b_n$, and that they obey the interlacing condition $\chi(a \prec b)$. Additionally, it is clear from (1.48) that $p(0) = 0$, and that if $p(b) = 0$, then $p(-b) = 0$. Thus, noting that p has simple poles at $\pm a_i$ for $i = 1, \dots, n$, it follows that it is possible to write

$$p(\lambda) = \lambda \prod_{i=1}^{n} \frac{\lambda^2 - b_i^2}{\lambda^2 - a_i^2} \tag{1.50}$$

Comparing the residue at a_i of p in (1.48) and (1.50), and elementary computation gives that

$$-q_i = \frac{\prod_{j=1}^{n} a_i^2 - b_j^2}{\prod_{j=1, j\neq i}^{n} a_i^2 - a_j^2} \tag{1.51}$$

The PDF for the variables $\{q_i\}_{i=1}^n$ is

$$\exp\left(-\sum_{i=1}^{n} q_i\right) \tag{1.52}$$

and we want to change variables to $\{b_i\}_{i=1}^n$. The Jacobian J for that transformation is, up to a possible sign,

$$J = \det\left[\frac{-2b_j q_i}{a_i^2 - b_j^2}\right]_{i,j=1,\dots,n} = \frac{\prod_{1\leq i<j\leq n}(a_i^2 - a_j^2)(b_i^2 - b_j^2)}{\prod_{1\leq i,j\leq n} a_i^2 - b_j^2} \prod_{i=1}^{n} 2b_i q_i$$

where the determinant is evaluated using the Cauchy double alternate identity (1.28). Inserting the expression for q_i from (1.51) simplifies this to

$$J = 2^n \prod_{1\leq i<j\leq n} \frac{b_i^2 - b_j^2}{a_i^2 - a_j^2} \prod_{i=1}^{n} b_i \tag{1.53}$$

By expanding (1.48) and (1.50) at infinity and comparing the $1/\lambda$ coefficient it follows that

$$-\sum_{i=1}^{n} q_i = \sum_{i=1}^{n} a_i^2 - b_i^2 \tag{1.54}$$

Inserting this in (1.52) and multiplying by the Jacobian (1.53) gives the sought fom (1.49). □

Lemma 1.11. *[31] Let $0 < a_1 < \cdots < a_n$ be fixed real numbers. Let $q_1, \dots, q_n$ be i.i.d.* exp(1) *random variables, specified by the PDF e^{-x} $(x > 0)$, and let q_0 be $\Gamma(1/2, 1)$ distributed, having PDF $(\pi x)^{-1/2} e^{-x}$ $(x > 0)$. The random rational function*

$$p(\lambda) = \lambda - \frac{q_0}{\lambda} - \sum_{i=1}^{n} \frac{\lambda q_i}{\lambda^2 - a_i^2} \tag{1.55}$$

has $n+1$ positive zeroes denoted $0 < b_0 < \cdots < b_n$ and their PDF is

$$\frac{2^{n+1}}{\sqrt{\pi}} \frac{\Delta(b^2)}{\Delta(a^2)} e^{-b_0^2} \prod_{i=1}^{n} \frac{e^{a_i^2 - b_i^2}}{a_i} \chi(b \prec a) \tag{1.56}$$

Proof. Following the logic of the beginning of Lemma 1.10, it is possible to write

$$p(\lambda) = \frac{\lambda^2 - b_0^2}{\lambda} \prod_{i=1}^{n} \frac{\lambda^2 - b_i^2}{\lambda^2 - a_i^2}$$

From here, it is convenient to introduce an arbitrary $a_0 = 0$, and then the proof is virtually unchanged from that of Lemma 1.10 with indices starting from zero instead of from one. The Jacobian expression from (1.53) can then be simplified as

$$J = 2^{n+1} \prod_{0 \le i < j \le n} \frac{b_i^2 - b_j^2}{a_i^2 - a_j^2} \prod_{i=0}^{n} b_i = 2^{n+1}(-1)^n b_0 \prod_{0 \le i < j \le n} \frac{b_i^2 - b_j^2}{a_i^2 - a_j^2} \prod_{i=1}^{n} \frac{b_i}{a_i^2} \tag{1.57}$$

Computing the residue at the origin of p in (1.55) and (1.50) gives

$$\prod_{i=0}^{n} b_i^2 = q_0 \prod_{i=1}^{n} a_i^2 \tag{1.58}$$

The expression corresponding to (1.54) is

$$-\sum_{i=0}^{n} q_i = -b_0 + \sum_{i=1}^{n} a_i - b_i \tag{1.59}$$

The PDF of the variables $\{q_i\}_{i=0}^{n}$ is

$$\frac{\exp(-\sum_{i=0}^{n} q_i)}{\sqrt{\pi q_0}}$$

Multiplying this with the Jacobian (1.57), inserting (1.58) and (1.59) gives the sought form (1.56).

□

Theorem 1.12. *[31] Let H be an $n \times n$ matrix from the anti-symmetric GUE ensemble. Let H_k be the $k \times k$ leading sub-block of H. Let $\lambda^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_{\lfloor k/2 \rfloor}^{(k)})$ be the positive eigenvalues of H_k, ordered so that $\lambda_i^{(k)} > \lambda_{i+1}^{(k)}$. Then the joint PDF of $\lambda^{(1)}, \ldots, \lambda^{(n)}$ is given by*

$$\frac{1}{C_n} \Delta\left((\lambda^{(n)})^2\right) \prod_{i=1}^{n/2} e^{-(\lambda_i^{(n)})^2} \prod_{k=1}^{n-1} \chi(\lambda^{(k)} \prec \lambda^{(k+1)}) \qquad \text{for } n \text{ even} \tag{1.60}$$

$$\frac{1}{C_n} \Delta\left((\lambda^{(n)})^2\right) \prod_{i=1}^{(n-1)/2} \lambda_i^{(n)} e^{-(\lambda_i^{(n)})^2} \prod_{k=1}^{n-1} \chi(\lambda^{(k)} \prec \lambda^{(k+1)}) \qquad \text{for } n \text{ odd} \tag{1.61}$$

where

$$C_{2n} = \frac{\pi^{n/2}}{2^{n^2}} \qquad C_{2n+1} = \frac{\pi^{n/2} n!}{2^{n(n+1)}} \tag{1.62}$$

Proof. Such a matrix H has the property that if λ is an eigenvalue of H, then so is $-\lambda$. Also, if the size of H is odd, this implies that one of the eigenvalues will be zero.

The proof is an inductive one. A 2×2 matrix from this ensemble is of the form $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ where $a \in \mathrm{N}[0, 1/\sqrt{2}]$. Its eigenvalues are $\pm a$, confirming the theorem in the case $n = 2$.

First, let n be even. Consider an $n \times n$ matrix A from this ensemble. The induction assumption is that its eigenvalue PDF is given by (1.60). Consider the $(n+1) \times (n+1)$ matrix given by bordering A,

$$\begin{pmatrix} A & w \\ w^* & 0 \end{pmatrix}$$

Here, w is a column vector of n purely imaginary numbers, all $\mathrm{N}[0, 1/\sqrt{2}]$. The star means transpose and complex conjugate.

The eigenvectors of A can be paired up in the following way: If v is an eigenvector corresponding to eigenvalue λ, then $\bar{v}$ is an eigenvector corresponding to eigenvalue $-\lambda$. Consider a normalised eigenvector, $|v| = 1$. Since v and $\bar{v}$ must be orthogonal to each other, $|\mathrm{Re}\, v|^2 = \frac{1}{4}(v+\bar{v}, v+\bar{v}) = \frac{1}{2}$, where $(\cdot, \cdot)$ denotes the inner product.

Let $C = [v_1, \bar{v_1}, v_2, \dots]$ be the matrix whose columns are eigenvectors of A. Then

$$\begin{pmatrix} C^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & w \\ w^* & 0 \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} D & C^* w \\ w^* C & 0 \end{pmatrix} \tag{1.63}$$

where D is a diagonal matrix of the eigenvalues. It follows from the above considerations of eigenvectors and an elementary calculation that $w^* C = (a_1, \bar{a_1}, a_2, \dots)$ where each a_i is a complex number, the real and imaginary part of which are $\mathrm{N}[0, 1/\sqrt{2}]$. Let $p_n(\lambda)$ be the characteristic polynomial of A and say that the eigenvalues of A are $\pm\mu_1, \dots, \pm\mu_{n/2}$. Of course the eigenvalues of A give the factorisation of p_n as

$$p_n(\lambda) = (\mu_1^2 - \lambda^2) \dots (\mu_{n/2}^n - \lambda^2)$$

Then it can be shown, say by expanding along the last row of the RHS of (1.63), that the characteristic polynomial of that larger matrix is such that

$$\frac{p_{n+1}(\lambda)}{p_n(\lambda)} = \lambda - \sum_{i=1}^{n/2} \frac{2 a_i \bar{a_i} \lambda}{\mu_i^2 - \lambda^2}$$

With a_i distributed as $\mathrm{N}[0, 1/\sqrt{2}] + i\mathrm{N}[0, 1/\sqrt{2}]$, it follows that $2a_i\bar{a_i}$ is $\exp(1)$ distributed. So we now need to find the PDF of the zeroes of this random rational function, which is precisely what is given by Lemma 1.10. Multiplying the expression that the induction assumption gives us for n with the conditional PDF from Lemma 1.10 proves the statement for H_{n+1} when n is even.

Assume now that n is odd. Do the same construction but the matrix A will now have one eigenvalue which is zero. Performing the same bordering as in (1.63), only this time $w^* C = (a_1, \bar{a_1}, \dots, a_n, \bar{a_n}, ib)$ where b is $\mathrm{N}[0, 1/2]$. As above the characteristic polynomials for the $n \times n$ matrix and the $(n+1) \times (n+1)$- matrix are related by

$$\frac{p_{n+1}(\lambda)}{p_n(\lambda)} = \lambda - \frac{b^2}{\lambda} - \sum_{i=1}^{(n-1)/2} \frac{2 a_i \bar{a_i} \lambda}{\lambda_i^2 - \lambda^2}$$

Apply this time Lemma 1.11 to prove the statement for H_{n+1} for n odd. □

Also of interest is the eigenvalue PDF of an $n \times n$ anti-symmetric GUE, independent of the eigenvalues of the sub-blocks. This can be obtained by integrating out the latter variables in (1.60).

Lemma 1.13. *Integrating out the eigenvalues* $\{\lambda^{(j)}\}_{j=1,\dots,m-1}$ *in* (1.60) *gives, for* m *even*

$$\frac{1}{B_m C_n} \Delta\left((\lambda^{(m)})^2\right) \Delta\left((\lambda^{(n)})^2\right) \prod_{l=1}^{n/2} e^{-(\lambda_l^{(n)})^2} \prod_{k=m}^{n-1} \chi(\lambda^{(k)} \prec \lambda^{(k+1)}) \qquad \text{for } n \text{ even}$$

$$\frac{1}{B_m C_n} \Delta\left((\lambda^{(m)})^2\right) \Delta\left((\lambda^{(n)})^2\right) \prod_{i=1}^{(n-1)/2} \lambda_i^{(n)} e^{-(\lambda_i^{(n)})^2} \prod_{k=m}^{n-1} \chi(\lambda^{(k)} \prec \lambda^{(k+1)}) \qquad \text{for } n \text{ odd}$$

while for m odd

$$\frac{1}{B_m C_n}\left((\lambda^{(m)})^2\right)\Delta\left((\lambda^{(n)})^2\right)\prod_{i=1}^{(m-1)/2}\lambda_i^{(m)}\prod_{i=1}^{n/2}e^{-(\lambda_i^{(n)})^2}\prod_{k=m}^{n-1}\chi(\lambda^{(k)}\prec\lambda^{(k+1)})\qquad\text{for } n \text{ even}$$

$$\frac{1}{B_m C_n}\left((\lambda^{(m)})^2\right)\Delta\left((\lambda^{(n)})^2\right)\prod_{i=1}^{(m-1)/2}\lambda_i^{(m)}\prod_{i=1}^{(n-1)/2}\lambda_i^{(n)}e^{-(\lambda_i^{(n)})^2}\prod_{k=m}^{n-1}\chi(\lambda^{(k)}\prec\lambda^{(k+1)})\qquad\text{for } n \text{ odd}$$

where C_n is as in (1.62) *and*

$$B_{2m}=\prod_{j=1}^{m-1}(2j)!\qquad B_{2m+1}=\prod_{j=1}^{m}(2j-1)!$$

Proof. As in Lemma 1.5, we proceed by induction. In the base case $m=1$ there is nothing to prove. Suppose now that the result is true for $m=l$ odd. Making use of (1.18) shows we want to integrate

$$\det\left[(\lambda_{(l+1)/2-i}^{(l)})^{2j-1}\right]_{i,j=1,\dots,(l-1)/2}\chi(\lambda^l\prec\lambda^{l+1})$$

over $\{\lambda_i^{(l)}\}_{i=1,\dots,(l-1)/2}$. The integration can be done row by row to give

$$\begin{aligned}\det\left[\int_{\lambda_{(l+1)/2+1-i}^{(l+1)}}^{\lambda_{(l+1)/2-i}^{(l+1)}}u^{2j-1}du\right]_{i,j=1}^{\frac{l-1}{2}} &= \det\left[\frac{1}{2j}\left((\lambda_{(l+1)/2-i}^{(l+1)})^{2j}-(\lambda_{(l+1)/2+1-i}^{(l+1)})^{2j}\right)\right]_{i,j=1}^{\frac{l-1}{2}}\\ &= \prod_{j=1}^{(l-1)/2}\frac{1}{2j}\Delta\left((\lambda_1^{(l+1)})^2,\dots,(\lambda_{(l+1)/2}^{(l+1)})^2\right)\end{aligned}\tag{1.64}$$

Substituting (1.64) back in the remaining terms of the appropriate equation in Lemma 1.13 (whether n is odd or even) establishes the case $m=l+1$ even. Repeating this process now for $m=l+1$, we want to integrate

$$\det\left[(\lambda_{(l+1)/2+1-i}^{(l+1)})^{2(j-1)}\right]_{i,j=1,\dots,(l+1)/2}\chi(\lambda^{l+1}\prec\lambda^{l+2})$$

over $\{\lambda_i^{(l+1)}\}_{i=1,\dots,(l+1)/2}$. The integration can be done row by row to give, with $\lambda_{(l+1)/2+1}^{(l+2)}:=0$,

$$\begin{aligned}\det\left[\int_{\lambda_{(l+1)/2+2-i}^{(l+2)}}^{\lambda_{(l+1)/2+1-i}^{(l+2)}}u^{2(j-1)}du\right]_{i,j=1}^{\frac{l+1}{2}} &= \det\left[\frac{1}{2j-1}\left((\lambda_{(l+1)/2+1-i}^{(l+1)})^{2j-1}-(\lambda_{(l+1)/2+2-i}^{(l+1)})^{2j-1}\right)\right]_{i,j=1}^{\frac{l+1}{2}}\\ &= \prod_{j=1}^{(l+1)/2}\frac{\lambda_j^{(l+2)}}{2j-1}\Delta\left((\lambda_1^{(l+2)})^2,\dots,(\lambda_{(l+1)/2}^{(l+2)})^2\right)\end{aligned}\tag{1.65}$$

Substituting (1.65) back in the remaining terms of the appropriate equation in Lemma 1.13 (whether n is odd or even) establishes the case $m=l+2$ odd and completes the proof. □

We read off from this result that the eigenvalue PDF of an $n\times n$ anti-symmetric GUE matrix is equal to

$$\begin{aligned}&\frac{1}{A_n}\prod_{i=1}^{n/2}e^{-(\lambda_i^{(n)})^2}\prod_{1\le j<k\le n/2}\left((\lambda_j^{(n)})^2-(\lambda_k^{(n)})^2\right)^2 &&\text{for } n \text{ even}\\ &\frac{1}{A_n}\prod_{i=1}^{(n-1)/2}(\lambda_i^{(n)})^2e^{-(\lambda_i^{(n)})^2}\prod_{1\le j<k\le (n-1)/2}\left((\lambda_j^{(n)})^2-(\lambda_k^{(n)})^2\right)^2 &&\text{for } n \text{ even}\end{aligned}\tag{1.66}$$

where

$$A_{2n} = \frac{\pi^{n/2}}{2^{n^2}} \prod_{i=0}^{n-1} (2i)! \qquad A_{2n+1} = \frac{\pi^{n/2}}{2^{n(n+1)}} \prod_{i=0}^{n-1} (2i+1)! \tag{1.67}$$

1.6 The half-hexagon

As the GUE* was shown to be a limit of the particle picture of the rhombus tiling of a hexagon, it is shown in [31] that the antisymmetric GUE is a limit of the particle picture of the rhomus tiling of a half-hexagon. An a, N half-hexagon is literally a $2a \times N \times N$ hexagon cut in half, and as with the rhombus tiling of the hexagon, we see from Figure 5 that a rhombus tiling of an a, N half-hexagon is in bijection with a set of a simple symmetric random walks conditioned never to intersect or go below 0, starting at $(0, 2i-1)$ and ending at $(2N, 2i-1)$ for $i = 1, \ldots, a$. As before, we are interested in the holes between the walkers. On line t there will be $r(t)$ particles,

$$r(t) = \begin{cases} \lfloor t/2 \rfloor & t \leq N \\ \lfloor (2N-t)/2 \rfloor & N < t \leq 2N \end{cases}$$

Let $x^{(t)} = (x_1^{(t)}, \ldots x_{r(t)}^{(t)})$ be the positions of the particles on line t, where by labelling convention, $x_i^{(t)} > x_j^{(t)}$ for $i < j$. Since no blue particle may be outside the hexagon, the particle positions must be positive, and the highest possible position for $x_1^{(t)}$, $g(t)$ say is given by

$$g(t) = \begin{cases} 2a + t - 1 & t \leq N \\ 2a + 2N - t - 1 & N < t \leq 2N \end{cases}$$

For combinatorial reasons, which are clear to see from the picture, the particles fulfill the interlacing requirement $\chi(x^{(t)} \prec x^{(t+1)})$ as defined in §6.1. We also have the restriction that

$$x_i^{(t)} - t \ \text{ odd } \ \forall\, i, t. \tag{1.68}$$

Given that every possible tiling is equally likely by definition, the probability of some configuration $\bar{x} = (x^{(0)}, \ldots, x^{(2N)})$ can be written

$$p(\bar{x}) = \frac{1}{C_{a,N}} \prod_{t=0}^{N-1} \chi(x^{(t)} \prec x^{(t+1)}) \prod_{t=N}^{2N-1} \chi(x^{(t+1)} \prec x^{(t)})$$

where the virtual particles $x_{r(t)+1}^{(t)} = (-1-(-1)^t)/2$ and $x_0^{(t)} = g(t) + 2$ have been included to ensure the particles lie inside the hexagon, and furthermore the condition (1.68) is required. $C_{a,N}$ is some normalization constant, which is equal to the number of possible configurations for an a, N half-hexagon. We now introduce a Lemma and a Theorem, analagous to Lemma 1.7 and Theorem 1.8 from the a, b, c hexagon case studied earlier.

Lemma 1.14. *[31] Let $t \leq N$. Given some configuration $x^{(t)}$, the number of configurations to the left of line t, i.e.*

$$G_t(x^{(t)}) = \sum_{x^{(0)}, \ldots, x^{(t-1)}} \prod_{n=0}^{t-1} \chi(x^{(n)} \prec x^{(n-1)})$$

with virtual particles as described above, is

$$G_t(x^{(t)}) = c_t \prod_{1 \leq i < j \leq t/2} \left((x_i^{(t)})^2 - (x_j^{(t)})^2 \right) \tag{1.69}$$

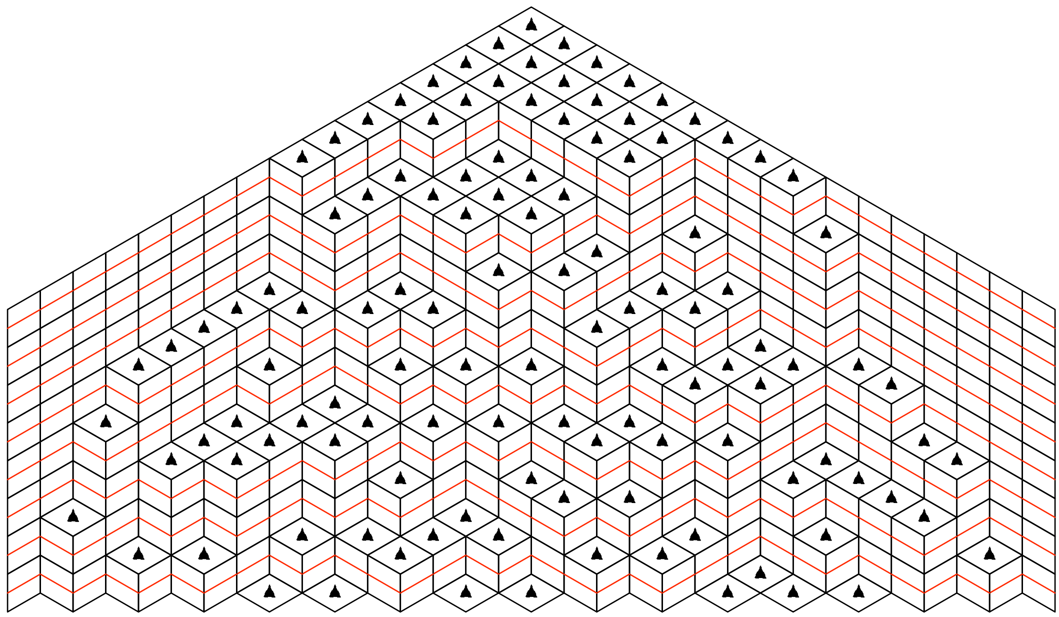


Figure 5: An example of a random tiling of an 8, 16 half-hexagon by rhombi, with the corresponding interlaced particles and non-intersecting walks shown.

for t even and

$$G_t(x^{(t)}) = c_t \prod_{1\leq i<j\leq (t-1)/2} \left((x_i^{(t)})^2 - (x_j^{(t)})^2\right) \prod_{i=1}^{(t-1)/2} x_i^{(t)} \tag{1.70}$$

for t odd, where

$$c_{2n} = \prod_{i=0}^{n-1} \frac{1}{2^{2i}(2i)!} \qquad c_{2n+1} = \frac{n!}{(2n)!} \prod_{i=0}^{n-1} \frac{1}{2^{2i}(2i)!} \tag{1.71}$$

Proof. The proof is an inductive one. The $t = 2$ case is true by virtue of the fact that $G_2(x) := 1$. Suppose (1.69) has been established for $t = 2n$. Then

$$G_{2n+1}(x_1,\dots,x_n) = \sum_{i=1}^{n} \sum_{\substack{x_i>y_i>x_{i+1} \\ y_i-x_i \text{ odd}}} c_{2n} \begin{vmatrix} 1 & y_1^2 & \dots & y_1^{2(n-1)} \\ \vdots & & & \vdots \\ 1 & y_n^2 & \dots & y_n^{2(n-1)} \end{vmatrix}$$

Using (1.34) and (1.35), we perform column operations to remove dependence on lower order terms, and perform row operations to clean up, to give

$$G_{2n+1}(x_1,\dots,x_n) = c_{2n} \begin{vmatrix} \frac{x_1}{2} & \frac{x_1^3}{6} & \dots & \frac{x_1^{2n-1}}{4n-2} \\ \vdots & & & \vdots \\ \frac{x_n}{2} & \frac{x_n^3}{6} & \dots & \frac{x_n^{2n-1}}{4n-2} \end{vmatrix}$$

which is equal to (1.70) with

$$c_{2n+1} = \frac{c_{2n} n!}{(2n)!} \tag{1.72}$$

Now suppose (1.70) has been established for $t = 2n+1$. Then, proceeding as before,

$$G_{2n+2}(x_1,\ldots,x_{n+1}) = \sum_{i=1}^{n} \sum_{\substack{x_i > y_i > x_{i+1} \\ y_i - x_i \text{ odd}}} c_{2n+1} \begin{vmatrix} y_1 & y_1^3 & \cdots & y_1^{2n-1} \\ \vdots & & & \vdots \\ y_n & y_n^3 & \cdots & y_n^{2n-1} \end{vmatrix}$$

which gives

$$G_{2n+2}(x_1,\ldots,x_{n+1}) = c_{2n+1} \begin{vmatrix} \frac{x_1^2 - x_2^2}{4} & \frac{x_1^4 - x_2^4}{8} & \cdots & \frac{x_1^{2n} - x_2^{2n}}{4n} \\ \vdots & & & \vdots \\ \frac{x_n^2 - x_{n+1}^2}{4} & \frac{x_n^4 - x_{n+1}^4}{8} & \cdots & \frac{x_n^{2n} - x_{n+1}^{2n}}{4n} \end{vmatrix}$$

Bordering this determinant and performing row and column operations gives (1.69) with

$$c_{2n+2} = \frac{c_{2n+1}}{4^n n!} \tag{1.73}$$

and (1.72) and (1.73) combine to give (1.71) □

Theorem 1.15. *The probability that on line t, $t \le N$, the particles are at positions $(x_1,\ldots,x_{r(t)})$ is*

$$p_{2n}(x_1,\ldots,x_n) = Z_{2n,a,N}^{-1} \prod_{1 \le i < j \le n} \left(x_i^2 - x_j^2\right)^2 \prod_{l=1}^{n} f_{2n}(x_l) \tag{1.74}$$

$$p_{2n+1}(x_1,\ldots,x_n) = Z_{2n+1,a,N}^{-1} \prod_{1 \le i < j \le n} \left(x_i^2 - x_j^2\right)^2 \prod_{l=1}^{n} x_l^2 f_{2n+1}(x_l) \tag{1.75}$$

for $t = 2n$ or $t = 2n+1$ respectively, where $Z_{t,a,N}$ is a normalizing constant and

$$f_t(x) = \prod_{i=1}^{N-t} (2a + 2i + t - 1 - x)(2a + 2i + t - 1 + x) \tag{1.76}$$

Proof. The proof is the same as that for Theorem 1.8. We introduce virtual particles above the hexagon so that the number of possible tilings to each side of line t is given by Lemma 1.14 for the appropriate line number. The area to the left of line t can be tiled in $G_t(x_1, x_{r(t)})$ ways. The area to the right of line t can be tiled in $G_{2N-t}(x_{t-N},\ldots,x_{-1},x_1,\ldots,x_{r(t)})$ ways, where we have introduced the virtual particles $x_{-i} = 2a + 2i + t - 1$. Then

$$p_t(x_1,\ldots,x_{r(t)}) = \frac{1}{C_{a,N}} G_t(x_1,\ldots,x_{r(t)}) G_{2N-t}(x_{t-N},\ldots,x_{-1},x_1,\ldots,x_{r(t)})$$

The theorem follows since the part of the Vandemonde that has to do with the virtual particles is, up to a constant, $\prod_{l=1}^{r(t)} f_t(x_l)$. □

We now define the random variable $X_t^{\text{HalfHex}} = (x^{(0)},\ldots,x^{(t)})$. From Lemma 1.14 and Theorem 1.15, and the requirement that every possible tiling of the half-hexagon is equally likely, we have that the PDF of X_t^{HalfHex} is

$$P_{(t)}^{\text{HalfHex}}(x^{(1)},\ldots,x^{(t)}) = \frac{p_t(x^{(t)})}{G_t(x^{(t)})} \prod_{n=0}^{t-1} \chi(x^{(n)} \prec x^{(n+1)}) \tag{1.77}$$

In an appropriate scaling limit this discrete distribtution reduces to the eigenvalue distribution for Theorem 1.12 for matrices from the antisymmetric GUE and their submatrices.

Proposition 1.16. *Under the rescaling* $x_i^{(j)} = \lambda_i^{(j)}\sqrt{4N(\alpha+1)\alpha}$*, where the* $x_i^{(j)}$ *are distributed as in* (1.77) *with* $a = \alpha N$*, one has*

$$P_{(t)}^{\mathrm{HalfHex}}(x^{(1)},\ldots,x^{(t)}) \to P_{\mathrm{AntiSymGUE},(t)}(\lambda_i^{(1)},\ldots,\lambda_i^{(t)})$$

where $P_{\mathrm{AntiSymGUE},(t)}$ *is given by* (1.60) *for* t *even and* (1.61) *for* t *odd, as* $N \to \infty$*, where the convergence is uniform on compact sets with respect to the* $x_i^{(j)}$*.*

Proof. We will go through the even case $t = 2n$, as the t odd case is trivially similar. From (1.77) we have

$$P_{(2n)}^{\mathrm{HalfHex}}(x^{(1)},\ldots,x^{(2n)}) = \frac{1}{c_{2n}Z_{2n,a,N}} \times \prod_{1\le i<j\le n}(x_i^{(2n)})^2-(x_j^{(2n)})^2 \prod_{i=1}^{n} f_{2n}(x_i^{(2n)}) \prod_{k=0}^{2n-1}\chi(x^{(k)} \prec x^{(k+1)})$$

From (1.76), for $a = \alpha N$, we have that

$$f_{2n}(x) = \prod_{i=1}^{N-2n}(2\alpha N + 2i + 2n - 1 - x)(2\alpha N + 2i + 2n - 1 + x)$$

We make the substitution $x = \sqrt{\beta N}y$,

$$\begin{aligned} f_{2n}(\sqrt{\beta N}y) &= \prod_{i=1}^{N-2n}(2\alpha N + 2i + 2n - \sqrt{\beta N}y - 1)(2\alpha N + 2i + 2n + \sqrt{\beta N}y - 1) \\ &= 2^{2N-4n}\frac{((\alpha+1)N - n - \frac{1}{2}\sqrt{\beta N}y - \frac{1}{2})!((\alpha+1)N - n + \frac{1}{2}\sqrt{\beta N}y - \frac{1}{2})!}{(\alpha N + n - \frac{1}{2}\sqrt{\beta N}y - \frac{1}{2})!(\alpha N + n + \frac{1}{2}\sqrt{\beta N}y - \frac{1}{2})!} \end{aligned}$$

Setting $\beta = 4(\alpha+1)\alpha$ and applying Stirling's formula (1.16) gives

$$f_{2n}\left(y\sqrt{4N(\alpha+1)\alpha}\right) = 2^{2N-4n}N^{2N-4n}e^{-y^2-2N}\frac{(\alpha+1)^{2(\alpha+1)N-2n}}{\alpha^{2\alpha N+2n}} + \mathrm{O}(N^{-1/2}) \tag{1.78}$$

For ease of notation, define $F_{2n,N}^{\mathrm{HalfHex}}$ so that the leading order term of the RHS of (1.78) is equal to $F_{2n,N}^{\mathrm{HalfHex}}e^{-y}$. Then, noting that

$$\prod_{k=1}^{2n}\bigwedge_{i=1}^{r(k)} dx_i^{(k)} = \frac{1}{2^{n^2}}\left(4N(\alpha+1)\alpha\right)^{\frac{n^2}{2}}\prod_{k=1}^{2n}\bigwedge_{i=1}^{r(k)} d\lambda_i^{(k)}$$

where the 2^{-n^2} comes from the fact that the original $x_i^{(k)}$'s possible positions were confined to every *second* integer, we have

$$P_{(2n)}^{\mathrm{HalfHex},N}(\lambda^{(1)},\ldots,\lambda^{(2n)}) = \frac{(F_{2n,N}^{\mathrm{HalfHex}})^n}{c_{2n}Z_{2n,\alpha N,N}}\frac{1}{2^{n^2}}\left(4N(\alpha+1)\alpha\right)^{n^2-n/2}\prod_{i=1}^{n}e^{-\lambda_i^{(2n)}} \times \prod_{1\le i<j\le n}\left((\lambda_i^{(2n)})^2-(\lambda_j^{(2n)})^2\right)\prod_{k=0}^{2n-1}\chi(\lambda^{(k)}\prec\lambda^{(k+1)}) + \mathrm{O}(N^{-1/2}) \tag{1.79}$$

Applying Stirling's formula (1.16) to $Z_{2n,\alpha N,N}$, which we find in Proposition 2.4 in §2.1, gives

$$Z_{2n,\alpha N,N} = \pi^{\frac{n}{2}}2^{2Nn-3n^2-2n}N^{2Nn-3n^2-n/2}e^{-2Nn}\frac{(\alpha+1)^{2Nn(\alpha+1)-n^2-n/2}}{\alpha^{2\alpha nN+n^2+n/2}}\prod_{i=0}^{n-1}(2i)! + \mathrm{O}(1/N)$$

which along with (1.78) gives

$$\frac{(F^{\text{Halfhex}}_{2n,N})^n}{c_{2n}Z_{2n,\alpha N,N}}\frac{1}{2^{n^2}}\left(4N(\alpha+1)\alpha\right)^{n^2-n/2}=\frac{2^{n^2}}{\pi^{\frac{n}{2}}}+\mathrm{O}(1/N)$$

and so the PDF (1.79) converges to the Antisymmetric GUE PDF in Theorem 1.12. □

So, the very same methods that we used to show that a scaling of the hexagon rhombus tiling positions tended toward GUE* eigenvalues have been used to show a link between a scaling of the half-hexagon rhombus tiling positions and the Antisymmetric GUE eigenvalues. As we also showed a link between the Aztec diamond and the GUE* eigenvalues, a logical step would be to look at a Half Aztec diamond case, and see if the same link can be shown there. This is taken up in §4.4.

1.7 Hexagon walks

Our theme to date has been statistical systems described by sets of interlacing variables which limit to the distribution of the eigenvalues of successive sub-blocks of the GUE*. In the next chapter this theme will be supplemented by the computation of the correlation functions for the interlacing variables. With this theme in mind, we return to the rhombus tiling of an $a \times b \times c$ hexagon first consider in §1.3, and consider in more detail the corresponding set of simple symmetric non-intersecting paths complementary to the particles (recall Figure 1). While we will find PDFs for the walkers explicitly, it would be possible to find a PDF for the walkers directly from this relationship, given that we have PDFs for the particle model (we will use this relationship as a check of our results, however a calculation for finding the paths in this way can be found in [43]). We will also use a similar relationship applied to joint PDFs in the Aztec diamond later to find a one-line PDF.

The walkers model for an $a \times b \times c$ hexagon consists of a symmetric random walkers, beginning at positions $(0, 2i)$ for $i = 0, \ldots, a-1$ at time $t = 0$, and with each step moving one position up or down and one position to the right. Hence a walker will be at some position (t, x) at time t, with $t + x$ even. Because of the starting positions, ending positions, and the restriction to only be able to move one position up or down with each right movement, the walkers must lie between $g(t)$ and $h(t)$ at time t, where g and h are as in (1.31).

To find a PDF for the positions of these walkers at time t, we consider the model of Gorin in [37], in which there are N 'walkers', beginning at $(0, i-1)_{i=1,\ldots,N}$ and ending at $(T, S+i-1)_{i=1,\ldots,N}$ and conditioned to never intersect, where each 'step' by the walkers is horizontally right or diagonally right-up. We note that this model is the same as our hexagon walker model, under the change of variables $z_{(\text{Gorin})} = (x_{(\text{Hex})} + t)/2$, with $N = a$, $S = c$ and $T = b + c$.

Proposition 1.17. *[37] The number of distinct configurations of non-intersecting paths (where each step is right or right-up) connecting points (t_1, a_i) and (t_2, b_i), where i varies from 1 to N, is equal to*

$$\det\left[\binom{t_2-t_1}{b_i-a_j}\right]_{i,j=1,\ldots,N} \tag{1.80}$$

Proof. This is a consequence of the well known Gessel-Viennot theorem [8], [35] relating to the number of non-intersecting paths on general acyclic graphs, with initial and final positions prescribed.

The theorem gives that the number is equal to $\det\left[G_{t_2-t_1}(b_i,a_j)\right]_{i,j=1,\dots,N}$ where $G_{t_2-t_1}(b_i,a_j)$ is the number of paths starting at a_j and finishing at b_i for a single walker. This latter number is equal to $\begin{pmatrix} t_2-t_1 \\ b_i-a_j \end{pmatrix}$. □

Using this proposition, it follows that the PDF for the positions of the N walkers in Gorin's model at time t is given by

$$P_t(z_1,\dots,z_N) = \frac{\det\left[\begin{pmatrix} t \\ z_i-j+1 \end{pmatrix}\right]_{i,j=1}^{N} \times \det\left[\begin{pmatrix} T-t \\ S+i-1-z_j \end{pmatrix}\right]_{i,j=1}^{N}}{\det\left[\begin{pmatrix} T \\ S+i-j \end{pmatrix}\right]_{i,j=1}^{N}} \tag{1.81}$$

Using the identity

$$\det\left[\begin{pmatrix} t \\ C+j-x_i \end{pmatrix}\right]_{i,j=1}^{N} = \prod_{i=1}^{N}\frac{(t+i-1)!}{(t+N-1)!}\begin{pmatrix} t+N-1 \\ C+N-x_i \end{pmatrix}\prod_{1\le i<j\le N}(x_j-x_i) \tag{1.82}$$

taken from [50], (1.81) becomes

$$\prod_{1\le i<j\le N}(z_i-z_j)^2\prod_{i=1}^{N}\frac{1}{z_i!(T-S-t+z_i)!(N+t-1-z_i)!(N+S-1-z_i)!}$$
$$\times\prod_{i=1}^{N}\frac{(T-S+i-1)!(t+i-1)!(T-t+i-1)!(N+S-i)!}{(T+i-1)!(i-1)!} \tag{1.83}$$

The determinant formula (1.80) can be generalized to the case of hexagon walkers, where each step is right-up or right-down, by the change of variables $z=(x+t)/2$. Additional changes of variables $N=a$, $S=c$ and $T=b+c$ in (1.83) gives the PDF for the positions of the a hexagon walkers $\{x_i^{(t)}\}_{i=1,\dots,a}$ in an $a\times b\times c$ hexagon

$$\prod_{1\le i<j\le N}(x_i-x_j)^2\prod_{i=1}^{a}\frac{1}{\left(\frac{1}{2}(x_i+t)\right)!\left(b+\frac{1}{2}(x_i-t)\right)!\left(a-1-\frac{1}{2}(x_i-t)\right)!\left(a+c-1-\frac{1}{2}(x_i+t)\right)!}$$
$$\times\prod_{i=1}^{a}\frac{(b+i-1)!(t+i-1)!(b+c-t+i-1)!(N+c-i)!}{2^i(b+c+i-1)!(i-1)!} \tag{1.84}$$

As a small aside, it is worth noting here that (1.84) is complementary to (1.36), in the sense that the positions of the walkers are precisely the unoccupied lattice sites in a configuration of the particles. Generally in this setting is it possible to predict the precise relation between the corresponding PDFs, according to a result of Borodin [4].

Proposition 1.18. *[4] Let $X_w^{(n)}$ denote a random subset of $L=\{l_1,\dots,l_M\}\subset\mathbb{R}$ with $|X_u^{(n)}|=n$ and PDF*

$$\Pr(X_w^{(n)}=\{x_1,\dots,x_n\}) = \frac{1}{C}\prod_{l=1}^{n}w(x_l)\prod_{1\le j<k\le n}(x_j-x_k)^2 \tag{1.85}$$

Suppose $u(x)$, $v(x)$ satisfy

$$u(x)v(x)\propto\prod_{i\in L\backslash x}\frac{1}{(x-i)^2} \tag{1.86}$$

Then

$$\Pr(X_u^{(n)}=\{x_1,\dots,x_n\}) = \Pr(L\backslash X_v^{(M-n)}=\{x_1,\dots,x_n\})$$

We can readily check that the results (1.36) and (1.84) are consistent with this proposition. Thus we see from (1.36) that the PDF for the holes are of the form (1.85) with $u(x) = w(x)$, where

$$u(x) = \prod_{k=1}^{|c-t|} (g(t) + 2k - x) \prod_{k=1}^{|b-t|} (x - h(t) + 2k)$$

for $g(t)$ and $h(t)$ as in (1.31). Furthermore, we see from (1.84) that the PDF for the walkers are of the form (1.85) with $v(x) = w(x)$, where

$$v(x) = \frac{1}{\left(\frac{1}{2}(x_i + t)\right)! \left(b + \frac{1}{2}(x_i - t)\right)! \left(a - 1 - \frac{1}{2}(x_i - t)\right)! \left(a + c - 1 - \frac{1}{2}(x_i + t)\right)!}$$

We read off from these that

$$u(x)v(x) \propto \left[\left(\frac{g(t) - x}{2} \right)! \left(\frac{x - h(t)}{2} \right)! \right]^{-2}$$

On the other hand, letting $L = \{h(t), h(t) + 2, \dots, g(t) - 2, g(t)\}$ as implied by the definition of our hexagon model, we see that the RHS of (1.86) is

$$\left[\left(\frac{g(t) - x}{2} \right)! \left(\frac{x - h(t)}{2} \right)! \right]^{-2} 2^{h(t) - g(t)}$$

in keeping with Proposition 1.18.

As we knew that these two PDFs were complementary by definition, Proposition 1.18 could also have been used to find the PDF for the walkers in the hexagon on any one line. However, it is a useful check that our change of variables to Gorin's model obtained the correct result.

The reason we used Gorin's model originally, rather than Borodin's formula for complements, is the ability to to extend the PDF to more than one line. Generally, if $\{x_i^{(t_k)}\}_{\substack{i=1,\dots,a \\ k=1,\dots,M}}$ represents the set of positions of the i-th walker at time t_k, the joint PDF for the $x_i^{(t_k)}$ is

$$P\left(x^{(t_1)}, \dots, x^{(t_M)}\right) = \frac{1}{C_{\mathrm{HexWalk},a,b,c}} \prod_{k=0}^{M} \det \left[\begin{pmatrix} t_{k+1} - t_k \\ \frac{1}{2}\left(x_i^{(t_{k+1})} - x_j^{(t_k)} + t_{k+1} - t_k\right) \end{pmatrix} \right]_{i,j=1}^{a} \qquad (1.87)$$

where $t_0 = 0$, $t_{M+1} = b + c$, $x_i^{(0)} = 2(a - i)$, $x_i^{(b+c)} = c - b + 2(a - i)$ and

$$C_{\mathrm{HexWalk},a,b,c} = \det \left[\begin{pmatrix} b + c \\ \frac{1}{2}(c - b + 2i - 2j) \end{pmatrix} \right]_{i,j=1}^{a} \qquad (1.88)$$

As the particle model taken from the hexagon tiling tended in a certain limit to the GUE* eigenvalue process, the walkers model also has a well known limit, simple Brownian motion on a line, conditioned never to intersect [22]. We consider a system of N Brownian walkers, conditioned never to intersect, to begin at time $t = 0$ at positions $\vec{x^{(0)}} = \{x_1^{(0)}, \dots, x_N^{(0)}\}$ and end at time $t = 2T$ at positions $\vec{x^{(2T)}} = \{x_1^{(2T)}, \dots, x_N^{(2T)}\}$. Figure 6 is an example of such a system with $N = 10$, $T = 1/2$ and $\vec{x}^{(0)} = \vec{x}^{(1)} = \vec{0}$. The PDF for N Brownian walkers going from $\vec{x}$ to $\vec{y}$ in time t without intersecting is

$$\begin{aligned} G_t(\vec{x}, \vec{y}) &= \det \left[\frac{1}{\sqrt{2\pi t}} e^{-(x_j - y_k)^2/2t} \right]_{j,k=1,\dots,N} \\ &= \left(\frac{1}{2\pi t} \right)^{N/2} e^{-\sum_{j=1}^{N} (x_j^2 + y_j^2)/2t} \det[e^{x_j y_k/t}]_{j,k=1\dots,N} \end{aligned} \qquad (1.89)$$

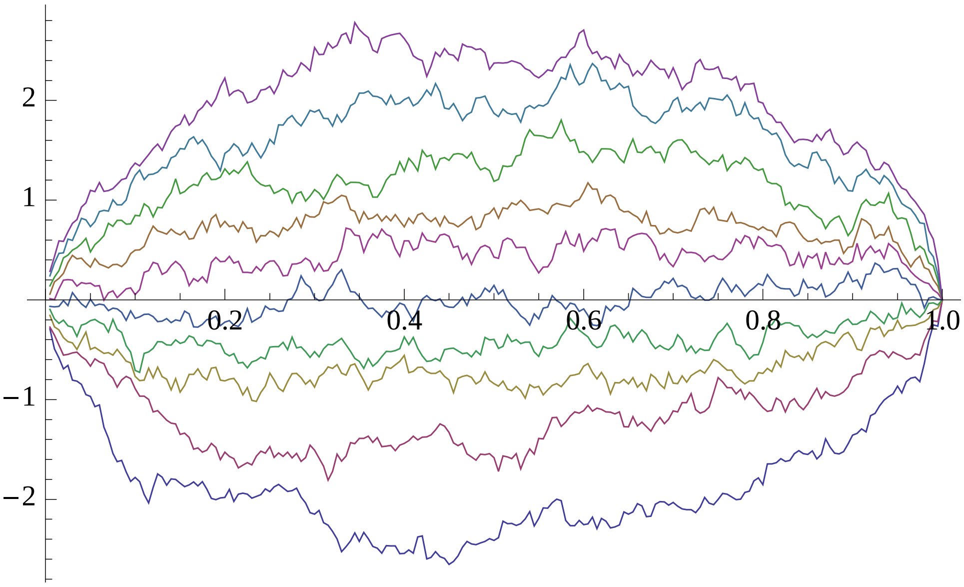


Figure 6: Ten Brownian walkers, conditioned never to intersect and to begin at the origin at $t = 0$ and end at the origin at $t = 1$. This is generated according to the analogy to the eigenvalues of complex Hermitian matrices with Brownian entries.

Here an entry (jk) of the first determinant is the PDF for a single Brownian motion going from x_j to y_k in time t. That N non-intersecting Brownian motions have PDF given by the determinant of this is a result due to Karlin and McGregor [45]. It is the continuous analogue of the Gessel-Viennot theorem (recall the proof of Proposition 1.17).

Given that the walkers start at $\vec{x}^{(0)}$ at time $t = 0$ and end at $\vec{x}^{(2T)}$ at time $t = 2T$ the joint probability density $P_{\vec{x}^{(0)},\vec{x}^{(2T)}}$ for arriving at $\{(\vec{x}^{(s)}, t = t_s)\}_{s=1,\ldots,M}$ along the way is

$$P_{\vec{x}^{(0)},\vec{x}^{(2T)}}(\vec{x}^{(1)}, \ldots, \vec{x}^{(M)}) = \frac{G_{t_1}(\vec{x}^{(0)}, \vec{x}^{(1)})G_{t_2-t_1}(\vec{x}^{(1)}, \vec{x}^{(2)}) \ldots G_{2T-t_M}(\vec{x}^{(M)}, \vec{x}^{(2T)})}{G_{2T}(\vec{x}^{(0)}, \vec{x}^{(2T)})} \tag{1.90}$$

Specialising to the case where $\vec{x}^{(0)} \to \vec{0}$, $\vec{x}^{(2T)} \to \vec{\alpha} = \{\alpha, \alpha, \ldots, \alpha\}$, (1.90) becomes

$$P_{\vec{0},\vec{\alpha}}(\vec{x}^{(1)}, \ldots, \vec{x}^{(M)}) = \frac{1}{(2\pi)^{N/2}\prod_{l=1}^{N-1} l!}\left(\frac{2T}{t_1(2T-t_M)}\right)^{N^2/2}\prod_{l=1}^{M-1} G_{t_{l+1}-t_l}(\vec{x}^{(l)}, \vec{x}^{(l+1)})$$
$$\times \prod_{j=1}^{N} e^{-(x_j^{(1)})^2/2t_1-(x_j^{(M)}-\alpha)^2/2(2T-t_M)} \prod_{1\le j<k\le N}(x_j^{(1)} - x_k^{(1)})(x_j^{(M)} - x_k^{(M)}) \tag{1.91}$$

Proposition 1.19. *Let* $\{x_i^{(N^* t_j)}\}_{\substack{i=1,\ldots,a\\ j=1,\ldots,M}}$ *represent the positions of the* i*-th walker on an* $a \times TN \times (TN + \alpha\sqrt{N})$ *hexagon at a certain line* $N^* t_j$*, where*

$$N^* = N + \frac{\alpha}{2T\sqrt{N}}$$

Then the scaled variables $y_i^{(t)} = x_i^{(N^* t)}/\sqrt{N}$ *have joint PDF as in* (1.91) *in the limit* $N \to \infty$*, where the convergence is uniform on compact sets with respect to the* $x_i^{(j)}$.

Proof. We wish to show that, with x related to y as described, (1.87) converges to (1.91). From the definitions of $x_i^{(0)}$ and $x_i^{(b+c)}$ for an $a \times b \times c$ hexagon,

$$y_i^{(0)} = \frac{x_i^{(0)}}{\sqrt{N}} = \frac{2(a-i)}{\sqrt{N}} \to 0 \qquad y_i^{(2T)} = \frac{x_i^{(2TN+\alpha\sqrt{N})}}{\sqrt{N}} = \frac{\alpha\sqrt{N}+2a-2i}{\sqrt{N}} \to \alpha$$

We now consider the functions that make up the determinants in (1.87) and (1.89). These are of the form

$$f_t(x_1,x_2) = \begin{pmatrix} t \\ \frac{1}{2}(x_2 - x_1 + t) \end{pmatrix} \qquad\qquad g_t(y_1,y_2) = \frac{1}{\sqrt{2\pi t}} e^{-(y_2-y_1)^2/2t}$$

for the discrete x picture and the continuous y picture respectively. Considering a scaled form of the discrete PDF and converting to the y scale, we have

$$2^{-tN^*} f_{tN^*}(x_1,x_2) = 2^{-t\left(N+\frac{\alpha}{2T\sqrt{N}}\right)} \frac{\left(tN + \frac{t\alpha}{2T\sqrt{N}}\right)!}{\left(\frac{1}{2}tN + \frac{1}{2}\sqrt{N}y + \frac{t\alpha}{4T\sqrt{N}}\right)!\left(\frac{1}{2}tN - \frac{1}{2}\sqrt{N}y + \frac{t\alpha}{4T\sqrt{N}}\right)!} \tag{1.92}$$

for $y = y_2 - y_1$. After multiplying by $\dfrac{dx}{dy} = \sqrt{N}/2$, noting that the factor of 2 comes from the fact that the $x_i^{(j)}$ are confined to every second lattice point, and applying Stirling's formula (1.16), (1.92) becomes

$$2^{-tN^*} f_{tN^*}(x_1,x_2) = \frac{1}{\sqrt{2\pi t}} e^{-(y_2-y_1)^2/2t} + \mathrm{O}(N^{-1/2}) = g_t(y_1,y_2) + \mathrm{O}(N^{-1/2})$$

Thus the joint PDF for the $x_i^{(j)}$,

$$\frac{\prod_{k=0}^{M} \det\left[f_{N^*(t_{k+1}-t_k)}\left(x_i^{(Nt_k)}, x_j^{(Nt_{k+1})}\right)\right]_{i,j=1\dots,a}}{\det\left[f_{N^*(t_{M+1}-t_0)}\left(x_i^{(Nt_0)}, x_j^{(Nt_{M+1})}\right)\right]_{i,j=1\dots,a}}$$

with $t_0 = 0$, $t_{M+1} = 2T$, converges to

$$\frac{2^{aN^*\sum_{k=0}^{M} t_{k+1}-t_k}}{2^{2TaN^*}} \frac{\prod_{k=0}^{M} \det\left[g_{t_{k+1}-t_k}\left(y_i^{(t_k)}, y_j^{(t_{k+1})}\right)\right]_{i,j=1\dots,a}}{\det\left[g_{2T}\left(y_i^{(0)}, y_j^{(2T)}\right)\right]_{i,j=1\dots,a}}$$

which after cancelling of terms, is equal to (1.90). That we have shown $y_i^{(0)} \to 0$ and $y_i^{(2T)} \to \alpha$ completes the proof, as (1.91) is the special case of (1.90) with $\vec{x}^{(0)} \to \vec{0}$ and $\vec{x}^{(2T)} \to \vec{\alpha} = \{\alpha, \alpha, \dots, \alpha\}$. □

The simplest example of (1.91) is the case $M = 1$, $\alpha = 0$, when it reads

$$P_{\vec{0},\vec{0}}(\vec{x}) = \frac{1}{(2\pi)^{N/2}\prod_{l=1}^{N-1} l!}\left(\frac{2T}{t(2T-t)}\right)^{N^2/2} \prod_{k=1}^{N} e^{\frac{-Tx_k^2}{t(2T-t)}} \prod_{1\le i<j\le N}(x_i - x_j)^2 \tag{1.93}$$

With the scalings $y_j = (T/t(2T-t))^{1/2} x_j$ this PDF is recognised as the eigenvalue PDF for $N \times N$ GUE matrices (1.19) with $h_i^M \to \sqrt{2}h_i^M$ and $M = N$. Thus there is again a random matrix analogy. In fact, by an appropriate extension of the GUE so that the independent elements themselves are Brownian variables this analogy can be extended to the case of multiple times.

To see this, consider the weight P_t on the space of complex Hermitian matrices proportional to $\exp\left(-\mathrm{Tr}(X - X^0)^2/2(t - t_0)\right)$, where $t > t_0$ and X^0 is a given Hermitian matrix. With μ labelling a position and part in the matrix, P_t satisfies the multi-dimensional heat equation [18]

$$\frac{\partial P_t}{\partial t} = \sum_{\mu} D_{\mu} \frac{\partial^2 P_t}{\partial X_{\mu}^2} \tag{1.94}$$

where $D_\mu = 1$ for the diagonal elements and $D_\mu = 1/2$ for the off diagonal elements. We therefore have that the real and imaginary parts of each independent entry is undergoing a Brownian motion.

Given the eigenvalues of X^0 as $\{x_j^0\}_{j=1,\dots,N}$ we would like to compute the eigenvalues $\{x_j\}_{j=1,\dots,N}$ of X. Since X is complex Hermitian it can be diagonalised $X = U\Lambda U^\dagger$ where $\Lambda = \mathrm{diag}(x_1, \dots, x_N)$ and U is the unitary matrix of eigenvectors. With (dX) denoting the product of independent differentials for X, a standard result in random matrix theory (see e.g. [27], Proposition 1.3.4) gives

$$(dX) = \prod_{1 \le j < k \le N} (x_j - x_k)^2 \prod_{i=1}^{N} dx_i (U^\dagger dU) \tag{1.95}$$

Here $(U^\dagger dU)$ is the so called Haar (uniform) measure on unitary matrices. It thus follows that, up to proportionality, the sought eigenvalue PDF is given by

$$\prod_{1 \le j < k \le N} (x_j - x_k)^2 \int \exp\left(-\mathrm{Tr}\left(U\Lambda U^\dagger - \Lambda^0\right)^2 / 2\left(t - t_0\right)\right) (U^\dagger dU) \tag{1.96}$$

$$= \prod_{1 \le j < k \le N} (x_j - x_k)^2 \prod_{i=1}^{N} e^{-\left(x_i^2 + (x_i^0)^2\right)/2(t - t_0)} \int \exp\left(\mathrm{Tr}\left(U\Lambda U^\dagger \Lambda^0\right) / \left(t - t_0\right)\right) (U^\dagger dU)$$

The latter matrix integral has a well known eigenvalue evaluation due to Harish-Chandra, and Itzykson and Zuber (see e.g. [27] Proposition 11.6.1, [47]).

Proposition 1.20. *Let $(U^\dagger dU)$ denote the Haar volume form for $N \times N$ unitary matrices, normalized so that $\int (U^\dagger dU) = 1$. Let A and B be $N \times N$ Hermitian matrices with eigenvalues $\{a_j\}$ and $\{b_j\}$. We have*

$$\int e^{\mathrm{Tr}(UAU^\dagger B)} (U^\dagger dU) = \prod_{i=1}^{N-1} i! \frac{\det\left[e^{a_i b_j}\right]_{j,k=1,\dots N}}{\prod_{1 \le j < k \le N} (a_k - a_j)(b_k - b_j)}$$

Using this in (1.96) shows the PDF for the eigenvalues $\{x_j\}$ to be proportional to

$$\prod_{1 \le j < k \le N} \frac{(x_j - x_k)}{(x_j^0 - x_k^0)} G_{t - t_0}\left(\vec{x}, \vec{x}^0\right)$$

where $G_t(\vec{x}, \vec{y})$ is specified by (1.89). It follows from this that the joint eigenvalue PDF $P^{\mathrm{eig}}_{\vec{x}^{(0)}, \vec{x}^{(2T)}}$ for there being eigenvalues at $\{(\vec{x}^{(s)}, t = t_s)\}_{s=1,\dots,M}$, given that the eigenvalues are at $\vec{x}^0$ at time $t = 0$, and at $\vec{x}^{(2T)}$ at time $t = 2T$, is proportional to

$$\prod_{1 \le j < k \le N} \frac{(x_j^{(2T)} - x_k^{(2T)})}{(x_j^{(0)} - x_k^{(0)})} \frac{1}{P(\vec{x}^{(2T)})} G_{t_1}(\vec{x}^{(0)}, \vec{x}^{(1)}) G_{t_2 - t_1}(\vec{x}^{(1)}, \vec{x}^{(2)}) \dots G_{2T - t_m}(\vec{x}^{(M)}, \vec{x}^{(2T)}) \tag{1.97}$$

(cf. (1.90)), where $P(\vec{x}^{(2T)})$ is the eigenvalue PDF for the $N \times N$ GUE* as specified by (1.19).

Taking the limit $\vec{x}^{(0)} \to \vec{0}$, $\vec{x}^{(2T)} \to \vec{\alpha} = \{\alpha, \dots, \alpha\}$ in (1.97), the joint PDF for the non-intersecting walkers (1.91) is reclaimed. In terms of underlying Brownian motion this means that all entries are equal to 0 at $t = 0$, while the diagonal entries are equal to α for $t = 2T$ and the off diagonal entries again equal to 0. Hence the Brownian entries are required to be Brownian bridges.

1.8 Half-hexagon walks

As described in §1.6, a rhombus tiling of an a, N half-hexagon is in bijection with a simple symmetric random walks conditioned never to intersect or go below 0, starting at $(0, 2i-2)$ and ending at $(2N, 2i-2)$ for $i = 1, \ldots, a$. As it was in the case of the walkers in an $a \times b \times c$ hexagon rhombus tiling described in the previous section, the positions of these walkers on line t is the complement of the positions of the particles on line t, as given by Theorem 1.15 and as such it would be possible to find the PDF for the positions of these walkers by applying Borodin's Proposition 1.18. We will instead follow the work of Forrester and Nordenstam in [31] and use a method of counting configurations, applying the following result by Krattenthaler, Guttmann and Viennot [51].

Proposition 1.21. *(Theorem 6 in [51]). Let* $e_1 < e_2 < \cdots < e_p$ *with* $e_i = m \mod 2$ *for* $i = 1, \ldots, p$. *The number of distinct configurations of non-intersecting paths, where each step is right up or right down and no path can touch or go below the* x*-axis, connecting points* $A_i = (0, 2i-2)$ *to* $E_i = (m, e_i)$, $i = 1, \ldots, p$

$$\prod_{i=1}^{p} \frac{(e_i+1)(m+2i-2)!}{(\frac{m+e_i}{2}+p)!(\frac{m-e_1}{2}+p-1)!} \prod_{1 \le i < j \le p} \left(\left(\frac{e_i+1}{2}\right)^2 - \left(\frac{e_j+1}{2}\right)^2 \right) \tag{1.98}$$

Using the fact that the probability of the walkers being at a certain configuration of positions on line t must be the product of the number of stars to the left and the number of stars to the right, divided by the total number of tilings of the half-hexagon, we arrive at the following result.

Lemma 1.22. *[29] Let* $\{x_i^{(t)}\}_{i=1,\ldots,a}$ *represent the positions of the* a *walkers on line* t *of an* a, N *half-hexagon. Then the* $x_i^{(t)}$ *have PDF*

$$Z_{a,N}^{-1} \prod_{1 \le i < j \le a} ((x_i+1)^2 - (x_j+1)^2))^2 \prod_{i=1}^{a} \frac{(x_i+1)^2}{(\frac{t+x_i}{2}+a)!(\frac{2N-t+x_i}{2}+a)!(\frac{t-x_i}{2}+a-1)!(\frac{2N-t-x_i}{2}+a-1)!} \tag{1.99}$$

for $0 \le x_1 < \cdots < x_a \le 2a+t-2$ *and* $x_i + t$ *even for all* i, *where* $Z_{a,N}$ *is given by*

$$Z_{a,N} = 2^{2a(a-1)} \prod_{i=1}^{a} \frac{(2i-1)!}{(N+a+i-1)!(N+a-i)!} \frac{(2N+2i-2)!}{(t+2i-2)!(2N-t+2i-2)!}$$

Proof. (1.99) is the result of taking (1.98) with $p = a$, $e_i = x_i$ and $m = t$, multiplying by (1.98) with $p = a$, $e_i = x_i$ and $m = 2N-t$ and dividing by (1.98) with $p = a$, $e_i = 2i-2$ and $m = 2N$. □

It is straightfoward to give a determinant formula analogous to (1.87) for the positions $\{x_i^{(t_k)}\}_{\substack{i=1,\ldots a \\ k=1,\ldots M}}$ of the walkers at multiple times. The main ingredient is the generalisation of Proposition 1.17 in the case of a wall.

As the hexagon walks described in §1.7 were shown to converge in a certain limit to a system of non-intersecting Brownian motions, it is also true that the half-hexagon walks have a limit in non-intersecting Brownian motions. However, because of the condition on the half-hexagon walkers that they must never go below 0, the convergence is instead to Brownian bridges near a wall.

Proposition 1.23. *[51] Let* $a_1 > a_2 > \cdots > a_M$, $b_1 > b_2 > \cdots > b_M$, *with* $a_i = t_1 \mod 2$ *and* $b_i = t_2 \mod 2$ *for* $i = 1, \ldots, M$. *The number of distinct configurations of non-intersecting paths,* *where each step is right up or right down and no path can touch or go below the x-axis, connecting points (t_1, a_i) and (t_2, b_i) $(i = 1, \dots M)$ is equal to*

$$\det\left[\begin{pmatrix} t_2 - t_1 \\ \frac{1}{2}(t_2 - t_1 - b_i + a_j) \end{pmatrix} - \begin{pmatrix} t_2 - t_1 \\ \frac{1}{2}(t_2 - t_1 + b_i + a_j) \end{pmatrix}\right]_{i,j=1,\dots,M}$$

Proof. Without the subtracted term the entry of the determinant count the number of single right up or right down paths connecting (t_1, a_i) and (t_2, b_i). The subtracted term counts the same with the finish point now at $(t_2, -b_i)$, so the full entry counts only those paths above $x = 0$. With the single walker counting function known, the case of N non-intersecting such paths follows from the Gessel-Viennot theorem. □

The joint PDF of the $x_i^{(t_k)}$ is therefore

$$P\left(x^{(t_1)}, \dots, x^{(t_M)}\right) = \frac{1}{C_{\text{HalfHexWalk},a,N}} \tag{1.100}$$

$$\times \prod_{k=0}^{M} \det\left[\begin{pmatrix} t_{k+1} - t_k \\ \frac{1}{2}\left(x_j^{(t_k)} - x_i^{(t_{k+1})} + t_{k+1} - t_k\right) \end{pmatrix} - \begin{pmatrix} t_{k+1} - t_k \\ \frac{1}{2}\left(x_j^{(t_k)} + x_i^{(t_{k+1})} + t_{k+1} - t_k\right) \end{pmatrix}\right]_{i,j=1}^{a}$$

where $t_0 = 0$, $t_{M+1} = 2N$, $x_i^{(0)} = 2(a - i)$, $x_i^{(2N)} = 2(a - i)$ and

$$C_{\text{HalfHexWalk},a,N} = \det\left[\begin{pmatrix} 2N \\ N + i - j \end{pmatrix} - \begin{pmatrix} 2N \\ N + 2a - i - j \end{pmatrix}\right]_{i,j=1}^{a} \tag{1.101}$$

We introduce a system of a Brownian walkers, conditioned never to intersect or go below 0, to begin at time $t = 0$ at positions $x^{(\vec{0})} = \{x_1^{(0)}, \dots, x_a^{(0)}\}$ and end at time $t = 2T$ at positions $x^{(\vec{2T})} = \{x_1^{(2T)}, \dots, x_a^{(2T)}\}$. The PDF for a Brownian walkers going from $\vec{x}$ to $\vec{y}$ in time t without intersecting or going below 0 is

$$\begin{aligned} G_t^{\text{Wall}}(\vec{x}, \vec{y}) &= \det\left[\frac{1}{\sqrt{2\pi t}}\left(e^{-(x_j - y_k)^2/2t} - e^{-(x_j + y_k)^2/2t}\right)\right]_{j,k=1,\dots,a} \\ &= \left(\frac{1}{2\pi t}\right)^{a/2} e^{-\sum_{j=1}^{a}(x_j^2 + y_j^2)/2t} \det[e^{x_j y_k/t} - e^{-x_j y_k/t}]_{j,k=1\dots,a} \end{aligned} \tag{1.102}$$

Given that the walkers start at $\vec{x}^{(0)}$ at time $t = 0$ and end at $\vec{x}^{(2T)}$ at time $t = 2T$ the joint probability density $P^{\text{Wall}}_{\vec{x}^{(0)}, \vec{x}^{(2T)}}$ for arriving at $\{(\vec{x}^{(s)}, t = t_s)\}_{s=1,\dots,M}$ along the way, without ever going below 0, is

$$P^{\text{Wall}}_{\vec{x}^{(0)}, \vec{x}^{(2T)}}(\vec{x}^{(1)}, \dots, \vec{x}^{(M)}) = \frac{G^{\text{Wall}}_{t_1}(\vec{x}^{(0)}, \vec{x}^{(1)}) G^{\text{Wall}}_{t_2 - t_1}(\vec{x}^{(1)}, \vec{x}^{(2)}) \dots G^{\text{Wall}}_{2T - t_M}(\vec{x}^{(M)}, \vec{x}^{(2T)})}{G^{\text{Wall}}_{2T}(\vec{x}^{(0)}, \vec{x}^{(2T)})} \tag{1.103}$$

(cf. (1.90)). Specialising to the case where $\vec{x}^{(0)} \to \vec{0}$ and $\vec{x}^{(2T)} \to \vec{0}$, (1.103) becomes

$$P^{\text{Wall}}_{\vec{0},\vec{0}}(\vec{x}^{(1)}, \dots, \vec{x}^{(M)}) = \frac{2^{a(a+1)}}{\pi^{a/2} \prod_{i=1}^{a}(2i-1)!} \left(\frac{T}{t_1(2T - t_M)}\right)^{a^2 + a/2} \prod_{l=1}^{M-1} G^{\text{Wall}}_{t_{l+1} - t_l}(\vec{x}^{(l)}, \vec{x}^{(l+1)}) \tag{1.104}$$

$$\times \prod_{j=1}^{N} x_j^{(1)} x_j^{(M)} e^{-(x_j^{(1)})^2/2t_1 - (x_j^{(M)})^2/2(2T - t_M)} \prod_{1 \le j < k \le N} \left((x_j^{(1)})^2 - (x_k^{(1)})^2\right)\left((x_j^{(M)})^2 - (x_k^{(M)})^2\right)$$

Proposition 1.24. *Let* $\{x_i^{(Nt_j)}\}_{\substack{i=1,\dots,a\\ j=1,\dots,M}}$ *represent the positions of the i-th walker on an* a, TN *half-hexagon lines* Nt_j*. Then the scaled variables* $y_i^{(t_j)} = x_i^{(Nt_j)}/\sqrt{N}$ *have joint PDF as in* (1.104) *in the limit* $N \to \infty$*, where the convergence is uniform on compact sets with respect to the* $x_i^{(j)}$*.*

Proof. We wish to show that, with x related to y as described, (1.100) converges to (1.104). From the definitions of $x_i^{(0)}$ and $x_i^{(2N)}$ for walkers in an a, N half-hexagon,

$$y_i^{(0)} = \frac{x_i^{(0)}}{\sqrt{N}} = \frac{2(a-i)}{\sqrt{N}} \to 0 \qquad y_i^{(2T)} = \frac{x_i^{(2TN)}}{\sqrt{N}} = \frac{2(a-i)}{\sqrt{N}} \to 0$$

We now consider the functions that make up the determinants in (1.100) and (1.102). These are of the form

$$\begin{aligned} f_t(x_1, x_2) &= \begin{pmatrix} t \\ \frac{1}{2}(t - x_2 + x_1) \end{pmatrix} - \begin{pmatrix} t \\ \frac{1}{2}(t + x_2 + x_1) \end{pmatrix} \\ g_t(y_1, y_2) &= \frac{1}{\sqrt{2\pi t}} \left(e^{-(y_2-y_1)^2/2t} - e^{-(y_2+y_1)^2/2t} \right) \end{aligned}$$

for the discrete x picture and the continuous y picture respectively. Considering a scaled form of the discrete PDF and converting to the y scale, we have

$$2^{-tN} f_{tN}(x_1, x_2) = 2^{-tN} \left[\begin{pmatrix} tN \\ \frac{1}{2}\left(tN - y_2\sqrt{N} + y_1\sqrt{N}\right) \end{pmatrix} - \begin{pmatrix} tN \\ \frac{1}{2}\left(tN + y_2\sqrt{N} + y_1\sqrt{N}\right) \end{pmatrix} \right] \tag{1.105}$$

After multiplying by $\dfrac{dx}{dy} = \sqrt{N}/2$, noting that the factor of 2 comes from the fact that the $x_i^{(j)}$ are confined to every second lattice point, and applying Stirling's formula (1.16), (1.105) becomes

$$2^{-tN} f_{tN}(x_1, x_2) = \frac{1}{\sqrt{2\pi t}} e^{-(y_2-y_1)^2/2t} + \mathrm{O}(N^{-1/2}) = g_t(y_1, y_2) + \mathrm{O}(N^{-1/2})$$

Thus the joint PDF for the $x_i^{(j)}$,

$$\frac{\prod_{k=0}^{M} \det\left[f_{N(t_{k+1}-t_k)}\left(x_i^{(Nt_k)}, x_j^{(Nt_{k+1})} \right) \right]_{i,j=1\dots,a}}{\det\left[f_{N(t_{M+1}-t_0)}\left(x_i^{(Nt_0)}, x_j^{(Nt_{M+1})} \right) \right]_{i,j=1\dots,a}}$$

with $t_0 = 0$, $t_{M+1} = 2T$, converges to

$$\frac{2^{aN\sum_{k=0}^{M} t_{k+1}-t_k}}{2^{2TaN}} \frac{\prod_{k=0}^{M} \det\left[g_{t_{k+1}-t_k}\left(y_i^{(t_k)}, y_j^{(t_{k+1})} \right) \right]_{i,j=1\dots,a}}{\det\left[g_{2T}\left(y_i^{(0)}, y_j^{(2T)} \right) \right]_{i,j=1\dots,a}}$$

which after cancelling of terms, is equal to (1.103). That we have shown $y_i^{(0)} \to 0$ and $y_i^{(2T)} \to 0$ completes the proof, as (1.104) is the special case of (1.103) with $\vec{x}^{(0)} \to \vec{0}$ and $\vec{x}^{(2T)} \to \vec{0}$. □

The simplest example of (1.104) is the case $M = 1$, when it reads

$$P_{\vec{0},\vec{0}}^{\text{Wall}}(\vec{x}) = \frac{2^{a(a+1)}}{\pi^{a/2} \prod_{i=1}^{a} (2i-1)!} \left(\frac{T}{t(2T-t)} \right)^{a^2+a/2} \prod_{i=1}^{a} e^{\frac{-T(x_i)^2}{t(2T-t)}} x_i^2 \prod_{1 \le i < j \le N} (x_i^2 - x_j^2)^2 \tag{1.106}$$

After the scaling $\sqrt{T/t(2T-t)}\,x_i \to x_i$, (1.106) is recognised as the second PDF of (1.66) with $n = 2a+1$. Moreover, as for the continuous non-intersecting Brownian walkers of §1.7, this random matrix analogy can be extended to the case of multiple times.

We consider the weight P_t on the space of $(2a+1)\times(2a+1)$ anti-symmetric Hermitian matrices (recall §1.5) proportional to $\exp\left(-\mathrm{Tr}(X-X^0)^2/4(t-t_0)\right)$, where $t>t_0$ and X^0 is a given anti-symmetric Hermitian matrix. We know from (1.94) that this implies the independant entries (which are all pure imaginary) undergo Brownian motion.

Given the positive eigenvalues of X^0 as $\{x_j^0\}_{j=1,\dots,a}$ we seek the eigenvalue PDF of the positive eigenvalues $\{x_j\}_{j=1,\dots,a}$ of X. We require the fact that X can be decomposed in the form $X = R\Lambda R^T$ where

$$\Lambda = \mathrm{diag}\left(\begin{bmatrix} 0 & ix_1 \\ -ix_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & ix_a \\ -ix_a & 0 \end{bmatrix}, 0\right)$$

and R is real orthogonal. Analagous to (1.95) we have (see e.g. [27] Exercises 1.3 Q5 (iii))

$$(dX) = \prod_{j=1}^{a} x_j^2 \prod_{1\le j<k\le a} \left(x_j^2 - x_k^2\right)^2 \prod_{i=1}^{a} dx_i (R^T dR)$$

and so up to proportionality the sought eigenvalue PDF is

$$\prod_{j=1}^{a} x_j^2 \prod_{1\le j<k\le a} \left(x_j^2 - x_k^2\right)^2 \prod_{i=1}^{a} e^{-\left(x_i^2+(x_i^0)^2\right)/2(t-t_0)} \int \exp\left(\mathrm{Tr}(R\Lambda R^T \Lambda^0)/2(t-t_0)\right)(R^T dR) \quad (1.107)$$

The theory of Harish-Chandra which gave Proposition 1.20 also gives a determinantal formula for the above matrix integral (see e.g [23])

Proposition 1.25. *Let $(R^T dR)$ denote the Haar volume for $(2a+1)\times(2a+1)$ orthogonal matrices, normalized so that $\int (R^T dR) = 1$. Let F and G be $(2a+1)\times(2a+1)$ anti-symmetric Hermitian matrices with positive eigenvalues $\{f_j\}_{j=1,\dots,a}$ and $\{g_j\}_{j=1,\dots,a}$. We have*

$$\int e^{\frac{1}{2}\mathrm{Tr}(RAR^T B)}(R^T dR) = 2^a \prod_{l=1}^{a} \frac{1}{(2l-1)!} \frac{\det\left[e^{f_i g_j} - e^{-f_i g_j}\right]_{i,j=1\dots,a}}{\tilde{\Delta}(f)\tilde{\Delta}(g)}$$

where

$$\tilde{\Delta}(x) = \prod_{1\le j<k\le a} (x_j^2 - x_k^2) \prod_{l=1}^{a} x_l$$

Using this in (1.107) shows the PDF for the eigenvalues $\{x_j\}$ to be proportional to

$$\prod_{l=1}^{a} \frac{x_l}{x_l^0} \prod_{1\le j<k\le a} \frac{\left((x_j)^2 - (x_k)^2\right)}{\left((x_j^0)^2 - (x_k^0)^2\right)} G_{t-t_0}^{\mathrm{Wall}}(\vec{x}, \vec{x}^0)$$

where $G_t^{\mathrm{Wall}}(\vec{x}, \vec{y})$ is specified by (1.102). It follows from this that the joint eigenvalue PDF $P^{\mathrm{eig,Wall}}_{\vec{x}^{(0)},\vec{x}^{(2T)}}$ for there being positive eigenvalues at $\{(\vec{x}^{(s)}, t=t_s)\}_{s=1,\dots,M}$, given that the positive eigenvalues are at $\vec{x}^0$ at time $t=0$, and at $\vec{x}^{(2T)}$ at time $t=2T$, is proportional to

$$\prod_{l=1}^{a} \frac{x_l^{(2T)}}{x_l^{(0)}} \prod_{1\le j<k\le a} \frac{\left((x_j^{(2T)})^2 - (x_k^{(2T)})^2\right)}{\left((x_j^{(0)})^2 - (x_k^{(0)})^2\right)} \frac{G_{t_1}(\vec{x}^{(0)}, \vec{x}^{(1)}) G_{t_2-t_1}(\vec{x}^{(1)}, \vec{x}^{(2)}) \dots G_{2T-t_m}(\vec{x}^{(M)}, \vec{x}^{(2T)})}{P(\vec{x}^{(2T)})} \quad (1.108)$$

(cf. (1.103)), where $P(\vec{x}^{(2T)})$ is the eigenvalue PDF for the $2a+1\times 2a+1$ Antisymmetric GUE as specified by (1.66).

Taking the limit $\vec{x}^{(0)}, \vec{x}^{(2T)} \to \vec{0}$ in (1.108), the joint PDF for the non-intersecting walkers with a wall (1.104) is reclaimed. In terms of underlying Brownian motion this means that all entries are equal to 0 at $t = 0$ and at $t = 2T$. Hence the Brownian entries are again required to be Brownian bridges, as with the case without a wall in §1.7. We remark that there is a larger class of random matrix ensembles consisting of Brownian entries, of which the eigenvalue interpretation as non-intersecting paths are also confined to the half space [48].

2 Correlation functions

Fundamental to the specification of the statistical properties of a particle system are the correlation functions. For a set of random variables $\{x_i^{(j)}\}$ with $i = 1 \ldots, r(j)$ and $j = 1, \ldots, \mathcal{L}$ for some fixed number of 'lines' $\mathcal{L}$, with joint PDF $p(x^{(1)}, \ldots, x^{(\mathcal{L})})$, we say there is a particle at (x, s) if, for any value of i, $x_i^{(s)} = x$. The un-normalized probability density for particles at $\{(x_j, s_j)\}_{j=1,\ldots,r}$ is given by the r-point correlation function, denoted $\rho_{(r)}(\{(x_j, s_j)\}_{j=1,\ldots,r})$. Specifically, these have the meaning that the ratio

$$\frac{\rho_{(r)}\left((x_1, s_1); (x_2, s_2); \ldots; (x_r, s_r)\right)}{\rho_{(r-1)}\left((x_1, s_1); (x_2, s_2); \ldots; (x_{r-1}, s_{r-1})\right)}$$

is the density at (x_r, s_r) given that there is a particle at (x_j, s_j) for $j = 1, \ldots, r-1$. In particular, the one-point correlation function, $\rho_{(1)}((x, s))$ is just the density at (x, s).

2.1 A single line

We will see that a feature of all the particle systems introduced in the previous chapter is that the r-point correlation function can be expressed as an $r \times r$ determinant. The case of single line will be considered first. Insight can be obtained by considering the function forms of the joint PDF on a single line for the models of the previous chapter. The joint particle PDF on a single line for the particles associated with the queueing process is given by (1.19), or alternatively according to the result below (1.29) this can be interpreted as the eigenvalue PDF for the $M \times M$ GUE*. The joint PDF on a single line for particles associated with the rhombi tiling of an $a \times b \times c$ hexagon is given by (1.36). For $n \times n$ anti-symmetric GUE matrices the eigenvalue PDF is given by (1.66), and this relates through a scaling limit to the single line PDF for the particles associated with the rhombi tiling of a half-hexagon. In the picture of lines rather than particles we also have the PDF (1.84) for a single line in the case of tilings of the full hexagon and (1.106) in the case of the half-hexagon. All these PDFs have the common structure

$$\frac{1}{C} \prod_{l=1}^{N} w(x_l) \prod_{1 \le j < k \le N} (x_j - x_k)^2 \tag{2.1}$$

(except (1.66) and (1.106), which differ slightly) for an appropriate support of the so called weight function $w(x)$, and with the ordering $x_1 > x_2 > \cdots > x_N$.

Crucial to the calculation of the correlation functions are the single variable orthogonal polynomials $\{p_n(x)\}_{n=0,1,\ldots}$ associated with $w(x)$. With the convention that they are monic (coefficient of leading monomial x^n unity), they are uniquely determined by the orthogonality

$$\int_{-\infty}^{\infty} p_n(x) p_m(x) w(x) dx = \delta_{m,n} \mathcal{N}_n \tag{2.2}$$

via a Gram-Schmidt construction. To see their utility, let us first show how they can be used to determine the normalization constant in (2.1).

Lemma 2.1. *In the general single line PDF* (2.1), C *is given by*

$$C = \prod_{i=0}^{N-1} \mathcal{N}_i \tag{2.3}$$

for $\mathcal{N}_n$ *as in* (2.2)

Proof. As (2.1) is a probability and $x_1 > x_2 > \cdots > x_N$, by noting that (2.1) is a symmetric function of the x_j's we must have

$$\frac{1}{C}\int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_N \prod_{l=1}^{N} w(x_l) \prod_{1\le j<k\le N} (x_j - x_k)^2 = N! \tag{2.4}$$

Recalling (1.18) and using column operations, noting that $p_n(x)$ is monic, we have

$$\prod_{1\le j<k\le N} (x_j - x_k) = \det\left[x_{N+1-i}^{j-1}\right]_{i,j=1,\ldots,N} = \det\left[p_{j-1}(x_{N+1-i})\right]_{i,j=1,\ldots,N} \tag{2.5}$$

It is also true that

$$\int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_N \left(\det\left[\sqrt{w(x_{N+1-i})}p_{j-1}(x_{N+1-i})\right]_{i,j=1}^{N}\right)^2 = \tag{2.6}$$

$$N!\det\left[\int_{-\infty}^{\infty} p_{i-1}(x)p_{j-1}(x)w(x)dx\right]_{i,j=1}^{N}$$

This follows by noting that since both determinants in the LHS are anti-symmetric in the x_j's, the second determinant can be replaced by $N!$ times its diagonal term (i.e. each one of the $N!$ terms that makes up the second determinant gives the same contribution to the integral). Applying (2.5) and (2.6), (2.4) can be rewritten

$$\det\left[\int_{-\infty}^{\infty} p_{i-1}(x)p_{j-1}(x)w(x)dx\right]_{i,j=1,\ldots,N} = C \tag{2.7}$$

Applying (2.2) to (2.7) gives (2.3). □

As an example, we will use Lemma 2.1 to check our results for the normalizations of both the one line GUE* PDF (1.19) , and the one line Brownian walkers PDF (1.93), both of which are of the form (2.1). We introduce the Hermite polynomials $H_n(x)$, which have orthogonality

$$\int_{-\infty}^{\infty} H_j(x)H_k(x)e^{-x^2}dx = h_k\delta_{j,k} \tag{2.8}$$

with $h_k = 2^k k!\sqrt{\pi}$. These polynomials can be generalised to find the monic orthogonal polynomials for any weight function of the form $e^{-\alpha x^2}$ with $\alpha > 0$. Let the polynomials $H_n^{(\alpha)}$ be defined

$$H_n^{(\alpha)}(x) = \frac{1}{(2\sqrt{\alpha})^n}H_n(\sqrt{\alpha}x)$$

These polynomials are monic by the fact that the Hermite polynomials have coefficient of leading monomial x^n equal to 2^n, and applying a simple change of variables to (2.8) gives us the orthogonality relationship for the $H_n^{(\alpha)}$

$$\int_{-\infty}^{\infty} H_j^{(\alpha)}(x)H_k^{(\alpha)}(x)e^{-\alpha x^2}dx = h_k^{(\alpha)}\delta_{j,k} \tag{2.9}$$

where

$$h_k^{(\alpha)} = \sqrt{\frac{\pi}{\alpha}}\frac{k!}{(2\alpha)^k} \tag{2.10}$$

In the case of the weight function $e^{-x^2/2}$, the weight function for the GUE* eigenvalue ensemble, the corresponding polynomials are sometimes known as the probabilitists' Hermite polynomials and are denoted $\tilde{H}(x)$ defined by

$$\tilde{H}_n(x) = (-1)^n e^{\frac{x^2}{2}}\frac{d^n}{dx^n}\left(e^{\frac{-x^2}{2}}\right) \tag{2.11}$$

These polynomials have orthogonality

$$\int_{-\infty}^{\infty} e^{-x^2/2} \tilde{H}_j(x) \tilde{H}_k(x) dx = \tilde{h}_k \delta_{j,k}$$

where $\tilde{h}_k = k!\sqrt{2\pi}$, and comparing with (2.9) shows that $\tilde{H}_n(x) \equiv H_n^{(1/2)}(x)$. Using these, Lemma 2.1 gives us another derivation of the normalization constant in (1.19)

$$\prod_{k=0}^{M-1} \tilde{h}_k = (2\pi)^{M/2} \prod_{l=1}^{M-1} l!$$

Similarly, setting

$$\alpha = \frac{T}{t(2T-t)}$$

and using (2.10), Lemma 2.1 gives us another derivation of the normalization constant in (1.93)

$$\prod_{k=0}^{N-1} h_k^{(\alpha)} = \prod_{k=0}^{N-1} \sqrt{\frac{\pi t(2T-t)}{T}} \left(\frac{t(2T-t)}{2T}\right)^k k! = (2\pi)^{N/2} \left(\frac{t(2T-t)}{2T}\right)^{N^2/2} \prod_{l=1}^{N-1} l!$$

While Lemma 2.1 as presented applies to PDFs of continuous variables, it turns out that the result holds also for discrete PDFs of the same form, where the monic polynomials $\{p_n(x)\}_{n=0,1,\ldots}$ have orthogonality

$$\sum_{x=-\infty}^{\infty} p_n(x) p_m(x) w(x) dx = \delta_{m,n} \mathcal{N}_n$$

where the sum is often restricted by defining $w(x) = 0$ outside a certain range (a, b). An example are the monic Hahn polynomials $Q_s^{(a,b,L)}(y)$, which correspond to the weight function

$$w_{a,b,L}(y) = \frac{(y+a)!}{y!a!} \frac{(L-y+b)!}{(L-y)!b} \chi_{0 \leq y \leq L} \tag{2.12}$$

and have orthogonality

$$\sum_{y=0}^{L} Q_s^{(a,b,L)}(y) Q_t^{(a,b,L)}(y) w_{a,b,L}(y) = H_s^{(a,b,L)} \delta_{s,t} \tag{2.13}$$

where

$$H_s^{(a,b,L)} = \frac{1}{(2s+a+b+1)} \frac{s!}{(L-s)!} \frac{(a+s)!(b+s)!}{a!b!} \frac{(s+a+b+L+1)!(s+a+b)!}{[(2s+a+b)!]^2} \tag{2.14}$$

(see [49] for further properties of the Hahn polynomials). These polynomials can be used to find the normalizations for the particle models corresponding to the hexagon, given by (1.36). We will show this for the case $a = b = c = N$, as was used in Proposition 1.9

Proposition 2.2. *For $a = b = c = N$, $Z_{t,a,b,c}$ in* (1.36) *is given by*

$$Z_{t,N,N,N} = \prod_{i=0}^{t-1} 2^{2N-2t+2i} [(N-t)!]^2 H_i^{(N-t,N-t,N+t-1)}$$

Proof. For $a = b = c = N$, (1.36) is of the form in (2.1) with $w(x) = f_t(x)$, and comparing the definition of $f_t(x)$ from (1.40) with (2.12) we have

$$f_t(x) = 2^{2(N-t)} (N-t)!^2 w_{N-t,N-t,N+t-1}\left(\frac{x+t}{2}\right)$$

where $w_{a,b,L}(x)$ is as in (2.14). Then considering (2.13), the discrete monic polynomials orthogonal with respect to $f_t(x)$ have squared norm

$$\mathcal{N}_i = 2^{2N-2t+2i}[(N-t)!]^2 H_i^{(N-t,N-t,N+t-1)}$$

and the result follows from Lemma 2.1 □

As was mentioned earlier, (1.66) and (1.106) have a slightly different form to that of (2.1), and require a slightly modified form of Lemma 2.1 to check their normalizations.

Lemma 2.3. *In the general single line PDF*

$$\frac{1}{C}\prod_{l=1}^{N} w(x_l) \prod_{1\le j<k\le N} \left((x_j^2-(x_k)^2)\right)^2 \tag{2.15}$$

where $w(x)$ is an even function and the x_i are restricted to the positive real line, C is given by

$$C = \frac{1}{2^N}\prod_{i=0}^{N-1}\mathcal{N}_{2i} \tag{2.16}$$

Similarly, in the general single line PDF

$$\frac{1}{C}\prod_{l=1}^{N} w(x_l)x_l \prod_{1\le j<k\le N} \left((x_j^2-(x_k)^2)\right)^2 \tag{2.17}$$

where $w(x)$ is an even function and the x_i are restricted to the positive real line, C is given by

$$C = \frac{1}{2^N}\prod_{i=0}^{N-1}\mathcal{N}_{2i+1} \tag{2.18}$$

Proof. We begin with the (2.15) case. As (2.15) is a probability and $x_1 > x_2 > \cdots > x_N > 0$, by noting that (2.1) is a symmetric function of the x_j's we must have

$$\frac{1}{C}\int_0^\infty dx_1 \ldots \int_0^\infty dx_N \prod_{l=1}^{N} w(x_l) \prod_{1\le j<k\le N} \left((x_j)^2-(x_k)^2\right)^2 = N! \tag{2.19}$$

Recalling (1.18) and using column operations, noting that $p_n(x)$ is monic, we have

$$\prod_{1\le j<k\le N}(x_j^2-x_k^2) = \det\left[x_{N+1-i}^{2(j-1)}\right]_{i,j=1,\ldots,N} = \det\left[p_{2(j-1)}(x_{N+1-i})\right]_{i,j=1,\ldots,N} \tag{2.20}$$

It is also true that

$$\int_0^\infty dx_1 \ldots \int_0^\infty dx_N \left(\det\left[\sqrt{w(x_{N+1-i})}p_{2(j-1)}(x_{N+1-i})\right]_{i,j=1}^{N}\right)^2 = \\ N!\det\left[\int_0^\infty p_{2(i-1)}(x)p_{2(j-1)}(x)w(x)dx\right]_{i,j=1}^{N} \tag{2.21}$$

This follows by noting that since both determinants in the LHS are anti-symmetric in the x_j's, the second determinant can be replaced by $N!$ times its diagonal term (i.e. each one of the $N!$ terms that makes up the second determinant gives the same contribution to the integral). Applying (2.20) and (2.21), (2.19) can be rewritten

$$\det\left[\int_0^\infty p_{2(i-1)}(x)p_{2(j-1)}(x)w(x)dx\right]_{i,j=1,\ldots,n} = C \tag{2.22}$$

Because $w(x)$ is an even function in x, $p_{2n}(x)$ is also an even function in x, and so the (2.22) can be rewritten

$$\det\left[\frac{1}{2}\int_{-\infty}^{\infty} p_{2(i-1)}(x)p_{2(j-1)}(x)w(x)dx\right]_{i,j=1,\ldots,n} = C \tag{2.23}$$

Applying (2.2) to (2.23) gives (2.16).

For the (2.17) case, the proof is the same as the above up to (2.22) after changing the order of the polynomials from $2(j-1)$ to $2j-1$. Then, we note that because $w(x)$ is an even function in x, $p_{2n-1}(x)$ is an odd function of x, and so $p_{2i-1}(x)p_{2j-1}(x)$ is an even function of x, and so a rewriting of the deterimant analogous to (2.23) is permitted, giving (2.18) □

We can use this Lemma to check the normalizations of (1.66). Comparing (1.66) to the forms found in Lemma 2.3, we see that

$$A_{2n} = \frac{1}{2^n}\prod_{i=0}^{n-1} h_{2i}^{(1)} \qquad A_{2n+1} = \frac{1}{2^n}\prod_{i=0}^{n-1} h_{2i+1}^{(1)} \tag{2.24}$$

where $h_n^{(1)}$ is as in (2.10) with $\alpha = 1$. Noting that $h_n^{(1)} = \sqrt{\pi}n!2^{-n}$, we see that (2.24) is equivalent to (1.67). We can also use this Lemma, along with the Hahn polynomials, to find the normalizations for the particle model corresponding to the half-hexagon found in Theorem 1.15. We will demonstrate here for the even numbered lines, as used in Proposition 1.16

Proposition 2.4. *$Z_{t,a,N}$ in Theorem 1.15 is given by*

$$\begin{aligned} Z_{2n,a,N} &= 2^{-n}\prod_{i=0}^{n-1} 2^{2N-4n+4i}\left((N-2n)!\right)^2 H_{2i}^{(N-2n,N-2n,2a+2n-1)} \\ Z_{2n+1,a,N} &= 2^{-n}\prod_{i=0}^{n-1} 2^{2N-4n+4i}\left((N-2n-1)!\right)^2 H_{2i+1}^{(N-2n-1,N-2n-1,2a+2n)} \end{aligned}$$

for $H_i^{(a,b,L)}$ as in (2.14)

Proof. (1.74) is of the form (2.15) and (1.75) is of the form (2.17) with $w(x) = f_t(x)$. Clearly by the definition (1.76), $f_t(x)$ is an even function in x, and and comparing with (2.12) we have

$$f_t(x) = 2^{2(N-t)}(N-t)!^2 w_{N-t,N-t,2a+t-1}\left(a + \frac{x-1+t}{2}\right)$$

where $w_{a,b,L}(x)$ is as in (2.14). Then considering (2.13), the discrete monic polynomials orthogonal with respect to $f_t(x)$ have squared norm

$$\mathcal{N}_i = 2^{2N-2t+2i}[(N-t)!]^2 H_i^{(N-t,N-t,2a+t-1)}$$

and the result follows from Lemma 2.3 □

As well as being related to the normalization of the PDF C, it is a well known result in random matrix theory (see e.g. [27]) that the correlation functions for the PDF (2.1) can be expressed in terms of these polynomials.

Proposition 2.5. *Let $X = \{x_i\}_{i=1\ldots,N}$ have PDF* (2.1)*. Then the r-point correlation function for $Y = \{y_1,\ldots,y_r\} \subset X$ is given by*

$$\rho_{(r)}(y_1,\ldots,y_r) = \det\left[K_N(y_i,y_j)\right]_{i,j=1,\ldots r} \tag{2.25}$$

where, for $\{p_n(x)\}_{n=0,1,\dots}$ *as in* (2.2)*,* K_N *is given by*

$$K_N(x,y) = \sqrt{w(x)w(y)} \sum_{n=0}^{N-1} \frac{p_n(x)p_n(y)}{\mathcal{N}_n} \tag{2.26}$$

Proof. We introduce the generalized partition function $\hat{Z}_N[a]$ defined by

$$\hat{Z}_N[a] := \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_N \prod_{l=1}^{N} w(x_l)a(x_l) \prod_{1\le j<k\le N} (x_j - x_k)^2 \tag{2.27}$$

for some arbitrary one-body potential $a(x)$. Then, using the functional differentiation formula

$$\frac{\delta}{\delta a(x)} \int_{-\infty}^{\infty} a(y)f(y)dy := f(x)$$

it follows from the definition of correlation functions that

$$\rho_{(r)}(y_1,\dots,y_r) = \frac{1}{\hat{Z}_N[1]} \frac{\delta^r}{\delta a(y_1)\dots\delta a(y_r)} \hat{Z}_N[a]\Big|_{a=1} \tag{2.28}$$

Our task then is to show that the RHS of (2.28) is equal to the RHS of (2.25). Applying the same arguments as in the proof of Lemma 2.1, in particular (2.5) and (2.6), to (2.27) we see that $\hat{Z}_N[a]$ can be expressed

$$\hat{Z}_N[a] = N! \det \left[\int_{-\infty}^{\infty} a(x)w(x)p_{i-1}(x)p_{j-1}(x)dx \right]_{i,j=1,\dots N} \tag{2.29}$$

Due to the orthogonality of $\{p_n(x)\}_{n=0,1,\dots}$ we see immediately from this that

$$\hat{Z}_N[a]\Big|_{a=1} = N! \prod_{i=0}^{N-1} \mathcal{N}_i$$

The remaining task is to apply the r functional differentiations, row by row, to (2.29). For a non-zero contribution this operation must act on r distinct rows. Setting $a = 1$, the remaining $N-r$ rows are non-zero only in the diagonal terms. Expanding the determinant by these elements then gives

$$\rho_{(r)}(y_1,\dots,y_r) = \sum_{\substack{j_1,\dots,j_r=1\\ j_1\neq\dots\neq j_r}}^{N} \prod_{l=1}^{r} \mathcal{N}_{j_l-1}^{-1} \det\left[w(y_\mu)p_{j_\mu-1}(y_\mu)p_{j_\gamma-1}(y_\mu)\right]_{\mu,\gamma=1,\dots,r}$$

We see that two rows of the determinant will be equal and thus the determinant will vanish if two of the j_i's are equal. Thus the constraint $j_1 \neq \dots \neq j_r$ can be relaxed. Futhermore, dividing the factor $\sqrt{w(y_\mu)}p_{j_\mu-1}(y_\mu)$ from the μ-th row and multiplying into the μ-th column gives that the determinant is equal to

$$\det\left[\sqrt{w(y_\mu)w(y_\gamma)}p_{j_\gamma-1}(y_\gamma)p_{j_\gamma-1}(y_\mu)\right]_{\mu,\gamma=1,\dots,r}$$

The summations can now be done column by column to give (2.25) ☐

Proposition 2.6. *With* K_N *as in* (2.26) *and* $x \neq y$*, we have*

$$K_N(x,y) = \frac{\sqrt{w(x)w(y)}}{\mathcal{N}_{N-1}} \frac{p_N(x)p_{N-1}(y) - p_N(y)p_{N-1}(x)}{x-y} \tag{2.30}$$

This is known as the Christoffel-Darboux summation formula.

Proof. We begin by noting that, since the p_n are monic, $p_n(x) - xp_{n-1}(x)$ is a polynomial of order $n-1$, and thus can be written

$$p_n(x) - xp_{n-1}(x) = \sum_{i=0}^{n-1} c_i p_i(x) \tag{2.31}$$

for some $c_0, c_1, \ldots, c_{n-1}$. Recalling (2.2), we multiply (2.31) by $p_m(x)w(x)$ and integrate, giving

$$\int_{-\infty}^{\infty} \Big(p_n(x) - xp_{n-1}(x)\Big) p_m(x)w(x)dx = \mathcal{N}_m c_m$$

For $m = 0, \ldots, n-3$, the LHS is equal to zero, so we have $c_0, \ldots, c_{n-3} = 0$. For $m = n-2$, use of (2.2) and (2.31) gives that the LHS is equal to $-\mathcal{N}_{n-1}$, and so (2.31) can be rewritten

$$p_n(x) = (x + b_n)p_{n-1}(x) - \frac{\mathcal{N}_{n-1}}{\mathcal{N}_{n-2}} p_{n-2}(x)$$

for some b_n. It follows directly from this that

$$p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y) = (x-y)p_n(x)p_n(y) + \frac{\mathcal{N}_n}{\mathcal{N}_{n-1}}\Big(p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)\Big)$$

Dividing by $\mathcal{N}_n(x-y)$, summing over n from 0 to $N-1$ (noting that the $n = 0$ case holds if we interpret $p_{-1}(x) \equiv 0$) and imputting (2.26) gives the result (2.30) □

By taking the limit $y \to x$ in Proposition 2.6, we obtain the form of $K_N(x, x)$,

$$K_N(x,x) = w(x) \sum_{n=0}^{N-1} \frac{p_n(x)p_n(x)}{\mathcal{N}_n} = \frac{w(x)}{\mathcal{N}_{N-1}} \Big(p'_N(x)p_{N-1}(x) - p_N(x)p'_{N-1}(x)\Big) \tag{2.32}$$

and considering Proposition 2.5 with $r = 1$, we see that (2.32) is infact the one-point correlation function $\rho_{(1)}(x)$, which as mentioned earlier is the density at x.

2.2 Large N limits - Region of support and density profile

While on the topic of single line PDFs, we will take some time to discuss some large N limit behaviour which is dependent only on the single line PDF. In particular, for a particle model $\{x_i^{(j)}\}_{\substack{i=1\ldots,r(j)\\ j=1,\ldots N}}$, we wish to evaluate the leading order region of support, and the density of particles on a line, as $N \to \infty$. The region of support is expressed by two curves, $c(S)$ and $d(S)$, where $S(j)$ is a continuous scaled value, usually proportional to j/N. These curves have the meaning that, to leading order, the particles $\{x_i^{(j)}\}_{i=1,\ldots,r(j)}$ will obey $c\,(S(j)) < x_i^{(j)} < d\,(S(j))$. To find the region of support we use what we call the 'log-gas' method.

The Boltzmann factor for a log-gas of N_p particles has the form

$$\prod_{1 \le i < j \le N_p} |x_i - x_j|^\beta \prod_{k=1}^{N_p} e^{-\beta V(x_k)} \tag{2.33}$$

where β denotes the inverse temperature and $V(x)$ is a one body potential, due to background charge density $-\rho(x)$. Explicitly,

$$V(x) := \int_c^d \rho(t) \log|t - x| dt, \tag{2.34}$$

where normalization of the density requires

$$\int_c^d \rho(t)dt = N_p \tag{2.35}$$

A hypothesis of the log-gas picture is that for large N_p and to leading order the particle charge density and background charge density cancel, so that the particle density is to leading order equal to $\rho(x)$.

For another viewpoint of the hypothesis, we note that (2.33) can be written

$$\exp\left(\frac{\beta}{2}\sum_{i\neq j}\log|x_i - x_j| - \beta\sum_{k=1}^{N_p}V(x_k)\right) \tag{2.36}$$

Suppose that for large N_p the inter-particle spacing goes to zero such that there is a continuous particle density $\tilde{\rho}(x)$ supported on $[c, d]$, which we normalize so that

$$\int_c^d \tilde{\rho}(x)dx = 1 \tag{2.37}$$

Then to leading order (2.36) assumes the form

$$\exp\left(\frac{\beta N_p^2}{2}\int_c^d dx\tilde{\rho}(x)\int_c^d dy\tilde{\rho}(y)\log|x-y| - \beta N_p\int_c^d \tilde{\rho}(x)V(x)dx\right) \tag{2.38}$$

Hypothesizing now that $\tilde{\rho}(x)$ maximizes (2.38) - thus giving the most probable configuration - by taking the functional derivative with respect to $\tilde{\rho(x)}$ and setting to zero we reclaim (2.34) with $\rho(t) = N_p\tilde{\rho}(t)$. The work of Boutet de Monvel, Pastur and Shcherbina [10] and Johansson [38] validates this hypothesis and thus allows $\tilde{\rho}(x)$ to be computed as the solution of the integral equation

$$V(x) := \int_c^d \tilde{\rho}(t)\log|t-x|dt, \tag{2.39}$$

subject to the constraint (2.37).

Generally the equation (2.39) does not have a unique solution unless $\rho(t)$ is bounded at the end points c and d [58]. Furthermore, the density being supported on a single interval $[c, d]$ is an assumption which can be rejected if it leads to an inconsistency (i.e. a negative value for the density).

The explicit form of $\rho(x)$ obtained by solving the integral equation (2.34) is known in terms of $V(x)$ to be [27]

$$\rho(y) = \frac{1}{\pi^2}\sqrt{(y-c)(d-y)}\int_c^d \frac{V'(y)-V'(t)}{y-t}\frac{dt}{\sqrt{(t-c)(d-t)}} \tag{2.40}$$

where the boundaries of the support c and d are determined by the equations [27]

$$\int_c^d \frac{V'(t)}{\sqrt{(d-t)(t-c)}}dt = 0 \tag{2.41}$$

$$\int_c^d \frac{tV'(t)}{\sqrt{(d-t)(t-c)}}dt = \pi N_p \tag{2.42}$$

The result of all this is the following Theorem

Theorem 2.7. *Let $\{x_i\}_{i=1,\dots,N}$ be distributed as in (2.1). Then, to leading order in N,*

$$c < x_i < d$$

for c, d the solutions to (2.41) and (2.42) with $N_p = N$ and

$$V(t) = \frac{-1}{2}\log_e(w(x))$$

Furthermore, the density of the x_i is given by (2.40)

Proof. We recognize that (2.33) has the same form as (2.1), with $\beta = 2$, $N = N_p$ and $e^{-2V(x)} = w(x)$. Thus, for N large, (2.1) can be considered a log-gas and the above theory applies. □

We will use this Theorem to evaluate the region of support for eigenvalues of the GUE* eigenvalue process.

Proposition 2.8. *Let $x_i^{(N)}$ represent the i-th largest eigenvalue taken from an $N \times N$ submatrix taken from the GUE*. In the limit $N \to \infty$, to leading order*

$$-\sqrt{4N} < x_i^{(N)} < \sqrt{4N} \tag{2.43}$$

for all $i = 1, \dots, N$. Furthermore, the density of eigenvalues is given by

$$\rho_{GUE^*}(x, N) = \frac{1}{2\pi}\sqrt{4N - x^2} \qquad\qquad |x| \leq \sqrt{4N} \tag{2.44}$$

Proof. Recalling that the PDF for eigenvalues taken from an $N \times N$ submatrix taken from the GUE* is given by (1.19) with $M = N$, we let

$$V(t) = \frac{-1}{2}\log_e(e^{-t^2/2}) = t^2/4$$

Then, using Theorem 2.7, we solve (2.41), (2.42) using $V'(t) = t/2$ to get $c = -\sqrt{4N}$, $d = \sqrt{4N}$. Applying these, along with $V'(t) = t/2$ to (2.40) completes the proof. □

In the random matrix literature this limiting density is referred to as the Wigner semi-circle law.

Another method for finding the limiting density and region of support is to use the Christoffel-Darboux summation formula (2.32) in the limit $N \to \infty$, since as mentioned (2.32) is literally the density at x. This method has the weakness that it requires knowledge of the asymptotic form of the appropriate polynomials p_n, and so in general we will use the log-gas method, but as an example we will rederive the result of Proposition 2.8 using this method, as an asymptotic form for the Hermite polynomials is known [65]

$$e^{-x^2/4}\tilde{H}_n(x) = 2^{-n/2}\frac{\Gamma(n+1)}{\Gamma(n/2+1)}\cos\left(x\sqrt{n+1/2} - n\pi/2\right) + \mathrm{O}\left(n^{-1/2}\right) \tag{2.45}$$

Rewriting this as

$$\tilde{H}_n(x) = A_n e^{-x^2/4}\cos\left(B_n x + C_n\right) + \mathrm{O}\left(n^{-1/2}\right)$$

we use (2.32) to express $K_N(x,x)$ as

$$\begin{aligned} K_N(x,x) = \frac{A_N A_{N-1}}{\tilde{h}_{N-1}}\Big(& B_{N-1}\cos(B_N x + C_N)\sin(B_{N-1}x + C_{N-1}) \\ & -B_N\cos(B_{N-1}x + C_{N-1})\sin(B_N x + C_N)\Big) + \mathrm{O}\left(N^{-1/2}\right) \end{aligned} \tag{2.46}$$

Use of Stirling's formula (1.16) gives that

$$\frac{A_N A_{N-1}}{\tilde{h}_{N-1}} = \frac{1}{\pi} + \mathrm{O}(1/N)$$

and a few simple trigonometric identities give that difference inside the large parenthesis in (2.46) is equal to

$$\sqrt{N - N\sin^2\left(\frac{x}{2\sqrt{N}} + \mathrm{O}(N^{-3/2})\right)} + \mathrm{O}(N^{-1/2})$$

and so

$$K_N(x,x) = \frac{\sqrt{4N - x^2}}{2\pi} + \mathrm{O}(N^{-1/2}) \tag{2.47}$$

as in (2.44). Setting the leading order term in (2.47) to 0 gives the boundaries of the region of support, which are $-\sqrt{4N}$ and $\sqrt{4N}$ as in (2.43).

2.3 Borodin method - Same number of particles on each line

Having discussed fully the single line PDF, we move now to the more difficult problem of finding the correlation functions for a joint multi-line PDF. To find these correlation functions, we employ the method developed by Borodin and Rains [8] and Borodin, Ferrari, Prahoffer and Sasamoto [6] which requires first confining the particles to lattice sites, and then later taking the continuum limit. The crux of the method is the following theorem.

Theorem 2.9. *Let $\mathcal{X} = \{x_i^{(j)}\}_{\substack{i=1,\ldots,r(j)\\ j=1,\ldots\mathcal{L}}}$ be a set of random variables distributed with PDF $p(\mathcal{X})$, with $\max_j r(j) = N$. We introduce $\mathcal{L}$ discretizations $\mathcal{M}_1, \ldots, \mathcal{M}_{\mathcal{L}}$ of an appropriate interval of the real line, weighted by the spacing of the lattice, such that all the particles in $\mathcal{X}$ are confined to the lattice points of the discretization. Let L be an $|\mathcal{M}| \times |\mathcal{M}|$ matrix with $\mathcal{M} := \{1, \ldots, N\} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_{\mathcal{L}}$ and let L_Y denote the restriction of L to the rows and columns corresponding to $Y \subset \mathcal{M}$. Suppose*

$$p(\mathcal{X}) = \frac{1}{C}\frac{\det L_{\{1,\ldots,N\}\cup\mathcal{X}}}{\det(\mathbf{1}^* + L)} \tag{2.48}$$

for some constant C, where $\mathbf{1}^*$ is the $|\mathcal{M}| \times |\mathcal{M}|$ identity matrix with the first N ones set to zero. The correlation function for particles at $Y \subset \mathcal{M}\backslash\{1, \ldots, N\}$ is given by

$$\rho(Y) = \det K_Y, \qquad K = \left(I - (\mathbf{1}^* + L)^{-1}\right)|_{\mathcal{M}\backslash\{1,\ldots,N\}}$$

Proof. The proof can be found in [8] □

Notice the appearance of determinants. First the joint PDF specifying the configurations (2.48) is expressed as a determinant, and more crucially the correlation function for $|Y|$ particles is given in terms of a $|Y| \times |Y|$ determinant. In such circumstances the statistical system is said to be a determinantal point process.

The simplest form of this method is the case where $r(j)$ is a constant with no dependence on j. This means that there are the same number of particles for each j, such as when the $x_i^{(j)}$ represent the positions of random walkers at some time j. We will demonstrate using the Brownian motion model described earlier with PDF (1.91).

To use the Borodin method to find correlation functions, we need the PDF to be in determinantal form. For the easiest case with $r(j) = N$ for all $j = 1, \ldots, M$, this means we want the PDF to be of the form

$$\frac{1}{C}\det[\phi_j(x_i^{(1)})]_{i,j=1,\ldots N}\prod_{l=1}^{M-1}\det\left[W_l(x_i^{(l)},x_j^{(l+1)})\right]_{i,j=1,\ldots,N}\det[\psi_j(x_i^{(M)})]_{i,j=1,\ldots N} \tag{2.49}$$

Theorem 2.10. *Let the particles* $\{x_i^{(j)}\}_{\substack{i=1,\ldots,N\\ j=1,\ldots,M}}$ *be distributed with PDF as in* (2.49). *Let*

$$\begin{aligned}
B(y,m) &= \int_{-\infty}^{\infty}\phi_m(z)W_{[1,t_y)}(z,y)dz\\
C(x,n) &= \int_{-\infty}^{\infty}W_{[t_x,M)}(x,z)\psi_n(x)dz\\
E(m,n) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_m(x)W_{[1,M-1)}(x,y)\psi_n(y)dxdy
\end{aligned}$$

where $W_{[i,j)} = W_i * \cdots * W_{j-1}$ *if* $j > i$ *(and* 0 *otherwise). Here* $*$ *represents the convolution*

$$(a*b)(x,z) = \int_{-\infty}^{\infty}a(x,y)b(y,z)dy \tag{2.50}$$

Then, if E *satisfies* $E(m,n) = \delta_{m,n}E(n,n)$, $E(n,n) \neq 0$,

$$\rho_{(r)}(\{(x_i,t_i)\}_{i=1,\ldots,r}) = \det\left[K(x_j,t_j;x_k,t_k)\right]_{j,k=1,\ldots,r}$$

where

$$K(y,t_y;x,t_x) = -W_{[t_x,t_y)}(x,y) + \sum_{n=1}^{N}\frac{B(y,n)C(x,n)}{E(n,n)} \tag{2.51}$$

Proof. To begin, we introduce M discretizations $\mathcal{M}_1, \ldots, \mathcal{M}_M$ as specified in Theorem 2.9, and define the matrices

$$\begin{aligned}
B &= \left[[\phi_i(x_j)]_{\substack{i=1,\ldots,N\\ x_j\in\mathcal{M}_1}}, 0_{N\times\mathcal{M}_2},\ldots,0_{N\times\mathcal{M}_M}\right] && (2.52)\\
C &= \left[0_{\mathcal{M}_1\times N},\ldots,0_{\mathcal{M}_{M-1}\times N},[\psi_j(x_i)]_{\substack{x_i\in\mathcal{M}_M\\ j=1,\ldots,N}}\right]^T && (2.53)\\
D &= I - \left[\delta_{i+1,j}[W_i(x,y)]_{\substack{x\in\mathcal{M}_i\\ y\in\mathcal{M}_{i+1}}}\right]_{i,j=1,\ldots,M} && (2.54)
\end{aligned}$$

Then the $|\mathcal{M}| \times |\mathcal{M}|$ matrix

$$L = \begin{bmatrix} 0_{N\times N} & B\\ C & D-I\end{bmatrix} \tag{2.55}$$

satisfies the conditions of Theorem 2.9 and $K = \left(I - (\mathbf{1}^* + L)^{-1}\right)|_{\mathcal{M}\setminus\{1,\ldots,N\}}$. But with L in the form of (2.55), we have from [8] that

$$\left(I - (\mathbf{1}^* + L)^{-1}\right)|_{\mathcal{M}\setminus\{1,\ldots,N\}} = I - D^{-1} + D^{-1}C\left(BD^{-1}C\right)^{-1}BD^{-1} \tag{2.56}$$

We now take the continuum limit of the discretizations $\mathcal{M}_1, \ldots, \mathcal{M}_M$ such that, for $j = 1, \ldots, M$,

$$\sum_{x_i\in\mathcal{M}_j}a(y,x_i)b(x_i,z) \to (a*b)(y,z)$$

Noting that the LHS is the form resulting from certain matrix multiplication, we use the defintion of D to compute

$$D^{-1} = I + \left[[W_{[i,j)}(x,y)]_{\substack{x\in\mathcal{M}_i \\ y\in\mathcal{M}_j}} \right]_{i,j=1,\dots,M} \tag{2.57}$$

We use this, along with the definitions of B and C found in (2.52) and (2.53), to evaluate

$$\begin{aligned} BD^{-1} &= \left[[B(y,m)]_{\substack{m=1,\dots,N \\ y\in\mathcal{M}_j}} \right]_{j=1,\dots,M} \\ D^{-1}C &= \left[[C(x,n)]_{\substack{x\in\mathcal{M}_i \\ n=1,\dots,N}} \right]_{i=1,\dots,M} \\ BD^{-1}C &= [E(i,j)]_{i,j=1,\dots,N} \end{aligned}$$

Then, if E satisfies $E(m,n) = \delta_{m,n}E(m,m)$,

$$\left(BD^{-1}C\right)^{-1} = \left[\frac{\delta_{i,j}}{E(i,i)} \right]_{i,j=1,\dots,N}$$

and so

$$D^{-1}C\left(BD^{-1}C\right)^{-1}BD^{-1} = \left[\sum_{n=1}^{N} \frac{B(y,n)C(x,n)}{E(n,n)} \right]_{x,y\in\mathcal{M}\setminus\{1,\dots,N\}}$$

and the result follows. □

We will now demonstrate the use of this Theorem by evaluating the correlation functions for N Brownian walkers conditioned never to intersect, all beginning at 0 and ending at some point α, as discussed in §1.7.

Proposition 2.11. *[46] Let the particles $\{x_i^{(j)}\}_{\substack{i=1,\dots,N \\ j=1,\dots,M}}$ be distributed with PDF as in* (1.91)*. Then the correlation functions are of the form*

$$\rho_{(r)}\left((x_1,t_1);\dots;(x_r,t_r)\right) = \det\left[K(x_i,t_i;x_j,t_j)\right]_{i,j=1,\dots,r}$$

with correlation kernel

$$K(y,t_y;x,t_x) = \begin{cases} \dfrac{1}{c_x}e^{-\left(\frac{x-a_x}{c_x}\right)^2} \displaystyle\sum_{n=0}^{N-1} \frac{1}{h_n}\left(\frac{z_y}{z_x}\right)^n H_n\left(\frac{x-a_x}{c_x}\right) H_n\left(\frac{y-a_y}{c_y}\right) & t_x \geq t_y \\ -\dfrac{1}{c_x}e^{-\left(\frac{x-a_x}{c_x}\right)^2} \displaystyle\sum_{n=N}^{\infty} \frac{1}{h_n}\left(\frac{z_y}{z_x}\right)^n H_n\left(\frac{x-a_x}{c_x}\right) H_n\left(\frac{y-a_y}{c_y}\right) & t_x < t_y \end{cases} \tag{2.58}$$

where $c_x = \sqrt{\dfrac{t_x(2T-t_x)}{T}}$, $a_x = \dfrac{t_x\alpha}{2T}$, $z_x = \sqrt{\dfrac{2T-t_x}{t_x}}$, $h_n = 2^n n!\sqrt{\pi}$ and the H_n are the Hermite polynomials described in (2.8)*.*

Proof. To convert (1.91) to the form in (2.49) required for Theorem 2.10, we define

$$p_{t_y-t_x}(x,y) := \frac{1}{\sqrt{2\pi(t_y-t_x)}} e^{-(x_j-y_k)^2/2(t_y-t_x)}$$

so that $G_t(\vec{x},\vec{y}) = \det[p_t(x_i,y_j)]_{i,j=1,\dots,N}$. Then $P_{\vec{0},\vec{\alpha}}(\vec{x}^{(1)},\dots,\vec{x}^{(M)})$ from (1.91) is of the form in (2.49), with $W_l = p_{t_{l+1}-t_l}$ and

$$\begin{aligned} \phi_j(x) &= e^{-\left(1-\frac{t_1}{2T}\right)\left(\frac{x-a_1}{c_1}\right)^2 - \frac{x a}{2T}} H_{j-1}\left(\frac{x-a_1}{c_1}\right) \\ \psi_j(x) &= e^{-\frac{t_M}{2T}\left(\frac{x-a_M}{c_M}\right)^2 + \frac{x a}{2T}} H_{j-1}\left(\frac{x-a_M}{c_M}\right) \end{aligned}$$

Using the scaled variables c_n, a_n and z_n we see that

$$p_{t_y-t_x}(x,y) = \sqrt{\frac{T}{t_y(2T-t_x)}} e^{\left(\frac{t_y}{2T}-1\right)\left(\frac{y-a_y}{c_x}\right)^2 - \frac{t_x}{2T}\left(\frac{x-a_x}{c_x}\right)^2 - \frac{\alpha}{2T}(y-x) + \frac{\alpha^2(t_y-t_x)}{8T^2}} \times \sum_{k=0}^{\infty} \frac{1}{h_k}\left(\frac{z_y}{z_x}\right)^k H_k\left(\frac{x-a_x}{c_x}\right) H_k\left(\frac{y-a_y}{c_y}\right)$$

and note that $(p_{t_y-t_x} * p_{t_z-t_y})(x,z) = p_{t_z-t_x}(x,z)$, and thus $W_{[i,j)}(x,y) = p_{t_j-t_i}(x,y)$. With B, C and E as described in Theorem 2.10, we evaluate

$$E(m,n) = \delta_{m,n} c_1 c_M \sqrt{\frac{T}{t_M(2T-t_1)}} e^{\frac{\alpha^2(t_M-t_1)}{8T^2}} h_{m-1}\left(\frac{z_M}{z_1}\right)^{m-1}$$

and

$$\frac{B(y,n)C(x,n)}{E(n,n)} = \sqrt{\frac{T}{t_y(2T-t_x)}} e^{\left(\frac{t_y}{2T}-1\right)\left(\frac{y-a_y}{c_x}\right)^2 - \frac{t_x}{2T}\left(\frac{x-a_x}{c_x}\right)^2 - \frac{\alpha}{2T}(y-x) + \frac{\alpha^2(t_y-t_x)}{8T^2}} \times \frac{1}{h_{n-1}}\left(\frac{z_y}{z_x}\right)^{n-1} H_{n-1}\left(\frac{x-a_x}{c_x}\right) H_{n-1}\left(\frac{y-a_y}{c_y}\right)$$

Then it is clear that, for K as in (2.58) and $F(t,x) = e^{-\alpha x/2T + \alpha^2 t/8T^2}$,

$$K(t_y, y; t_x, x) = \frac{F(t_x,x)}{F(t_y,y)}\left(-W_{[t_x,t_y)}(x,y) + \sum_{n=1}^{N} \frac{B(y,n)C(x,n)}{E(n,n)}\right)$$

and the proof is completed by applying Theorem 2.10 and noting that

$$\det\left[K(t_i,x_i;t_j,x_j)\right]_{i,j=1,\ldots,r} = \det\left[\frac{F(t_i,x_i)}{F(t_j,x_j)} K(t_i,x_i;t_j,x_j)\right]_{i,j=1,\ldots,r} \tag{2.59}$$

for any functions $F(t,x) \neq 0$, $K(t,x;s,y)$. □

A similar analysis applies to the Brownian walkers near a wall.

Proposition 2.12. *Let the particles* $\{x_i^{(j)}\}_{\substack{i=1,\ldots,N \\ j=1,\ldots,M}}$ *be distributed with PDF as in* (1.104). *Then the correlation functions are of the form*

$$\rho_{(r)}\left((x_1,t_1);\ldots;(x_r,t_r)\right) = \det\left[K(x_i,t_i;x_j,t_j)\right]_{i,j=1,\ldots,r}$$

with correlation kernel

$$K(y,t_y;x,t_x) = \begin{cases} \dfrac{1}{c_x} e^{-\left(\frac{x}{c_x}\right)^2} \displaystyle\sum_{n=0}^{N-1} \frac{2}{h_{2n+1}}\left(\frac{z_y}{z_x}\right)^{2n+1} H_{2n+1}\left(\frac{x}{c_x}\right) H_{2n-1}\left(\frac{y}{c_y}\right) & t_x \geq t_y \\ -\dfrac{1}{c_x} e^{-\left(\frac{x}{c_x}\right)^2} \displaystyle\sum_{n=N}^{\infty} \frac{2}{h_{2n+1}}\left(\frac{z_y}{z_x}\right)^{2n+1} H_{2n+1}\left(\frac{x}{c_x}\right) H_{2n-1}\left(\frac{y}{c_y}\right) & t_x < t_y \end{cases} \tag{2.60}$$

where $c_x = \sqrt{\frac{t_x(2T-t_x)}{T}}$, $z_x = \sqrt{\frac{2T-t_x}{t_x}}$, $h_n = 2^n n! \sqrt{\pi}$ *and* $H_n(x)$ *are the Hermite polynomials described in* (2.8).

Proof. As with the proof of Proposition 2.11, we wish to convert (1.104) to the form in (2.49). We define

$$p_{t_y-t_x}^{\text{Wall}}(x,y) := \frac{1}{\sqrt{2\pi(t_y-t_x)}}\left(e^{-(x_j-y_k)^2/2(t_y-t_x)} - e^{-(x_j+y_k)^2/2(t_y-t_x)}\right)$$

so that $G_t^{\rm Wall}(\vec{x},\vec{y}) = \det[p_t^{\rm Wall}(x_i,y_j)]_{i,j=1,\dots,N}$. Then $P_{\vec{0},\vec{0}}^{\rm Wall}(\vec{x}^{(1)},\dots,\vec{x}^{(M)})$ from (1.104) is of the form in (2.49), with $W_l = p_{t_{l+1}-t_l}^{\rm Wall}$ and

$$\begin{aligned}\phi_j(x) &= e^{-\left(1-\frac{t_1}{2T}\right)\left(\frac{x}{c_1}\right)^2} H_{2j-1}\left(\frac{x}{c_1}\right)\chi_{x>0}\\ \psi_j(x) &= e^{-\frac{t_M}{2T}\left(\frac{x}{c_M}\right)^2} H_{2j-1}\left(\frac{x}{c_M}\right)\chi_{x>0}\end{aligned}$$

Using the scaled variables c_n and z_n, we see that

$$p_{t_y-t_x}^{\rm Wall}(x,y) = \sqrt{\frac{T}{t_y(2T-t_x)}}e^{\frac{-t_x}{2T}\left(\frac{x}{c_x}\right)^2-\left(1-\frac{t_y}{2T}\right)\left(\frac{y}{c_y}\right)^2}\sum_{k=0}^{\infty}\frac{2}{h_{2k+1}}\left(\frac{z_y}{z_x}\right)^{2k+1}H_{2k+1}\left(\frac{x}{c_x}\right)H_{2k+1}\left(\frac{y}{c_y}\right)$$

and note that $(p_{t_y-t_x}^{\rm Wall} * p_{t_z-t_y}^{\rm Wall})(x,z) = p_{t_z-t_x}^{\rm Wall}(x,z)$, and thus $W_{[i,j)}(x,y) = p_{t_j-t_i}^{\rm Wall}(x,y)$. With B, C and E as described in Theorem 2.10, we evaluate

$$E(m,n) = \delta_{m,n}\frac{c_1c_Mh_{2m-1}}{2}\sqrt{\frac{T}{t_M(2T-t_1)}}\left(\frac{z_M}{z_1}\right)^{2m-1}$$

and

$$\begin{aligned}\frac{B(y,n)C(x,n)}{E(n,n)} &= \sqrt{\frac{T}{t_y(2T-t_x)}}e^{-\frac{t_x}{2T}\left(\frac{x}{c_x}\right)^2-\left(1-\frac{t_y}{2T}\right)\left(\frac{y}{c_x}\right)^2}\\ &\qquad\times\frac{2}{h_{n-1}}\left(\frac{z_y}{z_x}\right)^{2n-1}H_{2n-1}\left(\frac{x}{c_x}\right)H_{2n-1}\left(\frac{y}{c_y}\right)\end{aligned}$$

Then it is clear that, for K as in (2.60),

$$K(t_y,y;t_x,x) = -W_{[t_x,t_y)}(x,y) + \sum_{n=1}^{N}\frac{B(y,n)C(x,n)}{E(n,n)}$$

and the proof is completed by applying Theorem 2.10 □

2.4 Gorin method

In [37], Gorin finds the correlation functions for a discrete walkers model closely resembling the hexagon paths model mentioned in §1.7. While the calculation is approached slightly differently, his key proposition (Proposition 2.13 below), which he describes as a slight extension of a theorem by Eynard and Mehta [21], can be shown to be equivalent to Theorem 2.10.

Proposition 2.13. *[37] Assume that for every discrete time parameter t we are given an orthonormal system $\{f_n^t\}$ in linear space $l_2(\{0,1,\dots,L\})$, and a set of numbers $c_0^t, c_1^t, \dots$. Denote*

$$v_{t,t+1}(x,y) = \sum_{n\geq 0} c_n^t f_n^t(x) f_n^{t+1}(y) \tag{2.61}$$

Assume also that we are given a discrete time Markov process X_t taking values in N-tuples of elements of the set $\{0,1,\dots,L\}$, with one-dimensional distributions

$$\left(\det[f_{i-1}^t(x_j)]_{i,j=1}^N\right)^2 \tag{2.62}$$

and transition probabilities

$$\frac{1}{\prod_{n=0}^{N-1}c_n^t}\frac{\det[v_{t,t+1}(x_i,y_j)]_{i,j=1}^N\det[f_{i-1}^{t+1}(y_j)]_{i,j=1}^N}{\det[f_{i-1}^t(x_j)]_{i,j=1}^N} \tag{2.63}$$

Then the correlation functions are of the form

$$\rho_{(r)}\left((x_1,t_1);\ldots;(x_r,t_r)\right) = \det\left[K(x_i,t_i;x_j,t_j)\right]_{i,j=1,\ldots,r} \tag{2.64}$$

where the pair (x,t) represents $x \in X_t$, and, with $c_n^{s,t}$ defined by $c_n^{t,t} = 1$, $c_n^{s,t} = \prod_{i=s}^{t-1} c_n^i$, K is given by

$$K(x,s;y,t) = \begin{cases} \sum_{n=0}^{N-1} \frac{1}{c_n^{t,s}} f_n^s(x) f_n^t(y) & s \geq t \\ -\sum_{n\geq N} c_n^{s,t} f_n^s(x) f_n^t(y) & s < t \end{cases} \tag{2.65}$$

Proof. In [37], Gorin describes this Proposition as a slight extension of a theorem by Eynard and Mehta [21]. However we can also prove this using Theorem 2.10. Using (2.62) and (2.63), we find that the joint PDF for the particle process $\{x_i^{(j)}\}$, where $x_i^{(j)}$ represents the i-th largest term in the Markov process X_j, is given by

$$\frac{1}{C}\det[f_{i-1}^1(x_j^{(1)})]_{i,j=1}^N \prod_{t=1}^{M-1} \det\left[v_{t,t+1}(x_i^{(t)}, x_j^{(t+1)})\right]_{i,j=1}^N \det[f_{i-1}^M(x_j^{(M)})]_{i,j=1}^N$$

where $C = \prod_{n=0}^{N-1} c_n^{1,M-1}$. Clearly this is the same form (2.49) as required by Theorem 2.10, with $\phi_j = f_{j-1}^1$, $\psi_j = f_{j-1}^M$ and $W_t = v_{t,t+1}$. Then redefining convolution as

$$(a * b)(x,z) = \sum_{y=0}^{L} a(x,y)b(y,z)$$

since here our functions are discrete, we have by the same method that we obtained (2.51),

$$K(x,s;y,t) = -\chi_{t>s}(v_{s,s+t} * \cdots * v_{t-1,t})(x,y) + \sum_{n=1}^{N} \frac{B(y,n)C(x,n)}{E(n,n)} \tag{2.66}$$

where

$$\begin{aligned} B(y,m) &= (f_{m-1}^1 * v_{1,2} * \cdots * v_{t-1,t})(y) \\ C(x,m) &= (v_{s,s+1} * \cdots * v_{M-1,M} * f_{n-1}^M)(x) \\ E(m,n) &= (f_{m-1}^1 * v_{1,2} * \cdots * v_{M-1,M} * f_{n-1}^M) \end{aligned}$$

Applying these, the orthogonality of the f_n^t, and the definition of v (2.61) to (2.66) gives (2.65). □

We will use this proposition to briefly review Gorin's calculation of the correlations for the discrete walker model. The PDF for the positions of the N walkers at time t is given by (1.83), which after a some transformations on z, can be rewritten,

$$P_t(z_1,\ldots,z_N) = \frac{1}{Z}\prod_{1\leq i<j\leq N}(z_i' - z_j')^2 \prod_{i=1}^{N} w_{a,b,L}(z_i') \tag{2.67}$$

where $w_{a,b,L}(x)$ is the Hahn weight function as described in (2.12), associated with the Hahn polynomials $Q_n^{(a,b,L)}(x)$, and a, b and L depend on T, N, S and t. Important here is the t

dependence, as it implies the weight function changes on different lines (times) in a given system. Thus, we define

$$f_n^t(x) = \frac{H_n^t(x)\sqrt{w_t(x)}}{\sqrt{(H_n^t, H_n^t)}} \tag{2.68}$$

where and $H_n^t(x) = Q_n(x')$ and $w_t(x)$ is the Hahn weight for the appropriate t, N, T, S and transformation $x \to x'$, so that (2.67) can be rewritten

$$P_t(z_1, \ldots, z_N) = \left(\det[f_{i-1}^t(z_j)]_{i,j=1}^N\right)^2 \tag{2.69}$$

From here, two Lemmas from Gorin are required to complete the calculation and apply Proposition 2.13

Lemma 2.14. *[37] The one-step transition probabilities for the discrete walker model are given by*

$$P_{t,t+1}(x_1, \ldots, x_N; y_1, \ldots, y_N) = \tag{2.70}$$
$$\frac{(T-t-1)!}{(T-t+N-1)!} \prod_{i<j} \frac{(y_i - y_j)}{(x_i - x_j)} \prod_{i:y_i = x_i+1} (N + S - x_i - 1) \prod_{y_i = x+i} (x_i + T - t - S)$$

provided that each difference $y_i - x_i$ is equal to zero or one, otherwise the probabilities are equal to zero.

Proof. This formula can be checked by direct computation starting from the definition and using Proposition 1.17, which gives that the LHS of (2.70) must be equal to

$$\frac{\det\left[\begin{pmatrix} 1 \\ y_i - x_j \end{pmatrix}\right]_{i,j=1,\ldots,N} \det\left[\begin{pmatrix} T-t-1 \\ S+i-1-y_j \end{pmatrix}\right]_{i,j=1,\ldots,N}}{\det\left[\begin{pmatrix} T-t \\ S+i-1-x_j \end{pmatrix}\right]_{i,j=1,\ldots,N}}$$

and applying the determinantal identity (1.82) from [50] □

Lemma 2.15. *[37] With $w_t(x)$, $f_n^t(x)$ defined as in* (2.68)*,*

$$\sqrt{\frac{w_{t+1}(y)}{w_t(x)}} \sum_{k \geq 0} c_k^t f_k^t(x) f_k^{t+1}(y) = \frac{S+N-1-x}{\sqrt{(t+N)(T+N-t-1)}} \delta_{x+1}^y + \frac{T-t-S+x}{\sqrt{(t+N)(T+N-t-1)}} \delta_x^y$$

for

$$c_i^t = \sqrt{\left(1 - \frac{i}{t+N}\right)\left(1 - \frac{i}{T+N-t-1}\right)} \tag{2.71}$$

Proof. For a proof, see [37] □

An important result of this Lemma is that, with $v_{t,t+1}(x,y)$ as in (2.61),

$$v_{t,t+1}(x,y) = \sqrt{\frac{w_t(x)}{w_{t+1}(y)}} \left(\frac{S+N-1-x}{\sqrt{(t+N)(T+N-t-1)}} \delta_{x+1}^y + \frac{T-t-S+x}{\sqrt{(t+N)(T+N-t-1)}} \delta_x^y\right) \tag{2.72}$$

Proposition 2.16. *Let $\{x_i^{(t)}\}_{\substack{i=1,\ldots,N \\ t=1,\ldots,T-1}}$ represent the position of the i-th walker at time t in the discrete walkers model. Then, with $f_n^t(x)$ and c_n^t defined by* (2.68) *and* (2.71)*, Proposition 2.13 applies and the correlations are given by* (2.64) *with K given by* (2.65)

Proof. It is sufficient to show that the transition probabilities have the required form (2.63). Using (1.83), (2.69), (2.71) and (2.72), we evaluate

$$\frac{1}{\prod_{n=0}^{N-1} c_n^t} \frac{\det[v_{t,t+1}(x_i,y_j)]_{i,j=1}^N \det[f_{i-1}^{t+1}(y_j)]_{i,j=1}^N}{\det[f_{i-1}^t(x_j)]_{i,j=1}^N}$$
$$= \frac{(T-t-1)!}{(T-t+N-1)!} \prod_{i<j} \frac{(y_i-y_j)}{(x_i-x_j)} \det\left[(S+N-x_i-1)\delta_{x_i+1}^{y_j} + (T-t-S+x_i)\delta_{x_i}^{y_j}\right]$$

There are two cases: either every difference $y_i - x_i$ is equal to 0 or 1 or this is not true. In the latter case, the determinant is equal to zero. In the former case, the determinant is equal to the product of its diagonal elements. Thus in both cases, recalling Lemma 2.14,

$$\frac{1}{\prod_{n=0}^{N-1} c_n^t} \frac{\det[v_{t,t+1}(x_i,y_j)]_{i,j=1}^N \det[f_{i-1}^{t+1}(y_j)]_{i,j=1}^N}{\det[f_{i-1}^t(x_j)]_{i,j=1}^N} = P_{t,t+1}(x_1,\ldots,x_N;y_1,\ldots,y_N)$$

□

2.5 Borodin method - Differing numbers of particles on each line

For many models, including the GUE* eigenvalue process specified earlier, the number of particles per line is not constant ($r(j)$ has j dependence). This creates a problem for the Borodin method of evaluating correlation functions, as the determinantal form required is harder to obtain. Specifically, a problem arises in determining the size of the determinants involving $W_l(x_i^{(l)}, x_j^{(l+1)})$, since $r(l)$ may not be equal to $r(l+1)$. However, if $r(j)$ obeys

$$|r(j) - r(j+1)| \leq 1$$

for all values of j, then the problem can be solved by introducing virtual particles. The simplest example of this is when $r(j) = j$.

Theorem 2.17. *Let the particles* $\{x_i^{(j)}\}_{\substack{i=1,\ldots,j\\ j=1,\ldots,N}}$ *be distributed with PDF*

$$\frac{1}{C}\prod_{l=0}^{N-1} \det\left[W_l(x_i^{(l)}, x_j^{(l+1)})\right]_{i,j=1,\ldots,l+1} \det[\psi_j(x_i^{(N)})]_{i,j=1,\ldots N} \tag{2.73}$$

with virtual particles $x_{j+1}^{(j)} := \omega$ *for* $j = 0,\ldots,N-1$. *Define matrices*

$$\begin{aligned} B &= [E_k]_{k=1,\ldots,N} \\ C &= \left[0_{\mathcal{M}_1\times N},\ldots,0_{\mathcal{M}_{N-1}\times N},[\psi_j(x_i)]_{\substack{x_i\in\mathcal{M}_N\\ j=1,\ldots,N}}\right]^T \\ D &= I - \left[\delta_{i+1,j}[W_i(x,y)]_{\substack{x\in\mathcal{M}_i\\ y\in\mathcal{M}_{i+1}}}\right]_{i,j=1,\ldots,N} \end{aligned}$$

where the matrices E_k *are defined*

$$E_k = \left[\delta_{i,k} W_{k-1}\left(x_k^{(k-1)}, x_j^{(k)}\right)\right]_{\substack{i=1,\ldots,N\\ x_j\in\mathcal{M}_k}}$$

Furthermore, define matrices

$$\begin{aligned} M &= BD^{-1}C \\ M\Phi &= BD^{-1} \\ \Psi &= D^{-1}C \end{aligned}$$

Then for $Y = \bigcup_{i=1}^{r} y_i^{(t_i)}$

$$\rho_{(r)}((y_1,t_1);\dots;(y_r,t_r)) = \det\left[K(y_i,t_i;y_j,t_j)\right]_{i,j=1,\dots,r}$$

where

$$K(y,t;x,s) = -W_{[t,s)}(y,x) + (\Psi\Phi)_{y,t;x,s} \tag{2.74}$$

Proof. As with the proof of Theorem 2.10, the $|\mathcal{M}| \times |\mathcal{M}|$ matrix L as given in (2.55) satisfies the conditions of Theorem 2.9 and K is given by (2.56). From (2.57) and the definitions of Φ and Ψ, the result follows. □

We will now use this theorem to evaluate the correlation functions of the GUE* eigenvalue process. First, we need a Lemma which helps us convert the PDF in (1.29) to the determinantal form required

Lemma 2.18. *[32] Assume*

$$x_N < x_{N-1} < \cdots < x_1, \qquad y_N < y_{N-1} < \cdots < y_1$$

Then

$$\det[\chi_{x_j - y_k > 0}]_{j,k=1,\dots,N} = \chi_{y_N < x_N < y_{N-1} < x_{N-1} < \cdots < y_1 < x_1}$$

Proof. If $y_N < x_N < y_{N-1} < x_{N-1} < \cdots < y_1 < x_1$, then the determinant is diagonal with ones in the upper right half, with ones on the diagonal. It is clear that this is equal to 1, so RHS = LHS.

If $x_N < y_N < x_{N-1} < y_{N-1} < \cdots < x_1 < y_1$ then the determinant is diagonal with ones in the upper right half, with zeroes on the diagonal. It is clear that this is equal to 0, so RHS = LHS.

For all other possible orderings, there must a pair of consecutive x's or y's. Let this pair be called x_i, x_{i+1}, (or y_i, y_{i+1}). Since this pair is consecutive in the ordering, $x_i > y_j$ if and only if $x_{i+1} > y_j$ $\forall j$, so the determinant is unchanged if we swap the i-th row with the $i+1$-th row (or for y's, unchanged if we swap columns). Therefore, the determinant is equal to 0, and RHS = LHS. □

Proposition 2.19. *[43] Let the particles* $\{x_i^{(j)}\}_{\substack{i=1,\dots,j\\ j=1,\dots,N}}$ *represent the i-th largest eigenvalue of a* $j \times j$ *submatrix of the GUE*, and thus be distributed with PDF*

$$p_{GUE^*,N}\left(\{x_i^{(j)}\}\right) = \frac{1}{(2\pi)^{\frac{N}{2}}} \prod_{k=1}^{N} e^{\frac{-\left(x_k^{(N)}\right)^2}{2}} \Delta\left(x^{(N)}\right) \prod_{n=0}^{N-1} \chi\left(x^{(n)} \prec x^{(n+1)}\right) \tag{2.75}$$

where $\chi\left(x^{(n)} \prec x^{(n+1)}\right)$ *is the interlacing requirement defined in §6.1 (recall* (1.29)*). Then for* $Y = \bigcup_{i=1}^{r} y_i^{(t_i)}$

$$\rho_{(r)}((y_1,t_1),\dots,(y_r,t_r)) = \det\left[K(y_i,t_i;y_j,t_j)\right]_{i,j=1,\dots,r} \tag{2.76}$$

with correlation kernal

$$K(y,t;x,s) = \begin{cases} e^{-(x^2+y^2)/4} \displaystyle\sum_{l=1}^{s} \frac{\tilde{H}_{t-l}(y)\tilde{H}_{s-l}(x)}{\tilde{h}_{s-l}} & s \le t \\ -e^{-(x^2+y^2)/4} \displaystyle\sum_{l=-\infty}^{0} \frac{\tilde{H}_{t-l}(y)\tilde{H}_{s-l}(x)}{\tilde{h}_{s-l}} & s > t \end{cases} \tag{2.77}$$

where $\tilde{H}_n(x)$ *are the probabilists' Hermite polynomials as described in* (2.11).

Proof. Using Lemma 2.18, (2.75) can be written as

$$\frac{1}{C}\prod_{k=1}^{N-1}\det\left[\chi_{x_i^k<x_j^{k+1}}\right]_{i,j=1,\ldots,k+1}\det\left[e^{\frac{-\left(x_i^{(N)}\right)^2}{2}}\tilde{H}_{N-j}\left(x_i^{(N)}\right)\right]_{i,j=1,\ldots,N}$$

with virtual particles $x_{i+1}^{(i)}=\omega\to-\infty$ Thus, (2.75) can be written in the form required for Theorem 2.17 with

$$\begin{aligned}W_i(y,x) &= \chi_{y<x}\\ \psi_j(x) &= e^{-x^2/2}\tilde{H}_{N-j}(x)\end{aligned}$$

From this, we evaluate

$$W_{[i,j)}(y,x)=\frac{1}{(j-i-1)!}\chi_{y<x}(x-y)^{j-i-1} \tag{2.78}$$

To make use of Theorem 2.17, we must find the matrices Φ and Ψ. Noting that D is the same as in Theorem 2.10, we use (2.57) and the definition of C to evaluate

$$\Psi:=D^{-1}C=\left[\left[(W_{[t,N)}*\psi_n)(y)\right]_{\substack{y\in\mathcal{M}_t\\ n=1,\ldots,N}}\right]^T_{t=1,\ldots,N} \tag{2.79}$$

for $*$ as in (2.50). We define matrices $\Psi^t=[\Psi_n^t(x)]_{\substack{x\in\mathcal{M}_t\\ n=1,\ldots,N}}$ such that $\Psi=[\Psi^1,\ldots,\Psi^N]^T$. Then, by using integration by parts on (2.79), we have

$$\Psi_n^t(y)=\begin{cases} e^{-y^2/2}\tilde{H}_{t-n}(y) & \text{if} \quad n\le t\\ \dfrac{1}{(n-t-1)!}\displaystyle\int_y^\infty (x-y)^{n-t-1}e^{-x^2/2}dx & \text{if} \quad n>t\end{cases} \tag{2.80}$$

We now go about finding Φ. With D^{-1} again being as in (2.57), we evaluate

$$\begin{aligned}BD^{-1} &= \left[E_s+\sum_{k=1}^{s-1}E_kW_{[k,s)}\right]_{s=1,\ldots,N}\\ &= \left[\left[\left(W_{m-1}*W_{[m,s)}\right)(\omega,x)\right]_{\substack{m=1,\ldots,N\\ x\in\mathcal{M}_s}}\right]_{s=1,\ldots,N}\end{aligned}$$

We now define polynomials $\{\Phi_n^t(x)\}_{n=1,\ldots,t}$ by

$$\int_{-\infty}^{\infty}\Phi_i^t(x)\Psi_j^t(x)dx=\delta_{i,j} \tag{2.81}$$

where the $\Psi_n^t(x)$ are as in (2.80). This gives us

$$\Phi_n^t(x)=\frac{\tilde{H}_{t-n}(x)}{\tilde{h}_{t-n}}$$

For each $s=1,\ldots,N$, let

$$\left(W_{m-1}*W_{[m,s)}\right)(\omega,x)=\sum_{l=1}^{s}\left(A_s\right)_{m,l}\Phi_l^s(x) \tag{2.82}$$

for some $N\times N$ matrix A_s. Then by (2.81), we can multiply by $\Psi_n^s(x)$ and integrate to show

$$\left(A_s\right)_{m,n}=\left(W_{m-1}*W_{[m,s)}*\Psi_n^s\right)(\omega)$$

Evaluating the RHS of this gives

$$(A_s)_{m,n} = \begin{cases} \lim_{\omega \to -\infty} \int_\omega^\infty \frac{(x-\omega)^{n-m}}{(n-m)!} e^{-x^2/2} dx & m \le n \\ 0 & m > n \end{cases}$$

So the A_s matrices have no s dependence and can be replaced by $A = A_N$ with the form

$$(A)_{m,n} = \left(W_{m-1} * W_{[m,N)} * \Psi_n^N\right)(a)$$

and from (2.82) we now have $BD^{-1} = A\Phi$ for $\Phi = [\Phi^1, \ldots, \Phi^N]$ with

$$(\Phi^s)_{l,x} = \begin{cases} \Phi_l^s(x) & l \le s \\ 0 & l > s \end{cases}$$

Finally, $M := BD^{-1}C$ is defined by

$$(M)_{m,n} = \left(W_{m-1} * W_{[m,N)} * \psi_n\right)(\omega)$$

and thus, since $\Psi_n^N(x) = \psi_n(x)$, $A = M$. Thus, $BD^{-1} = M\Phi$. We now have the conditions for Theorem 2.17 and so from (2.74), (2.76) is true with

$$K(y,t;x,s) = -\frac{1}{(s-t-1)!}\chi_{y<x}(x-y)^{s-t-1} + \sum_{l=1}^{s} \Phi_l^s(x)\Psi_l^t(y)$$

If $s \le t$,

$$-\frac{1}{(s-t-1)!}\chi_{y<x}(x-y)^{s-t-1} + \sum_{l=1}^{s} \Phi_l^s(x)\Psi_l^t(y) = e^{-y^2/2} \sum_{l=1}^{s} \frac{\tilde{H}_{s-l}(x)\tilde{H}_{t-l}(y)}{\tilde{h}_{s-l}} \tag{2.83}$$

If $s > t$, let $s = t + 1 + k$, $k \ge 0$. Then

$$\sum_{l=1}^{s} \Phi_l^s(x)\Psi_l^t(y) = e^{-y^2/2} \sum_{l=1}^{t} \frac{\tilde{H}_{s-l}(x)\tilde{H}_{t-l}(y)}{\tilde{h}_{s-l}} + \sum_{l=0}^{k} \frac{\tilde{H}_{k-l}(x)}{l!\tilde{h}_{k-l}} \int_y^\infty (z-y)^l e^{-z^2/2} dz \tag{2.84}$$

and

$$\begin{aligned} \frac{1}{k!}\chi_{y<x}(x-y)^k &= \frac{1}{k!}\sum_{l=0}^{\infty} \frac{\tilde{H}_l(x)}{\tilde{h}_l} \int_y^\infty e^{-z^2/2}(z-y)^k \tilde{H}_l(z) dz \\ &= e^{-y^2/2} \sum_{l=-\infty}^{t} \frac{\tilde{H}_{s-l}(x)\tilde{H}_{t-l}(y)}{\tilde{h}_{s-l}} + \sum_{l=0}^{k} \frac{\tilde{H}_{k-l}(x)}{l!\tilde{h}_{k-l}} \int_y^\infty e^{-z^2/2}(z-y)^l dz \end{aligned} \tag{2.85}$$

so

$$-\frac{1}{(s-t-1)!}\chi_{y<x}(x-y)^{s-t-1} + \sum_{l=1}^{s} \Phi_l^s(x)\Psi_l^t(y) = -e^{-y^2/2} \sum_{l=-\infty}^{0} \frac{\tilde{H}_{s-l}(x)\tilde{H}_{t-l}(y)}{\tilde{h}_{s-l}}$$

Since the form of K found in (2.83) and (2.85) differ from that in (2.77) only by the function $e^{(x^2-y^2)/4}$ which causes no change to the determinant in (2.76) (recall (2.59)), the proof is complete.

□

2.6 Borodin method - Differing numbers of particles on each line near a wall

The formalism of Theorem 2.17 applies when the number of particles on line j is equal to j for $j = 1, \ldots, N$. The antisymmetric GUE process of §1.5 gives rise to a particle system for which the number of particles on line j is equal to $\lfloor j/2 \rfloor$ for $j = 1, \ldots, N$. In such circumstance, the correlations can be computed using a modification of Theorem 2.17.

Theorem 2.20. *Let the particles* $\{x_i^{(j)}\}_{\substack{i=1,\dots,\lfloor j/2\rfloor \\ j=1,\dots,N}}$ *be distributed with PDF*

$$\frac{1}{C}\prod_{l=1}^{N-1}\det\left[W_l\left(x_i^{(l)},x_j^{(l+1)}\right)\right]_{i,j=1,\dots,\lfloor\frac{l+1}{2}\rfloor}\det\left[\psi_j\left(x_i^{(N)}\right)\right]_{i,j=1,\dots\lfloor\frac{N}{2}\rfloor} \tag{2.86}$$

with virtual particles $x_j^{(2j-1)} := 0$ *for* $j = 1, 2, \dots, \lfloor N/2\rfloor$. *Define matrices*

$$\begin{aligned} B &= [E_k^*]_{k=1,\dots,N} \\ C &= \left[0_{\mathcal{M}_1\times\lfloor\frac{N}{2}\rfloor},\dots,0_{\mathcal{M}_{N-1}\times\lfloor\frac{N}{2}\rfloor},[\psi_j(x_i)]_{\substack{x_i\in\mathcal{M}_N \\ j=1,\dots,\lfloor N/2\rfloor}}\right]^T \\ D &= I-\left[\delta_{i+1,j}[W_i(x,y)]_{\substack{x\in\mathcal{M}_i \\ y\in\mathcal{M}_{i+1}}}\right]_{i,j=1,\dots,N} \end{aligned} \tag{2.87}$$

where the matrices E_k^* *are defined*

$$E_k^* = \begin{cases} \left[\delta_{i,k/2}W_{k-1}\left(x_{k/2}^{(k-1)},x_j^{(k)}\right)\right]_{\substack{i=1,\dots,\lfloor N/2\rfloor \\ x_j\in\mathcal{M}_k}} & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases}$$

Furthermore, define matrices

$$\begin{aligned} M &= BD^{-1}C \\ M\Phi &= BD^{-1} \\ \Psi &= D^{-1}C \end{aligned}$$

Then for $Y = \bigcup_{i=1}^{r} y_i^{(t_i)}$

$$\rho_{(r)}((y_1,t_1);\dots;(y_r,t_r))) = \det\left[K(y_i,t_i;y_j,t_j)\right]_{i,j=1,\dots,r}$$

where

$$K(y,t;x,s) = -W_{[t,s)}(y,x) + (\Psi\Phi)_{y,t;x,s} \tag{2.88}$$

Proof. The proof is identical to that of Theorem 2.17 □

We will now use Theorem 2.20 to evaluate the correlation functions of the anti-symmetric GUE eigenvalue process.

Proposition 2.21. *[29] Let* H *be an* $N\times N$ *matrix from the anti-symmetric GUE ensemble. Let the particles* $\{x_i^{(j)}\}_{\substack{i=1,\dots,\lfloor j/2\rfloor \\ j=1,\dots,N}}$ *represent the* i*-th largest positive eigenvalue of the* $j\times j$ *leading sub-block of* H*, and thus be distributed with PDF*

$$\begin{aligned} &\frac{1}{C_N}\Delta\left((x^{(N)})^2\right)\prod_{i=1}^{N/2}e^{-\left(x_i^{(N)}\right)^2}\prod_{k=1}^{N-1}\chi\left(x^{(k)}\prec x^{(k+1)}\right) && \text{for } N \text{ even} \\ &\frac{1}{C_N}\Delta\left((x^{(N)})^2\right)\prod_{i=1}^{(N-1)/2}x_i^{(N)}e^{-\left(x_i^{(N)}\right)^2}\prod_{k=1}^{N-1}\chi\left(x^{(k)}\prec x^{(k+1)}\right) && \text{for } N \text{ odd} \end{aligned} \tag{2.89}$$

where $\chi\left(x^{(k)}\prec x^{(k+1)}\right)$ *is the interlacing requirement defined in §6.1 (recall (1.60)). Then, for* $Y = \bigcup_{i=1}^{r} y_i^{(t_i)}$,

$$\rho_{(r)}((y_1,t_1),\dots,(y_r,t_r)) = \det\left[K(y_i,t_i;y_j,t_j)\right]_{i,j=1,\dots,r} \tag{2.90}$$

with correlation kernal

$$K(y,t;x,s) = \begin{cases} 2e^{-(x^2+y^2)/2} \sum_{l=1}^{\lfloor \frac{s}{2} \rfloor} \frac{H_{t-2l}(y)H_{s-2l}(x)}{h_{s-2l}} & s \le t \\ -2e^{-(x^2+y^2)/2} \sum_{l=-\infty}^{0} \frac{H_{t-2l}(y)H_{s-2l}(x)}{h_{s-2l}} & s > t \end{cases} \tag{2.91}$$

where $H_n(x)$ are the Hermite polynomials as described in (2.8).

Proof. Using Lemma 2.18, (2.89) can be written as

$$\frac{1}{C} \prod_{k=1}^{N-1} \det \left[\chi_{x_i^{(k)} < x_j^{(k+1)}} \right]_{i,j=1,\dots,\lfloor \frac{k+1}{2} \rfloor} \det \left[e^{-\left(x_i^{(N)}\right)^2} H_{N-2j}\left(x_i^{(N)}\right) \right]_{i,j=1,\dots,\lfloor \frac{N}{2} \rfloor}$$

with virtual particles $x_i^{(2i-1)} = 0$ for $i = 1, 2, \dots, \lfloor N/2 \rfloor$. Thus, (2.89) can be written in the form (2.86) required for Theorem 2.20 with

$$\begin{aligned} W_i(y,x) &= \chi_{y<x} \\ \psi_j(x) &= e^{-x^2} H_{N-2j}(x) \end{aligned}$$

From here, we follow the proof of Proposition 2.19 and attempt to find the matrices Φ and Ψ. Noting that D is the same as in Theorem 2.10 and W is the same as in Proposition 2.19 we use (2.57), (2.78) and (2.87) to evaluate

$$\Psi := D^{-1}C = \left[\left[(W_{[t,N)} * \psi_n)(y) \right]_{\substack{y \in \mathcal{M}_t \\ n=1,\dots,N/2}} \right]^T_{t=1,\dots,N/2} \tag{2.92}$$

where here (and throughout this particular proof) the convolution $*$ is defined

$$(a * b)(x,z) = \int_0^\infty a(x,y)b(y,z)dy$$

We define matrices $\Psi^t = [\Psi_n^t(x)]_{\substack{x \in \mathcal{M}_t \\ n=1,\dots,\lfloor N/2 \rfloor}}$ such that $\Psi = [\Psi^1, \dots, \Psi^N]^T$. Then, by using the fact that the Hermite polynomials can be expressed by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2} \right) \tag{2.93}$$

and integration by parts on (2.92), we have

$$\Psi_n^t(y) = \begin{cases} e^{-x^2/2} H_{t-2n}(y) & \text{if} \quad n \le t/2 \\ \dfrac{1}{(2n-t-1)!} \displaystyle\int_y^\infty (x-y)^{2n-t-1} e^{-x^2} dx & \text{if} \quad n > t/2 \end{cases} \tag{2.94}$$

We now go about finding Φ. With D^{-1} again being as in (2.57), we evaluate

$$\begin{aligned} BD^{-1} &= \left[E_s + \sum_{k=1}^{s-1} E_k W_{[k,s)} \right]_{s=1,\dots,N} \\ &= \left[\left[\left(W_{2m-1} * W_{[2m,s)} \right)(0,x) \right]_{\substack{m=1,\dots,\lfloor N/2 \rfloor \\ x \in \mathcal{M}_s}} \right]_{s=1,\dots,N} \end{aligned}$$

We now define polynomials $\{\Phi_n^t(x)\}_{n=1,\dots,\lfloor t/2 \rfloor}$ by

$$\int_0^\infty \Phi_i^t(x) \Psi_j^t(x) dx = \delta_{i,j} \tag{2.95}$$

where the $\Psi_n^t(x)$ are as in (2.94). This gives us

$$\Phi_n^t(x) = \frac{2H_{t-2n}(x)}{h_{t-2n}}$$

For each $s = 1, \dots, N$, let

$$\left(W_{2m-1} * W_{[2m,s)}\right)(0, x) = \sum_{l=1}^{\lfloor s/2 \rfloor} (A_s)_{m,l} \, \Phi_l^s(x) \tag{2.96}$$

for some $\lfloor N/2 \rfloor \times \lfloor N/2 \rfloor$ matrix A_s. Then by (2.95), we can multiply by $\Psi_n^s(x)$ and integrate to show

$$(A_s)_{m,n} = \left(W_{2m-1} * W_{[2m,s)} * \Psi_n^s\right)(0)$$

Evaluating the RHS of the above equation gives

$$(A_s)_{m,n} = \begin{cases} \displaystyle\int_0^\infty \frac{x^{n-m}}{(n-m)!} e^{-x^2/2} dx & m \leq n \\ 0 & m > n \end{cases}$$

So the A_s matrices have no s dependence and can be replaced by $A = A_N$ with the form

$$(A)_{m,n} = \left(W_{m-1} * W_{[m,N)} * \Psi_n^N\right)(a)$$

and from (2.96) we now have $BD^{-1} = A\Phi$ for $\Phi = [\Phi^1, \dots, \Phi^N]$ with

$$(\Phi^s)_{l,x} = \begin{cases} \Phi_l^s(x) & l \leq s/2 \\ 0 & l > s/2 \end{cases}$$

Finally, $M := BD^{-1}C$ is defined by

$$(M)_{m,n} = \left(W_{2m-1} * W_{[2m,N)} * \psi_n\right)(0)$$

and thus, since $\Psi_n^N(x) = \psi_n(x)$, $A = M$. Thus, $BD^{-1} = M\Phi$. We now have the conditions for Theorem 2.17 and so from (2.88), (2.90) is true with

$$K(y, t; x, s) = -\frac{1}{(s-t-1)!} \chi_{y<x} (x-y)^{s-t-1} + \sum_{l=1}^{\lfloor s/2 \rfloor} \Phi_l^s(x) \Psi_l^t(y) \tag{2.97}$$

If $s \leq t$,

$$-\frac{1}{(s-t-1)!} \chi_{y<x} (x-y)^{s-t-1} + \sum_{l=1}^{s} \Phi_l^s(x) \Psi_l^t(y) = 2e^{-y^2} \sum_{l=1}^{\lfloor s/2 \rfloor} \frac{H_{s-2l}(x) H_{t-2l}(y)}{h_{s-2l}} \tag{2.98}$$

If $s > t$,

$$\sum_{l=1}^{s} \Phi_l^s(x) \Psi_l^t(y) = 2e^{-y^2} \sum_{l=1}^{\lfloor t/2 \rfloor} \frac{H_{s-2l}(x) H_{t-2l}(y)}{h_{s-2l}} + 2 \sum_{l=\lfloor t/2 \rfloor + 1}^{\lfloor s/2 \rfloor} \frac{H_{s-2l}(x)}{h_{s-2l}} \int_y^\infty \frac{(z-y)^{2l-t-1} e^{-z^2}}{(2l-t-1)!} dz$$

We now consider

$$-\frac{1}{(s-t-1)!} (\text{sgn } x)^t \chi_{y<|x|} (|x| - y)^{s-t-1}$$

which for $x > 0$ is equal to the first term in (2.97). This can be expanded in terms of the Hermite polynomials

$$\begin{aligned}
&\frac{-1}{(s-t-1)!}(\mathrm{sgn}\ x)^t \chi_{y<|x|}(|x|-y)^{s-t-1} \\
=&\frac{-1}{(s-t-1)!}\sum_{j=0}^{\infty}\frac{H_j(x)}{h_j}\left[(-1)^t\int_{-\infty}^{y}e^{-z^2}H_j(z)(-z-y)^{t-s-1}dz+\int_{y}^{\infty}e^{-z^2}H_j(z)(z-y)^{t-s-1}dz\right] \\
=&\frac{-2}{(s-t-1)!}\sum_{j=0}^{\infty}\frac{H_{2j+\epsilon}(x)}{h_{2j+\epsilon}}\int_{y}^{\infty}e^{-z^2}(z-y)^{t-s-1}H_{2j+\epsilon}(z)dz \qquad (2.99)
\end{aligned}$$

where $\epsilon = 0, 1$ for t even, odd. Making use of (2.93) and integration by parts, (2.99) can be written

$$-2e^{-y^2}\sum_{l=-\infty}^{\lfloor t/2\rfloor}\frac{H_{s-2l}(x)H_{t-2l}(y)}{h_{s-2l}} - 2\sum_{l=\lfloor t/2\rfloor+1}^{\lfloor s/2\rfloor}\frac{H_{s-2l}(x)}{h_{s-2l}}\int_{y}^{\infty}\frac{(z-y)^{2l-t-1}e^{-z^2}}{(2l-t-1)!}dz$$

and so

$$-\frac{1}{(s-t-1)!}\chi_{y<x}(x-y)^{s-t-1}+\sum_{l=1}^{s}\Phi_l^s(x)\Psi_l^t(y) = -2e^{-y^2}\sum_{l=-\infty}^{0}\frac{H_{s-2l}(x)H_{t-2l}(y)}{h_{s-2l}} \qquad (2.100)$$

Since the form of K found in (2.98) and (2.100) differ from that in (2.91) only by the function $e^{(x^2-y^2)/2}$ which causes no change to the determinant in (2.90), the proof is complete. □

2.7 Scaling limits

Having now discussed the correlation functions for particle systems $\{x_i^{(j)}\}_{\substack{i=1,\dots,r(j)\\ j=1,\dots,N}}$ for fixed N, we move on to discussion of limits in cases where $N \to \infty$. In the simplest case of the particle density on a single line N in §2.2, we bypassed the explicit formula and instead used a log-gas argument to compute the leading support of the density profile, and its functional form for large N. This relied crucially on the inter-particle spacing going to zero as $N \to \infty$. However, this limit is not the most natural from a statistical mechanical viewpoint, where the so called thermodynamic limit requires that the inter-particle spacing be of order unity in the large N limit. This can be brought about by the linear change of variables

$$s_i = N + T_N t_i, \qquad x_i = M_{N,t_i} + \frac{X_i}{\sigma_N}$$

where T_N, M_{N,t_i} and σ_N are scale factors, and we would like to compute the scaled correlation functions

$$\begin{aligned}
&\bar{\rho}_{(r)}((X_1,t_1);\dots;(X_r,t_r)) = \qquad (2.101)\\
&\qquad \lim_{N\to\infty}\left(\frac{1}{\sigma_N}\right)^r \rho_{(r)}\left(\left(M_{N,t_i}+\frac{X_1}{\sigma_N}, N+T_N t_1\right);\dots;\left(M_{N,t_i}+\frac{X_r}{\sigma_N}, N+T_N t_r\right)\right)
\end{aligned}$$

The choices of T_N, M_{N,t_i} and σ_N depend on the different scaling regimes: the bulk or the edge. These must be distinguished because the (unscaled) inter-particle spacing is different in each. Furthermore, if the system is effectively confined by a wall (as in the half-hexagon for example), we must distinguish this edge (the so called hard edge) from the "free boundary" case (the so called soft edge).

In the bulk scaling limit, the particles are on lines a finite distance apart, with the interparticle spacing on the lines scaled to be unity. We require too that the particles be away from the boundary of support, (that they be 'in the bulk'), which we do by requiring them to be in the neighbourhood of the midpoint of the support (c_N, d_N). Thus, when evaluating the correlation function for particles in the bulk, we set the scale factors $T_N = 1$, $M_N = (c_N + d_N)/2$, and $\sigma_N = \rho(M_N, N)$.

In the soft edge scaling limit, the particles are in the neighbourhood of the largest particle on their line, and on lines of order $N^{2/3}$ apart. It is called soft edge because we will be centering the particles at the position of the largest particle to leading order, our upper bound of the area of support d_N, not the absolute maximum position. We generally require also that the inter-particle spacing is of order unity. Thus the scaling factors are a little less well defined, with $T_N = kN^{2/3}$ for some constant k, $M_N = d_{s_i}$ and σ_N differing depending on the particle system being looked at. In general the important thing is being able to find an asymptotic form of the orthogonal polynomials that make up the correlation functions when the particles are in the neighbourhood of the soft edge d_{s_i}.

The hard edge scaling limit (when applicable) is similar to the bulk scaling limit in that well defined limiting expressions apply when the particles are on lines a finite distance apart. Furthermore, the interparticle spacings on the lines have to be scaled so that they are order unity.

We will now evaluate both the bulk and soft edge scaled correlation functions for the GUE* eigenvalue process (there is no hard edge for this model).

Proposition 2.22. *Define pairs* (x_i, s_i) *by*

$$s_i = N + t_i, \qquad x_i = \frac{\pi X_i}{\sqrt{N}} \tag{2.102}$$

and let the scaled correlation function $\bar{\rho}_{\mathrm{Bulk},(r)}$ *be defined*

$$\bar{\rho}_{\mathrm{Bulk},(r)}\left((X_1, t_1); \dots; (X_r, t_r)\right) := \lim_{N\to\infty} \left(\frac{\pi}{\sqrt{N}}\right)^r \rho_{(r)}((x_1, s_1); \dots; (x_r, s_r)) \tag{2.103}$$

for $\rho_{(r)}$ *as in* (2.76)*. Then* $\bar{\rho}_{\mathrm{Bulk},(r)}$ *is given by*

$$\bar{\rho}_{\mathrm{Bulk},(r)}((X_1, t_1); \dots; (X_r, t_r)) = \det\left[\bar{K}_{\mathrm{Bulk}}(X_i, t_i; X_j, t_j)\right]_{i,j=1,\dots,r} \tag{2.104}$$

where

$$\bar{K}_{\mathrm{Bulk}}(Y, t; X, s) = \begin{cases} \displaystyle\int_0^1 v^{t-s} \cos\left(v\pi(X - Y) + \frac{\pi}{2}(t - s)\right) dv & s \le t \\ \displaystyle -\int_1^\infty v^{t-s} \cos\left(v\pi(X - Y) + \frac{\pi}{2}(t - s)\right) dv & s > t \end{cases} \tag{2.105}$$

Proof. Considering (2.102), (2.103) and (2.104) and recalling (2.59), to establish (2.105) it is enough to show that

$$\bar{K}_{\mathrm{Bulk}}(Y, t; X, s) = \lim_{N\to\infty} \frac{C_{s,X}}{C_{t,Y}} \frac{\pi}{\sqrt{N}} K\left(\frac{\pi Y}{\sqrt{N}}, N + t; \frac{\pi X}{\sqrt{N}}, N + s\right) \tag{2.106}$$

for K as in (2.77), where $C_{s,X}$ is some non-zero function of s and X. From (2.77),

$$K\left(\frac{\pi Y}{\sqrt{N}}, N + t; \frac{\pi X}{\sqrt{N}}, N + s\right) = \begin{cases} \displaystyle e^{-\frac{\pi^2(X^2+Y^2)}{4N}} \sum_{l=1}^{N+s} \frac{\tilde{H}_{N+s-l}\left(\frac{\pi X}{\sqrt{N}}\right) \tilde{H}_{N+t-l}\left(\frac{\pi Y}{\sqrt{N}}\right)}{\tilde{h}_{N+s-l}} & s \le t \\ \displaystyle -e^{-\frac{\pi^2(X^2+Y^2)}{4N}} \sum_{l=-\infty}^{0} \frac{\tilde{H}_{N+s-l}\left(\frac{\pi X}{\sqrt{N}}\right) \tilde{H}_{N+t-l}\left(\frac{\pi Y}{\sqrt{N}}\right)}{\tilde{h}_{N+s-l}} & s > t \end{cases} \tag{2.107}$$

We use the uniform asymptotic expansion (2.45) and a simple trigonometric identity to deduce that, with $z := (N-l)/N$,

$$e^{-\frac{\pi^2(X^2+Y^2)}{4N}} \frac{\tilde{H}_{N+s-l}\left(\frac{\pi X}{\sqrt{N}}\right)\tilde{H}_{N+t-l}\left(\frac{\pi Y}{\sqrt{N}}\right)}{\tilde{h}_{N+s-l}} = \frac{(zN)^{\frac{t-s-1}{2}}}{2\pi}\cos\left(\sqrt{z}(X-Y)\pi + \frac{\pi}{2}(t-s)\right) + \mathrm{O}(N^{-1/2})$$

With this change of variables, we recognize (2.107) as a Riemann sum and to leading order the RHS can be approximated by

$$\frac{N^{\frac{t-s+1}{2}}}{2\pi} \times \begin{cases} \int_0^1 z^{\frac{t-s-1}{2}}\cos\left(\sqrt{z}(X-Y)\pi + \frac{\pi}{2}(t-s)\right)dz & s \le t \\ -\int_1^\infty z^{\frac{t-s-1}{2}}\cos\left(\sqrt{z}(X-Y)\pi + \frac{\pi}{2}(t-s)\right)dz & s > t \end{cases}$$

Changing variables to $v = \sqrt{z}$ shows that (2.106) holds with $C_{s,X} = N^{s/2}$. □

Proposition 2.23. *Define pairs* (x_i, s_i)

$$s_i = N + 2N^{2/3}t_i, \qquad x_i = \sqrt{4s_i} + \frac{X_i}{N^{1/6}} \tag{2.108}$$

and let the scaled correlation function $\bar{\rho}_{\mathrm{SE},(r)}$ *be defined*

$$\bar{\rho}_{\mathrm{SE},(r)}((X_1,t_1);\dots;(X_r,t_r)) := \lim_{N\to\infty}\left(\frac{1}{N^{1/6}}\right)^r \rho_{(r)}\left((x_1,s_1);\dots;(x_r,s_r)\right) \tag{2.109}$$

for $\rho_{(r)}$ *as in* (2.76). *Then* $\bar{\rho}_{\mathrm{SE},(r)}$ *is of the form*

$$\bar{\rho}_{\mathrm{SE},(r)}((X_1,t_1);\dots;(X_r,t_r)) = \det\left[\bar{K}_{\mathrm{SE}}(X_i,t_i;X_j,t_j)\right]_{i,j=1,\dots,r} \tag{2.110}$$

where

$$\bar{K}_{\mathrm{SE}}(Y,t;X,s) = \begin{cases} \int_0^\infty e^{u(s-t)}\mathrm{Ai}(X+u)\mathrm{Ai}(Y+u)du & s \le t \\ -\int_{-\infty}^0 e^{u(s-t)}\mathrm{Ai}(X+u)\mathrm{Ai}(Y+u)du & s > t \end{cases} \tag{2.111}$$

Proof. Considering (2.108), (2.109) and (2.110) and recalling (2.59), to establish (2.111) it is enough to show that

$$\bar{K}_{\mathrm{SE}}(Y,t;X,s) = \lim_{N\to\infty}\frac{1}{N^{1/6}}\frac{D_{s,X}}{D_{t,Y}}K\left(\sqrt{4N_t} + \frac{Y}{N^{1/6}}, N_t; \sqrt{4N_s} + \frac{X}{N^{1/6}}, N_s\right) \tag{2.112}$$

for K is as in (2.77), where $N_s = N + 2N^{2/3}s$, and $D_{s,X}$ is some non-zero function of s and X. From (2.77)

$$K(y,N_t;x,N_s) = e^{-\frac{(x^2+y^2)}{4}}\begin{cases} \sum_{l=1}^{N_s}\frac{\tilde{H}_{N_s-l}(x)\tilde{H}_{N_t-l}(y)}{\tilde{h}_{N_s-l}} & s \le t \\ -\sum_{l=-\infty}^{0}\frac{\tilde{H}_{N_s-l}(x)\tilde{H}_{N_t-l}(y)}{\tilde{h}_{N_s-l}} & s > t \end{cases} \tag{2.113}$$

To proceed, we make use of the uniform large N expansion [57]

$$e^{-x^2/4}\tilde{H}_N(x) = (2\pi)^{1/4}(N!)^{1/2}N^{-1/12}\left(\mathrm{Ai}(X) + \mathrm{O}(N^{-2/3})\begin{cases}\mathrm{O}(e^{-X}), & X>0\\ \mathrm{O}(1), & X<0\end{cases}\right)$$

valid for $x = \sqrt{4N} + XN^{-1/6}$. This can be rewritten

$$e^{-x^2/4}\tilde{H}_{N-k}(x) = (2\pi)^{1/4}[(N-k)!]^{1/2}(N-k)^{-1/12} \\ \times \left(\mathrm{Ai}\left(X + \frac{k}{N^{1/3}}\right) + \mathrm{O}(N^{-2/3}) \begin{cases} \mathrm{O}(e^{-kN^{-1/3}}), & k \geq 0 \\ \mathrm{O}(1), & k < 0 \end{cases} \right)$$

Thus, with $x = \sqrt{4N_s} + XN^{-1/6}$, $y = \sqrt{4N_t} + YN^{-1/6}$,

$$e^{-(x^2+y^2)/4}\frac{\tilde{H}_{N_s-l}(x)\tilde{H}_{N_t-l}(y)}{\tilde{h}_{N_s-l}} = \sqrt{\frac{(N_t-l)!}{(N_s-l)!}}((N_s-l)(N_t-l))^{-1/12}$$

$$\times \left[\mathrm{Ai}\left(X\left(\frac{N_s}{N}\right)^{1/6} + \frac{l}{N_s^{1/3}} \right) \mathrm{Ai}\left(Y\left(\frac{N_t}{N}\right)^{1/6} + \frac{l}{N_t^{1/3}} \right) + \mathrm{O}(N^{-2/3}) \begin{cases} \mathrm{O}(e^{-kN^{-1/3}}), & k \geq 0 \\ \mathrm{O}(1), & k < 0 \end{cases} \right]$$

Using this, we make a change of variables $l = uN^{1/3}$. We now recognize (2.113) as a Riemann sum and so

$$\lim_{N\to\infty} \frac{1}{N^{1/6}}\frac{D_{s,X}}{D_{t,Y}} K\left(\sqrt{4N_t} + \frac{Y}{N^{1/6}}, N_t; \sqrt{4N_s} + \frac{X}{N^{1/6}}, N_s\right) =$$

$$\lim_{N\to\infty} \frac{D_{s,X}}{D_{t,Y}} \begin{cases} \displaystyle\int_0^\infty \sqrt{\frac{(N_t-l)!}{(N_s-l)!}}\mathrm{Ai}(X+u)\mathrm{Ai}(Y+u)du & s \leq t \\ \displaystyle -\int_{-\infty}^0 \sqrt{\frac{(N_t-l)!}{(N_s-l)!}}\mathrm{Ai}(X+u)\mathrm{Ai}(Y+u)du & s > t \end{cases}$$

Use of Stirling's formula

$$(aN + bN^{2/3} + cN^{1/3} + d)! = \sqrt{2\pi aN}(aN)^{aN+bN^{2/3}+cN^{1/3}+d}e^{-aN+\frac{b^2}{2a}N^{1/3}+\frac{bc}{a}} + \mathrm{O}(N^{-1/3}) \quad (2.114)$$

gives that

$$\sqrt{\frac{(N+2N^{2/3}t - uN^{1/3})!}{(N+2N^{2/3}s - uN^{1/3})!}} = N^{N^{2/3}(t-s)}e^{t^2-s^2}e^{u(s-t)} + \mathrm{O}(N^{-1/3})$$

and so (2.112) holds with $D_{s,X} = e^{s^2}N^{N^{2/3}s}$. □

We turn our attention to the hard edge scaling limit. As already commented, this applied in the neighbourhood of a hard wall boundary condition. This then is not relevant to the GUE* eigenvalue process, but will apply instead to the Antisymmetric GUE eigenvalue process, which has a strict condition that the particles be greater than 0 (since by definition the particles represent the positive eigenvalues of an Antisymmetric GUE submatrix). We will evaluate the hard edge correlation functions for the Antisymmetric GUE eigenvalue process. To do this, we first require an asymptotic formula for the Laguerre polynomials.

Theorem 2.24. *(Theorem 8.22.4 from [65]). Let $\{L_n^{(a)}\}_{n=0,1...}$ be the family of polynomials of order n obeying the orthogonality relationship*

$$\int_0^\infty e^{-x}x^a L_n^{(a)}(x)L_m^{(a)}(x)dx = \Gamma(a+1)\begin{pmatrix} n+a \\ n \end{pmatrix}\delta_{m,n} \quad (2.115)$$

then

$$e^{-x/2}x^{a/2}L_n^{(a)}(x) = \left(\frac{2n+a+1}{2}\right)^{-a/2}\frac{\Gamma(n+a+1)}{n!}J_a\left(\sqrt{2x(2n+a+1)}\right) + x^{5/4}\mathrm{O}(n^{a/2-3/4}) \quad (2.116)$$

where J_a is the Bessel function

$$J_a(x) = \sum_{v=1}^{\infty} \frac{(-1)^v (x/2)^{a+2v}}{v!\Gamma(v+a+1)} \tag{2.117}$$

Using this, and the knowledge that the Hermite polynomials can be exactly represented by the Laguerre polynomials, we proceed.

Proposition 2.25. *Define pairs (x_i, s_i) by*

$$s_i = 2N + t_i, \qquad x_i = \frac{X_i}{2\sqrt{N}} \tag{2.118}$$

for $X_i > 0$, and let the scaled correlation function $\bar{\rho}_{\mathrm{HE},(r)}$ be defined

$$\bar{\rho}_{\mathrm{HE},(r)}\left((X_1, t_1); \ldots; (X_r, t_r)\right) := \lim_{N\to\infty} \left(\frac{1}{2\sqrt{N}}\right)^r \rho_{(r)}((x_1, s_1); \ldots; (x_r, s_r)) \tag{2.119}$$

for $\rho_{(r)}$ as in (2.90)*. Then $\bar{\rho}_{\mathrm{HE},(r)}$ is given by*

$$\bar{\rho}_{\mathrm{HE},(r)}((X_1, t_1); \ldots; (X_r, t_r)) = \det\left[\bar{K}_{\mathrm{HE}}(X_i, t_i; X_j, t_j)\right]_{i,j=1,\ldots,r} \tag{2.120}$$

where

$$\bar{K}_{\mathrm{HE}}(Y, t; X, s) = \begin{cases} \dfrac{\sqrt{XY}}{2} \displaystyle\int_0^1 w^{\frac{t-s}{2}} J_{-1/2+\epsilon_s}(X\sqrt{w}) J_{-1/2+\epsilon_t}(Y\sqrt{w}) dw & s \le t \\ -\dfrac{\sqrt{XY}}{2} \displaystyle\int_1^\infty w^{\frac{t-s}{2}} J_{-1/2+\epsilon_s}(X\sqrt{w}) J_{-1/2+\epsilon_t}(Y\sqrt{w}) dw & s > t \end{cases} \tag{2.121}$$

Where $\epsilon_s = 1$ if s is odd, and 0 otherwise.

Proof. Considering (2.118), (2.119) and (2.120) and recalling (2.59), to establish (2.121) it is enough to show that

$$\bar{K}_{\mathrm{HE}}(Y, t; X, s) = \lim_{N\to\infty} P_s P_t \frac{E_{s,X}}{E_{t,Y}} \frac{1}{2\sqrt{N}} K\left(\frac{Y}{2\sqrt{N}}, 2N + t; \frac{X}{2\sqrt{N}}, 2N + s\right) \tag{2.122}$$

for K as in (2.91), where $E_{s,X}$ is some non-zero function of s and X and P_s is some non zero function of s obeying $P_s^2 = 1$. From (2.91),

$$K\left(\frac{Y}{2\sqrt{N}}, N + t; \frac{X}{2\sqrt{N}}, N + s\right) = \begin{cases} 2e^{-\frac{X^2+Y^2}{4N}} \displaystyle\sum_{l=1}^{\lfloor \frac{2N+s}{2} \rfloor} \frac{H_{2N+t-2l}\left(\frac{Y}{2\sqrt{N}}\right) H_{2N+s-2l}\left(\frac{X}{2\sqrt{N}}\right)}{h_{2N+s-2l}} & s \le t \\ -2e^{-\frac{X^2+Y^2}{4N}} \displaystyle\sum_{l=-\infty}^{0} \frac{H_{2N+t-2l}\left(\frac{Y}{2\sqrt{N}}\right) H_{2N+s-2l}\left(\frac{X}{2\sqrt{N}}\right)}{h_{2N+s-2l}} & s > t \end{cases} \tag{2.123}$$

From here, we use the fact that the Hermite polynomials can be reduced to Laguerre polynomials [65]

$$\begin{aligned} H_{2m}(x) &= (-1)^m 2^{2m} m! L_m^{(-1/2)}(x^2) \\ H_{2m+1}(x) &= (-1)^m 2^{2m+1} m! x L_m^{(1/2)}(x^2) \end{aligned}$$

Combining this with (2.116), and using Stirling's formula (1.16), we evaluate

$$\begin{aligned} \frac{e^{-(x^2+y^2)/2} H_{2N-2l+s}(x) H_{2N-2l+s}(y)}{h_{2N-2l+s}} &= (-1)^{\frac{s+t-\epsilon_s-\epsilon_t}{2}} 2^{t-s} \sqrt{xy} \\ &\times (N-l)^{t/2-s/2} J_{-1/2+\epsilon_s}\left(\sqrt{x^2(4N-4l)}\right) J_{-1/2+\epsilon_t}\left(\sqrt{y^2(4N-4l)}\right) \left(1 + \mathrm{O}(n^{-1/2})\right) \end{aligned}$$

Letting $w = (N-l)/N$, we recognize (2.123) as a Riemann sum, which can be approximated to leading order by

$$(-1)^{\frac{s+t-\epsilon_s-\epsilon_t}{2}} N^{\frac{t-s+1}{2}} 2^{t-s}\sqrt{XY} \begin{cases} \int_0^1 w^{\frac{t-s}{2}} J_{-1/2+\epsilon_s}(X\sqrt{w}) J_{-1/2+\epsilon_t}(Y\sqrt{w}) dw & s \le t \\ -\int_1^\infty w^{\frac{t-s}{2}} J_{-1/2+\epsilon_s}(X\sqrt{w}) J_{-1/2+\epsilon_t}(Y\sqrt{w}) dw & s > t \end{cases}$$

So (2.122) holds with $P_s = (-1)^{\frac{\epsilon_s - s}{2}}$, $E_{s,x} = (2\sqrt{N})^{t-s}$. □

We remark that the kernel $\bar{K}_{\rm Bulk}$ was first identified in the study of the so called bead model (see the next section), while $\bar{K}_{\rm SE}$ is the dynamical correlation for the scaled largest eigenvalues in the Brownian motion model of complex Hermitian matrices introduced in §1.7 (see [27, 11.7]). The kernerl $\bar{K}_{\rm HE}$ was first calculated in the context of the Antisymmetric GUE eigenvalue process in [29], although the expression given there involves trigonometric rather than Bessel functions. This is consistent with the facts that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x \tag{2.124}$$

3 Hexagon bead model

In [11], Boutillier introduces what he calls a bead model. He describes an infinite collection of parallel threads lying on the plane, and defines a bead configuration as a collection of points on these threads, which in turn can be interpreted as a configuration of points on $\mathbb{Z} \times \mathbb{R}$, with the restrictions that

- The configuratin must be locally finite : the number of beads in each finite interval of a thread must be finite
- The configuration must interlace : Between two consecutive beads on a thread, there must be exactly one bead on each neighbouring thread.

Boutillier's motivation was that these bead configurations have a direct mapping to certain dimer models on a honeycomb lattice, namely that the horizontal lines of a dimer model on a honeycomb lattice obey the above restrictions and so can be mapped to the 'beads'. We note that the eigenvalues of the GUE* that we considered earlier also obey the above conditions, however with some additional 'boundary conditions'. Where Boutillier's bead model is unbounded, in the GUE* eigenvalue process we generally consider only N lines, with j particles on the j-th line. Boutillier uses a dimer viewpoint because it is more efficient in the large N bulk viewpoint, as the bead model is in essence a description of the bulk scaling regime for the GUE* described in §2.7. As we began with the finite N case, we approach the problem of the bulk from a different direction, and as such have used a different method to recreate Boutillier's result.

We now consider a different set of boundary conditions on a bead configuration, motivated as in [11] by a statistical mechanical model. Just as the GUE* was a limit of the first n lines of an $N \times N \times N$ hexagon for large N, these new boundary conditions give rise to a model that is the limit of a complete $N \times p \times q$ hexagon for large N, and call this model a p-q hexagon bead model. It turns out that this model is equivalent to a semi-continuous lattice path model proposed in [2] to model the bus transportation system in Cuernavaca (Mexico). The analysis of the GUE* eigenvalue process, finding the single line PDF, density profile, correlation functions and scaled correlation functions, can also be done for this hexagon bead model, and while we will again expect the bulk calculation to match that of Boutillier's model, under other scaling regimes we will expect the boundary conditions to come into play.

Most of this chapter is based on the publication 'A finitization of the bead process' by Fleming, Forrester and Nordenstam [26] supplemented by the recent preprint 'Fluctuation universality for a class of directed solid-on-solid models' by Fleming and Forrester [25] for the material of §3.5 and §3.6.

3.1 Definition

We now wish to define these boundary conditions. Let $\{x_i^{(j)}\}_{\substack{i=1,\dots,r(j)\\ j=1,\dots,f(p,q)}}$ represent the i-th largest 'bead' on the j-th 'thread' of a p-q hexagon bead model. As we want this bead model to be a scaled limit of the particles in a $N \times p \times q$ hexagon, we must have the same number of beads as particles, and so recalling §1.3, $f(p,q) = p+q-1$ and

$$r(t) = \begin{cases} t, & t \le p \\ p, & p \le t \le q \\ p+q-t, & q \le t \end{cases} \tag{3.1}$$

As the bulk of this model must correspond to the Boutillier bead model, the only restriction on bulk particles is the interlacing requirement, so choosing a scale factor so that the beads must lie in the interval $[-1,1]$, we have the joint PDF for the $\{x_i^{(j)}\}_{\substack{i=1,\dots,r(j)\\ j=1,\dots,p+q-1}}$

$$p_{Hexbead}(\{x_i^{(j)}\}) = \frac{1}{C_{p,q}} \chi_{-1<x_p^{(p)}<x_1^{(q)}<1} \prod_{t=1}^{p+q-2} \chi\left(x^{(t)} \prec x^{(t+1)}\right) \tag{3.2}$$

for $\chi\left(x^{(t)} \prec x^{(t+1)}\right)$ the interlacing requirement defined in §6.1. While it is quite clear that this PDF is a scaled limit of the particles in a rhombus tiling of an $N \times p \times q$ hexagon, it is also possible to obtain it as the joint eigenvalue PDF of a sequence of random matrices.

Our construction is based on theory related to certain random corank 1 projections contained in [3, 28] which will now be revised. Let

$$M = \Pi A \Pi, \qquad \Pi = \mathbb{I} - \vec{x}\vec{x}^\dagger$$

where

$$A = \mathrm{diag}\Big((a_1)^{s_1}, (a_2)^{s_2}, \dots, (a_n)^{s_n}\Big).$$

Here the notation $(a)^p$ means a is repeated p times, and it is assumed $a_1 > a_2 > \cdots > a_n$, while $\vec{x}$ is a normalized complex Gaussian vector of the same number of rows as A. The eigenvalues a_i of A occur in M with multiplicity $s_i - 1$. Zero is also an eigenvalue of M. The remaining $n-1$ eigenvalues of M occur at the zeros of the random rational function

$$\sum_{i=1}^{n} \frac{q_i}{x - a_i}$$

where $(q_1, \dots, q_n)$ has the Dirichlet distribution $\mathrm{D}[s_1, \dots, s_n]$. With these $n-1$ eigenvalues denoted $\lambda_1 > \cdots > \lambda_{n-1}$, it follows from this latter fact that their joint distribution is equal to

$$\frac{\Gamma(s_1 + \cdots + s_n)}{\Gamma(s_1)\cdots\Gamma(s_n)} \chi(\lambda \prec a) \frac{\prod_{1\le j<k\le n-1}(\lambda_j - \lambda_k)}{\prod_{1\le j<k\le n}(a_j - a_k)^{s_j+s_k-1}} \prod_{j=1}^{n-1}\prod_{p=1}^{n} |\lambda_j - a_p|^{s_p - 1} \tag{3.3}$$

After this revision, we begin the construction by forming $M_1 = \Pi A_1 \Pi$, where $A_1 = \mathrm{diag}((1)^q, (-1)^p)$. It follows from the above that M_1 has one eigenvalue $\lambda_1^{(1)}$ different from -1 and 1 satisfying $-1 < \lambda_1^{(1)} < 1$, and this eigenvalue has PDF proportional to

$$(1 + \lambda_1^{(1)})^{p-1}(1 - \lambda_1^{(1)})^{q-1}$$

Now, for $r = 2, \dots, p$ inductively generate $\{\lambda_i^{(r)}\}_{i=1,\dots,r}$ as the eigenvalues different from -1 and 1 of the matrix

$$M_r = \Pi A_r \Pi$$

where

$$A_r = \mathrm{diag}\Big((1)^{q-r+1}, \lambda_1^{(r-1)}, \dots, \lambda_{r-1}^{(r-1)}, (-1)^{p-r+1}\Big)$$

It follows from (3.3) that the PDF of $\{\lambda_j^{(r)}\}_{j=1,\dots,r}$, for given $\{\lambda_j^{(r-1)}\}_{j=1,\dots,r-1}$, is proportional to

$$\chi(\lambda^{(r-1)} \prec \lambda^{(r)}) \frac{\prod_{i<j}^{r}(\lambda_i^{(r)} - \lambda_j^{(r)})}{\prod_{i<j}^{r-1}(\lambda_i^{(r-1)} - \lambda_j^{(r-1)})} \frac{\prod_{k=1}^{r}(1-\lambda_k^{(r)})^{q-r}(1+\lambda_k^{(r)})^{p-r}}{\prod_{k=1}^{r-1}(1-\lambda_k^{(r-1)})^{q-r+1}(1+\lambda_k^{(r-1)})^{p-r+1}} \tag{3.4}$$

Forming the product of (3.4) over $r = 1, \dots, p$ gives the joint PDF of $\cup_{s=1}^{p}\{\lambda_i^{(s)}\}$ which is therefore proportional to

$$\prod_{r=2}^{p} \chi(\lambda^{(r-1)} \prec \lambda^{(r)}) \prod_{i<j}^{p}(\lambda_i^{(p)} - \lambda_j^{(p)}) \prod_{k=1}^{p}(1-\lambda_k^{(q)})^{q-p}. \tag{3.5}$$

Next, for $r = 1, \dots, q-p$, inductively generate $\{\lambda_i^{(p+r)}\}_{i=1,\dots,p}$ as the eigenvalues different from 0 and 1 of

$$M_{p+r} = \Pi A_{p+r} \Pi$$

where

$$A_{p+r} = \text{diag}\Big((1)^{q-p-r+1}, \lambda_1^{(p+r-1)}, \dots, \lambda_p^{(p+r-1)}\Big).$$

We have from (3.3) that the PDF of $\{\lambda_j^{(p+r)}\}_{j=1,\dots,p}$, for given $\{\lambda_j^{(p+r-1)}\}_{j=1,\dots,p}$, is proportional to

$$\chi(\lambda^{(p+r-1)} \prec \lambda^{(p+r)} \cup \{-1\}) \frac{\prod_{i<j}^{p}(\lambda_i^{(p+r)} - \lambda_j^{(p+r)})}{\prod_{i<j}^{p}(\lambda_i^{(p+r-1)} - \lambda_j^{(p+r-1)})} \frac{\prod_{k=1}^{p}(1-\lambda_k^{(p+r)})^{q-p-r}}{\prod_{k=1}^{p}(1-\lambda_k^{(p+r-1)})^{q-r+1}} \tag{3.6}$$

The joint PDF of $\cup_{s=1}^{q}\{\lambda_j^{(s)}\}$ is obtained by multiplying the product of (3.6) over $r = 1, \dots, q-p$ by (3.5). It is therefore proportional to

$$\chi_{-1<\lambda_p^{(p)}} \prod_{r=2}^{q} \chi(\lambda^{(r-1)} \prec \lambda^{(r)}) \prod_{i<j}^{p}(\lambda_i^{(q)} - \lambda_j^{(q)}). \tag{3.7}$$

The final step is to inductively generate $\{\lambda_i^{(q+r)}\}_{i=1,\dots,p-r}$ $(r = 1, \dots, p-1)$ as the eigenvalue different from 0 of

$$M_{q+r} = \Pi A_{q+r} \Pi$$

where

$$A_{q+r} = \text{diag}\Big(\lambda_1^{(q+r-1)}, \dots, \lambda_p^{(q+r-1)}\Big)$$

According to (3.3), the PDF of $\{\lambda_j^{(q+r)}\}_{j=1,\dots,p-r}$ for given $\{\lambda_j^{(q+r-1)}\}_{j=1,\dots,p-r+1}$ is proportional to

$$\chi\left(\lambda^{(q+r-1)} \prec \lambda^{(q+r)} \cup \{-1,1\}\right) \frac{\prod_{i<j}^{p-r}(\lambda_i^{(q+r)} - \lambda_j^{(q+r)})}{\prod_{i<j}^{p-r-1}(\lambda_i^{(q+r-1)} - \lambda_j^{(q+r-1)})} \frac{\prod_{k=1}^{p-r}(1-\lambda_k^{(q+r)})^{p-q-r}}{\prod_{k=1}^{p-r-1}(1-\lambda_k^{(q+r-1)})^{q-r+1}}$$

Forming the product over $r = 1, \dots, p-1$ and multiplying by the conditional PDF (3.7) gives that the joint PDF of $\cup_{s=1}^{q+p-1}\{\lambda_j^{(s)}\}$ is proportional to

$$\chi_{-1<\lambda_p^{(p)}<\lambda_1^{(q)}<1} \prod_{r=2}^{q+p-1} \chi(\lambda^{(r-1)} \prec \lambda^{(r)})$$

and thus to the our bead model (3.2).

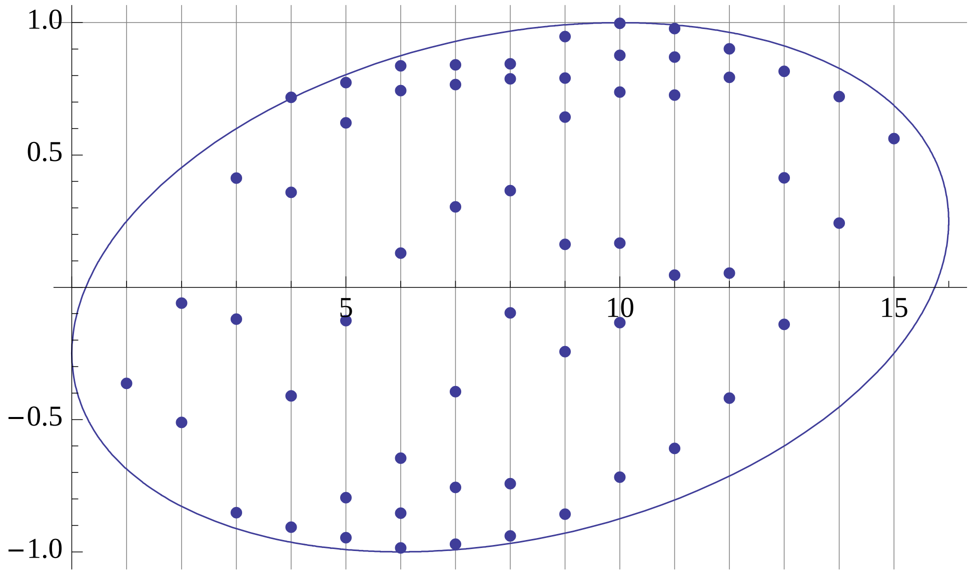


Figure 7: A configuration of the finite bead process with $p = 6$, $q = 10$, generated using random matrices as described in §3.1. The bounding curves are the limiting shape as described in Proposition 3.3. Comparing with Figure 1 allows us to better see the limiting relationship between tilings of the hexagon and the bead process.

3.2 Correlation functions

Recalling §2.1, we start by finding the correlation functions on a single line. To begin, we need to find the single line PDF

$$p_{(s)}(x^{(s)}) = \prod_{t\neq s}\prod_{i=1}^{r(t)}\int_{-\infty}^{\infty} dx_i^{(t)} p_{Hexbead}(\{x_i^{(j)}\})$$

which will give us a little more information about the general shape the particle model will take, as well as some other useful facts that we will get to later, so we go about finding this now. Using the results from (1.21) and Lemma 2.18, along with the identities

$$\int_{-1}^{1}\cdots\int_{-1}^{1}\prod_{i=1}^{p}(1+x_i)^m\Delta(x_1,\ldots,x_p)\det[\chi(x_i<y_j)]_{i,j=1,\ldots,p}dx_1\ldots dx_p$$
$$= \frac{m!}{(m+p)!}\prod_{i=1}^{p}(1+y_i)^{m+1}\Delta(y_1,\ldots,y_p)$$

$$\int_{-1}^{1}\cdots\int_{-1}^{1}\prod_{i=1}^{p}(1-x_i)^n\Delta(x_1,\ldots,x_p)\det[\chi(y_i<x_j)]_{i,j=1,\ldots,p}dx_1\ldots dx_p$$
$$= \frac{n!}{(n+p)!}\prod_{i=1}^{p}(1-y_i)^{n+1}\Delta(y_1,\ldots,y_p)$$

$$\int_{-1}^{1}\cdots\int_{-1}^{1}\prod_{i=1}^{r}(1+x_i)^m(1-x_i)^n\Delta(x_1,\ldots,x_r)\chi(x_1>y_1>x_2>\cdots>y_{r-1}>x_r)dx_1\ldots dx_r$$
$$= \frac{m!n!2^{m+n+1}}{(m+n+r)!}\prod_{i=1}^{r-1}(1+y_i)^{m+1}(1-y_i)^{n+1}\Delta(y_1,\ldots,y_{r-1})$$

we evaluate

$$C_{p,q} = 2^{pq} \prod_{i=0}^{p-1} i! \prod_{i=0}^{q-1} i! \prod_{i=0}^{q+p-1} \frac{1}{i!}$$

and see that the one line PDF can be represented

$$p_{(s)}(x^{(s)}) = \frac{1}{C_{p,q,s}} \Delta \left(x^{(s)}\right)^2 \prod_{i=1}^{r(s)} \left(1 - x_i^{(s)}\right)^{|q-s|} \left(1 + x_i^{(s)}\right)^{|p-s|} \tag{3.8}$$

where, for $r = r(s)$, $a = |q - s|$, $b = |p - s|$,

$$C_{p,q,s} = 2^{r(a+b+r)} \prod_{i=0}^{r-1} \frac{i!(a+i)!(b+i)!}{(a+b+r+i)!} \tag{3.9}$$

We note (3.8) is of the form (2.1) and so we can use the orthogonal polynomials associated with the weight function $(1-x)^a(1+x)^b$ to find check the normalization factor $C_{p,q,s}$ and find the single line correlation functions. To this end, we introduce the Jacobi polynomials

$$P_n^{(a,b)}(x) = \frac{(-1)^n}{2^n n!} \frac{1}{(1-x)^a(1+x)^b} \frac{d^n}{dx^n} \left((1-x)^{a+n}(1+x)^{b+n}\right) \tag{3.10}$$

These polynomials have orthogonality

$$\int_{-1}^{1} (1-x)^a(1+x)^b P_j^{(a,b)}(x) P_k^{(a,b)}(x) dx = \mathcal{N}_j^{(a,b)} \delta_{jk}$$

where

$$\mathcal{N}_n^{(a,b)} = \frac{2^{a+b+1}}{2n+a+b+1} \frac{(n+a)!(n+b)!}{n!(n+a+b)!} \tag{3.11}$$

These polynomials are not in fact monic, but have leading coefficient

$$A_n^{(a,b)} = \frac{(2n+a+b)!}{2^n n!(n+a+b)!}$$

Thus, with a, b, r, defined as in (3.9), by Lemma 2.1 we must have

$$C_{p,q,s} = \prod_{i=0}^{r-1} \frac{N_i^{(a,b)}}{\left(A_i^{(a,b)}\right)^2}$$

which is indeed true for $C_{p,q,s}$ as in (3.9).

By applying Proposition 2.5, we obtain the correlation function for the N particles ($N = r(s)$) with PDF (3.8) on line s of the p, q hexagon bead model

$$\rho_{(r)}(y_1, \ldots, y_r) = \det \left[K_N(y_i, y_j)\right]_{i,j=1,\ldots,r} \tag{3.12}$$

for

$$K(x,y) = ((1-x)(1-y))^{\frac{|q-s|}{2}} ((1+x)(1+y))^{\frac{|p-s|}{2}} \sum_{i=0}^{N-1} \frac{P_i^{(|q-s|,|p-s|)}(x) P_i^{(|q-s|,|p-s|)}(y)}{\mathcal{N}_i^{(|q-s|,|p-s|)}} \tag{3.13}$$

To find the correlation functions for particles on differing lines however, we will use Borodin's Theorem 2.9, as we did with the non-intersecting Brownian motions in §2.3 and the eigenvalues of the GUE* in §2.5. There is a difficulty however - neither Theorem 2.10 nor Theorem 2.17 can be used, as the hexagon bead model has neither the same number of particles on each line or a differing number of particles on each line, it is a combination of the two. Its outside regions resemble the GUE* eigenvalue process, however there is also a middle section that contains the same number of particles on each line. It is this fact that adds the most difficulty to finding the correlation functions of the hexagon bead model.

While (3.2) cannot be expressed in the form (2.49) or (2.73), it can still be expressed in determinantal form. Using Lemma 2.18

$$p_{Hexbead}(\{x_i^{(j)}\}) = \frac{1}{C_{p,q}} \prod_{t=0}^{q-1} \det[\chi(x_i^{(t)} < x_j^{(t+1)})]_{i,j=1}^{r(t+1)} \times \prod_{t=q}^{q+p-1} \det[\chi(x_i^{(t)} < x_{j-1}^{(t+1)})]_{i,j=1}^{r(t)}, \quad (3.14)$$

where we have introduced virtual particles

$$x_{t+1}^{(t)} = -1 \quad (t = 0, \ldots, p-1) \qquad x_0^{(t)} = 1 \quad (t = q+1, \ldots, q+p)$$

Following Theorem 2.9, we introduce $p+q-1$ discretisations $\mathcal{M}_1 \ldots, \mathcal{M}_{p+q-1}$ of the interval $(-1, 1)$, weighted by the spacing of the lattice, such that all the particles are confined to the lattice points of the discretization. Then, with $\mathcal{M} := \{1, \ldots, p\} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_{p+q-1}$, we must construct some $|\mathcal{M}| \times |\mathcal{M}|$ matrix L satisfying (2.48).

Similar to our B matrix in Theorem 2.17, we construct matrices B and C, with

$$\begin{aligned} B &= [B_1, \ldots, B_{p+q-1}] = [E_1, \ldots, E_p, 0, \ldots, 0] \\ C &= [C_1, \ldots, C_{p+q-1}]^T = [0, \ldots, 0, F_p \ldots, F_1] \end{aligned} \quad (3.15)$$

where

$$\begin{aligned} E_k &= [\delta_{i=k}\chi(-1 < x_j)]_{\substack{i=1,\ldots,p \\ x_j \in \mathcal{M}_k}} \\ F_k &= [\delta_{j=k}\chi(x_i < 1)]_{\substack{x_i \in \mathcal{M}_{p+q-k} \\ j=1,\ldots,p}} \end{aligned}$$

We also construct D as in (2.54)

$$D = I - \left[\delta_{i+1,j}[W_i(x,y)]_{\substack{x \in \mathcal{M}_i \\ y \in \mathcal{M}_{i+1}}}\right]_{i,j=1,\ldots,p+q-1} \quad (3.16)$$

with $W_i(x,y) = \chi(x < y)$ for all i so that the $|\mathcal{M}| \times |\mathcal{M}|$ matrix

$$L = \begin{bmatrix} 0_{p\times p} & B \\ C & D - I \end{bmatrix}$$

satisfies (2.48), so by Theorem 2.9 and (2.56), the correlation functions are given by

$$\rho(Y) = \det K_Y, \qquad K = I - D^{-1} + D^{-1}C\left(BD^{-1}C\right)^{-1} BD^{-1}$$

Proposition 3.1. *With matrices B, C and D defined as above, and $M := BD^{-1}C$, the matrix K given by*

$$K = I - D^{-1} + D^{-1}CM^{-1}BD^{-1} \quad (3.17)$$

is defined by

$$(K)_{y,t;x,s} = \frac{-(x-y)^{s-t-1}}{(s-t-1)!}\chi_{x>y} + \sum_{l=1}^{p} \left(\Psi^t\right)_{y,l} \left(\Phi^s\right)_{l,x} \quad (3.18)$$

where

$$(\Phi^s)_{l,x} = \begin{cases} 2^{p-s}\mathcal{N}_{p-l}^{(q-p,0)}\dfrac{(p-l)!}{(s-l)!}\dfrac{P_{s-l}^{(q-s,p-s)}(x)}{\mathcal{N}_{s-l}^{(q-s,p-s)}} & s \le p \\ \dfrac{(p-l)!}{(s-l)!}(1+x)^{s-p}P_{p-l}^{(q-s,s-p)}(x) & p < s \le q \\ (-2)^{q-s}\dfrac{(p+q-s-l)!}{(q-l)!}(1+x)^{s-p}(1-x)^{s-q}P_{p+q-s-l}^{(s-q,s-p)}(x) & q < s,\ l \le p+q-s \\ \dfrac{(-2)^{l-p}}{(q-l)!}\displaystyle\int_{-1}^{x}\dfrac{(x-z)^{l+s-p-q-1}}{(l+s-p-q-1)!}(1+z)^{q-l}(1-z)^{p-l}dz & q < s,\ l > p+q-s \end{cases} \tag{3.19}$$

$$\left(\Psi^t\right)_{y,l} = \begin{cases} \dfrac{2^{l-p}}{\mathcal{N}_{p-l}^{(0,q-p)}}\dfrac{1}{(p-l)!}\displaystyle\int_{y}^{1}\dfrac{(z-y)^{l-t-1}}{(l-t-1)!}(1-z)^{q-l}(1+z)^{p-l}dz & t < p,\ l > t \\ \dfrac{2^{t-p}}{\mathcal{N}_{p-l}^{(0,q-p)}}\dfrac{(t-l)!}{(p-l)!}(1-y)^{q-t}(1+y)^{p-t}P_{t-l}^{(q-t,p-t)}(y) & t < p,\ l \le t \\ \dfrac{1}{\mathcal{N}_{p-l}^{(0,q-p)}}\dfrac{(q-l)!}{(p+q-t-l)!}(1-y)^{q-t}P_{p-l}^{(q-t,t-p)}(y) & p \le t < q \\ (-2)^{t-q}\dfrac{(q-l)!}{(p+q-t-l)!}\dfrac{P_{p+q-t-l}^{(t-q,t-p)}(y)}{\mathcal{N}_{p+q-t-l}^{(t-q,t-p)}} & q \le t \end{cases} \tag{3.20}$$

for $P_n^{(a,b)}(x)$ *the Jacobi polynomials defined in* (3.10)

Proof. From the definition of D (3.16), we evaluate

$$D^{-1} = I + \left[[W_{[i,j)}(x,y)]_{\substack{x\in\mathcal{M}_i \\ y\in\mathcal{M}_j}}\right]_{i,j=1,\ldots,p}$$

where $W_{[i,j)}$ is as in (2.78). We now compute that the s-th block of the block row vector BD^{-1} is equal to

$$E_s + \sum_{i=1}^{s-1} E_i W_{[i,s)} \quad \text{for} \quad 1 \le s \le p \tag{3.21}$$

$$\sum_{i=1}^{p} E_i W_{[i,s)} \quad \text{for} \quad p < s \le p+q-1 \tag{3.22}$$

Similarly the t-th block of the block column vector $D^{-1}C$ is equal to

$$\sum_{i=1}^{p} W_{[t,p+q-i)}F_i \quad \text{for} \quad 1 \le t < q \tag{3.23}$$

$$F_{p+q-t} + \sum_{i=1}^{p+q-t-1} W_{[t,p+q-i)}F_i \quad \text{for} \quad q \le t \le p+q-1 \tag{3.24}$$

Evaluating the m,x entry of (3.21) gives

$$\begin{aligned} \left(E_s + \sum_{i=1}^{s-1} E_i W_{[i,s)}\right)_{m,x} &= \left(W_{m-1} * W_{[m,s)}\right)(-1,x) \\ &= \frac{(1+x)^{s-m}}{(s-m)!} \end{aligned} \tag{3.25}$$

where the convention $1/a! = 0$ for $a \in \mathbb{Z}_{<0}$ is to be used when $m > s$. Recalling that the single line PDFs had a Jacobi weight factor $(1-x)^{|q-s|}(1+x)^{|p-s|}$, we express (3.25) as a sum of the polynomials (3.10), with $a = q - s$, $b = p - s$, giving

$$\begin{aligned}\frac{(1+x)^{s-m}}{(s-m)!} &= \frac{1}{(s-m)!}\sum_{l=m}^{s}\frac{P_{s-l}^{(q-s,p-s)}(x)}{\mathcal{N}_{s-l}^{(q-s,p-s)}}\int_{-1}^{1}(1+z)^{p-m}(1-z)^{q-s}P_{s-l}^{(q-s,p-s)}(z)dz \\ &= \sum_{l=1}^{s}\frac{P_{s-l}^{(q-s,p-s)}(x)}{\mathcal{N}_{s-l}^{(q-s,p-s)}}\int_{-1}^{1}\frac{(1-z)^{q-l}(1+z)^{p-m}dz}{2^{s-l}(s-l)!(l-m)!}\end{aligned}$$

where the second equality follows from using (3.10) and integration by parts $s - l$ times. If we define B_0 and Φ^s $(s \leq p)$ by

$$\begin{aligned}(B_0)_{m,l} &= \frac{1}{2^{p-l}(p-l)!(l-m)!\mathcal{N}_{p-l}^{(q-p,0)}}\int_{-1}^{1}(1-z)^{q-l}(1+z)^{p-m}dz \\ (\Phi^s)_{l,x} &= \frac{2^{p-l}(p-l)!}{2^{s-l}(s-l)!}\frac{\mathcal{N}_{p-l}^{(q-p,0)}}{\mathcal{N}_{s-l}^{(q-s,p-s)}}P_{s-l}^{(q-s,p-s)}(x)\end{aligned}$$

then

$$\frac{(1+x)^{s-m}}{(s-m)!} = \sum_{l=1}^{p}(B_0)_{m,l}(\Phi^s)_{l,x} \tag{3.26}$$

and therefore

$$E_s + \sum_{i=1}^{s-1}E_iW_{[i,s)} = B_0\Phi^s$$

To find Φ^{p+j}, we must consider the m, x entry of (3.22),

$$\left(\sum_{i=1}^{p}E_iW_{[i,p+j)}\right)_{m,x} = \frac{(1+x)^{p+j-m}}{(p+j-m)!}$$

We begin with the $s = p$ case of (3.26)

$$\frac{(1+x)^{p-m}}{(p-m)!} = \sum_{l=1}^{p}(B_0)_{m,l}P_{p-l}^{(q-p,0)}(x) \tag{3.27}$$

and introduce the operator $J[f(x)] = \int_{-1}^{x} f(z)dz$, noting that

$$J^j[f(x)] = \frac{1}{(j-1)!}\int_{-1}^{x}(x-z)^{j-1}f(z)dz$$

Applying J^j to both sides of (3.27) gives

$$\frac{(1+x)^{p-m+j}}{(p-m+j)!} = \sum_{l=1}^{p}\frac{(B_0)_{m,l}}{(j-1)!}\int_{-1}^{x}(x-z)^{j-1}P_{p-l}^{(q-p,0)}(z)dz \tag{3.28}$$

Thus, defining Φ^s for $s > p$ by

$$\left(\Phi^{p+j}\right)_{l,x} = \frac{1}{(j-1)!}\int_{-1}^{x}(x-z)^{j-1}P_{p-l}^{(q-p,0)}(z)dz \tag{3.29}$$

gives us our desired result, $BD^{-1} = B_0\Phi$.

We now set about finding C_0, recalling the requirement that $B_0C_0 = M(:= BD^{-1}C)$. Using (3.15) and (3.22) we have

$$M = \sum_{j=1}^{p}\left(\sum_{i=1}^{p}E_iW_{[i,p+q-j)}\right)F_j$$

so

$$(M)_{m,n} = \frac{2^{p+q-n-m+1}}{(p+q-n-m+1)!}$$

This is equal to the LHS of (3.28) with $j = q+1-n$ and $x = 1$, so

$$(M)_{m,n} = \sum_{l=1}^{p} \frac{(B_0)_{m,l}}{(q-n)!} \int_{-1}^{1} (1-z)^{q-n} P_{p-l}^{(q-p,0)}(z) dz$$

Using integration by parts this becomes

$$(M)_{m,n} = \sum_{l=1}^{p} \left(\frac{-1}{2}\right)^{p-l} \frac{(B_0)_{m,l}\,(p-n)!}{(q-n)!(p-l)!(l-n)!} \int_{-1}^{1} (1-z)^{q-n}(1+z)^{p-l} dz$$

Thus, we set

$$(C_0)_{l,n} = \left(\frac{-1}{2}\right)^{p-l} \frac{(p-n)!}{(q-n)!(p-l)!(l-n)!} \int_{-1}^{1} (1-z)^{q-n}(1+z)^{p-l} dz$$

so that $B_0 C_0 = M$ as required.

Finally, we evaluate the y, n entry of (3.24)

$$\begin{aligned}\left(F_{p+q-t} + \sum_{i=1}^{p+q-t-1} W_{[t,p+q-i)} F_i\right)_{y,n} &= \left(W_{[t,p+q-n)} * W_{p+q-n}\right)(y,1) \\ &= \frac{(1-y)^{p+q-t-n}}{(p+q-t-n)!}\end{aligned}$$

Expressing this as a sum of Jacobi polynomials, for $a = t-q$, $b = t-p$ gives

$$\begin{aligned}\frac{(1-y)^{p+q-t-n}}{(p+q-t-n)!} &= \frac{1}{(p+q-t-n)!} \sum_{l=n}^{p+q-t} \frac{P_{p+q-t-l}^{(t-q,t-p)}(y)}{\mathcal{N}_{p+q-t-l}^{(t-q,t-p)}} \\ &\quad \times \int_{-1}^{1} (1-z)^{p-n}(1+z)^{t-p} P_{p+q-t-l}^{(t-q,t-p)}(z) dz \\ &= \sum_{l=n}^{p+q-t} \left(\frac{-1}{2}\right)^{p+q-t-l} \frac{P_{p+q-t-l}^{(t-q,t-p)}(y)}{\mathcal{N}_{p+q-t-l}^{(t-q,t-p)}} \int_{-1}^{1} \frac{(1-z)^{p-n}(1+z)^{q-l}}{(l-n)!} dz \\ &= \sum_{l=1}^{p} \left(\frac{-1}{2}\right)^{q-t} \frac{(C_0)_{l,n}\,(q-l)!}{(p+q-t-l)!} \frac{P_{p+q-t-l}^{(t-q,t-p)}(y)}{\mathcal{N}_{p+q-t-l}^{(t-q,t-p)}}\end{aligned} \tag{3.30}$$

Thus, we define Ψ^t for $q \leq t$ by

$$\left(\Psi^t\right)_{y,l} = \left(\frac{-1}{2}\right)^{q-t} \frac{(q-l)!}{(p+q-t-l)!} \frac{P_{p+q-t-l}^{(t-q,t-p)}(y)}{\mathcal{N}_{p+q-t-l}^{(t-q,t-p)}}$$

To find Ψ^{q-k}, we must consider the y, n entry of (3.23)

$$\left(\sum_{i=1}^{p} W_{[q-k,p+q-i)} F_i\right)_{y,n} = \frac{(1-y)^{p+k-n}}{(p+k-n)!}$$

We begin with the $t = q$ case of (3.30)

$$\frac{(1-y)^{p-n}}{(p-n)!} = \sum_{l=1}^{p} (C_0)_{l,n} \frac{(q-l)!}{(p-l)!} \frac{P_{p-l}^{(0,q-p)}(y)}{\mathcal{N}_{p-l}^{(0,q-p)}} \tag{3.31}$$

and introduce the operator $K\left[g(y)\right] = \int_y^1 g(z)dz$, noting that

$$K^k\left[g(y)\right] = \frac{1}{(k-1)!}\int_y^1 (z-y)^{k-1} g(z)dz$$

Applying K^k to both sides of (3.31) gives

$$\frac{(1-y)^{p-n+k}}{(p-n+k)!} = \frac{(q-l)!}{(p-l)!}\sum_{l=1}^{p}(C_0)_{l,n}\int_y^1 \frac{(z-y)^{k-1}}{(k-1)!}\frac{P_{p-l}^{(0,q-p)}(z)}{\mathcal{N}_{p-l}^{(0,q-p)}}dz$$

Thus, defining Ψ^t for $t < q$ by

$$\left(\Psi^{q-k}\right)_{y,l} = \frac{(q-l)!}{(p-l)!}\int_y^1 \frac{(z-y)^{k-1}}{(k-1)!}\frac{P_{p-l}^{(0,q-p)}(z)}{\mathcal{N}_{p-l}^{(0,q-p)}}dz \tag{3.32}$$

gives our desired result, $D^{-1}C = \Psi C_0$, and therefore $D^{-1}CM^{-1}BD^{-1} = \Psi\Phi$. Applying the identities

$$\frac{d}{dx}\left((1-x)^a P_n^{(a,b)}(x)\right) = -(n+a)(1-x)^{a-1}P_n^{(a-1,b+1)}(x) \tag{3.33}$$

$$\frac{d}{dx}\left((1+x)^b P_n^{a,b)}(x)\right) = (n+b)(1+x)^{b-1}P_n^{(a+1,b-1)}(x) \tag{3.34}$$

along with (3.10) to (3.29) and (3.32) gives the forms in (3.19) and (3.20). □

Proposition 3.2. *For a system of particles* $\mathcal{X} = \{x_i^{(j)}\}_{\substack{i=1,\dots,r(j)\\ j=1,\dots p+q-1}}$ *with PDF as in* (3.14)*, the correlation function for a subset* $Y \subset \mathcal{X}$ *where* $Y = \bigcup_{i=1}^{n} y_i^{(t_i)}$ *is represented by*

$$\rho_{(n)}((y_1,t_1),\dots,(y_n,t_n)) = \det\left[K(y_i,t_i;y_j,t_j)\right]_{i,j=1,\dots,n} \tag{3.35}$$

where

$$K(y,t;x,s) = \begin{cases} (1-y)^{|q-t|}(1+x)^{|p-s|}\displaystyle\sum_{l=1}^{p}\frac{A(t,l)}{A(s,l)}\frac{P_{r(s)-l}^{(|q-s|,|p-s|)}(x)P_{r(t)-l}^{(|q-t|,|p-t|)}(y)}{\mathcal{N}_{r(s)-l}^{(|q-s|,|p-s|)}} & s \le t \\ -(1-y)^{|q-t|}(1+x)^{|p-s|}\displaystyle\sum_{l=-\infty}^{0}\frac{A(t,l)}{A(s,l)}\frac{P_{r(s)-l}^{(|q-s|,|p-s|)}(x)P_{r(t)-l}^{(|q-t|,|p-t|)}(y)}{\mathcal{N}_{r(s)-l}^{(|q-s|,|p-s|)}} & s > t \end{cases} \tag{3.36}$$

for

$$A(s,l) = \begin{cases} 2^s(s-l)! & s \le p \\ 2^p\dfrac{(p-l)!(q-l)!}{(p+q-s-l)!} & p < s \le q \\ (-1)^{q+s}2^{p+q-s}(s-l)! & q < s \end{cases} \tag{3.37}$$

and $r(s)$ *as in* (3.1)

Proof. From Theorem 2.9 and Proposition 3.1, we have that, with the matrix K as in (3.18),

$$\rho_{(n)}(y_1^{(t_1)},\dots,y_n^{(t_n)}) = \det\left[(K)_{y_i,t_i;y_j,t_j}\right]_{i,j=1,\dots,n}$$

Our aim is to show that, with $(K)_{y,t;x,s}$ as in (3.18) and $K(y,t;x,s)$ as above,

$$(K)_{y,t;x,s} = g(y,t)h(x,s)K(y,t;x,s)$$

where

$$g(y,t) = (1-y)^{(q-t)/2-|q-t|/2}(1+y)^{(p-t)/2+|p-t|/2}$$
$$h(x,s) = (1-x)^{(s-q)/2+|s-q|/2}(1+x)^{(s-p)/2-|s-p|/2}$$

since the fact that

$$\prod_{i=1}^{n} g(y_i,t_i)h(y_i,t_i) = 1$$

implies that

$$\det\left[g(y_i,t_i)h(y_j,t_j)K(y_i,t_i;y_j,t_j)\right]_{i,j=1,\dots,n} = \det\left[K(y_i,t_i;y_j,t_j)\right]_{i,j=1,\dots,n}$$

So, our task is to show that

$$g(y,t)h(x,s)K(y,t;x,s) = \sum_{l=1}^{p} \left(\Psi^t\right)_{y,l} \left(\Phi^s\right)_{l,x} - \frac{(x-y)^{s-t-1}}{(s-t-1)!}\chi_{x>y} \tag{3.38}$$

For $s \le t$, this follows directly from the definitions (3.19), (3.20) and (3.36). For $s > t$ there are six possible cases to go through

- $t < s \le p$
- $t \le p < s \le q$
- $t \le p < q < s$
- $p < t < s \le q$
- $p < t \le q < s$
- $q < t < s$

In each of the cases, we expand the right most term in (3.38) in terms of Jacobi polynomials

$$\frac{(x-y)^{s-t-1}}{(s-t-1)!}\chi_{x>y} = \sum_{l=-\infty}^{f(s)} c_{f(s)-l}(y) \frac{P_{f(s)-l}^{(|q-s|,|p-s|)}(x)}{\mathcal{N}_{f(s)-l}^{(|q-s|,|p-s|)}} (1-x)^{(s-q)\chi_{s>q}}(1+x)^{(s-p)\chi_{s>p}} \tag{3.39}$$

We then use the orthogonality of the Jacobi polynomials, as well as integration by parts and the identities (3.10), (3.33), (3.34) and

$$\frac{d}{dx}\left(P_n^{(a,b)}(x)\right) = \frac{1}{2}(n+a+b+1)P_{n-1}^{(a+1,b+1)}(x)$$

to evaluate $c_{f(s)-l}(y)$, and then compare with the definitions (3.19) and (3.20) to show that (3.38). As an example, we will demonstrate in the $t < s \le p$ case. Then we evaluate

$$c_{s-l}(y) = \begin{cases} (1-y)^{q-t}(1+y)^{p-t}\displaystyle\sum_{l=1}^{t} 2^{t-s}\frac{(t-l)!}{(s-l)!}P_{t-l}^{(q-t,p-t)}(y) & l \le t \\ \dfrac{1}{2^{s-l}(s-l)!}\displaystyle\int_y^1 \frac{(z-y)^{l-t-1}}{(l-t-1)!}(1-z)^{q-l}(1+z)^{p-l}dz & l > t \end{cases} \tag{3.40}$$

From Proposition 3.1,

$$\begin{aligned} \sum_{l=1}^{p}\left(\Psi^t\right)_{y,l}\left(\Phi^s\right)_{l,x} &= (1-y)^{q-t}(1+y)^{p-t}\sum_{l=1}^{t} 2^{t-s}\frac{(t-l)!}{(s-l)!}\frac{P_{t-l}^{(q-t,p-t)}(y)P_{s-l}^{(q-s,p-s)}(x)}{\mathcal{N}_{s-l}^{(q-s,p-s)}} \\ &+ \sum_{l=t+1}^{s} 2^{l-s}\frac{P_{s-l}^{(q-s,p-s)}(x)}{\mathcal{N}_{s-l}^{(q-s,p-s)}}\frac{1}{(s-l)!}\int_y^1 \frac{(z-y)^{l-t-1}}{(l-t-1)!}(1-z)^{q-l}(1+z)^{p-l}dz \end{aligned} \tag{3.41}$$

and so comparing (3.39) with $c_{s-l}(y)$ as in (3.40) to (3.41) gives (3.38). □

By comparing (3.36) to (3.13), we see that, if $t_1 = t_2 = \cdots = t_n = s$, $\rho_{(n)}((y_1,t_1);\dots;(y_n,t_n))$ is as evaluated in (3.12), as expected.

3.3 Region of support and density profile

As we did with the GUE* eigenvalue process in §2.2, we will use a log-gas method to find the region of support and density of the particles in the hexagon bead model as the number of lines grows large. Specifically, we set $q = kp$ for some factor $k \geq 1$, and work in the limit $p \to \infty$. Noting the form of the single line PDF (3.8), we consider a log-gas with N_p particles and Boltzmann factor as in (2.33) with $\beta = 2$ and

$$V(t) = \frac{-N_p}{2} \log_e \left((1-t)^a (1+t)^b \right) \tag{3.42}$$

To evaluate a region of support (c, d), we must solve (2.41), (2.42) for c and d. (2.41) gives

$$\int_c^d \frac{a}{1-t} \frac{dt}{\sqrt{(d-t)(t-c)}} = \int_c^d \frac{b}{1+t} \frac{dt}{\sqrt{(d-t)(t-c)}}$$

while (2.42) gives

$$\int_c^d t \left(\frac{a}{1-t} - \frac{b}{1+t} \right) \frac{dt}{\sqrt{(d-t)(t-c)}} = 2\pi$$

Using these, along with the identity

$$\int_c^d \frac{dt}{\sqrt{(d-t)(t-c)}} = \pi$$

gives

$$\int_c^d \frac{a}{1-t} \frac{dt}{\sqrt{(d-t)(t-c)}} = \int_c^d \frac{b}{1+t} \frac{dt}{\sqrt{(d-t)(t-c)}} = \frac{(a+b+2)\pi}{2} \tag{3.43}$$

Using the identity

$$\int \frac{dx}{x\sqrt{Ax^2 + Bx + C}} = \frac{-1}{\sqrt{C}} \operatorname{arcsinh} \left(\frac{Bx + 2C}{|x|\sqrt{4AC - B^2}} \right) + \text{Constant}$$

we find that, for $c, d \in (-1, 1)$,

$$\int_c^d \frac{1}{1+t} \frac{dt}{\sqrt{(d-t)(t-c)}} = \frac{\pi}{\sqrt{(1+c)(1+d)}} \tag{3.44}$$

$$\int_c^d \frac{1}{1-t} \frac{dt}{\sqrt{(d-t)(t-c)}} = \frac{\pi}{\sqrt{(1-c)(1-d)}} \tag{3.45}$$

and using these along with (3.43), we have

$$c + d = \frac{2(a^2 - b^2)}{(a+b+2)^2} \tag{3.46}$$

$$cd = \frac{2(a^2 + b^2)}{(a+b+2)^2} - 1 \tag{3.47}$$

Proposition 3.3. *Let $x_i^{(pS)}$ represent the i-th largest particle on the pS-th line in a p-q hexagon bead model, with $q = kp$. In the limit $p \to \infty$, to leading order,*

$$\left| x_i^{(pS)} - \frac{(1-k)(k+1-2S)}{(1+k)^2} \right| < \frac{4\sqrt{kS(1+k-S)}}{(1+k)^2} \tag{3.48}$$

for all $i = 1, \ldots, N$. Furthermore, the density of the eigenvalues is given by

$$\rho_{Hexbead}(y) = \frac{p(k+1)\sqrt{(d-y)(y-c)}}{2\pi(1-y^2)} \tag{3.49}$$

where c and d are given by

$$c = \frac{(1-k)(1+k-2S) - 4\sqrt{kS(1+k-S)}}{(1+k)^2} \tag{3.50}$$

$$d = \frac{(1-k)(1+k-2S) + 4\sqrt{kS(1+k-S)}}{(1+k)^2} \tag{3.51}$$

Proof. We consider a log-gas of the form (2.33) with V as in (3.42) and with $a = |kp - Sp|/N_p$, $b = |p - Sp|/N_p$, where $N_p = r(Sp)$ for r as in (3.1). Then this log-gas is a representation of the single line PDF for the particles $x_i^{(pS)}$, (see (3.8)), and so the solutions to (3.44) and (3.45) will be the leading order region of support for these particles. We have three cases to consider:

- If $S < 1$, then $N_p = pS$, and so $a = (k - S)/S$ and $b = (1 - S)/S$.
- If $1 \leq S < k$, then $N_p = p$, and so $a = k - S$ and $b = S - 1$.
- If $k \leq S < k + 1$, then $N_p = (1 + k - S)p$, and so $a = (S - k)/(1 + k - S)$ and $b = (S - 1)/(1 + k - S)$.

In all three cases, inputting the appropriate form of a and b into (3.46) and (3.47) gives

$$c + d = \frac{2(1-k)(k+1-2S)}{(1+k)^2} \tag{3.52}$$

$$cd = \frac{k^2 + (1-2S)^2 - 2k(1+2S)}{(1+\alpha)^2} \tag{3.53}$$

So conveniently, the same rule for c and d is used for all three sections, and solving (3.52) and (3.53), we see that they are as in (3.50) and (3.51). It is clear that $c < x < d$ is equivalent to (3.48), and imputting these values of c and d into (2.40), as well as using (3.44), (3.45) and the fact that in all three cases, $N_p(a + b + 2) = p(k + 1)$, gives (3.49). □

3.4 Bulk scaling

Having found the region of support and density profile, we now have everything we need to work in the bulk regime. As in §3.3, we let the ratio $q/p = k$ be constant, and relabel the lines $s = Sp$, so that $S \in (0, 1 + k)$ is a continuous variable in the limit $p \to \infty$. Then, as outlined in §2.7, our aim here is to compute the scaled correlation kernal

$$\bar{K}(Y,t;X,s) := \lim_{p\to\infty} \frac{1}{\rho(X_S)} K\left(X_S + \frac{Y}{\rho(X_S)}, pS + t; X_S + \frac{X}{\rho(X_S)}, pS + s\right) \tag{3.54}$$

where X_S is the midpoint of the region of support $(c(S), d(S))$ where c, d are as in (3.50), (3.51), and $\rho(X_S)$ is the density at X_S as found by (3.49). Evaluating these, we have that

$$X_s = \frac{(1-k)(1+k-2S)}{(1+k)^2}$$

and

$$\rho(X_S) = \frac{p}{2\pi} \frac{4\sqrt{kS(1+k-S)}}{(1+k)(1-X_S^2)} := pu_S$$

The strategy to be adopted is to make use of a known asymptotic expansion for the large n form of $P_n^{\alpha+an,\beta+bn}(x)$ with x such that the leading behaviour is oscillatory (outside the interval (c, d) there is exponential decay) [14]. Based on experience with similar calculations [30, 29, 31], as well as our result in the GUE* case in §2.7, we expect that this will show that to leading order the sums are Riemann sums, and so turn into integrals in the limit $p \to \infty$.

Proposition 3.4. *Let a, b, x be given, and define the parameters $\Delta, \rho, \theta, \gamma$ according to*

$$\Delta = [a(x+1) + b(x-1)]^2 - 4(a+b+1)(1-x^2)$$
$$\frac{2e^{i\rho}}{\sqrt{(1+a+b)(1-x^2)}} = \frac{a(x+1)+b(x-1)+i\sqrt{-\Delta}}{(1+a+b)(1-x^2)} \qquad -\pi < \rho \leq \pi,$$
$$\sqrt{\frac{2(a+1)}{(1-x)(1+a+b)}}e^{i\theta} = \frac{(a+b+2)x-(3a+b+2)-i\sqrt{-\Delta}}{2(x-1)(1+a+b)} \qquad -\pi < \theta \leq \pi$$
$$\sqrt{\frac{2(b+1)}{(1+x)(1+a+b)}}e^{i\gamma} = \frac{(a+b+2)x+(a+3b+2)-i\sqrt{-\Delta}}{2(x+1)(1+a+b)} \qquad -\pi < \gamma \leq \pi \tag{3.55}$$

For $\Delta < 0$ we have the large n asymptotic expansion

$$P_n^{(\alpha+an,\beta+bn)}(x) = \left(\frac{4}{\pi n\sqrt{-\Delta}}\right)^{\frac{1}{2}} \left[\frac{2(a+1)}{(1-x)(1+a+b)}\right]^{\frac{n}{2}(a+1)+\frac{\alpha}{2}+\frac{1}{4}} \tag{3.56}$$
$$\times \left[\frac{2(b+1)}{(1+x)(1+a+b)}\right]^{\frac{n}{2}(b+1)+\frac{\beta}{2}+\frac{1}{4}} \left[\frac{(1-x^2)(a+b+1)}{4}\right]^{\frac{n}{2}+\frac{1}{4}}$$
$$\times \cos\left([n(a+1)+\alpha+\frac{1}{2}]\theta + [n(b+1)+\beta+\frac{1}{2}]\gamma - (n+\frac{1}{2})\rho + \frac{\pi}{4}\right)\left(1+O\left(\frac{1}{n}\right)\right)$$

valid for general $\alpha, \beta \in \mathbb{R}$ for $a, b \geq 0$. As noted in [15], results from [34, 9] imply that the $O(1/n)$ term holds uniformly in the parameters.

Actually (3.56) differs from the form reported in [14], with our $\sqrt{-\Delta}$ in the denominator of the first term on the RHS, whereas it is in the numerator of the corresponding term in [14], and furthermore some signs and factors of 2 in the cosine are in disagreement. One check is to exhibit the symmetry of the Jacobi polynomials

$$P_n^{(c,d)}(-x) = (-1)^n P_n^{(d,c)}(x) \tag{3.57}$$

For this we examine the effect on the parameters (3.55) under the mappings

$$x \mapsto -x, \qquad a \mapsto b, \qquad b \mapsto a \tag{3.58}$$

We see that Δ is unchanged, while

$$\rho \mapsto \pi - \rho, \qquad \theta \mapsto -\gamma, \qquad \gamma \mapsto -\theta. \tag{3.59}$$

Making the substitutions (3.58), (3.59) in (3.56), along with $\alpha \mapsto \beta$, $\beta \mapsto \alpha$ we see that indeed the RHS is consistent with (3.57). Another check is to specialize to the case $a = b = 0$. Then with $x = \cos\phi$, $0 \leq \phi \leq \pi$ we can check from (3.55) that $\sqrt{-\Delta} = 2\sin\phi$, $\rho = \pi/2$, $\theta = \pi/2 - \phi/2$, $\gamma = -\phi/2$ and so

$$P_n^{(\alpha,\beta)}(\cos\phi) \sim \left(\frac{1}{\pi n}\right)^{1/2} \frac{1}{(\sin\phi/2)^{\alpha+1/2}(\cos\phi/2)^{\beta+1/2}} \cos\Big((n+(\alpha+\beta+1)/2)\theta - (\alpha+1/2)\pi/2\Big)$$

which agrees with the result in Szegö's book [65]. We give our working in §6.2.

According to (3.36) the particular Jacobi polynomials appearing in the summation specifying K in (3.54) depends on the range of values of the continuum line label S. Let us suppose that $1 \leq S \leq k$, and so the number of particles on each line is p. In this case, and with $s \leq t$

$$K(y, pS+t; x, pS+s) = \sum_{n=0}^{p-1} c_n(x,y) P_n^{(an-s,bn+s)}(x) P_n^{(an-t,bn+t)}(y) \tag{3.60}$$

where

$$c_n(x,y) = \frac{(1-y)^{an-t}(1+x)^{bn+s}}{\mathcal{N}_n^{(an-s,bn+s)}} \frac{(an+n-s)!}{(an+n-t)!} \tag{3.61}$$

and with $w := n/p$,

$$a := \frac{k-S}{w}, \qquad b = \frac{S-1}{w}$$

The expression in the case $s > t$ is the same except that the range of the sum is now over p to ∞ instead of 0 to $p-1$. With x and y given by

$$x = X_S + \frac{X}{pu_S} \qquad y = X_S + \frac{Y}{pu_S}$$

we want to replace the summand by its large n asymptotic form. Because of this form of x and y, the parameters $\Delta, \rho, \theta, \gamma$ in Proposition 3.4, as they apply to the Jacobi polynomials in (3.36), all have expansions in inverse powers of $1/p$.

Lemma 3.5. *Let*

$$x = X_S + \frac{X}{pu_S} \tag{3.62}$$

The quantities (3.55) *have the large* p *expansion*

$$\begin{aligned} \Delta &= \Delta_0 + \Delta_1 X/p + \mathrm{O}(1/p^2) \\ \rho &= \rho_0 + \rho_1 X/p + \mathrm{O}(1/p^2) \\ \theta &= \theta_0 + \theta_1 X/p + \mathrm{O}(1/p^2) \\ \gamma &= \gamma_0 + \gamma_1 X/p + \mathrm{O}(1/p^2) \end{aligned}$$

where

$$\begin{aligned} \Delta_0 &= [a(X_S+1) + b(X_S-1)]^2 - 4(a+b+1)(1-X_S^2) \\ \Delta_1 &= \frac{2}{u_S}\left(a^2 - b^2 + (2+a+b)^2 X_S\right) \\ e^{i\rho_0} &= \frac{a(1+X_S) - b(1-X_S) + i\sqrt{-\Delta_0}}{2\sqrt{(1+a+b)(1-X_S^2)}} \\ e^{i\theta_0} &= \frac{-(a+b+2)X_S + (3a+b+2) + i\sqrt{-\Delta_0}}{2\sqrt{2(1+a)(1+a+b)(1-X_S)}} \\ e^{i\gamma_0} &= \frac{(a+b+2)X_S + (a+3b+2) - i\sqrt{-\Delta_0}}{2\sqrt{2(1+b)(1+a+b)(1+X_S)}} \\ \rho_1 &= \frac{-1}{u_S\sqrt{-\Delta_0}} \frac{a(1+X_S) + b(1-X_s)}{1-X_S^2} \\ \theta_1 &= \frac{-1}{2u_S\sqrt{-\Delta_0}} \frac{a(1+X_S) - (b+2)(1-X_S)}{1-X_S} \\ \gamma_1 &= \frac{1}{2u_s\sqrt{-\Delta_0}} \frac{(a+2)(1+X_S) - b(1-X_S)}{1+X_S} \end{aligned}$$

The use of this result is that it allows the terms in the summation (3.60) to be exhibited to have a Riemann sum form.

Proposition 3.6. *Let* x *and* y *be as in* (3.62), *and let* $w_0 < w \leq 1$ *correspond to* $\Delta(w) < 0$. *Let* w *be fixed, and* $n = wp \to \infty$. *To leading order in contribution to the summation in* (3.60), *we can replace*

$$c_n(x,y) P_n^{(an-s,bn+s)}(x) P_n^{(an-t,bn+t)}(y)$$

with

$$p^{t-s}e^{\frac{(X-Y)(wa(1+X_S)+wb(1-X_S))}{2u_S(1-X_S^2)}}\frac{(2+a+b)}{\pi\sqrt{-\Delta_0}}$$
$$\times\mathrm{Re}\Big[\exp\left(\frac{iw\sqrt{-\Delta_0}}{2u_S(1-X_S^2)}(X-Y)\right)\left(\frac{wa(1+X_S)+wb(1-X_S)+iw\sqrt{-\Delta_0}}{2(1-X_S^2)}\right)^{t-s}\Big]$$

For $0 \le w \le w_0$*, corresponding to* $\Delta \ge 0$*, to the same order the summand can be replaced by zero.*

Proof. Using Proposition 3.4 and applying Stirling's approximation to (3.61) we have

$$c_n(x,y)P_n^{(an-s,bn+s)}(x)P_n^{(an-t,bn+t)}(y) = \left(\frac{1-y}{1-x}\right)^{an/2}\left(\frac{1+x}{1+y}\right)^{bn/2}\frac{(1-x^2)^{s/2}}{(1-y^2)^{t/2}} \tag{3.63}$$
$$\times\frac{2(2+a+b)}{\pi}\left(\frac{1}{\sqrt{\Delta_x\Delta_y}}\right)^{1/2}[n^2(a+1)(b+1)]^{(t-s)/2}\cos(A_{x,s})\cos(A_{y,t})\left(1+\mathrm{O}(n^{-1})\right)$$

where

$$A_{x,s} = [n(a+1)-s+\frac{1}{2}]\theta_x + [n(b+1)+s+\frac{1}{2}]\gamma_x - (n+\frac{1}{2})\rho_x + \pi/4$$

for Δ_x, ρ_x, θ_x, γ_x, as in Lemma 3.5. Letting $n = pw$, $x = X_S + \dfrac{X}{pu_S}$ and $y = X_S + \dfrac{Y}{pu_S}$, we have

$$\left(\frac{1-y}{1-x}\right)^{an/2}\left(\frac{1+x}{1+y}\right)^{bn/2}\frac{(1-x^2)^{s/2}}{(1-y^2)^{t/2}} = e^{\frac{(X-Y)(wa(1+X_S)+wb(1-X_S))}{2u_S(1-X_S^2)}}\left(\frac{1}{1-X_S^2}\right)^{(t-s)/2} + \mathrm{O}(p^{-1})$$

and

$$A_{x,s} - A_{y,t} = w[(a+1)\theta_1 + (b+1)\gamma_1 - \rho_1](X-Y) + (t-s)(\theta_0-\gamma_0) + \mathrm{O}(p^{-1})$$

From Lemma 3.5,

$$(a+1)\theta_1 + (b+1)\gamma_1 - \rho_1 = \frac{\sqrt{-\Delta_0}}{2u_S(1-X_S^2)}$$

so, using a simple trigonmetric identity, the RHS of (3.63) can be rewritten

$$\frac{(2+a+b)}{\pi\sqrt{-\Delta_0}}e^{\frac{(X-Y)(wa(1+X_S)+wb(1-X_S))}{2u_S(1-X_S^2)}}\left(\frac{p^2w^2(a+1)(b+1)}{1-X_S^2}\right)^{(t-s)/2}$$
$$\times\cos\left(\frac{w\sqrt{-\Delta_0}}{2u_S(1-X_S^2)}(X-Y)+(t-s)(\theta_0-\gamma_0)\right)\left(1+\mathrm{O}(p^{-1})\right)$$

From Lemma 3.5

$$\left(\frac{w^2(a+1)(b+1)}{1-X_S^2}\right)^{(t-s)/2}e^{i(t-s)(\theta_0-\gamma_0)} = \left(\frac{wa(1+X_S)+wb(1-X_S)+iw\sqrt{-\Delta_0}}{2(1-X_S^2)}\right)^{t-s}$$

so using the fact that $\mathrm{Re}(e^{ix}) = \cos x$ gives the result □

Proposition 3.7. *Let* $\bar{K}(Y,t;X,s)$ *be as in* (3.54) *and let* S *be such that* $1 \le S \le k$*. Also, let*

$$v := \frac{wa(1+X_S)+wb(1-X_S)}{2\pi u_S(1-X_S^2)} = \frac{k-1}{k+1}\sqrt{\frac{(1+k-S)S}{k}} \tag{3.64}$$

We have

$$\bar{K}(Y,t;X,s) = \frac{F(Y,t)}{F(X,s)}K^*(Y,t;X,s) \tag{3.65}$$

where

$$F(X,s) = (p\pi u_S)^s \, e^{-\pi v X} \tag{3.66}$$

and

$$K^*(Y,t;X,s) = \begin{cases} \displaystyle\int_0^1 \left(v^2+z^2\right)^{\frac{t-s}{2}} \cos\left(\pi z(X-Y) + (t-s)\arctan(\frac{z}{v})\right) dz & s \le t \\ \displaystyle -\int_1^\infty \left(v^2+z^2\right)^{\frac{t-s}{2}} \cos\left(\pi z(X-Y) + (t-s)\arctan(\frac{z}{v})\right) dz & s > t \end{cases} \tag{3.67}$$

Proof. Recalling the definition of $\bar{K}$ (3.54), we begin with the case $s \le t$, and so converting the RHS of (3.60) into a Riemann sum, using Proposition 3.6 and noting that v has no w dependence, we have

$$\bar{K}(Y,t;X,s) = \frac{1}{\rho(X_S)} p^{t-s+1} e^{\pi v(X-Y)} \left(\pi u_S\right)^{t-s} \mathcal{I} \tag{3.68}$$

where

$$\mathcal{I} = \int_{w_0}^1 \frac{(2+a+b)}{\pi\sqrt{-\Delta_0}} \mathrm{Re}\Big[\exp\left(\frac{iw\sqrt{-\Delta_0}}{2u_S(1-X_S^2)}(X-Y)\right)\left(v + \frac{iw\sqrt{-\Delta_0}}{2\pi u_S(1-X_S^2)}\right)^{t-s}\Big] dw$$

We change variables to

$$z = \frac{w\sqrt{-\Delta_0}}{2\pi u_S(1-X_S^2)}$$

Then $z=0$ when $w=w_0$ by the defintion of w_0, and $z=1$ when $w=1$ because $\sqrt{-\Delta_0}|_{w=1} = 2\pi u_S(1-X_S^2)$. Also

$$dz = \frac{a+b+2}{\sqrt{-\Delta_0}\pi u_S} dw$$

so we have

$$\mathcal{I} = u_S \int_0^1 \mathrm{Re}\Big[e^{iz\pi(X-Y)}\left(v+iz\right)^{t-s}\Big] dw$$

Using the identies that $x+yi = \sqrt{x^2+y^2}e^{i\arctan y/x}$, and $\mathrm{Re}(e^{ix}) = \cos(x)$, the result follows.

For $s<t$, (3.60) is to replaced by

$$K(y, pS+t; x, pS+s) = -\sum_{n=p}^{\infty} c_n(x,y) P_n^{(an-s,bn+s)}(x) P_n^{(an-t,bn+t)}(y)$$

Thus, up to a minus sign, the asymptotic form is given by (3.68), but with the terminal of integration in $\mathcal{I}$ now from 1 to ∞. This gives result for $s<t$. □

Corollary 3.8. *With the scaled correlation function* $\bar{\rho}_{\mathrm{Bulk}}$ *specified by*

$$\begin{aligned} &\bar{\rho}_{\mathrm{Bulk},(r)}((X_1,t_1);\dots;(X_r,t_r)) \\ &\quad = \lim_{p\to\infty}\left(\frac{1}{\rho(X_S)}\right)^r \rho_{(r)}\left((X_S + \frac{X_1}{\rho(X_S)}, pS+t_1);\dots;(X_S+\frac{X_r}{\rho(X_S)}, pS+t_r)\right) \end{aligned}$$

and K^* *specified by* (3.67), *we have*

$$\bar{\rho}_{\mathrm{Bulk},(r)}((X_1,t_1);\dots;(X_r,t_r)) = \det[K^*(X_i,t_i;X_j,t_j)]_{i,j=1,\dots,r} \tag{3.69}$$

Proof. In the region $1 \le S \le k$, this is an immediate consequence of Proposition 3.7. The correlation function in the region $k \le S < 1+k$ follows from the form in the region $0<S<1$ upon making the mappings

$$(X_i,t_i) \mapsto (-X_i,-t_i), \qquad S \mapsto 1+k-S \tag{3.70}$$

which from (3.67) is indeed a symmetry of (3.54). To show that (3.69) is valid for $0 < S < 1$ requires deriving the analogue Proposition 3.7 in this case. Recalling (3.36), and with $s \leq t$, we must then obtain the large p form of

$$K(y, pS+t; x, pS+s) = \sum_{n=0}^{pS+s-1} c_n(x,y) P_n^{(an-s,bn-s)}(x) P_{n+t-s}^{(an-t,bn-t)}(y)$$

where

$$c_n(x,y) = 2^{t-s} \frac{(1-y)^{an-t}(1+x)^{bn-s}}{\mathcal{N}_n^{(an-s,bn-s)}} \frac{(n+t-s)!}{n!}$$

and with $w := n/p$,

$$a := \frac{k-S}{w}, \qquad b = \frac{1-S}{w}$$

This is done using the same general strategy as for $1 \leq S \leq k$, obtaining a result consistent with (3.67). □

At the beginning of this section, it was remarked that the bead model was introduced by Boutillier [11] as a continuum limit of a dimer model on the honeycomb lattice. The corresponding scaled correlation was calculated to be of the form (3.69) but with K^* replaced by J_γ, where

$$J_\gamma(Y,t;X,s) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-1}^{1} \left(e^{iz(X-Y)}(\gamma + iz\sqrt{1-\gamma^2})^{t-s}\right) dz, & s \leq t \\ -\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}\ [-1,1]} \left(e^{iz(X-Y)}(\gamma + iz\sqrt{1-\gamma^2})^{t-s}\right) dz, & s > t \end{cases} \tag{3.71}$$

The parameter γ, $|\gamma| < 1$, represents an anisotropy in the underlying $a \times b \times c$-hexagon, with $\gamma = 0$ corresponding to the symmetrical case.

We observe that the factor $(\gamma + it\sqrt{1-\gamma^2})^{t-s}$ in the integrands of (3.71) can be replaced by $(1 + it\sqrt{1-\gamma^2}/\gamma))^{t-s}$ without changing the value of the determinant. Changing scale $J_\gamma(Y,t;X,s) \mapsto \pi J_\gamma(\pi Y, t; \pi X, s)$, and comparing the resulting form of (3.71) with (3.67) shows the two results to be the same, upon the identification

$$\frac{\gamma}{\sqrt{1-\gamma^2}} = v \tag{3.72}$$

although the quantity on the RHS is always positive. This latter feature is a consequence of the calculations relating to our finitized bead model being carried out under the assumption that $q \geq p$. From the symmetry of the hexagon, the case $q < p$ is obtained by simply replacing X, Y in (3.71) by $-X, -Y$, or equivalently replacing v in (3.67) by $-v$. This then allows us to extend (3.72) to the region $-1 < \gamma < 0$, by replacing the γ in the denominator by $|\gamma|$.

It is also worth mentioning the interpretation of $\bar{\rho}_{\text{Bulk},(r)}$ when $k = 1$. In this case, $v = 0$, and the term $\arctan(z/v)$ is not strictly well defined. However, if we consider the case $k = 1 + \epsilon$, $\lim_{\epsilon \to 0} \arctan(z/v) = \pi/2$ and so (3.67) is equivalent to the bulk GUE* kernel in (2.105).

3.5 Soft edge scaling

Following the work on the GUE*, we now go about finding a form of the correlation functions for the hexagon bead model in the soft edge scaling limit. To proceed further we require the following asymptotic form of the Jacobi polynomials,

Proposition 3.9. *[44] Define variables* κ_n, γ *and* ψ *by*

$$\kappa_n = 2n + a + b + 1, \quad \cos\psi = \frac{a-b}{\kappa_n}, \quad \cos\gamma = \frac{a+b}{\kappa_n}$$

Using these variables, we now define

$$M_n = -\cos(\psi+\gamma), \qquad \sigma_n^3 = \frac{2\sin^4(\psi+\gamma)}{\kappa_n^2 \sin\psi \sin\gamma}$$

Then, for $x = M_n + \sigma_n X$,

$$P_n^{(a,b)}(x) = \sqrt{\frac{\kappa_n \sigma_n \mathcal{N}_n^{(a,b)}}{(1-x)^a(1+x)^b(1-x^2)}} \left(\mathrm{Ai}(X) + \mathrm{O}(N^{-2/3}) \begin{cases} e^{-X/2}, & X > 0 \\ 1, & X < 0 \end{cases} \right) \tag{3.73}$$

To use this asymptotic form, we consider particles on line $pS + \tau t p^{2/3}$ for some real values S, t, τ, and for large p. Then, as before we keep the ratio $q/p = k$ constant as we take p large, and work in the 'middle section', with $S \in (1, k)$. Then we have $a(t) = p(k-S) - \tau t p^{2/3}$, $b(t) = p(S-1) + \tau t p^{2/3}$. We note that with this setting,

$$\lim_{p\to\infty} M_p = \frac{(1-k)(1+k-2S) + 4\sqrt{kS(1+k-S)}}{(1+k)^2} \tag{3.74}$$

which is exactly our d value from (3.51). This shows that this asymptotic form of the Jacobi polynomials has the values centered around the largest particle on the line, to leading order.

Proposition 3.10. *Define pairs* (x_i, S_i) *by*

$$x_i = x_0(S_i) + X_i \sigma p^{-2/3}, \qquad S_i = S + t_i \tau p^{-1/3} \tag{3.75}$$

where

$$x_0(S_i) = \frac{(1-k)(1+k-2S_i) + 4\sqrt{kS_i(1+k-S_i)}}{(1+k)^2}$$

and

$$\sigma = \lim_{p\to\infty} p^{2/3}\sigma_p, \qquad \tau = \frac{S(1+k-S)\sigma^2}{1 - x_0(S)^2}$$

and let the scaled correlation function $\bar{\rho}_{SE}$ *be specified by*

$$\bar{\rho}_{SE,(r)}((X_1,t_1);\ldots;(X_r,t_r)) = \lim_{p\to\infty} (p^{-2/3}\sigma)^r \rho_{(r)}\left((x_1,pS_1);\ldots;(x_r,pS_r)\right) \tag{3.76}$$

where $\rho_{(r)}$ *is given by* (3.35) *with* $q = kp$. *Then we have*

$$\bar{\rho}_{SE,(r)}((X_1,t_1);\ldots;(X_r,t_r)) = \det[\bar{K}_{SE}(X_i,t_i;X_j,t_j)]_{i,j=1,\ldots,r} \tag{3.77}$$

where

$$\bar{K}_{SE}(Y,t;X,s) = \begin{cases} \displaystyle\int_0^\infty e^{u(s-t)} \mathrm{Ai}\,(u+X)\,\mathrm{Ai}\,(u+Y)\,du & s \le t \\ \displaystyle -\int_{-\infty}^0 e^{u(s-t)} \mathrm{Ai}\,(u+X)\,\mathrm{Ai}\,(u+Y)\,du & s > t \end{cases} \tag{3.78}$$

Moreover,

$$\begin{aligned} \lim_{p\to\infty} \left(p^{-2/3}\sigma\right)^r \int_{y_1}^1 dx_1 \ldots \int_{y_r}^1 dx_r \; \rho_{(r)}\left((x_1,pS_1);\ldots;(x_r,pS_r)\right) \\ = \int_{Y_1}^\infty dx_1 \ldots \int_{Y_k}^\infty dx_k \; \bar{\rho}_{SE,(r)}((X_1,t_1);\ldots;(X_r,t_r)) \end{aligned} \tag{3.79}$$

Proof. Considering (3.76) and (3.77), our aim here is to compute the scaled correlation kernal

$$K^*(Y,t;X,s) := \lim_{p\to\infty} \frac{\sigma}{p^{2/3}} K\left(x, pS_x; y, pS_y\right)$$

where K is as in Proposition 3.2, and, recalling (2.59), show that

$$\bar{K}_{SE}(Y,t;X,s) = \frac{A(t,Y)}{A(s,X)} K^*(Y,t;X,s) \tag{3.80}$$

for some non zero function $A(s,X)$. For $1 \le S \le k$, and with $s \le t$, K is given by

$$K(y,pS_y;x,pS_x) = (1-y)^{a_y}(1+x)^{b_x} \sum_{l=1}^{p} \frac{(p+a_x-l)!}{(p+a_y-l)!} \frac{P_{p-l}^{(a_x,b_x)}(x) P_{p-l}^{(a_y,b_y)}(y)}{\mathcal{N}_{p-l}^{(a_x,b_x)}} \tag{3.81}$$

where $a_x = p(k-S_x)$, $b_x = p(S_x-1)$. Looking at (3.73), it is true that for $x = x_0(S_x) + \sigma p^{-2/3} X$,

$$P_{p-l}^{(a,b)}(x) = \sqrt{\frac{\kappa_{p-l}\sigma_{p-l}(S_x)\mathcal{N}_{p-l}^{(a,b)}}{(1-x)^a(1+x)^b(1-x^2)}} \left[\mathrm{Ai}(X^*) + \mathrm{O}(p^{-2/3}) \begin{cases} e^{-X^*/2}, & X^*>0 \\ 1, & X^*<0 \end{cases} \right]$$

where

$$X^* = \frac{x_0(S_x) - M_{p-l}(S_x) + X\sigma p^{-2/3}}{\sigma_{p-l}(S_x)}$$

and so

$$K\left(x,pS_x;y,pS_y\right) = \sqrt{\frac{(1-y)^{a_y}(1+x)^{b_x}}{(1+y)^{b_x}(1-x)^{a_y}}} \sum_{l=1}^{p} \kappa_{p-l} \sqrt{\frac{(p+a_x-l)!(p+b_y-l)!}{(p+a_y-l)!(p+b_x-l)!}} \tag{3.82}$$

$$\times \sqrt{\frac{\sigma_{p-l}(S_x)\sigma_{p-l}(S_y)}{(1-x^2)(1-y^2)}} \left[\mathrm{Ai}(X^*)\mathrm{Ai}(Y^*) + \mathrm{O}(p^{-2/3}) \begin{cases} e^{-X^*/2}, & X^*>0 \\ 1, & X^*<0 \end{cases} \begin{cases} e^{-Y^*/2}, & Y^*>0 \\ 1, & Y^*<0 \end{cases} \right]$$

Noting that

$$M_{p-l}(S_x) = x_0(S_x) + \frac{l}{p} M_1(S) + \mathrm{O}(1/p + l/p^{4/3} + l^2/p^2)$$

where

$$M_1(S) = \frac{4(1-k)(1+k-2S)}{(1+k)^3} + \frac{2\left[k^2(S^2+5S-2) - k(6S^2-5S+1) - k^3(1+S) - S(1-S)\right]}{(1+k)^3\sqrt{kS(1+k-S)}} \tag{3.83}$$

and that $\sigma_{p-l} = \sigma p^{-2/3}[(1+\mathrm{O}(p^{-1/3}+lp^{-1})]$, we let $l = wp^{-1/3}$, so that X^* is to leading order $\mathrm{O}(1)$. With this substitution, we recognise (3.82) as a Riemann sum and convert to an integral. Use of Stirling's formula (2.114) shows that

$$\kappa_{p-l}\sqrt{\frac{\sigma_{p-l}(S_x)\sigma_{p-l}(S_y)}{(1-x^2)(1-y^2)}} \sqrt{\frac{(p+a_x-l)!(p+b_y-l)!}{(p+a_y-l)!(p+b_x-l)!}}$$
$$= \frac{B(s,X)}{B(t,Y)} \frac{\sigma p^{1/3}(1+k)}{1-(x_0(S))^2} e^{\frac{w\tau(k+1)(s-t)}{S(1+k-S)}} \left(1 + \mathrm{O}(p^{-1/3})\right)$$

where

$$B(s,X) = \left(p^2 S(1+k-S)\right)^{-p^{2/3}\tau s/2} \exp\left(\frac{\tau s(2S-k-1)}{4S(1+k-S)} p^{1/3} - \frac{(1+k-2S)(1+k)\tau^3 s^3}{12(1+k-S)^2 S^2}\right)$$

Thus

$$K^*(Y,t;X,s) = \tag{3.84}$$
$$\frac{A(s,X)}{A(t,Y)} \int_0^\infty e^{\frac{w\tau(k+1)(s-t)}{S(1+k-S)}} \frac{(k+1)\sigma^2}{1-(x_0(S))^2} \mathrm{Ai}\left(\frac{-M_1(S)w}{\sigma} + X\right) \mathrm{Ai}\left(\frac{-M_1(S)w}{\sigma} + Y\right) dw$$

where

$$A(s,X) = \sqrt{\frac{(1+x)^{b_x}}{(1-x)^{a_x}}}B(s,X)$$

Making the change of variables $u = -M_1 w/\sigma$, noting from (3.83) that

$$M_1(S) = \frac{-(k+1)\sigma^3}{1-(x_0(S))^2}$$

and using the definition of τ, (3.84) becomes

$$K^*(Y,t;X,s) = \frac{A(s,X)}{A(t,Y)}\int_0^\infty e^{u(s-t)}\operatorname{Ai}(u+X)\operatorname{Ai}(u+Y)\,du \tag{3.85}$$

satisfying (3.80). When $s > t$, (3.81) is the same but for a change of sign and that the sum instead goes from $-\infty$ to 0. This results in the integral in (3.85) instead being over all negative u and again (3.80) is satisfied.

As with the bulk scaling, the correlation function in the region $k \le S < 1+k$ follows from the form in the region $0 < S \le 1$ upon making the mappings (3.70). To show that the scaled correlations for $0 < S \le 1$ are of the form (3.77), the same method as above is applied, however

$$K(y,t;x,s) = (1-y)^{a_y}(1+x)^{b_x}2^{p^{2/3}\tau(t-s)}\sum_{l=1}^{pS+s\tau p^{2/3}}\frac{(p-b_x-l)!}{(p-b_y-l)!}\frac{P_{p-l}^{(a_x,b_x)}(x)P_{p-l}^{(a_y,b_y)}(y)}{\mathcal{N}_{p-l}^{(a_x,b_x)}}$$

for $s \le t$ is used, rather than (3.81). The calculation is sufficiently similar and will not be shown here. □

3.6 Directed solid-on-solid model interpretation

At the end of §1.3 it was remarked that the rhombus tiling of an $a\times b\times c$ hexagon has an equivalent interpretation as a $b \times c$ grid of stacked cubes, with the precise relationship between positions of the particles specifying the tiling and the heights of the cubes being given by (1.46). Likewise the present hexagon bead model allows an interpretation in terms of heights and so specifies a certain directed solid-on-solid model.

Consider a $p \times q$ integer grid $\{(i,j) : 1 \le i \le p,\ 1 \le j \le q\}$. At each site (i,j) associate a height variable $x_{i,j}$ chosen from the uniform distribution $[0,2]$, and require that

$$x_{i,1} < x_{i,2} < \cdots < x_{i,q}, \qquad x_{1,j} < x_{2,j} < \cdots < x_{c,j} \tag{3.86}$$

so that heights increase along rows and up columns. Rotate the rectangular grid 45^o anti-clockwise and mark in lines parallel to the y axis through the lattice points, so obtaining $p+q-1$ lines. By an appropriate relabelling of the heights in terms of their ordering on the particular lines it is clear that the ordering (3.86) correspond to the interlacing on neighbouring lines $x^{(t)} \prec x^{(t+1)}$. Furthermore, choosing each $x_{i,j}$ from the uniform distribution on $[0,2]$ is equivalent to constraining the particles to have co-ordinates between -1 and 1 in the hexagon bead model.

The directed solid-on-solid interpretation calls for the computation of some quantities which relate directly to the height variables. Thus it is natural to seek the limiting shape of the height profile as the number of lines forms a continuum, and it is similarly natural to seek the correlations between heights along particular rows or columns. We turn our attention to the limiting shape.

In the large p limit, with $q = kp$ for some fixed ratio $k \geq 1$, the density profile on a line $n = pS$, $S \in [0, 1+k]$, for the hexagon bead model was found in Proposition 3.3, with the support of the density the interval $[c_S, d_S]$ with c and d given by (3.50) and (3.51). To interpret these results in terms of the directed solid-on-solid model, we first agree to scale the $p \times q$ integer grid by $1/p$, so that it is a $p \times q$ grid in the rectangle $[0, k] \times [0, 1]$. Then, recalling the relationship between the hexagon bead model and a 45^o rotation of the $p \times q$ grid, we define a mapping

$$(x, y) = (X(S), Y(S)) + \bar{r}(S)(t, t) \tag{3.87}$$

where

$$X(S), Y(S), \bar{r}(S) = \begin{cases} 0, 1-S, S & 0 \leq S \leq 1 \\ S-1, 0, 1 & 1 \leq S \leq k \\ S-1, 0, 1+k-S & k \leq S \leq 1+k \end{cases}$$

We note that $\bar{r}(S) = \dfrac{1}{p} r(pS)$ for $r(t)$ the number of particles on line t as in (3.1), and the positions $(X(S), Y(S))$ make up the left and bottom boundary of the $[0, k] \times [0, 1]$ rectangle. The inverse of (3.87) is given by

$$S = (1 + x - y), \qquad t = \begin{cases} \dfrac{x}{1+x-y} & x \leq y \leq 1 \\ y & x - k + 1 \leq y \leq x \\ \dfrac{y}{k - x + y} & 0 \leq y \leq x - k + 1 \end{cases} \tag{3.88}$$

The interpretation of this mapping is that the expected height at position (x, y), $h(x, y)$ say, is exactly one more than the expected position of the $r(S)t$-th lowest particle on line pS in the hexagon bead model, where t and S are given by (3.88). Thus, for t and S given by (3.88), $h(x, y) = f(S(x, y), t(x, y))$ for some function f. By the interpretation of the mapping, we can find $f(S, t)$ by solving

$$\int_{-1}^{f(S,t)-1} \rho_{Hexbead}(u) du = r(S)t$$

since the LHS gives the number of particles lower than $f(S, t) - 1$, which we require to be $r(S)t$. Recalling (3.49), this can be rewritten

$$\int_{c_S}^{f(S,t)-1} \frac{(k+1)\sqrt{(d_S - u)(u - c_S)}}{2\pi(1 - u^2)} du = \bar{r}(S)t \tag{3.89}$$

for c_S, d_S given by (3.50), (3.51).

In the simplest case $k = 1$, the rectangle becomes the unit square, with a symmetric height profile $h(x, y) = h(y, x)$ (recall (3.86)). Here it suffices to solve the $S \leq 1$ case of (3.89)

$$\int_{-D}^{f(S,t)-1} \frac{\sqrt{D^2 - u^2}}{\pi(1 - u^2)} du = St$$

where $D = \sqrt{S(2 - S)}$. Use of computer algebra allows us to conclude

$$St = \frac{1}{\pi}\left(\arcsin v + \pi/2 + \sqrt{1 - D^2}\left(\arctan\left(\frac{1 - D^2}{1 - v^2} v\right) + \pi/2\right)\right) \tag{3.90}$$

where $v = (f(S, t) - 1)/D$. Thus, for $x \leq y$, we obtain an equation for $h(x, y)$ by substituting

$$S = 1 + x - y, \qquad t = \frac{x}{1 + x - y} \tag{3.91}$$

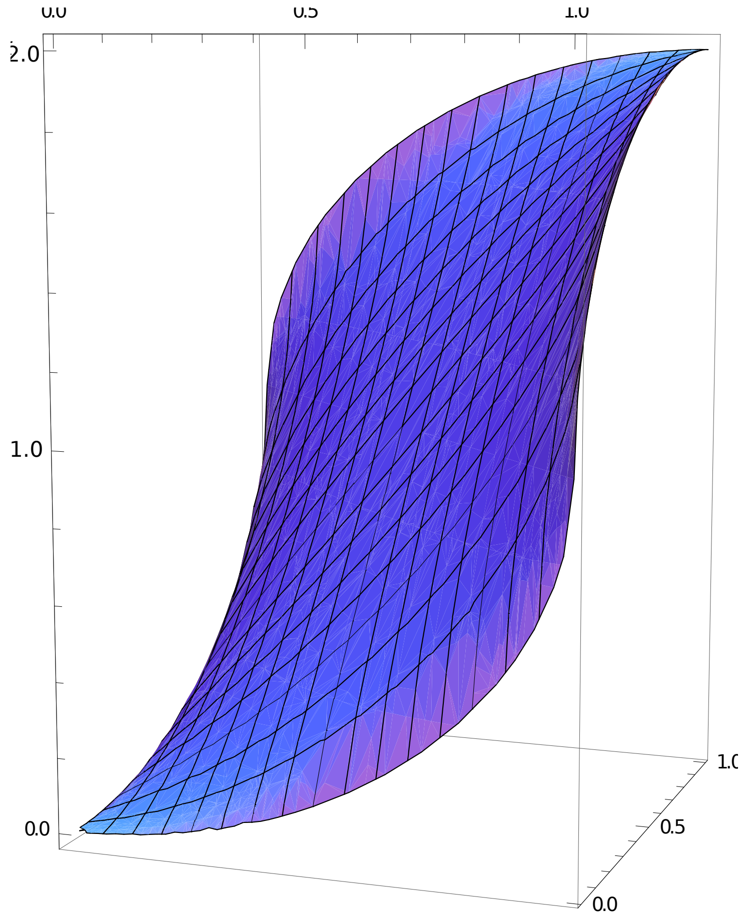


Figure 8: A plot of the surface $h(x, y)$ in the case $k = 1$, as calculated using (3.90) and (3.91)

into (3.90), which is solved using a root finding routine. This solves for the $y < x$ cases as well, since $h(y, x) = h(x, y)$. A graph of the resulting surface is plotted in Figure 8

In considering the solid-on-solid model, it is natural to consider the distributions of heights along a particular row or column. Recalling the relationship between the heights interpretation and the bead model, this encourages us to find the distributions of the largest particle on lines $1, \dots, q$, as these particles correspond to the heights along the back row of the $p \times q$ solid-on-solid model. To begin, we introduce the notation $E_0(\{l_i; (u_i, 1)\}_{i=1,\dots,n})$ to denote the probability that there is no particle in the interval $(u_i, 1)$ on line l_i for $i = 1, \dots, n$. The PDF $p^{\max}(\{l_i, u_i\}_{i=1,\dots,n})$ for the largest particle on line l_i being u_i for $i = 1, \dots, n$ then follows by partial differentiation by

$$p^{\max}(\{l_i, u_i\}_{i=1,\dots,n}) = \frac{\partial^n}{\partial u_1 \dots \partial u_n} E_0(\{l_i; (u_i, 1)\}_{i=1,\dots,n})$$

According to [27, §8.1], it is possible to express $E_0(\{l_i; (u_i, 1)\}_{i=1,\dots,n})$ in terms of correlation functions

$$E_0(\{l_i; (u_i, 1)\}_{i=1,\dots,n}) = \sum_{m_1=0}^{r(l_1)} \cdots \sum_{m_n=0}^{r(l_n)} \frac{(-1)^{m_1+\cdots+m_n}}{m_1! \dots m_n!} \tag{3.92}$$
$$\times \int_{u_1}^1 dy_1^{(l_1)} \dots \int_{u_1}^1 d_{m_1}^{(l_1)} \dots \int_{u_n}^1 dy_1^{(l_n)} \dots \int_{u_n}^1 d_{m_n}^{(l_n)} \rho_{(\sum_{i=1}^n m_n)} \left(\cup_{i=1}^n \cup_{j=1}^{m_i} \{(l_i, y_j^{(l_i)})\} \right)$$

for $r(t)$ the number of particles on line t given by (3.1), where the term $m_1 = \cdots = m_n = 0$ is taken to equal unity. It is furthermore the case that, in general, if the k-point correlation function $\rho_{(k)}$ has a determinantal form

$$\rho_{(k)}((x_1, l_1); \ldots (x_k, l_k)) = \det[K(x_i, l_i; x_j, l_j)]_{i,j=1,\ldots,k}$$

the multiple sum (3.92) can be summed [27, §9.1]. This can be done by defining the $n \times n$ matrix Fredholm integral operator $K(\{l_i; (u_i, 1)\}_{i=1,\ldots,n})$ with kernel

$$\bar{K}(x, y; \{l_i; (u_i, 1)\}_{i=1,\ldots,n}) = \left[\chi_{x \in (u_i,1)} K(x, l_i; y, l_j) \chi_{y \in (u_j,1)}\right]_{i,j=1,\ldots,n} \tag{3.93}$$

We then have that

$$E_0(\{l_i; (u_i, 1)\}_{i=1,\ldots,n}) = \det\left(\mathbf{1} - K(\{l_i; (u_i, 1)\}_{i=1,\ldots,n})\right) \tag{3.94}$$

where the meaning of the determinant can be taken as the product over the eigenvalues of the operator. Thus for the case of the p, q bead model, E_0 is given by (3.94) for the Fredholm integral operator $K(\{l_i; (u_i, 1)\}_{i=1,\ldots,n})$ with kernel defined by (3.93), with $K(y, t; x, s)$ given by (3.36).

We can also compute the scaled limit of E_0 by making use of Proposition 3.10. We know from [64], [7] that the convergence of the integrals (3.79) implies that the scaled limit of ρ can be applied term-by-term in (3.92).

Corollary 3.11. *Let x_i and S_i be related to X_i and t_i respectively as in* (3.75)*. We have*

$$\lim_{N \to \infty} E_0(\{pS_i; (x_i, 1)\}_{i=1,\ldots,n}) = \det\left(\mathbf{1} - K_{SE}(\{t_i, (X_i, \infty)\}_{i=1,\ldots,n})\right) \tag{3.95}$$

where $K_{SE}(\{t_i, (X_i, \infty)\}_{i=1,\ldots,n})$ is the $n \times n$ matrix Fredholm integral operator with kernel

$$\bar{K}_{SE}(X, Y; \{t_i; (X_i, \infty)_{i=1,\ldots,n}) = \left[\chi_{X \in (X_i,\infty)} \bar{K}_{SE}(X, t_i; Y, t_j) \chi_{Y \in (X_j,\infty)}\right]_{i,j=1,\ldots,n}$$

for $\bar{K}_{SE}(Y, t; X, s)$ as in (3.78)

The expression (3.95) is precisely that for the cumulative distribution function of the scaled largest eigenvalue in the Dyson Brownian motion model of complex Hermitian matrices [59] (see [27, Ch. 11] for an account of the model). It characterizes as well the fluctuations in a large class of growth models of the so called KPZ universality class [59]. As a recent development this same functional form has been shown to result from the continuum KPZ equation itself, when solved for so called narrow wedge initial conditions [62].

3.7 Hard edge

In our finitized bead model the particles are confined to the interval $(-1, 1)$ in the y direction according to the definition of the model. We know from Proposition 3.3 that the support of the density does not in general extend to these boundaries. An exception, with respect to the upper boundary $y = 1$, is the line number $n = q$. This gives rise to a hard edge boundary for particles on this and neighbouring lines, and consequently we expect

$$\bar{\rho}_{HE,(r)}((X_1, t_1); \ldots; (X_r, t_r)) = \lim_{p \to \infty} \kappa^r \rho_{(r)}((1 - \kappa X_1, q + t_1) : \ldots; (1 - \kappa X_r, q + t_r)) \tag{3.96}$$

for some scale value κ dependent on p To evaluate the explicit form of (3.96) we require a corresponding asymptotic formula of the Jacobi polynomials.

Theorem 3.12. *(Theorem 8.11 from [65]) Let* $P_N^{(a,b)}(x)$ *be the Jacobi polynomials* (3.10) *for some fixed real* a, b. *Then*

$$\lim_{N\to\infty} N^{-a} P_N^{(a,b)}\left(1-\frac{x^2}{2N^2}\right) = \left(\frac{x}{2}\right)^{-a} J_a(x) \tag{3.97}$$

where J_a *is the Bessel function* (2.117)

We note here that the Jacobi polynomials in the correlation functions (3.36) at line $t = q+t_i$ are of the form $P_{p-l}^{(|t_i|,q-p+t_i)}$. Using our usual construction for scaled limits with $q = kp$ and $p \to \infty$, this means that the b term is growing like p. However, (3.97) is defined only for set a and b. Since the RHS of (3.97) has no b dependence (as expected), it is likely that it is only required that the a term is fixed, and that N dependence in the b term is allowed. However, as we have no proof of this fact, we will instead use the simplest case $q = p$ (ie $k = 1$) so that the b term is fixed with respect to p as well.

Proposition 3.13. *Define pairs* (x_i, S_i) *by*

$$x_i = 1 - \frac{X_i}{2p^2} \qquad S_i = p + t_i$$

for some positive real X_i, *and let the scaled correlation function* $\bar{\rho}_{HE}$ *be specified by*

$$\bar{\rho}_{HE,(r)}((X_1,t_1);\ldots;(X_r,t_r)) = \lim_{p\to\infty}\left(\frac{1}{2p^2}\right)^r \rho_{(r)}\left((x_1,S_1);\ldots;(x_r,S_r)\right) \tag{3.98}$$

where $\rho_{(r)}$ *is given by* (3.35) *with* $q = p$. *Then we have*

$$\bar{\rho}_{HE,(r)}((X_1,t_1);\ldots;(X_r,t_r)) = \det[K_{HE}(X_i,t_i;X_j,t_j)]_{i,j=1,\ldots,r} \tag{3.99}$$

where

$$\bar{K}_{HE}(Y,t;X,s) = \begin{cases} \displaystyle\int_0^1 \frac{u^{\frac{1}{2}(t-s)}}{4} J_{|s|}\left(\sqrt{uX}\right) J_{|t|}\left(\sqrt{uY}\right) du & s \le t \\ \displaystyle -\int_1^\infty \frac{u^{\frac{1}{2}(t-s)}}{4} J_{|s|}\left(\sqrt{uX}\right) J_{|t|}\left(\sqrt{uY}\right) du & s > t \end{cases} \tag{3.100}$$

(compare with (2.121)*)*

Proof. Considering (3.98) and (3.99) and recalling (2.59), to establish (3.100) it is enough to show that

$$\bar{K}_{HE}(Y,t;X,s) = \lim_{p\to\infty} \frac{1}{2p^2}\frac{h(X,s)}{h(Y,t)} K\left(1-\frac{Y}{2p^2}, p+t; 1-\frac{X}{2p^2}, p+s\right) \tag{3.101}$$

for K as in (3.36) with $q = p$, where $h(X,s)$ is some non-zero function of X and s. From (3.36) with $q = p$,

$$K\left(1-\frac{Y}{2p^2}, p+t; 1-\frac{X}{2p^2}, p+s\right) = \left(\frac{Y}{2p^2}\right)^{|t|}\left(2-\frac{X}{2p^2}\right)^{|s|} \tag{3.102}$$
$$\times \begin{cases} \displaystyle\sum_{l=1}^{p} \frac{A(p+t,l)}{A(p+s,l)} \frac{P_{p-|s|-l}^{(|s|,|s|)}\left(1-\frac{X}{2p^2}\right) P_{p-|t|-l}^{(|t|,|t|)}\left(1-\frac{Y}{2p^2}\right)}{\mathcal{N}_{p-|s|-l}^{(|s|,|s|)}} & s \le t \\ \displaystyle -\sum_{l=-\infty}^{0} \frac{A(p+t,l)}{A(p+s,l)} \frac{P_{p-|s|-l}^{(|s|,|s|)}\left(1-\frac{X}{2p^2}\right) P_{p-|t|-l}^{(|t|,|t|)}\left(1-\frac{Y}{2p^2}\right)}{\mathcal{N}_{p-|s|-l}^{(|s|,|s|)}} & s > t \end{cases}$$

where $A(s,l)$ is given by (3.37). Using a slightly modified form of (3.97),

$$\lim_{p\to\infty} p^{-|s|} P^{(|s|,|s|)}_{p-l-|s|}\left(1-\frac{X}{2p^2}\right) = \left(\frac{\sqrt{X}}{2}\right)^{-|s|} J_{|s|}\left(\sqrt{X}\left(1-\frac{l+|s|}{p}\right)\right)$$

for $X>0$. We now make a change of variables $l = wp$. With this substitution, we recognise (3.102) as a Riemann sum and to leading order the RHS is equal to

$$2^{2|s|}p^{|s|-|t|+1}Y^{|t|/2}X^{-|s|/2}\begin{cases} \displaystyle\int_0^1 \frac{J_{|s|}\left(\sqrt{X}(1-w)\right)J_{|t|}\left(\sqrt{Y}(1-w)\right)}{\mathcal{N}^{(|s|,|s|)}_{p(1-w)-|s|}}\frac{A(p+t,wp)}{A(p+s,wp)}dw & s\le t \\ \displaystyle -\int_{-\infty}^0 \frac{J_{|s|}\left(\sqrt{X}(1-w)\right)J_{|t|}\left(\sqrt{Y}(1-w)\right)}{\mathcal{N}^{(|s|,|s|)}_{p(1-w)-|s|}}\frac{A(p+t,wp)}{A(p+s,wp)}dw & s>t \end{cases} \tag{3.103}$$

From (3.37) we have

$$A(p+t,wp) = 2^{p-|t|}(p-wp+t)!(-1)^{\frac{1}{2}(t+|t|)}$$

and from (3.11) we have

$$\mathcal{N}^{(|s|,|s|)}_{p(1-w)-|s|} = \frac{2^{2|s|+1}}{2p(1-w)+1}\frac{((p-wp)!)^2}{(p-wp-s)!(p-wp+s)!}$$

and so applying Stirling's formula (1.16) we have

$$\frac{1}{\mathcal{N}^{(|s|,|s|)}_{p(1-w)-|s|}}\frac{A(p+t,wp)}{A(p+s,wp)} = (-1)^{\frac{1}{2}(t+|t|-s-|s|)}2^{-|s|-|t|}\left(p(1-w)\right)^{t-s+1} + \mathrm{O}(1/p) \tag{3.104}$$

Then, combining (3.103) and (3.104),

$$\lim_{p\to\infty}\frac{1}{2p^2}\frac{h(X,s)}{h(Y,t)}K\left(1-\frac{Y}{2p^2},p+t;1-\frac{X}{2p^2},p+s\right) = \begin{cases} \displaystyle\int_0^1 \frac{(1-w)^{t-s+1}}{2}J_{|s|}\left((1-w)\sqrt{X}\right)J_{|t|}\left((1-w)\sqrt{Y}\right)dw & s\le t \\ \displaystyle -\int_{-\infty}^0 \frac{(1-w)^{t-s+1}}{2}J_{|s|}\left((1-w)\sqrt{X}\right)J_{|t|}\left((1-w)\sqrt{Y}\right)dw & s>t \end{cases}$$

for $h(Y,t) = (-1)^{\frac{1}{2}(t+|t|)}2^{-|t|}p^{t-|t|}Y^{|t|/2}$. After a change of variables $u=(1-w)^2$ we see that this satisfies to (3.101). □

4 Aztec diamond

A common theme of this thesis has been the link between random tilings of geometric shapes and interlacing particle models, by equating the positions of certain types or orientations of tiles with particles. It was mentioned in §1.4 that any particular tiling of an Aztec diamond by 2×1 dominoes has an underlying particle picture associated with it, and it was shown in [43] that these particles have the same behaviour as those of the GUE* eigenvalue process in a certain scaled limit, using the correlation functions for the particles, found in [41]. It is also shown in [41] that as N grows large, the 'free area' of the Aztec diamond (which in the particle picture corresponds to the positions of the particles for the first $N/2$ lines and the positions where the particles arent for lines $k > N/2$) takes the shape of a circle. In this section we will recreate these result, along with finding the known result of the number of possible tilings of an Aztec diamond, $2^{N(N+1)/2}$, by finding the joint PDF of these particles. Furthermore we introduce a half Aztec diamond, and show that the corresponding particle system has the same behaviour as the antisymetric GUE eigenvalue process in a certain scaled limit, and that the shape of the free area is that of a semi-circle as N grows large, using the same methods. These results have been reported in the publication 'Interlaced particle systems and tilings of the Aztec diamond' by Fleming and Forrester [24]

4.1 The joint PDF and finding A_N

As mentioned in §1.4, any tiling of an Aztec diamond of order N by 2×1 dominoes has a corresponding particle configuration. In an appropriate co-ordinate system, these particles occupy distinct positions $x_1^{(k)} > \cdots > x_k^{(k)}$ restricted to the lattice points $0, 1, 2, \ldots, N$ on line k ($k = 1, \ldots, N$), and obey the interlacing condition (1.47). A crucial point in relation to our study is the inverse mapping, from the particles to the tiles. It turns out that the mapping is not a bijection, and that certain tilings give rise to the same particle system. To see this, consider a particle at $x_i^{(k-1)}$ on line $k-1$. Suppose furthermore that one of the inequalities in (1.47) is an equality. Then there is just a single possible domino orientation corresponding to $x_i^{(k-1)}$. On the other hand, if the interlacing condition (1.47) holds with strict inequalities there are precisely two domino orientations corresponding to $x_i^{(k-1)}$. We will often refer to such dominoes as squares, because of the available space they take up in the tiling regardless of which orientation they have (See Figure 9). Importantly, this means that unlike with the hexagon where the joint PDF was a product of interlacing conditions with no other $x_i^{(j)}$ dependence, given a random tiling of the Aztec diamond with every possible tiling equally likely, the corresponding particle system must be weighted according to the number of particles for which the interlacing condition is strict. We introduce $\alpha(x^{(k)})$, the number of particles on line k for which one of the inequalities in (1.47) is an equality. Then

$$\alpha(x^{(k)}) = \sum_{i=1}^{k} \delta_{x_i^{(k)}, x_i^{(k+1)}} + \delta_{x_i^{(k)}, x_{i+1}^{(k+1)}}$$

and the PDF for the particles on a random tiling of an Aztec diamond of order N is given by

$$P^{\text{Aztec}}(x^{(1)}, \ldots, x^{(N)}) = \frac{2^{N(N+1)/2}}{A_N} \prod_{k=1}^{N} 2^{-\alpha(x^{(k)})} \chi^*(x^{(k)} \prec x^{(k+1)}) \tag{4.1}$$

where we have introduced the virtual particles $x_i^{(N+1)} = N + 1 - i$, A_N is the number of possible tilings of an Aztec diamond of order N, and $\chi^*(x^{(k)} \prec x^{(k+1)})$ is the condition that the interlacing

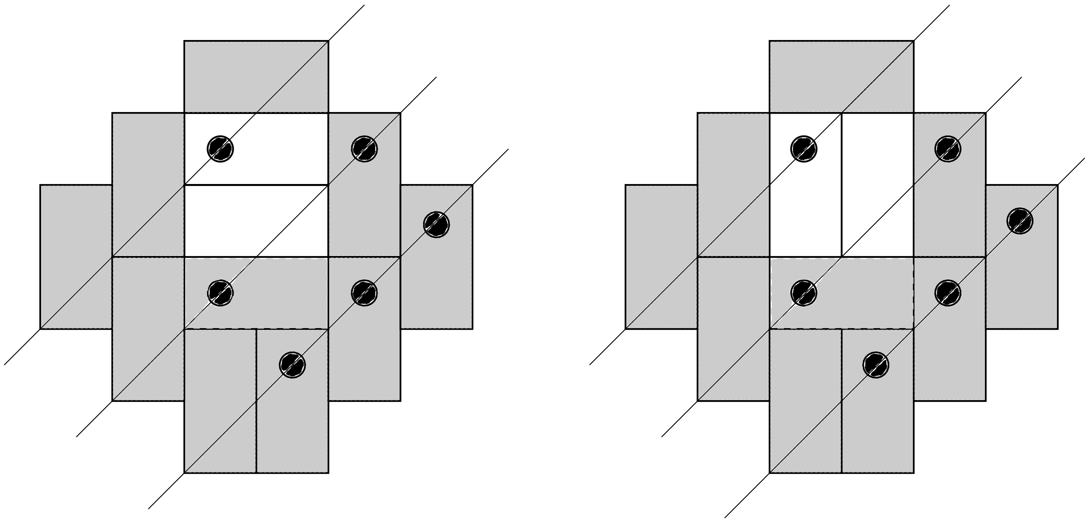


Figure 9: An example of two different tilings of an Aztec diamond of order 3 that have the same corresponding particle system. Note that the particle associated with the changed tiles obeys a strict interlacing with its two neighbours.

condition (1.47) holds.

Proposition 4.1. *For the particle system corresponding to uniform random tilings of the Aztec diamond of order N, when only considering lines $m,\dots,N$,* (4.1) *becomes*

$$P^{\text{Aztec}}_{(m,N)}(x^{(m)},\dots,x^{(N)}) = \frac{\Delta(x^{(m)})}{D_{m,N}} \prod_{k=m}^{N} 2^{-\alpha(x^{(k)})}\chi^*(x^{(k)} \prec x^{(k+1)}) \tag{4.2}$$

where

$$D_{m,N} = A_N 2^{-N(N+1)/2} \prod_{i=1}^{m-1} i! \tag{4.3}$$

Proof. Induction on m. The $m=1$ case is true from (4.1). Assuming the case is true for case m, consider case $m+1$.

$$\begin{aligned} P^{\text{Aztec}}_{(m+1,N)}(x^{(m+1)},\dots,x^{(N)}) &= \sum_{i=1}^{m} \sum_{x_i^{(m)}=0}^{N} \frac{\Delta(x^{(m)})}{D_{m,N}} \prod_{k=m}^{N} 2^{-\alpha(x^{(k)})}\chi^*(x^{(k)} \prec x^{(k+1)}) \\ &= \frac{1}{D_{m,N}} \prod_{k=m+1}^{N} \frac{\chi^*(x^{(k)} \prec x^{(k+1)})}{2^{\alpha(x^{(k)})}} \det\left[\sum_{t=a_i}^{b_i} \frac{t^{j-1}}{2^{\delta_{t,a_i}+\delta_{t,b_i}}}\right]_{i,j=1,\dots,m} \end{aligned}$$

where we have set $a_i = x^{(m+1)}_{m-i+2}$, $b_i = x^{(m+1)}_{m-i+1}$. The sum in the determinant is a polynomial function of a_i and b_i with highest degree term $(b_i^j - a_i^j)/j$. Since the lower degree terms will have the same dependence on a_i, b_i for each row i, they can be cancelled out by column operations. Thus

$$\begin{aligned} P^{\text{Aztec}}_{(m+1,N)}(x^{(m+1)},\dots,x^{(N)}) &= \frac{1}{D_{n,N}} \prod_{k=m+1}^{N} \frac{\chi^*(x^{(k)} \prec x^{(k+1)})}{2^{\alpha(x^{(k)})}} \det\left[\frac{b_i^j - a_i^j}{j}\right]_{i,j=1,\dots,m} \\ &= \frac{\Delta(x^{(m+1)})}{m! D_{m,N}} \prod_{k=m+1}^{N} 2^{-\alpha(x^{(k)})}\chi^*(x^{(k)} \prec x^{(k+1)}) \end{aligned}$$

where the determinant evaluation follows by noting that it must contain $\Delta(x^{(m+1)})$ as a factor, and is of the same degree as $\Delta(x^{(m+1)})$. The case $m+1$ has thus been established, provided $D_{m+1,N} = m!D_{m,N}$, which is indeed a property of (4.3). □

We use Proposition 4.1 along with the same steps of its proof, to see that

$$\sum_{i=1}^{N} \sum_{x_i^{(N)}=0}^{N} P_{(N)}^{\text{Aztec}}(x^{(N)}) = \frac{2^{N(N+1)/2}}{A_N \prod_{j=1}^{N} j!} \Delta(x^{(N+1)}) \tag{4.4}$$

Recalling that $x_i^{(N+1)} = N + 1 - i$ and noting that the LHS of (4.4) must be 1 by definition of probabilities, we conclude that

$$A_N = 2^{N(N+1)/2}$$

This present derivation of A_N using the particle picture appears to be new.

4.2 The one-line PDF

Consider a random tiling of an Aztec diamond of order N, shaded as in Figure 4. Imagine if instead we defined the particles as everywhere a line intersected an unshaded square. It is clear to see that this random particle system is the same as the original particle system, up to a relabelling of the lines $k \to N+1-k$. In our original particle model, this second set of particles fills exactly the lattice spaces that are not filled by the original set of particles. Thus, returning to our original set of particles with positions $x_i^{(k)}$, if we introduce new particles $y_i^{(k)}$ defined such that

$$\begin{aligned} x^{(k)} \cup y^{(N+1-k)} &= \{0, 1, \dots N\} \\ x^{(k)} \cap y^{(N+1-k)} &= \emptyset. \end{aligned}$$

then we must have

$$P_{(1,n)}^{\text{Aztec}}(x^{(1)}, \dots, x^{(n)}) = P_{(N+1-n,N)}^{\text{Aztec}}(y^{(N+1-n)}, \dots, y^{(N)}) \tag{4.5}$$

We use this fact to prove the following proposition

Proposition 4.2. *Let $\{x_i^{(j)}\}_{\substack{i=1,\dots,j \\ j=1\dots,N}}$ be the positions of the i-th particle on the j-th line in the particle system as described above of a random tiling of the Aztec diamond by 2×1 dominoes. The joint PDF for the $x_i^{(j)}$ with $j = 1, \dots, n$ is given by*

$$P_{(1,n)}^{\text{Aztec}}(x^{(1)}, \dots, x^{(n)}) = \frac{\Delta(x^{(n)})}{2^{N+(N-n)(n-1)}} \prod_{i=1}^{n} \frac{(N+1-i)!}{x_i^{(n)}!(N - x_i^{(n)})!} \prod_{k=1}^{n-1} \frac{\chi^*(x^{(k)})}{2^{\alpha(x^{(k)})}} \tag{4.6}$$

Proof. From (4.2)

$$P_{(N+1-n,N)}^{\text{Aztec}}(y^{(N+1-n)}, \dots, y^{(N)}) = \frac{\Delta(y^{(N+1-n)})}{D_{N+1-n,N}} \prod_{k=N+1-n}^{N} 2^{-\alpha(y^{(k)})} \chi^*(y^{(k)} \prec y^{(k+1)}) \tag{4.7}$$

Using the fact that

$$\Delta(\{0, 1, \dots N\}) = \prod_{i=1}^{N} i!$$

we have

$$\Delta(y^{(k)}) = \frac{\Delta(x^{(N+1-k)}) \prod_{i=1}^{N} i!}{\prod_{i=1}^{N+1-k} x_i^{(N+1-k)}!(N - x_i^{(N+1-k)})!} \tag{4.8}$$

Recalling the definition of α, we note that on any pair of lines k, $k+1$ there will be:

- $\alpha(y^{(k)})$ lattice positions with two y particles
- $\alpha(x^{(N-k)})$ lattice positions with two x particles
- $k-\alpha(y^{(k)}) = N-k-\alpha(x^{(N-k)})$ lattice positions remaining

Hence

$$\alpha(y^{(k)}) = 2k - N + \alpha(x^{(N-k)}) \tag{4.9}$$

So we have

$$\begin{aligned} P^{\text{Aztec}}_{(N+1-n,N)}(y^{(N+1-n)},\dots,y^{(N)}) &= \frac{1}{D_{N+1-n,N}} \frac{\Delta(x^{(n)})\prod_{i=1}^{N} i!}{\prod_{i=1}^{n} x_i^{(n)}!(N-x_i^{(n)})!} \\ &\times \prod_{k=N+1-n}^{N} 2^{-2k+N-\alpha(x^{(N-k)})}\chi^*(x^{(N-k)} \prec x^{(N-k+1)}) \end{aligned}$$

which is equal to the RHS of (4.7). The relation (4.5) completes the proof. □

Corollary 4.3. *Let* $\{x_i^{(j)}\}_{\substack{i=1,\dots,j\\ j=1\dots,N}}$ *be as in Proposition 4.2. Then the* $x_i^{(m)}$ *have PDF*

$$p(x^{(m)}) = \frac{\Delta(x^{(m)})^2}{2^{N+(N-m)(m-1)}\prod_{i=1}^{m} x_i^{(m)}!(N-x_i^{(m)})!} \prod_{i=0}^{m-1} \frac{(N-i)!}{i!} \tag{4.10}$$

Proof. (4.10) is a direct result or applying the methods of the proof of Proposition 4.1 to Proposition 4.2 for $m = n$. □

This has been derived using different arguments in [40] (in particular the weighted particle system is not specified by (4.1)), where it is recognised as a particular example of a discrete orthogonal polynomial unitary ensemble based on a particular Krawtchouk weight with $p = \frac{1}{2}$. In the case $p = \frac{1}{2}$, the monic Krawtchouk polynomials obey the discrete orthogonality equation

$$\sum_{x=0}^{N} \frac{1}{x!(N-x)!} p_{a,N}(x) p_{b,N}(x) = \frac{2^N a!}{2^{2a}(N-a)!}\delta_{a,b} \tag{4.11}$$

(see e.g. [49]) and so we can use these to check the normalization constant in (4.10) by noting it is of the form in (2.1) with

$$w(x) = \frac{1}{x!(N-x)!}$$

and using Lemma 2.1. Thus, with (4.10) written

$$p(x^{(m)}) = \frac{1}{C}\Delta(x^{(m)})^2 \prod_{i=1}^{m} \frac{1}{x_i^{(m)}!(N-x_i^{(m)})!}$$

we must have, by Lemma 2.1,

$$C = \prod_{i=0}^{m-1} \frac{2^{N-2i} i!}{(N-i)!}$$

which is indeed consistent with (4.10)

4.3 Large N

As mentioned earlier, it was shown in [43] that in a certain scaled limit, the particle system for the Aztec diamond converges to the GUE* eigenvalue process. We will show this convergence by scaling the joint PDF found in Proposition 4.1.

Proposition 4.4. *Let the points $z_i^{(j)} := (2x_i^{(j)} - N)/\sqrt{N}$ be a rescaling of the points $x_i^{(j)}$ in the Aztec diamond particle system as described above, where N is the order of the Aztec diamond. Given that the $x_i^{(j)}$ have PDF $P^{\text{Aztec}}_{(1,n)}$ as described in* (4.6)*, one has*

$$P^{\text{Aztec}}_{(1,n)}(x^{(1)}, \dots, x^{(n)}) \to P_{GUE^*,n}(z^{(1)}, \dots, z^{(n)})$$

where $P_{GUE^*,M}$ is as in (1.29)*, as $N \to \infty$, where the convergence is uniform on compact sets.*

Proof. Changing variables from x to z in (4.6) and applying Stirling's approximation (1.16), noting that

$$\prod_{k=1}^{n} \bigwedge_{i=1}^{k} dx_i^{(k)} = \left(\frac{N}{4}\right)^{n(n+1)/4} \prod_{k=1}^{n} \bigwedge_{i=1}^{k} dz_i^{(k)}$$

gives

$$P^{\text{Aztec}}_{(1,n)}(x^{(1)}, \dots, x^{(n)}) = \frac{\Delta(z^{(n)})}{(2\pi)^{n/2}} e^{-\sum_{i=1}^{n} \frac{1}{2}(z_i^{(n)})^2} \prod_{j=1}^{n-1} \chi(z^{(j)} \prec z^{(j+1)}) + \mathrm{O}(N^{-1/2})$$

Comparing this with (1.29) we see that the leading order term is equal to $P_{GUE^*,n}(z^{(1)}, \dots, z^{(n)})$. □

In addition to the link to the GUE* eigenvalue process, it was shown in [41] that in a tiling of an Aztec diamond of order N, the 'free area', the area of the Aztec diamond containing both horizontal and vertical tiles, converges in leading order to a perfect circle as $N \to \infty$. We will recreate this result in the particle interpretation, by finding the region of support of the particles by applying the log-gas method as used in §2.2 and §3.3 to the one line PDF for the particles from Corollary 4.3.

Proposition 4.5. *Let $x_i^{(sN)}$ represent the the i-th largest particle on the sN-th line in the particle model associated with a tiling of an Aztec diamond of order N, where $0 \le s \le \frac{1}{2}$. To leading order*

$$\frac{N}{2}\left(1 - \sqrt{1-(1-2s)^2}\right) \le x_i^{(sN)} \le \frac{N}{2}\left(1 + \sqrt{1-(1-2s)^2}\right) \tag{4.12}$$

Proof. We will use the same log-gas method as for the GUE* and hexagon bead model. As written, the one line PDF (4.10) in Corollary 4.3 is a lattice gas variant of the log-gas (2.33) in the case $\beta = 2$. In the limit $N \to \infty$, the lattice gas approaches the continuum log-gas upon the substitution

$$x_i^{(sN)} = N y_i^{(sN)} \tag{4.13}$$

where, to leading order in N, $0 \le y_i^{(sN)} \le 1$. Thus, we consider a log-gas with $N_p = sN$ particles and Boltzmann factor as in (2.33) with $\beta = 2$ and

$$e^{-2V(t)} = \frac{1}{(Nt)!(N - Nt)!}$$

and, as before, we aim to solve (2.41), (2.42) for c and d in the limit $N \to \infty$. Noting that $V(t) = V(1-t)$, (2.41) gives that for some $0 \le W \le 1/2$, $c = 1/2 - W$ and $d = 1/2 + W$. Noting that

$$\lim_{N\to\infty} \frac{2V'(y)}{N} = \log\left(\frac{y}{1-y}\right)$$

we compute that, in the limit $N \to \infty$, to leading order (2.42) reduces to

$$\int_{\frac{1}{2}-W}^{\frac{1}{2}+W} \frac{t\log(t/(1-t))}{\sqrt{(\frac{1}{2}+W-t)(t+W-\frac{1}{2})}}dt = 2\pi s$$

A change of variables $t = u + 1/2$ reduces this further, to give

$$\int_{-W}^{W} \frac{u\log(1+2u)}{\sqrt{W^2-u^2}}du = \pi s. \tag{4.14}$$

This integral can be computed exactly for $0 \le s \le 1/2$, giving

$$1 - \sqrt{1-4W^2} = 2s$$

and thus, to leading order in N,

$$\frac{1}{2}\left(1-\sqrt{1-(1-2s)^2}\right) \le y_i^{(sN)} \le \frac{1}{2}\left(1+\sqrt{1-(1-2s)^2}\right)$$

Reversing the change of variables (4.13) gives (4.12). □

Corollary 4.6. *[41] The so called 'free area', or disordered region, of a tiling of the Aztec diamond of order N, is to leading order a perfect circle as $N \to \infty$.*

Proof. By the definition the region of support and the relationship between the tiling and the particle model, the disordered region of the Aztec diamond is the intersection of the region of support of the particles, which correspond to the shaded tiles, and the region of support of the 'holes', the lattice points which contain no particle, which correspond to the unshaded tiles (see Figure 4). Recalling the relationship between the particles and the holes, in particular (4.5), we must have that for $1/2 \le s \le 1$, the region of support of the particles is $[0, N]$, and for all $0 \le s \le 1$, the region of support of the the holes on line sN is given by the region of support of the particles on line $(1-s)N$. Thus, for $c(sN)$, $d(sN)$ the upper and lower bounds of the disordered region of the sN-th line of a tiling of the Aztec diamond of order N,

$$c(sN) = N(1-\sqrt{1-(1-2s)^2})/2$$
$$d(sN) = N(1+\sqrt{1-(1-2s)^2})/2$$

The graphs of these two functions on the s, y-plane give a circle centred at $(N/2, N/2)$ with radius $N/2$. □

4.4 Half Aztec diamond

Consider an Aztec diamond of order $N = 2(M+1)$, and define a restriction on the tiling of this Aztec diamond such that in the particle picture as defined above, a particle at x on line j implies no particle at x on line $N+1-j$. Because of the interlacing restriction, this means that in the tiling picture the whole middle column between lines $k = M+1$ and $M+2$ will consist of free squares as seen in Figure 9 (the interlacing will be strict between these two lines). If we delete everything to the right of these squares (so that we are left with just the lines $k = 1, \dots, M$) we are left with what we will call a half Aztec diamond of order M.

Consider an Aztec diamond of order $N = 2(M+1)$ tiled such that its middle column is all squares as above (lines $M+1$ and $M+2$ interlace strictly) and that the tiling is symmetric about this column of squares. We will call any such tiling of an Aztec diamond symmetric. With A^*_N the number of symmetric tilings of an Aztec diamond of order N and H_M is the number of tilings of a half Aztec diamond of order M, it is clear that we must have

$$A^*_{2(M+1)} = H_M 2^{M+1}$$

We would like to initiate our study of the half Aztec diamond by using the particle picture to compute H_M. We begin by noting that the joint PDF for the weighted particle system in a half Aztec diamond is

$$P^{\text{HAztec}}(x^{(1)}, \dots, x^{(M)}) = \frac{2^{M(M+1)/2}}{H_M} \prod_{k=1}^{M} 2^{-\alpha(x^{(k)})} \chi^*(x^{(k)} \prec x^{(k+1)}) \tag{4.15}$$

(cf. (4.1)) with the restriction that the particles on line $M+1$ are of the form $x_i^{(M+1)} = 2M+3-2i$. Using the same method as in the derivation of Proposition 4.1 it follows from (4.15) that

$$\sum_{i=1}^{M} \sum_{x_i^{(M)}=0}^{2M+2} P^{\text{HAztec}}_{(M)}(x^{(M)}) = \frac{2^{M(M+1)/2}}{H_M \prod_{j=1}^{M} j!} \Delta(x^{(M+1)})$$

Noting that the LHS must equal 1, we evaluate the RHS with $x_i^{(M+1)} = 2M+3-2i$ to find

$$H_M = 2^{M(M+1)} \tag{4.16}$$

and thus $C^*_{2N} = 2^{N^2}$. Note that

$$\lim_{N\to\infty} \frac{1}{N^2} \log A_N = 2 \lim_{N\to\infty} \frac{1}{N^2} \log H_M \Big|_{M=N/2-1}$$

as to be expected from the interpretations of these quantities as entropies for the tiling problem. Thus to leading order the entropy is proportional to the volume, and the value of the full Aztec diamond is twice that of the half Aztec diamond.

There is a second particle system associated with symmetric tilings. This is obtained by rotating the half Aztec diamond — which has M vertical lines — by 90^o to obtain a half Aztec diamond positioned with long side horizontal and thus having $N = 2(M+1)$ vertical lines. The first of these is empty of particles and last one is full. Ignoring these two lines we have $2M$ lines where successive lines $2n-1$ and $2n$ ($n = 1, \dots, M$) have n particles. See Figure (10) for an example. We would like to develop the properties of this particle system.

Analogous to (4.15), although with H_M substituted by its evaluation (4.16), the joint PDF for this weighted particle system is

$$P^{\text{HAztec2}}(x^{(1)}, \dots, x^{(2M)}) = \prod_{k=1}^{2M} 2^{-\alpha(x^{(k)})} \chi^*(x^{(k)} \prec x^{(k+1)}) \tag{4.17}$$

where the virtual particles $x_i^{(2M+1)} = M+2-i$ have been introduced.

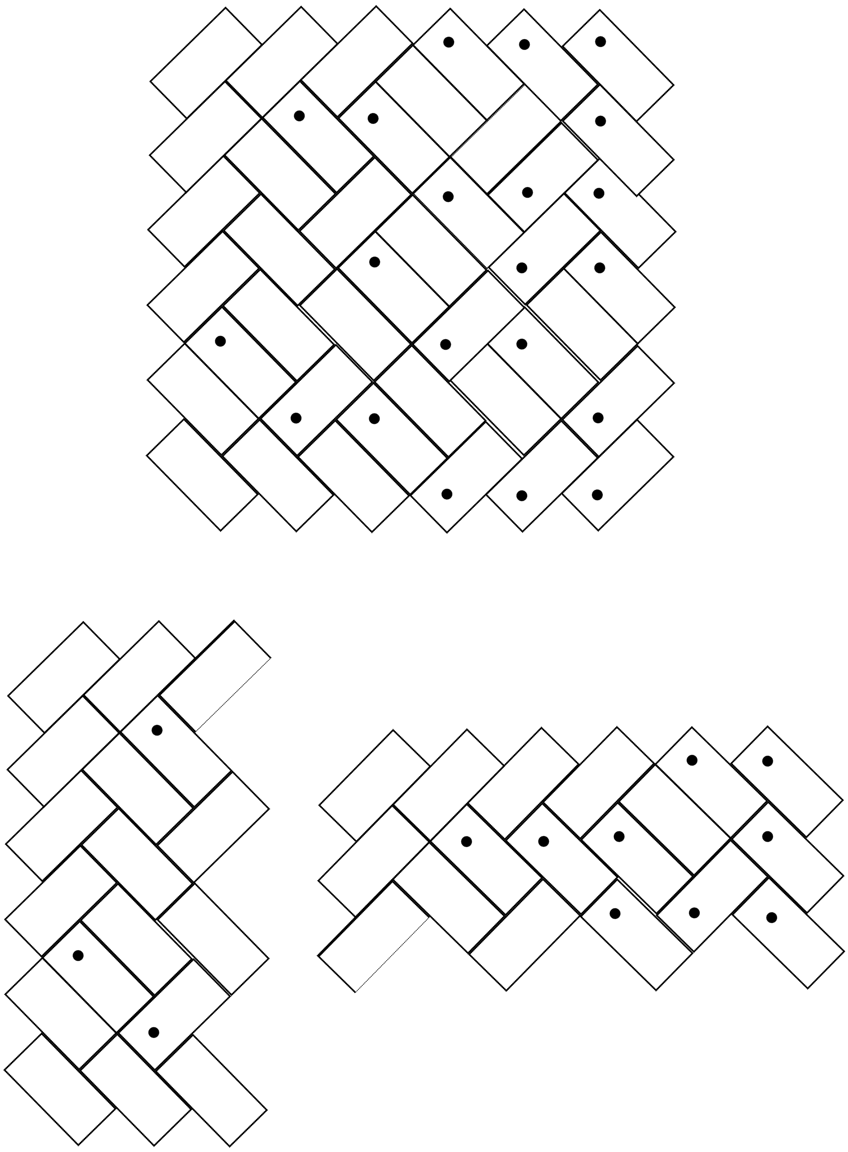


Figure 10: Top: a tiling of an Aztec diamond of order 6 with the symmetry restriction described at the beginning of §4.4. Bottom left: The result after deleting lines 3 through 6, a half Aztec diamond of order 2. Bottom right: The same half Aztec diamond rotated 90^o, with particles as described above (4.17)

Proposition 4.7. *Let*

$$H_{2m-1,M} = \prod_{i=1}^{m-1} (2i)! \qquad H_{2m,M} = \prod_{i=1}^{m-1} (2i+1)! \tag{4.18}$$

For the particle system corresponding to a uniform random tilings of the half Aztec diamond of order M as described above, when only considering lines $2n-1, \ldots, 2M$, (4.17) becomes

$$P^{\text{HAztec2}}_{(2n-1,2M)}(x^{(2n-1)}, \ldots, x^{(2M)}) = \frac{\Delta\left((x^{(2n-1)} - \frac{1}{2})^2\right)}{H_{2n-1,M}} \prod_{k=2n-1}^{2M} 2^{-\alpha(x^{(k)})} \chi^*(x^{(k)} \prec x^{(k+1)})$$

while when only considering lines $2n, \ldots, 2M$, (4.17) becomes

$$P^{\text{HAztec2}}_{(2n,2M)}(x^{(2n)}, \ldots, x^{(2M)}) = \frac{\Delta\left((x^{(2n)} - \frac{1}{2})^2\right) \prod_{i=1}^{n} (x_i^{(2n)} - \frac{1}{2})}{H_{2n,M}} \prod_{k=2n}^{2M} 2^{-\alpha(x^{(k)})} \chi^*(x^{(k)} \prec x^{(k+1)})$$

Proof. We proceed as in the proof of Proposition 4.1. The $2n-1 = 1$ case is true from (4.17). Assume the $n = 2m-1$ case is true. Then summing (4.7) on the $(2m-1)$-th line gives

$$\sum_{x_1^{(2m-1)} = x_2^{(2m)}}^{x_1^{(2m)}} \cdots \sum_{x_m^{(2m-1)} = 1}^{x_m^{(2m)}} \frac{\Delta\left((x^{(2m-1)} - \frac{1}{2})^2\right)}{H_{2m-1,M}} \prod_{k=2m-1}^{2M} 2^{-\alpha(x^{(k)})} \chi^*(x^{(k)} \prec x^{(k+1)})$$
$$= \frac{1}{H_{2m-1,M}} \det\left[d_{i,j}\right]_{i,j=1,\ldots,m} \prod_{k=2m}^{2M} 2^{-\alpha(x^{(k)})} \chi^*(x^{(k)} \prec x^{(k+1)})$$

where, with $a_i = x^{(2m)}_{m-i+2}$, $b_i = x^{(2m)}_{m-i+1}$

$$d_{i,j} = \begin{cases} \sum_{t=1}^{b_1} 2^{-\delta_{t,b_1}} (t - \frac{1}{2})^{2(j-1)}, & i = 1 \\ \sum_{t=a_i}^{b_i} 2^{-\delta_{t,a_i} - \delta_{t,b_i}} (t - \frac{1}{2})^{2(j-1)}, & i = 2, \ldots, m \end{cases}$$

This implies

$$P^{\text{HAztec2}}_{(2m,2M)}(x^{(2m)}, \ldots, x^{(2M)}) = \frac{2^m m!}{H_{2m-1,M}(2m)!}$$
$$\times \det\left[\left(x^{(2m)}_{m-i-1} - \frac{1}{2}\right)^{2(j-1)}\right]_{i,j=1,\ldots,m} \prod_{i=1}^{m} \left(x_i^{(2m)} - \frac{1}{2}\right) \prod_{k=2m}^{2M} 2^{-\alpha(x^{(k)})} \chi^*(x^{(k)} \prec x^{(k+1)})$$

which recalling (4.18) establishes the case $n = 2m$. Repeating this process. summing now (4.7) on line $2m$, gives

$$P^{\text{HAztec2}}_{(2m+1,2M)}(x^{(2m+1)}, \ldots, x^{(2M)}) = \frac{\Delta\left((x^{(2n-1)} - \frac{1}{2})^2\right)}{2^m m! H_{2m,M}} \prod_{k=2n-1}^{2M} 2^{-\alpha(x^{(k)})} \chi^*(x^{(k)} \prec x^{(k+1)})$$

and this (again recalling (4.18)) establishes the case $n = 2m+1$. □

As above with the Aztec diamond, the 'holes', or possible positions $\{1, \ldots, M+1\}$ that do not contain a particle in this half Aztec diamond picture, form a second system with the same probabilistic law as the first read right to left rather than left to right. Using this, we find a result analogous to Proposition 4.2.

Proposition 4.8. *Let* $\{x_i^{(j)}\}_{\substack{i=1,\dots,\lceil j/2\rceil \\ j=1\dots,2M}}$ *be the positions of the i-th particle on the j-th line in the particle system as described above of a random tiling of a half Aztec diamond of order M by* 2×1 *dominoes. Then, for*

$$G_{1,2n-1} = \frac{2^{2n(M-n+1)}}{\prod_{j=1}^{n}(2M-2n+2j+1)!} \qquad G_{1,2n} = \frac{2^{2n(M-n)}}{\prod_{j=1}^{n}(2M-2n+2j)!} \tag{4.19}$$

the joint PDF for the $x_i^{(j)}$ *with* $j=1,\dots,2n-1$ *is given by*

$$\begin{aligned} P^{\text{HAztec2}}_{(1,2n-1)}(x^{(1)},\dots,x^{(2n-1)}) =& \frac{1}{G_{1,2n-1}} \prod_{i=1}^{n} \frac{1}{(x_i^{(2n-1)}+M)!(M+1-x_i^{(2n-1)})!} \\ &\times \prod_{1\le i<j\le n}\left((x_i^{(2n-1)}-\frac{1}{2})^2-(x_j^{(2n-1)}-\frac{1}{2})^2\right)\prod_{k=1}^{2n-2}2^{-\alpha(x^{(k)})}\chi^*(x^{(k)}\prec x^{(k+1)}) \end{aligned} \tag{4.20}$$

while the joint PDF for the $x_i^{(j)}$ *with* $j=1,\dots,2n$ *is given by*

$$\begin{aligned} P^{\text{HAztec2}}_{(1,2n)}(x^{(1)},\dots,x^{(2n)}) =& \frac{1}{G_{1,2n}} \prod_{i=1}^{n} \frac{(x_i^{(2n)}-\frac{1}{2})}{(x_i^{(2n)}+M)!(M+1-x_i^{(2n)})!} \\ &\times \prod_{1\le i<j\le n}\left((x_i^{(2n)}-\frac{1}{2})^2-(x_j^{(2n)}-\frac{1}{2})^2\right)\prod_{k=1}^{2n-1}2^{-\alpha(x^{(k)})}\chi^*(x^{(k)}\prec x^{(k+1)}) \end{aligned} \tag{4.21}$$

Proof. As with the proof of Proposition 4.2, we introduce particles $y_i^{(k)}$ defined such that

$$\begin{aligned} x^{(k)}\cup y^{(2M+1-k)} &= \{1,\dots M\} \\ x^{(k)}\cap y^{(2M+1-k)} &= \emptyset \end{aligned}$$

so that we have

$$P^{\text{HAztec2}}_{(1,n)}(x^{(1)},\dots,x^{(n)}) = P^{\text{HAztec2}}_{(2M+1-n,2M)}(y^{(2M+1-n)},\dots,y^{(2M)}) \tag{4.22}$$

Noting that

$$\prod_{1\le i<j\le n}\left((x_i-\frac{1}{2})^2-(x_j-\frac{1}{2})^2\right) = \prod_{1\le i\le j<n}(x_i-x_j)(x_i+x_j-1)$$

we use the fact that

$$\prod_{1\le i<j\le M+1}(i+j-1) = \prod_{i=1}^{M}\frac{(2i)!}{i!}$$

to see that

$$\begin{aligned} &\prod_{1\le i\le j<n}(y_i^{(n)}+y_j^{(n)}-1) \\ &= \prod_{1\le i\le j<n}(x_i^{(2M-n+1)}+x_j^{(2M-n+1)}-1)\prod_{i=1}^{r}\frac{(2x_i^{(2M-n+1)}-1)(x_i^{(2M-n+1)}-1)!}{(x_i^{(2M-n+1)}+M)!}\prod_{j=1}^{M}\frac{(2j)!}{j!} \end{aligned} \tag{4.23}$$

where $r=|x^{(2M-n+1)}|$. It is also not hard to compute that

$$\prod_{i=1}^{n}(y_i^{(2n)}-\frac{1}{2}) = \frac{(2M+2)!}{2^{2M+2}(M+1)!}\frac{1}{\prod_{i=1}^{M-n+1}(x_i^{(2M-2n+1)}-\frac{1}{2})} \tag{4.24}$$

Taking $P^{\text{HAztec2}}_{(2M+2-2n,2M)}(y^{(2M+2-2n)},\dots,y^{(2M)})$ as in (4.7) and applying (4.23), (4.24) and slightly modified versions of (4.8) and (4.9) gives the RHS of (4.20).

Similarly, taking $P^{\text{HAztec2}}_{(2M+1-2n,2M)}(y^{(2M+1-2n)},\dots,y^{(2M)})$ as in (4.7) and applying (4.23) and slightly modified versions of (4.8) and (4.9) gives the RHS of (4.21). The relation (4.22) completes the proof. □

Corollary 4.9. *Let the $x_i^{(j)}$ be as in Proposition 4.8. Then, for $H_{n,M}$ as in (4.18) and $G_{1,n}$ as in (4.19), the $x_i^{(2n-1)}$ have PDF*

$$P^{\text{HAztec2}}_{(2n-1)}(x^{(2n-1)}) = \frac{1}{G_{1,2n-1}H_{2n-1,M}}\prod_{i=1}^{n}\frac{1}{(x_i^{(2n-1)}+M)!(M+1-x_i^{(2n-1)})!} \times \prod_{1\le i<j\le n}\left((x_i^{(2n-1)}-\frac{1}{2})^2-(x_j^{(2n-1)}-\frac{1}{2})^2\right)^2 \tag{4.25}$$

while the $x_i^{(2n)}$ have PDF

$$P^{\text{HAztec2}}_{(2n)}(x^{(2n)}) = \frac{1}{G_{1,2n}H_{2n,M}}\prod_{i=1}^{n}\frac{(x_i^{(2n)}-\frac{1}{2})^2}{(x_i^{(2n)}+M)!(M+1-x_i^{(2n)})!}\prod_{1\le i<j\le n}\left((x_i^{(2n)}-\frac{1}{2})^2-(x_j^{(2n)}-\frac{1}{2})^2\right)^2 \tag{4.26}$$

We remarked above that the one-line PDF (4.10) corresponds to a discrete orthogonal polynomial unitary ensemble based on a particular Krawtchouk weight. Again, we may use he monic Krawtchouk polynomials (4.11) to check the normalizations in (4.25) and (4.26). From (4.11) we see that

$$\sum_{x=-M}^{M+1}\frac{1}{(x+M)!(M+1-x)!}p_{a,2M+1}(x+M)p_{b,2M+1}(x+M) = \frac{2^{2M+1}a!}{2^{2a}(2M+1-a)!}\delta_{a,b}$$

Using the fact that this weight function is even about the point $x=1/2$, a simple mapping from $x \to x+\frac{1}{2}$ allows us to use Lemma 2.3, and so we must have

$$G_{1,2n}H_{2n,M} = \prod_{i=0}^{n-1}\frac{2^{2M}(2i)!}{2^{4i}(2M+1-2i)!}$$
$$G_{1,2n}H_{2n,M} = \prod_{i=0}^{n-1}\frac{2^{2M}(2i+1)!}{2^{4i+2}(2M-2i)!}$$

which is consistent with the definitions of H and G, (4.18) and (4.19).

As the Aztec diamond particle system converged to the GUE* eigenvalue process in a certain limit, this half Aztec diamond particle system can be shown to coverge to the Antisymmetric GUE eigenvalue process in a certain limit.

Proposition 4.10. *Let the points $\lambda_i^{(j+1)} := x_i^{(j)}/\sqrt{M}$ be a rescaling of the points $x_i^{(j)}$ in the half Aztec diamond picture as described above, where M is the order of the half Aztec diamond. Given that the $x_i^{(j)}$ have PDF $P^{\text{HAztec2}}_{(1,n)}$ as described in Proposition 4.8, one has*

$$P^{\text{HAztec2}}_{(1,n)}(x^{(1)},\dots,x^{(n)}) \to P_{\text{AntiSymGUE},(n)}(\lambda_i^{(1)},\dots,\lambda_i^{(n+1)})$$

where $P_{\text{AntiSymGUE},(n)}$ is given by (1.60) for n even and (1.61) for n odd, as $M\to\infty$, where the convergence is uniform on compact sets.

Proof. The proof involves a simple change of variables and use of Stirling's formula (1.16), analogous to the proof of Proposition 4.4. □

We will conclude by using the log-gas method outlined in §2.2 to show that the free area of the half Aztec diamond is a half-circle.

Proposition 4.11. *Let $x_i^{(sM)}$ represent the the i-th largest particle on the sN-th line in the particle model associated with a tiling of a Half Aztec diamond of order M, where $0 \le s \le 1$. To leading order*

$$0 \le x_i^{(sM)} \le M\sqrt{1-(1-s)^2} \tag{4.27}$$

Proof. The lower bound is implied. To find the upper bound, we proceed using the same method as for the full Aztec diamond case in §4.3. Here however, we will be dealing with Boltzmann factors of the form

$$\prod_{1\le i<j\le N_p} |x_i^2 - x_j^2|^\beta \prod_{k=1}^{N_p} e^{-\beta V(x_k)} \tag{4.28}$$

so (2.34) becomes

$$V(x) := \int_0^a \rho(t) \log |x^2 - t^2| dt.$$

If we define $\rho(-x) := \rho(x)$, then this can be expressed

$$V(x) = \int_{-a}^a \rho(y) \log |x-y| dy$$

similar to (2.34), although now (2.35) becomes

$$\int_{-a}^a \rho(y) dy = 2N_p. \tag{4.29}$$

In the limit $M \to \infty$, (4.25) and (4.26) approach a continuum log-gas (4.28) upon the substitution $x = Mz$ where, to leading order, $0 \le z \le 1$.

In terms of the co-ordinate $z_i = z_i^{(2n-1)}$, the one body factor in (4.28) reads

$$e^{-2V(z)} = \frac{1}{\left(M(1+z)+\frac{1}{2}\right)!\left(M(1-z)+\frac{1}{2}\right)!} \tag{4.30}$$

On line sM, $N_p = sM/2$ and, from (4.30),

$$\lim_{M\to\infty} \frac{2V'(z)}{M} = \log\left(\frac{1+z}{1-z}\right). \tag{4.31}$$

According to (2.42) but taking into account (4.29), a is given by solving

$$\int_{-a}^a \frac{t\log\left(\frac{1+t}{1-t}\right)}{\sqrt{a^2-t^2}} dt = 2\pi s,$$

(cf. (4.14)) and the integral can be evaluated to give

$$1 - \sqrt{1-a^2} = s$$

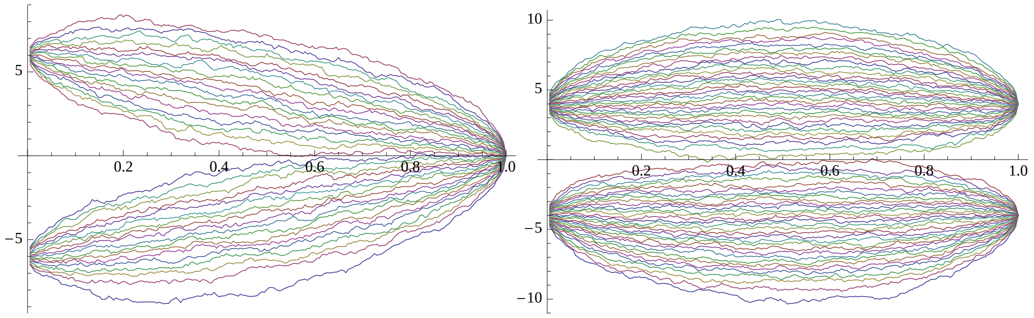
Rearranging this and then converting from z to x gives (4.27).

In terms of the co-ordinate $z_i = z_i^{(2n)}$, the one body factor in (4.28) reads

$$e^{-2V(z)} = \frac{Mz}{\big(M(1+z)+\frac{1}{2}\big)!\,\big(M(1-z)+\frac{1}{2}\big)!}$$

This gives the same equation for $V'(z)$ as in (4.31), and thus proceeding as above leads to the same upper bound (4.27). □

Corollary 4.12. *The so called 'free area', or disordered region, of a tiling of a Half Aztec diamond of order M, is to leading order a perfect half-circle as $M \to \infty$.*



(a) Pearcey

(b) tacnode

Figure 11: Examples of the Pearcey and tacnode systems with 36 and 64 walkers respectively

5 Future directions

In this section a brief account of some topics relating to those of this thesis, and which have received recent attention in the literature, will be reviewed.

We begin by considering two variants of the non-intersecting Brownian walker model displayed in Figure 6. The first of these involves partitioning the walkers into two halves. The first set of walkers are to start from $x = -\sqrt{N}$ and the second from $x = \sqrt{N}$. Both sets are conditioned not to intersect, and to return to the origin at time $2T$. Analysis of the neighbourhood of $t = T$ and the x_j near zero gives rise to what is referred to in the literature as the Pearcey kernel, first identified in [12, 13] (see Figure 11a).

As a model of non-intersecting Brownian walkers the study of Daems and Kuijlaars [16] extend this setting by formulating the correlation kernel in the case of a prescribed number of starting and finishing points in terms of certain multiple orthogonal polynomials. Of this extension, the most interesting is when there are two starting points and two end points, both symmetrical about the origin (see Figure 11b). The starting positions can be chosen so that the envelopes meet at a single point. The correlation kernel about the meeting point of the two envelopes was subsequently analyzed in Adler, Ferrari and van Moerbeke [1], Borodin and Duits [5] and most recently by Johansson [42]. Its limiting form is referred to as the tacnode kernel.

In following the limiting relationship between Brownian walkers and hexagon tiling problems as discussed in §1.7, we would imagine that both the singularities in the Brownian walkers problems giving rise to the Pearcey and tacnode kernels can also be realized as the limits of hexagon like tiling problems. Rather than the tiling of a full hexagon, sections must be removed to generate the seperate starting points and ending points required in the Pearcey and tacnode systems. In the case of the Pearcey process this was noted in the equivalent form of a directed solid-on-solid model in the work of Okounkov and Reshetikhin, [56]. A natural candidate for the realisation of the tacnode is an indented hexagon (see Figure 12).

Another topic of recent attention is that of the dynamics of particle systems. In [53], Nordenstam interprets the shuffling algorithm of [60] as a dynamics on the interlaced particle system in tilings of the Aztec diamond. This dynamics has the feature that the positions of the particles on line $j+1$ at time $t+1$ depend only on the positions of the particles on lines j and $j+1$ at time t. If we represent the position of the i-th largest particle in column j at time t as $x_i^j(t)$, then

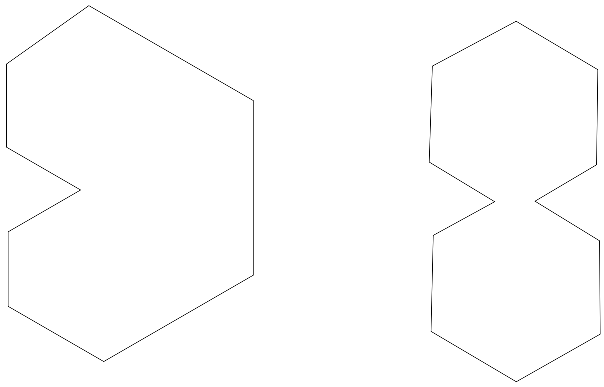


Figure 12: The basic shapes which would lead to the Pearcey and tacnode systems respectively

$x_i^j(j) = j - i + 1$ and $x_i^j(t+1) = x_i^j(t) + y_{i,j,t}$ where $y_{i,j,t} = 0$ or 1, that is the particles 'first appear' filling the lowest positions of their line, and then either stay where they are or move up one place. When both $y_{i,j,t} = 0$ and $y_{i,j,t} = 1$ result in $x_i^j(t+1)$ obeying the interlacing condition $x_i^{j-1}(t) < x_i^j(t+1) \leq x_{i-1}^{j-1}(t)$, then $y_{i,j,t}$ takes either value with probability 0.5. When only one of $y_{i,j,t} = 0$ or $y_{i,j,t} = 1$ results in the interlacing being obeyed, then that is the value taken. As an explicit example, here is a realisation of the shuffling algorithm at $t = 3$, highlighting the different situations in which $y_{i,j,t}$ can or must take different values.

$$\begin{array}{ccccl} & \bullet & \bullet & \leftarrow \text{Must move,} & y_{1,3,3} = 1 \\ \bullet & & \bullet & \leftarrow \text{Free to move,} & y_{2,3,3} = 0 \text{ or } 1 \\ & \bullet & \bullet & \leftarrow \text{Can't move,} & y_{3,3,3} = 0 \end{array}$$

We introduce new variables $X_i^j(t) = x_i^j(t+j)$, and consider the the trajectories in t for given i and j. These obey the interlacing condition

$$X_{i+1}^j(t) \leq X_i^{j-1}(t) < X_i^j(t)$$

This turns out to be a discrete counterpart to an interlaced Brownian motion model studied by Warren [66], which involves Brownian paths in a way that is similar to but different from the non-intersecting paths discussed in §1.7. The idea is to begin with one Brownian path (corresponding to the trajectory $X_1^1(t)$ in the discrete Aztec diamond model), then insert two Brownian motions, one conditioned to be always above the first path, and the other conditioned to be always below the first path (these Brownian motions correspond to the discrete trajectories $X_1^2(t)$ and $X_2^2(t)$. Next, the first Brownian motion is ignored, and three Brownian motions are inserted, conditioned to intersect with the previous two Brownian motions (corresponding to discrete trajectories $X_1^3(t)$, $X_2^3(t)$ and $X_3^3(t)$) and then the second two Brownian motions are ignored and four Brownian motions are inserted, conditioned to intersect with the previous three Brownian motions, and so on. The distribution of the n Brownian motions corresponding to the trajectories of $X_i^n(t)$, $i = 1, \ldots, n$ is that of n non-intersecting Brownian paths as evaluated in §1.7. Furthermore, transition probabilities between the Brownian motions corresponding to $X^j(t)$ and $X^{j+1}(t)$ are shown by Warren to have determinantal formulas.

6 Appendix

6.1 $\chi(\mu \prec \kappa)$

Throughout this thesis, we often have the requirement that two sets, μ and κ say, interlace, which we will represent by multiplying by $\chi(\mu \prec \kappa)$. Here we will give a precise meaning.

Let $\mu = \{\mu_1, \ldots, \mu_m\}$, $\kappa = \{\kappa_1, \ldots, \kappa_k\}$ be two sets of m and k real numbers respectively, labelled so that $\mu_1 > \mu_2 > \cdots > \mu_m$ and $\kappa_1 > \kappa_2 > \cdots > \kappa_m$.

- If $|k - m| \geq 2$, $\chi(\mu \prec \kappa) = 0$.
- If $k = m$,

$$\chi(\mu \prec \kappa) = \begin{cases} 1 & \text{if} \quad \kappa_1 > \mu_1 > \kappa_2 > \mu_2 > \cdots > \kappa_m > \mu_m \\ 0 & \quad\text{otherwise} \end{cases}$$

- If $k = m + 1$,

$$\chi(\mu \prec \kappa) = \begin{cases} 1 & \text{if} \quad \kappa_1 > \mu_1 > \kappa_2 > \mu_2 > \cdots > \kappa_m > \mu_m > \kappa_{m+1} \\ 0 & \quad\text{otherwise} \end{cases}$$

- If $k = m - 1$,

$$\chi(\mu \prec \kappa) = \begin{cases} 1 & \text{if} \quad \mu_1 > \kappa_1 > \mu_2 > \kappa_2 \cdots > \kappa_{m-1} > \mu_m \\ 0 & \quad\text{otherwise} \end{cases}$$

An important feature of this is that, where $|k - m| = 1$, $\chi(\mu \prec \kappa) = \chi(\kappa \prec \mu)$

6.2 Chen Ismail correction

Chen and Ismail [14] use the asymptotic method of Darboux (see e.g. [65]) applied to the generating function

$$\sum_{n=0}^{\infty} P_n^{(\alpha+an,\beta+bn)}(x)t^n = (1+\xi)^{\alpha+1}(1+\eta)^{\beta+1}[1 - a\xi - b\eta - (1+a+b)\xi\eta]^{-1} =: f(t), \tag{6.1}$$

where ξ and η depend on x and t according to

$$\xi = \frac{1}{2}(x+1)t(1+\xi)^{1+a}(1+\eta)^{1+b} \text{ and } \eta = \frac{x-1}{x+1}\xi \tag{6.2}$$

The basic idea is to identify and analyze the neighbourhood of the t-singularities of $f(t)$, to replace $f(t)$ in (6.1) by its leading asymptotic form $g(t)$ in the neighbourhood of the singularities (referred to as the comparison function), and finally to expand the latter about the origin to equate coefficients of t^n and so read off the asymptotic form of $P_n^{\alpha+an,\beta+bn}(x)$.

It is shown in [14] that the t-singularities of $f(t)$ occur at

$$t_\pm = \frac{b(x-1) + a(x+1) \pm i\sqrt{-\Delta}}{(1+a+b)(1-x^2)}[1+\xi_\pm]^{-1-a}[1+\eta_\pm]^{-1-b} \tag{6.3}$$

where Δ is given by (3.55) and

$$\xi_\pm = \frac{b(x-1) + a(x+1) \pm i\sqrt{\Delta}}{2(1+a+b)(1-x)}, \qquad \eta_\pm = \frac{x-1}{x+1}\xi_\pm \tag{6.4}$$

The comparison function is computed as

$$g(t) = B_+(t_+ - t)^{-1/2} + B_-(t_- - t)^{-1/2} \tag{6.5}$$

where

$$B_\pm = \lim_{t\to t_\pm}(t_\pm - t)^{1/2} f(t). \tag{6.6}$$

Since we know from [14] that

$$\frac{dt}{d\xi}\Big|_{\xi=\xi_\pm} = 0, \qquad \frac{d^2t}{d\xi^2}\Big|_{\xi=\xi_\pm} = \frac{\pm 2i\sqrt{-\Delta}}{(1+x)^2(1+\xi_\pm)^{2+a}(1+\eta_\pm)^{2+b}} =: 2A_\pm,$$

we calculate from (6.6) that

$$\begin{aligned} B_\pm &= \mp\frac{(1+\xi_\pm)^{\alpha+1}(1+\eta_+)^{\beta+1}(x+1)\sqrt{-A_\pm}}{i\sqrt{-\Delta}} \\ &= e^{\mp\pi i/4}(-\Delta)^{-1/4}(1+\xi_\pm)^{\alpha-a/2}(1+\eta_\pm)^{\beta-b/2} \end{aligned} \tag{6.7}$$

The first of the formulas in (6.7) is not reported in [14], while the second is their (2.14), (2.15) but with our $e^{\mp\pi i/4}(-\Delta)^{-1/4}$ replaced by $-(\Delta)^{-1/4}i$ and $(-\sqrt{\Delta})^{-1/2}i$ respectively.

The coefficient of t^n in (6.5), and thus the leading asymptotic form of $P_n^{(\alpha+an,\beta+bn)}(x)$ according to the method of Darboux, is equal to

$$(-1)^n\begin{pmatrix} -1/2 \\ n \end{pmatrix}\left(B_+t_+^{-n-\frac{1}{2}} + B_-t_-^{-n-\frac{1}{2}}\right) \tag{6.8}$$

To simplify this, we note from (3.55) and (3.27) that

$$\begin{aligned} 1+\xi_\pm &= \Big(\frac{2(a+1)}{(1-x)(a+b+1)}\Big)^{1/2}e^{\pm i\theta} \\ 1+\eta_\pm &= \Big(\frac{2(b+1)}{(1+x)(a+b+1)}\Big)^{1/2}e^{\pm i\gamma} \\ \frac{2\xi_\pm}{x+1} &= \frac{2e^{\pm i\rho}}{\sqrt{(a+b+1)(1-x^2)}} \end{aligned}$$

These substituted in (6.3) and (6.7) give

$$\begin{aligned} & B_+t_+^{-n-\frac{1}{2}} + B_-t_-^{-n-\frac{1}{2}} \\ &= \left(\frac{1}{\sqrt{-\Delta}}\right)^{\frac{1}{2}} 2\cos h(\theta,\gamma,\rho)\left[\frac{2(a+1)}{(1-x)(1+a+b)}\right]^{\frac{n}{2}(a+1)+\frac{\alpha}{2}+\frac{1}{4}} \\ &\times\left[\frac{2(b+1)}{(1+x)(1+a+b)}\right]^{\frac{n}{2}(b+1)+\frac{\beta}{2}+\frac{1}{4}}\left[\frac{(1-x^2)(a+b+1)}{4}\right]^{\frac{n}{2}+\frac{1}{4}} \end{aligned}$$

where

$$h(\theta,\gamma,\rho) = [n(a+1)+\alpha+\frac{1}{2}]\theta + [n(b+1)+\beta+\frac{1}{2}]i\gamma - (n+\frac{1}{2})\rho + \frac{\pi}{4}$$

This, together with the expansion

$$(-1)^n\begin{pmatrix} -1/2 \\ n \end{pmatrix} = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} = \frac{n^{-\frac{1}{2}}}{\sqrt{\pi}}$$

substituted in (6.8) gives (3.56).

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