The University of Melbourne,
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Robinson-Schensted-Knuth Correspondence

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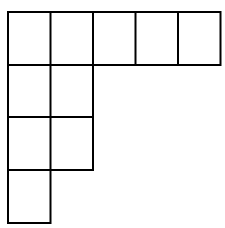
Introduction

The theory of correspondences associated with Young tableaux is a mainstream topic in combinatorics (see, e.g., [18]). It is this development, and its implications in relation to probabilistic models of growth processes, which will be reported here.

In the course of this study, close ties with the theory of symmetric functions become evident. Indeed it is possible to give combinatorial derivations, in the context of probabilistic models, of a number of identities satisfied by Schur polynomials ([11],[12]). These symmetric function identities are in turn known to be intimately related to the representation theory of the symmetric group (see, e.g., [11], [16]), and it is in this topic that Young tableaux were introduced by Young in 1900 (see, e.g., [15]).

The first combinatorial result relating to Young tableaux appears in a paper by Robinson published in 1938 [14]. Define a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ to be a weakly decreasing sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Define the diagram of a partition to consist of n rows such that the j^{th} row contains λ_j squares/boxes, left justified. For example, the diagram of the partition $(5, 2, 2, 1)$ is:



Hence the diagram consists of $N = |\lambda| = \sum_{j=1}^{n} \lambda_j$ boxes. A standard Young tableau is a numbering of the boxes in the diagram of λ by the numbers $\{1, 2, \dots, N\}$, such that the numbers are strictly increasing, both along rows and down columns. The partition λ is called the shape of the tableau, and N is called the content. For example, a standard Young tableau of shape $\lambda = (5, 2, 2, 1)$ and content 10 is:

1	3	6	8	9
2	5			
4	7			
10				

Robinson gave, in what has been said to have been expressed in vague terms by Knuth [10], a bijective correspondence between permutations of $\{1, 2, \ldots, N\}$ written as two-line arrays, and pairs of standard tableaux with the same shape, and of content N.

For example, in the case of permutations of $\{1, 2, 3\}$, the correspondence gives:

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix} \longleftrightarrow \quad \begin{ytableau} 1 & 2 & 3 \end{ytableau} \qquad \begin{ytableau} 1 & 2 & 3 \end{ytableau}$$

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} \longleftrightarrow \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \qquad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$$

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix} \longleftrightarrow \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \qquad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}$$

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix} \longleftrightarrow \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \qquad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$$

$$\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix} \longleftrightarrow \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \qquad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}$$

$$\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix} \longleftrightarrow \quad \begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau} \qquad \begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau}$$

Independently, and some years later, Schensted [17] found the same correspondence, and focussed attention on a particular feature, namely that of the longest increasing subsequence. This is the maximum length of the subsequences in the second line of the two-line array which are strictly increasing. In the second permutation above, the longest increasing subsequence is $(1, 2)$, and so the maximum length is 2. This is the same as λ_1 (ie, the length of the first row) in the corresponding tableau shape, and Schensted showed that this will be true in general.

As a mark of the pioneering work of Robinson and Schensted, the bijection between permutations and standard tableaux is now referred to as the Robinson-Schensted correspondence.

In 1970 Knuth generalised the Robinson-Schensted correspondence [10] (Knuth is well known to all producers of mathematical theses for his typesetting program TEX!) The generalisation, now referred to as the Robinson-Schensted-Knuth correspondence, gives a bijection between non-negative integer matrices and semi-standard tableaux.

A consequence of the Robinson-Schensted-Knuth correspondence is that it allows us to express the probability of all increasing subsequences having length less than or equal to a given value as a summation over Schur polynomials. It was realised by Gessel [6], and later extended by Rains [13], that the summations can in turn be expressed as multidimensional integrals relating to random matrix averages over the classical groups. Methods of random matrix theory allow for an asymptotic analysis of the multidimensional integrals. As a result it has been possible to evaluate the limiting probability density function for the longest increasing subsequence length [1], [2].

Johansson [9] interpreted a geometrical picture of the Robinson-Schensted correspondence due to Viennot [19] as a growth model. In the same paper the full Robinson-Schensted-Knuth correspondence was also presented as a growth model. Beginning with this work there have been a number of papers dealing with the application of ideas relating to the Robinson-Schensted-Knuth correspondence to physical models of growth and directed percolation.

In this thesis the theme is to study the details of the Robinson-Schensted-Knuth correspondence in the matrix-ball picture of Fulton [5], to relate this to growth models of non-intersecting paths, and to show how the correspondence can be used to obtain probabilistic formulae for the growth model. In addition to the case of general non-negative integer matrices, the special case of square matrices symmetric about the diagonal or about the anti-diagonal, as well as the special case of zero-one matrices, all have features which warrant a separate treatment. Our aim is to develop the theory in each of the cases to the same extent as is possible for the general case. For the most part this program is carried through, although in the course of the study some aspects of the theory which appear incomplete became apparent. These are reported as possible extensions of the present study in Chapter 5.

Chapter 1

Robinson-Schensted-Knuth correspondence - construction

The Robinson-Schensted-Knuth correspondence is a one-to-one correspondence between matrices with non-negative integer entries and pairs of semi-standard tableaux of the same shape. A semi-standard tableau, similar to the standard tableau introduced earlier, is a numbering of the diagram of λ which must be strictly increasing down columns, but only weakly increasing across rows.

We can construct this correspondence in several ways, of which two geometrical methods will be considered here - matrix-ball construction and the sequence of growth models of non-intersecting weighted paths.

1.1 Matrix-ball construction

We will adopt a notation to describe the relative position of two boxes in a diagram, or to compare positions of entries in a matrix. We say that a box B is **West** of C if the column of B is strictly to the left of the column of C, and we say that B is **west** of C if the column of B is weakly to the left of (ie left of or equal to) the column of C.

We use the corresponding notations for other compass directions, and we combine them, using capital and small letters to denote strict or weak inequalities. For example, we say B is **southEast** of C if the row of B is below or equal to the row of C, and the column of B is strictly to the right of the column of C.

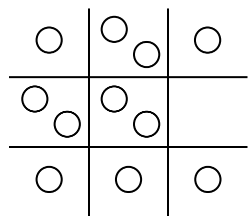
The following matrix-ball construction, particular to the RSK correspondence, uses a **northwest** ordering. It is possible to obtain variations of the RSK correspondence (ie alternative correspondences between matrices and tableaux) by using a different ordering (as explored in Appendix 1).

Given an $m \times n$ matrix $A = (a(i,j))$ with non-negative integer entries, put $a(i,j)$ balls in the position (i,j), arranging balls diagonally within each position from NorthWest to SouthEast, ie from top left to bottom right.

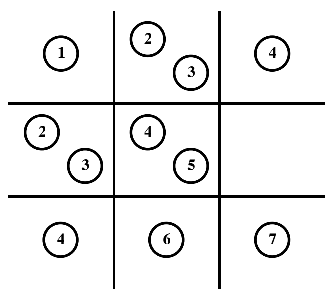
For example, the matrix A:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

leads to the following arrangement of balls:



Now we number each ball with a positive integer, such that each ball is numbered k if $(k-1)$ is the largest number occurring in a ball northwest of its position. Hence a ball with no balls northwest of it is numbered '1' - in this case, the ball in position $(1,1)$. In this way from A we get the configuration of numbered balls in a matrix, $A^{(1)}$:

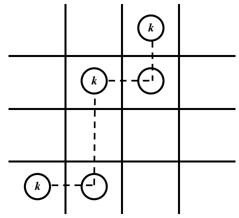


From $A^{(1)}$ we can work out the first row of the semi-standard tableaux (P,Q) corresponding to A. The first row of P is found from $A^{(1)}$ by listing the left-most columns where each number first occurs, and the first row of Q is found by listing the top-most row where each number first occurs.

In this example, the first row of P is $(1,1,1,1,2,2,3)$, meaning that the numbers 1-4 first appear in the first column, 5 and 6 in the second column, and 7 in the third. Similarly, the first row of Q is $(1,1,1,1,2,3,3)$.

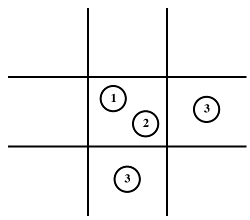
To find the second row of the tableaux, we form a new matrix of balls. Whenever there are $\ell > 1$ balls with the same number, k, in the given matrix (in this case $A^{(1)}$), put $\ell - 1$ balls in the new matrix. To do this, consider the ℓ balls in $A^{(1)}$, and put a ball to the right of each of these balls, except the last, directly under the next ball.

For example:

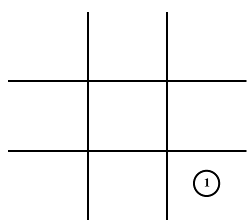


Note that the original ℓ balls labelled k do *not* appear in the new matrix, they are simply used as a guide to construct it.

When this has been done for all k, we have a new matrix of balls which we then number using the same technique as for $A^{(1)}$. Continuing with the example, this gives us $A^{(2)}$:



which gives us the second rows of P and Q, by the same procedure as before. So the second row of P is (2,2,2), and the second row of Q is (2,2,2). We continue this process until no two balls in $A^{(p)}$ have the same label. For the example in question, this means going as far as $A^{(3)}$:



which gives the third row of P as (3) and the third row of Q as (3). So for the matrix A, the corresponding semi-standard tableaux (P, Q) are:

1	1	1	1	2	2	3
2	2	2				
3						

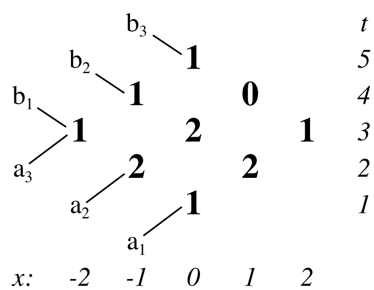
1	1	1	1	2	3	3
2	2	2				
3						

1.2 Growth model of non-intersecting lattice paths

Given an $n \times n$ non-negative integer matrix, $X = [x_{i,j}]_{i,j=1,\ldots,n}$, we can generate a sequence of growth models from which we obtain the tableaux corresponding to X. In this instance we count the rows of the matrix from the bottom, so in order to use the same example as for the matrix-ball construction, we will interchange the order of the rows to give X:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

We rotate X 45° anti-clockwise and label the horizontal rows of the rotated matrix by $t = 1, 2, \ldots, 2n-1$ and the vertical columns by $x = 0, \pm 1, \ldots, \pm(n-1)$, where $x = 0$ corresponds to the diagonal $i = j$ of X:

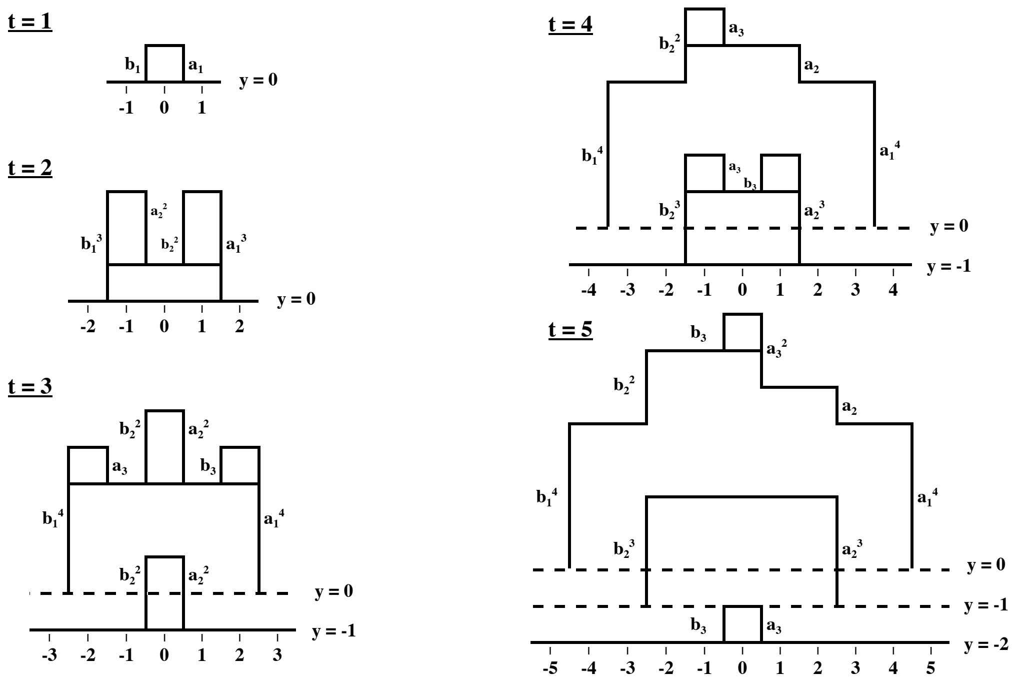


The entries $x_{i,j}$ in the matrix for successive t values represent the heights of weighted *nucleation events* at time t. A nucleation event is a column of unit width, height $x_{i,j}$ and weight $(a_i b_j)^{x_{i,j}}$, centred about the corresponding x-coordinate, which is placed on top of earlier nucleation events and their growth. So at $t = 1$, there is a nucleation event centred at $x = 0$, consisting of a column of width 1, height x_{11} (in X above, $x_{11} = 1$).

As $t \mapsto t+1$, the profile of all nucleation events recorded 'grows' one unit in the $-x$ direction and one unit in the $+x$ direction. So at $t = 2$, the nucleation event centred at $x = 0$ is 3 units wide, and on top of this profile we add two more nucleation events, centred at $x = -1$ and $x = 1$, of unit width, height x_{21}, x_{12} and weight $(a_2 b_1)^{x_{21}}$, $(a_1 b_2)^{x_{12}}$ respectively.

This process is continued until $t = 2n-1$, with the previous nucleation events growing one unit in each direction and then adding on any new nucleation events for time t. When there is some overlap due to growth of adjacent nucleation events, the overlap is moved to the line immediately below, where it continues to undergo the same process.

For the matrix X, the corresponding sequence of growth models is:



To obtain the pair of tableaux associated with X, take the paths at $t = 2n-1$ and think of each path as being a pair of paths - one path for $x < 0$ and another for $x > 0$. The path at $y = 1-n$ gives the n^{th} row of the tableaux, where the power of b_k (where $x < 0$) and a_k (where $x > 0$) tells us the number of times the integer k is present in the n^{th} row of tableaux P and Q respectively.

From the above growth models, we obtain (P, Q):

1	1	1	1	2	2	3
2	2	2				
3						

1	1	1	1	2	3	3
2	2	2				
3						

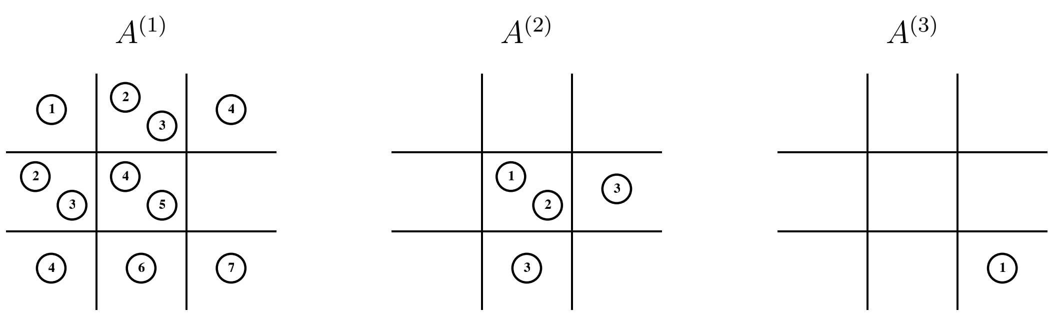
which is the same as the pair of tableaux obtained using matrix-ball construction. This will be the case in general, as becomes clear from the following discussion.

1.3 Relationship between matrix-ball and growth model constructions

1.3.1 Matrix-ball to growth model

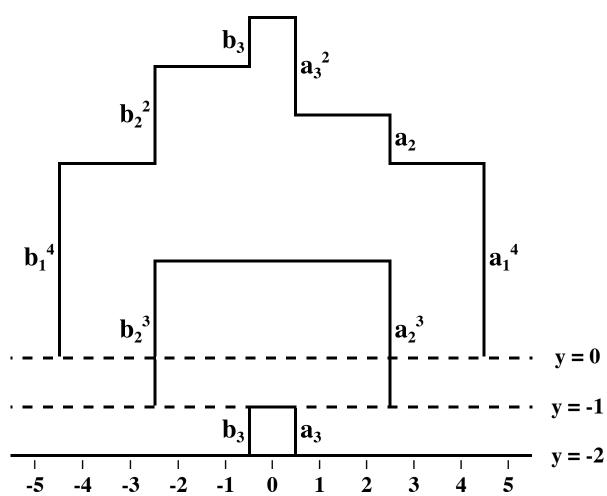
From the matrix-ball form, it is quite straightforward to obtain the set of paths for $t = 2n-1$. We relate $A^{(k)}$ to the path at $y = 1-k$. From the $n \times n$ matrix $A^{(k)}$, the i^{th} row and the j^{th} column tell us the power of a_i and b_j respectively in the path at $y = 1-k$. This is done by finding the maximum number in a ball in the i^{th} row (or j^{th} column), and subtracting from it the maximum number occurring in previous rows (or columns). The result is the power to which a_i (or b_j) is raised in the path at $y = 1-k$.

For the example, we use $A^{(1)}$ to find the path at $y = 0$, $A^{(2)}$ for the path at $y = -1$ and $A^{(3)}$ for the path at $y = -2$.



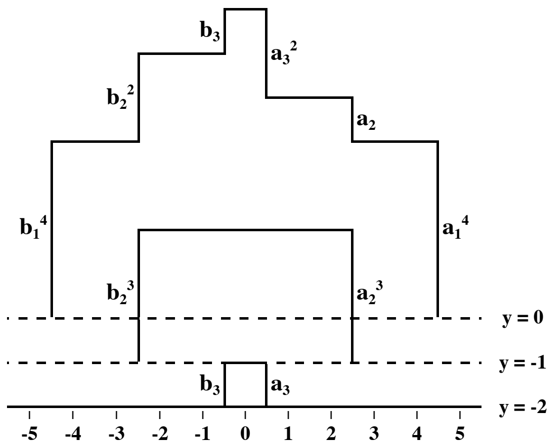
From $A^{(1)}$ we find that in the path at $y = 0$, a_1 is raised to the power 4, b_1 to the power 4, a_2 to the power 1, b_2 to the power 2, a_3 to the power 2 and b_3 to the power 1. Similarly we find from $A^{(2)}$ that the path at $y = -1$ has ${a_2}^3$, ${b_2}^3$, ${a_3}^0$ and ${b_3}^0$, and $A^{(3)}$ has ${a_3}^1$ and ${b_3}^1$.

So from the matrix-ball constructions $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ we get the paths:



1.3.2 Growth model to matrix-ball

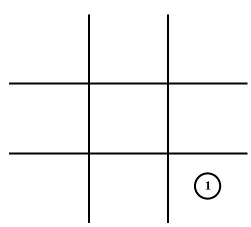
This is less straightforward than going from the matrix-ball form to the growth model. Given the set of non-intersecting paths:



it is possible to derive the corresponding matrix-ball form, more or less by reversing the method used to obtain the paths from the matrix-ball form.

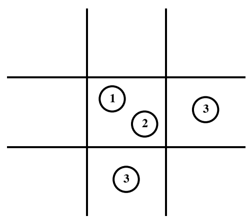
From the path at $y = 1 - k$ we can find the matrix-ball form $A^{(k)}$, in the following way: the sum of the heights of all paths at $y \leq 1 - k$ gives the total number of balls in $A^{(k)}$, the height of the path at $y = 1 - k$ gives the maximum number permitted on any ball in $A^{(k)}$, and the height at $x = 2n + \frac{1}{2} - 2i$ and $x = -2n - \frac{1}{2} - 2j$ gives the maximum number permitted in a ball at or above the i^{th} row and at or to the left of the j^{th} column. From this we can deduce the matrix-ball form, by beginning with the lowest path, (hence the greatest value of k, say ℓ)

For the example, we use the path at $y = -2$ to find the matrix-ball form $A^{(3)}$. The sum of the heights of all paths at $y \leq -2$ is 1, and the height at $y = -2$ is 1, so there is one ball in $A^{(3)}$, numbered up to 1. ${b_3}^1$ and ${a_3}^1$ tell us that in both the third row and the third column the maximum number present is 1, whereas for all other rows and columns there are no balls present. This leads to the matrix ball form $A^{(3)}$:



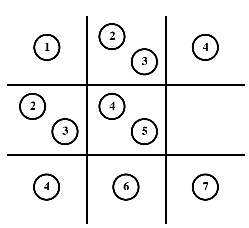
The matrix-ball form $A^{(\ell)}$ helps in deducing the next matrix, $A^{(\ell-1)}$, using the idea that for every ball present in $A^{(\ell)}$ there must be two balls in $A^{(\ell-1)}$ with the same number, one to the North and one to the West of the position of the ball in $A^{(\ell)}$.

So for the example we use the path at $y = -1$ combined with $A^{(3)}$ to find the matrix-ball form $A^{(2)}$: The sum of the heights of all paths at $y \leq -1$ is 4, and the height at $y = -1$ is 3, so there are four balls in $A^{(2)}$, numbered up to 3. ${a_2}^3$ and ${b_2}^3$ tell us that the maximum number present in the second column and also in the second row is 3, with no balls present on the first row or first column. From $A^{(3)}$ we know that there will be a ball at either $A^{(2)}_{1,3}$ or $A^{(2)}_{2,3}$, and one at either $A^{(2)}_{3,1}$ or $A^{(2)}_{3,2}$. As we know there are no balls in the first row or the first column, this gives us one at $A^{(2)}_{2,3}$ and one at $A^{(2)}_{3,2}$. With only two more balls to be placed on the matrix, in order to satisfy all conditions, we end up with $A^{(2)}$:



Now we use the path at $y = 0$ combined with $A^{(2)}$ to find the matrix-ball form $A^{(1)}$. We find from the path at $y = 0$ that $A^{(1)}$ will have 11 balls, numbered up to 7, with the first row containing the numbers 1-4, the second row containing 5 and numbers < 5, and the third row containing 6 & 7 and numbers < 6; the first column has numbers 1-4, the second column 5 & 6 and numbers < 5, and the third column 7 and numbers < 7. From $A^{(2)}$ we know that there must be (at least) two balls in each of $A^{(1)}_{1,2}$ and $A^{(1)}_{2,1}$, also there must be one ball in each of $A^{(1)}_{1,3}$ and $A^{(1)}_{3,1}$, and a ball in either $A^{(1)}_{2,1}$ or $A^{(1)}_{2,2}$ and either $A^{(1)}_{1,2}$ or $A^{(1)}_{2,2}$, with common numbers so that they will produce the balls in $A^{(2)}$.

From all this we can deduce $A^{(1)}$:



Chapter 2

Concepts relevant to the RSK correspondence

Associated with semi-standard tableaux are a class of symmetric functions referred to as Schur polynomials. These polynomials, combined with the RSK correspondence, allow for the derivation of a formula for the probability of the maximum height in the growth model being less than a given value.

Another relevant concept is that of recurrence relations satisfied by path displacements in the RSK correspondence. These displacements satisfy formulae which can be interpreted from the perspective of both the growth models and the matrix-balls.

2.1 Schur polynomials

For a given partition λ, with at most N rows, we can use semi-standard tableaux to define a symmetric polynomial, in the following way:

For a numbering T of a Young diagram there is a monomial, x^T, defined as:

$$x^T = \prod_{i=1}^{N} (x_i)^{\text{number of times } i \text{ occurs in } T}$$

For example, for the semi-standard tableau:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}$$

the corresponding monomial is $x^T = x_1^4 x_2^5 x_3^2$

The *Schur polynomial* $s_\lambda(x_1, \ldots, x_N)$ is the sum of all monomials coming from semi-standard tableaux of shape λ using the numbers from 1 to m:

$$s_\lambda(x_1, \ldots, x_N) \;=\; \sum_{\text{tableaux of shape } \lambda} x^T \;=\; \sum_{\lambda} x_1^{\#1\text{'s}} \ldots x_N^{\#N\text{'s}}$$

2.1.1 Symmetry of Schur polynomials

A special property of the Schur polynomials is that they can be expressed as a ratio of determinants, ie:

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det[x_j^{N-k+\lambda_k}]_{j,k=1,\ldots,N}}{\det[x_j^{N-k}]_{j,k=1,\ldots,N}} \tag{2.1}$$

For notational convenience this will be established for the case $N = 3$. In fact, $N = 3$ is representative of the general N case.

We want to show that:

$$s_\lambda(x_1, x_2, x_3) = \frac{\begin{vmatrix} x_1^{2+\lambda_1} & x_1^{1+\lambda_2} & x_1^{\lambda_3} \\ x_2^{2+\lambda_1} & x_2^{1+\lambda_2} & x_2^{\lambda_3} \\ x_3^{2+\lambda_1} & x_3^{1+\lambda_2} & x_3^{\lambda_3} \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix}} \tag{2.2}$$

Our strategy is to show that both sides of (2.2) satisfy the recurrence:

$$s_\lambda(x_1, x_2, x_3) = \sum_{\substack{\mu_1, \mu_2 \text{ such that} \\ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3}} s_\mu(x_1, x_2) x_3^{|\lambda| - |\mu|}$$

together with the initial condition $s_\lambda(x_1, x_2, x_3) = 1$ for $\lambda = \emptyset$.

For the LHS of (2.2), we refer to the definition of the Schur polynomial, from which the recurrence follows, as the portion of the tableau containing only 1's and 2's forms a tableau consisting of 2 parts, and fully contained in λ.

To show that the determinant formula satisfies the recurrence, we begin by setting $x_3 = 1$ and subtracting the last row from all others, in each determinant:

$$s_\lambda(x_1, x_2, 1) = \frac{\begin{vmatrix} x_1^{\lambda_1+2} - 1 & x_1^{\lambda_2+1} - 1 & x_1^{\lambda_3} - 1 \\ x_2^{\lambda_1+2} - 1 & x_2^{\lambda_2+1} - 1 & x_2^{\lambda_3} - 1 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^2 - 1 & x_1 - 1 & 0 \\ x_2^2 - 1 & x_2 - 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}}$$

Now we divide the first row by $x_1 - 1$ and the second by $x_2 - 1$, which gives:

$$s_\lambda(x_1, x_2, 1) = \frac{\begin{vmatrix} \sum_{\ell=0}^{\lambda_1+1} x_1^\ell & \sum_{\ell=0}^{\lambda_2} x_1^\ell & \sum_{\ell=0}^{\lambda_3-1} x_1^\ell \\ \sum_{\ell=0}^{\lambda_1+1} x_2^\ell & \sum_{\ell=0}^{\lambda_2} x_2^\ell & \sum_{\ell=0}^{\lambda_3-1} x_2^\ell \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 + 1 & 1 & 0 \\ x_2 + 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}}$$

Next subtract the second column of each determinant from the first, and then subtract the third column from the second:

$$s_\lambda(x_1, x_2, 1) = \frac{\begin{vmatrix} \sum_{\ell=\lambda_2+1}^{\lambda_1+1} x_1^\ell & \sum_{\ell=\lambda_3}^{\lambda_2} x_1^\ell & \sum_{\ell=0}^{\lambda_3-1} x_1^\ell \\ \sum_{\ell=\lambda_2+1}^{1+\lambda_1} x_2^\ell & \sum_{\ell=\lambda_3}^{\lambda_2} x_2^\ell & \sum_{\ell=0}^{\lambda_3-1} x_2^\ell \\ 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 & 0 \\ x_2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}}$$

Expanding by the final row, and manipulating the sums on the numerator shows:

$$s_\lambda(x_1,x_2,1) \;=\; \frac{\begin{vmatrix} \sum\limits_{\mu_1=\lambda_2}^{\lambda_1} x_1^{\mu_1+1} & \sum\limits_{\mu_2=\lambda_3}^{\lambda_2} x_1^{\mu_2} \\ \sum\limits_{\mu_1=\lambda_2}^{\lambda_1} x_2^{\mu_1+1} & \sum\limits_{\mu_2=\lambda_3}^{\lambda_2} x_2^{\mu_2} \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}$$

$$= \sum_{\mu_1=\lambda_2}^{\lambda_1} \sum_{\mu_2=\lambda_3}^{\lambda_2} \frac{\begin{vmatrix} x_1^{\mu_1+1} & x_1^{\mu_2} \\ x_2^{\mu_1+1} & x_2^{\mu_2} \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}$$

and so:

$$s_\lambda(x_1,x_2,1) \;=\; \sum_{\mu_1=\lambda_2}^{\lambda_1} \sum_{\mu_2=\lambda_3}^{\lambda_2} s_\mu(x_1,x_2) \tag{2.3}$$

We note from the determinant formula that:

$$s_\lambda(cx_1, cx_2, \ldots, cx_n) = c^{|\lambda|} s_\lambda(x_1, x_2, \ldots, x_n)$$

So if we replace $x_1 \mapsto \dfrac{x_1}{x_3}$ and $x_2 \mapsto \dfrac{x_2}{x_3}$ in (2.3), we have:

$$s_\lambda(x_1,x_2,x_3) = \sum_{\mu_2=\lambda_3}^{\lambda_2} s_\mu(x_1,x_2) x_3^{|\lambda|-|\mu|}$$

which is the sought recurrence. The determinant formula also gives the sought initial condition, $s_{(0,0,0)}(x_1,x_2,x_3) = 1$.

Another property which follows from (2.1) is the basic fact that the Schur polynomials are symmetric functions, ie:

$$s_\lambda(x_1,x_2,\ldots,x_n) = s_\lambda(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \quad \text{for any permutation } \sigma \in S_n$$

which, in terms of tableaux, means that the number of tableaux, on a given shape λ, with m_1 1's, m_2 2's, …, m_n n's is the same as the number of tableaux on λ with $m_{\sigma(1)}$ 1's, $m_{\sigma(2)}$ 2's, …, $m_{\sigma(n)}$ n's, for any permutation $\sigma \in S_n$.

2.2 Recurrences for λ_ℓ

For a given matrix X, consider λ_1, the length of the first row in the tableau, and more generally λ_ℓ. A recurrence can be obtained by considering the case where we block out the final row/column of X.

We denote by $\lambda_\ell(n_1, n_2)$ the maximum displacement of the level-ℓ path of the growth models from matrix X, where we have truncated the matrix X to the first n_1 rows and n_2 columns. In the context of the matrix-balls, this is equivalent to the highest numbering of a ball in the matrix $A^{(\ell)}$.

We first consider the maximum displacement of the level-1 path, which can be split into the displacement due to the final nucleation event, and the displacement due to previous events. The final nucleation step adds a height of x_{n_1,n_2} to the displacement at $x = 0$, and the previous events add the maximum of the height at $x = -1$ and the height at $x = 1$ from the previous time step. So:

$$\lambda_1(n_1, n_2) = \max\big(h_1(n_1, n_2 - 1), h_1(n_1 - 1, n_2)\big) + x_{n_1,n_2}$$

However, the maximum of the height at $x = -1$ in the previous step is equivalent to the displacement at $x = 0$ of the level-1 path for the growth model of the matrix truncated to n_1 rows and $(n_2 - 1)$ columns, so $h_1(n_1, n_2 - 1) = \lambda_1(n_1, n_2 - 1)$, and similarly for $h_1(n_1 - 1, n_2)$. Hence:

$$\lambda_1(n_1, n_2) = \max\big(\lambda_1(n_1, n_2 - 1), \lambda_1(n_1 - 1, n_2)\big) + x_{n_1,n_2} \tag{2.4}$$

where we require $\lambda_1(0, j) = \lambda_1(i, 0) = 0$.

This is immediately translatable to the case of the matrix-balls, as the highest numbering of a ball in the matrix $A^{(1)}$ is equal to the number of balls in the bottom right square, x_{n_1,n_2}, plus the highest numbering of a ball in previous rows or columns - $H_1(n_1, n_2-1)$ and $H_1(n_1-1, n_2)$ respectively. But these are equivalent to the highest numbering of a ball in the matrix $A^{(1)}$, found by truncating the original matrix to n_1 rows and (n_2-1) columns, and (n_1-1) rows and n_2 columns respectively. So we have:

$$\lambda_1(n_1, n_2) = \max\big(\lambda_1(n_1, n_2 - 1), \lambda_1(n_1 - 1, n_2)\big) + x_{n_1,n_2}$$

as found before. Again we require $\lambda_1(0, j) = \lambda_1(i, 0) = 0$

We now consider $\lambda_\ell(n_1, n_2)$, for $\ell > 1$. Again, we will derive a recurrence from the growth models. The maximum displacement of the level-ℓ path is equal to the sum of the displacement due to the most recent overlap and the displacement due to previous overlaps - equivalent to the maximum displacement at $x = 1$ or $x = -1$.

So, similarly to in the λ_1 case, we obtain the recurrence:

$$\lambda_\ell(n_1, n_2) = \max\big(\lambda_\ell(n_1, n_2 - 1), \lambda_\ell(n_1 - 1, n_2)\big) + x^{(\ell-1)}_{n_1,n_2} \tag{2.5}$$

where $x^{(\ell-1)}_{n_1,n_2}$ denotes the height of the overlap from the level-$(\ell - 1)$ path. This overlap is equivalent to the minimum displacement of the level-$(\ell - 1)$ path at $x = -1$ or $x = 1$, less its displacement at $x = 0$ at the previous time step. So we have the following relationship:

$$x^{(\ell-1)}_{n_1,n_2} = \min\big(h_{(\ell-1)}(n_1, n_2 - 1), h_{(\ell-1)}(n_1 - 1, n_2)\big) - h_{(\ell-1)}(n_1 - 1, n_2 - 1)$$

Relating this to the growth models of truncated matrices gives:

$$x^{(\ell-1)}_{n_1,n_2} = \min\big(\lambda_{(\ell-1)}(n_1, n_2 - 1), \lambda_{(\ell-1)}(n_1 - 1, n_2)\big) - \lambda_{(\ell-1)}(n_1 - 1, n_2 - 1) \tag{2.6}$$

Combining equations (2.5) and (2.6), we get:

$$\begin{aligned}\lambda_\ell(n_1, n_2) \quad = \quad & \max\big(\lambda_\ell(n_1, n_2 - 1), \lambda_\ell(n_1 - 1, n_2)\big) \\ & + \min\big(\lambda_{(\ell-1)}(n_1, n_2 - 1), \lambda_{(\ell-1)}(n_1 - 1, n_2)\big) \\ & - \lambda_{(\ell-1)}(n_1 - 1, n_2 - 1)\end{aligned} \tag{2.7}$$

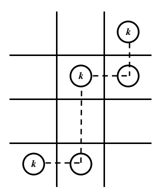
When combined with the boundary conditions $\lambda_\ell(0, j) = \lambda_\ell(i, 0) = 0$, and $\{\lambda_1(n_1, n_2)\}_{i,j=1,\ldots,n}$, the above equation uniquely specifies $\{\lambda_\ell(n_1, n_2)\}_{i,j=1,\ldots,n}$

The recurrence for λ_ℓ, $\ell > 1$, can also be seen from the matrix-ball perspective. $\lambda_\ell(n_1, n_2)$ is equivalent to the highest numbering of a ball in the matrix-ball form $A^{(\ell)}$, from the truncated $n_1 \times n_2$ matrix. This can be expressed as the number of balls in position (n_1, n_2) plus the highest numbering of a ball in any position except (n_1, n_2). The latter is the same as the maximum of the numberings in the $n_1 \times (n_2 - 1)$ and the $(n_1 - 1) \times n_2$ truncated matrices. So we have:

$$\lambda_\ell(n_1, n_2) = \max\big(\lambda_\ell(n_1, n_2 - 1), \lambda_\ell(n_1 - 1, n_2)\big) + x^{(\ell-1)}_{n_1,n_2}$$

where $x^{(\ell-1)}_{n_1,n_2}$ denotes the number of balls in position (n_1, n_2) in $A^{(\ell)}$, as a result of balls in $A^{(\ell-1)}$, which is the same as (2.5).

We can derive a formula for $\ell > 1$ by considering how the matrix-ball construction leads to balls in position (n_1, n_2). Each ball in position (n_1, n_2) of $A^{(\ell)}$ must have one ball North and one ball West of its position in $A^{(\ell-1)}$ with a common number. So the number of balls in position (n_1, n_2) of $A^{(\ell)}$ is determined by the number of balls in the n_1^{th} row and the n_2^{th} column which share a common number, that is not present in any other row or column. If it were also present in any of the $(n_1 - 1) \times (n_2 - 1)$ positions, then we would have the following case:



where none of the resultant balls are in position (n_1, n_2).

In order to count the numbered balls common to the n_1^{th} row and the n_2^{th} column, we take the smallest of the maximum number permitted in the $n_1 \times (n_2 - 1)$ matrix and the maximum in the $(n_1 - 1) \times n_2$ matrix, and subtract the maximum in the $(n_1 - 1) \times (n_2 - 1)$ matrix. This gives:

$$x^{(\ell-1)}_{n_1,n_2} = \min\big(\lambda_{(\ell-1)}(n_1, n_2 - 1), \lambda_{(\ell-1)}(n_1 - 1, n_2)\big) - \lambda_{(\ell-1)}(n_1 - 1, n_2 - 1)$$

which is the same as (2.6). As we have the same boundary conditions, $\lambda_\ell(0, j) = \lambda_\ell(i, 0) = 0$, this results in an identical recurrence to that found from the growth model perspective.

2.3 Probabilistic formulae

With the right assignment of probability to each entry, we can make use of the Schur polynomials to derive some probabilistic formulae for the general case. Let each entry k in position (i,j) of the matrix occur with geometric probability:

$$(1-a_ib_j)(a_ib_j)^k$$

Hence the probability of the matrix having entries $[x_{ij}]_{i,j=1,\ldots,n}$ is:

$$\prod_{i,j=1}^{n}(1-a_ib_j)(a_ib_j)^{x_{ij}}$$

We can use this to find the probability of λ_1 being less than or equal to a given value, h, ie find $\Pr(\lambda_1 \le h)$. We do this by effectively changing coordinates. We consider all weighted matrices which, in correspondence with pairs of tableaux, have the same value of λ, for some $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

The weight of all such pairs of tableaux is equal to:

$$\left(\sum_{\text{tableaux of shape }\lambda} a_1^{\#1\text{s}}a_2^{\#2\text{s}}\ldots a_n^{\#n\text{'s}}\right)\left(\sum_{\text{tableaux of shape }\lambda} b_1^{\#1\text{s}}b_2^{\#2\text{s}}\ldots b_n^{\#n\text{'s}}\right)$$

which, by the definition of the Schur polynomials, is equivalent to:

$$s_\lambda(a_1,a_2,\ldots,a_n)s_\lambda(b_1,b_2,\ldots,b_n)$$

So the probability that X corresponds to a pair of tableaux with shape λ is:

$$\prod_{i,j=1}^{n}(1-a_ib_j)s_\lambda(a_1,a_2,\ldots,a_n)s_\lambda(b_1,b_2,\ldots,b_n)$$

And hence $\Pr(\lambda_1 \le h) = \prod_{i,j=1}^{n}(1-a_ib_j)\sum_{\lambda:\lambda_1\le h} s_\lambda(a_1,a_2,\ldots,a_n)s_\lambda(b_1,b_2,\ldots,b_n)$

As we require $\Pr(\lambda_1^{(1)} \le h) \to 1$ as $h \to \infty$, this gives:

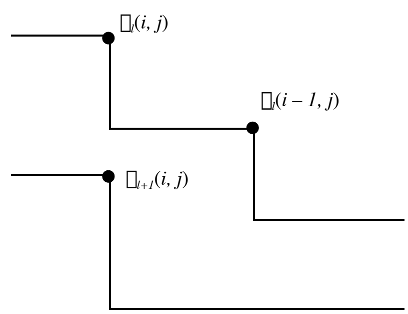
$$\sum_\lambda s_\lambda(a_1,a_2,\ldots,a_n)s_\lambda(b_1,b_2,\ldots,b_n) = \frac{1}{\prod_{i,j=1}^{n}(1-a_ib_j)} \tag{2.8}$$

The result is the Cauchy identity, from the theory of symmetric functions [12], hence we have arrived at a direct proof of it via the Robinson-Schensted-Knuth correspondence.

2.4 Alternative interpretation of λ

We have so far considered $\lambda_\ell(n_1, n_2)$ through two different interpretations: as the maximum height of the level-ℓ path resulting from applying the growth process to the truncation X_{n_1,n_2} of the original matrix X, or as the highest numbering of a ball in $A^{(\ell)}$ (again where $A^{(\ell)}$ comes from the truncated matrix X_{n_1,n_2})

However, we can also interpret $\lambda_\ell(n_1, j)$ as the displacement of the level-ℓ path at $x = -2n_{\max} - \frac{1}{2} + 2j$, while $\lambda_\ell(i, n_2)$ can be interpreted as the displacement of the level-ℓ path at $x = 2n_{\max} + \frac{1}{2} - 2i$, where $n_{\max} = \max(n_1, n_2)$. This leads to some general properties of λ.



From this interpretation it follows (refer to the figure above):

$$\lambda_\ell(i, j) = 0 \qquad \text{for} \quad \ell > n_{\max} \tag{2.9}$$

$$\lambda_\ell(i, j) \geq \lambda_\ell(i - 1, j) \geq \lambda_{\ell+1}(i, j) \tag{2.10}$$

$$\lambda_\ell(i, j) \geq \lambda_\ell(i, j - 1) \geq \lambda_{l+1}(i, j) \tag{2.11}$$

We can relate this easily to the case of the matrix-balls. We interpret $\lambda_\ell(n_1, j)$ as the maximum numbering permitted in a ball in or to the left of the j^{th} column, and $\lambda_\ell(i, n_2)$ as the maximum permitted in a ball in or above the i^{th} row, of $A^{(\ell)}$. Equation (2.9) then follows by the matrix-ball construction, as $A^{(\ell)}$ cannot exist for $\ell > n_{\max}$, because the matrix-ball form at $A^{(\ell)}$ will not have balls in the first $\ell - 1$ rows or columns.

The first inequalities of both (2.10) and (2.11) follow by the system of numbering used for the construction, as the maximum number in the $(j-1)^{th}$ column cannot be greater than that in the j^{th}, and equivalently with rows. The second inequalities of (2.10) and (2.11) follow by the idea that for a given ball in $A^{(\ell+1)}$, there are at least two balls northwest of its position in $A^{(\ell)}$, and hence the numbering of $A^{(\ell)}$ in row i or column j cannot be less than that for $A^{(\ell+1)}$.

We can also note:

$$\sum_{\ell=1}^{n}\big(\lambda_\ell(n,j)-\lambda_\ell(n,j-1)\big)=\sum_{i=1}^{n}x_{ij} \tag{2.12}$$

$$\sum_{\ell=1}^{n}\big(\lambda_\ell(i,n)-\lambda_\ell(i-1,n)\big)=\sum_{j=1}^{n}x_{ij} \tag{2.13}$$

which can be shown by the rules of both the growth model and the matrix-balls.

We will first consider these in the context of the growth model. The LHS of (2.12) is equivalent to the total number of "up-steps" occurring at $x=-2n-\frac{1}{2}+2j$. The RHS of (2.12) is equal to the sum of the j^{th} column of X. In terms of the growth model, these n entries in the j^{th} column will correspond to nucleation events at times $t=j,j+1,\ldots,j+n-1$, centred at $x=j-1,j-2,\ldots j-n$ respectively (ie, the entry from the j^{th} column at $t=j+k-1$ will be centred at $x=j-k$, $k=1,\ldots,n$).

An event centred at $x=j-k$ (hence with initial left up-step at $x=j-k-\frac{1}{2}$) at time $t=j+k-1$ will grow a total of $(2n-1)-(j+k-1)=2n-j-k$ units to the left (not necessarily all at level-ℓ, as it may overlap with another nucleation event and continue its growth at a different level). So all entries from the j^{th} column will contribute to the up-steps occurring at different levels at $x=-2n-\frac{1}{2}+2j$, and furthermore they will be the sole contributors. Hence the number of up-steps at $x=-2n-\frac{1}{2}+2j$ is equal to the sum of the entries in the j^{th} column of X, and (2.12) is shown.

The LHS of (2.13), similarly, is equal to the total number of up-steps occurring at $x=2n+\frac{1}{2}-2i$, and the RHS is the sum of the i^{th} row of X. The n entries in the i^{th} row will correspond to nucleation events at times $t=i,i+1,\ldots,j+n-1$, centred at $x=-i+1,-i+2,\cdots-i+n$ respectively. As an event centred at $x=-i+k$ (and hence with initial right up-step at $x=-i+k+\frac{1}{2}$), at time $t=i+k-1$ will grow a total of $2n-i-k$ units to the right, all up-steps at $x=2n+\frac{1}{2}-2i$ can be entirely attributed to the entries in the i^{th} row, and (2.13) is shown.

Both (2.12) and (2.13) can be interpreted much more naturally from the perspective of the matrix-balls. The LHS of (2.12) is equivalent to adding the differences in ball numberings from the $(j-1)^{th}$ column to the j^{th} for all $A^{(\ell)}$. But, by a significant feature of the matrix-ball construction, this is simply equal to the number of balls in the j^{th} column at $A^{(1)}$, which is equivalent to the sum of the entries in the j^{th} column of X, which is the RHS. Similarly the LHS of (2.13) is equivalent to adding the differences in ball numberings from the $(i-1)^{th}$ row to the i^{th} for all $A^{(\ell)}$, which gives us the number of balls in the i^{th} row at $A^{(1)}$, and hence is equal to the RHS.

Chapter 3

Effect of matrix symmetries

A consideration of some simple matrix symmetries - about the diagonal and about the anti-diagonal - with respect to the Robinson-Schensted-Knuth correspondence leads to some useful formulae. A study of the probabilistic formulae, similar to in the general case, produces an analogue of the Cauchy identity. Also, in the case of symmetry about the diagonal, we obtain a useful summation formula for the λ on the diagonal, as a consequence of the general formulae from section 2.3.

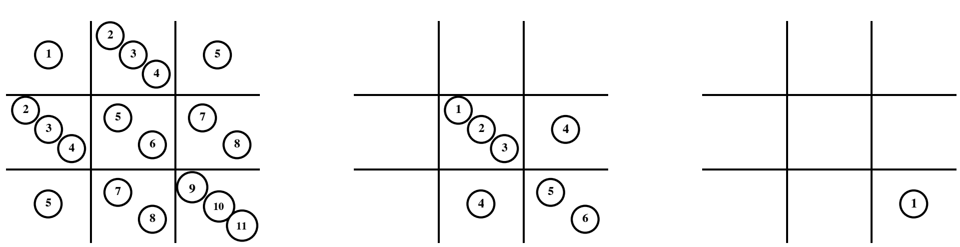
3.1 Symmetry about the diagonal

First consider the case of matrices symmetric about the diagonal (for the matrix-ball construction, this is from the top left to the bottom right, but for the growth model, as we number rows from the bottom, this is from the bottom left to the top right).

3.1.1 Matrix-ball construction

For example, performing the matrix-ball construction for the matrix A:

$$\begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$



We can see the effect of this symmetry immediately - each $A^{(p)}$ is symmetric about the diagonal also, so the row and column numbering will be equivalent.

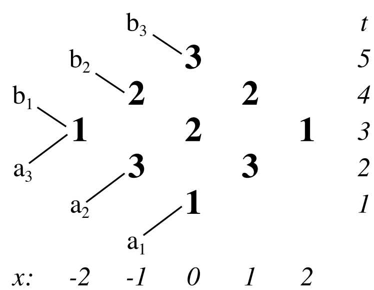
Hence, as is true in general, this results in two identical tableaux for P and Q:

1	1	1	1	1	2	2	2	3	3	3
2	2	2	2	3	3					
3										

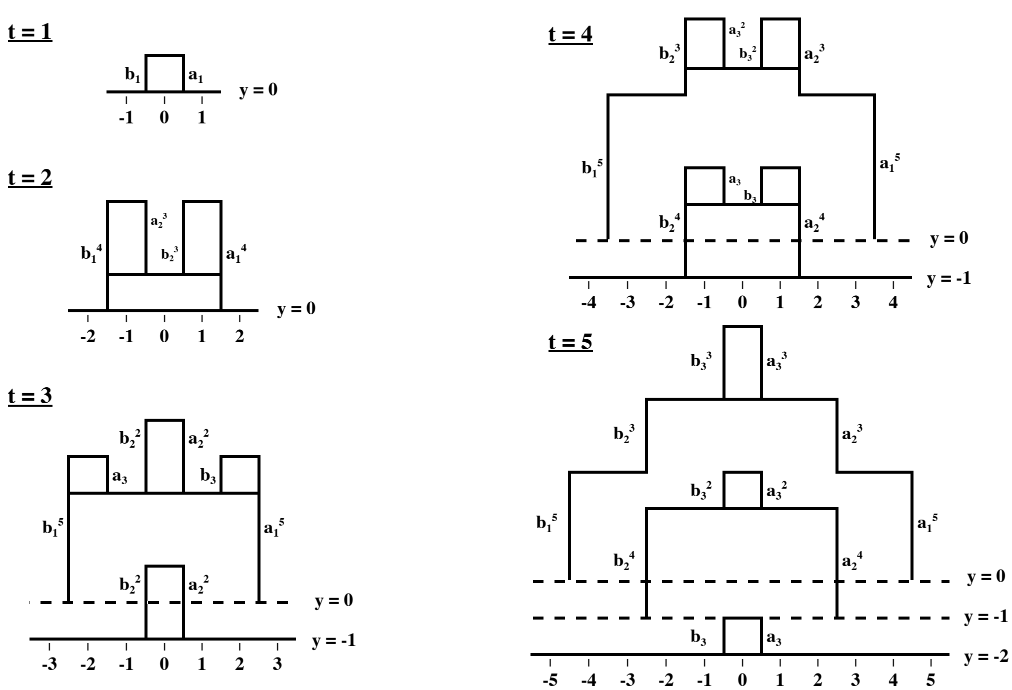
1	1	1	1	1	2	2	2	3	3	3
2	2	2	2	3	3					
3										

3.1.2 Growth models

Using the same example as for the matrix-balls, with the order of rows reversed, rotating the matrix and labelling the columns and rows gives:



from which we find the sequence of growth models:



Again, the effect of the symmetry is immediately obvious. The growth model at time $t = 5$ consists of n paths, all symmetric about $x = 0$, so we must get two identical tableaux. Obtaining the tableaux from the growth model at $t = 5$ confirms this, and we see the corresponding tableaux (P, Q) are, as found with the matrix-ball construction:

1	1	1	1	1	2	2	2	3	3	3
2	2	2	2	3	3					
3										

1	1	1	1	1	2	2	2	3	3	3
2	2	2	2	3	3					
3										

In summary, the application of the RSK correspondence to a matrix which is symmetric about the diagonal gives two identical semi-standard tableaux, ie $P = Q$.

3.1.3 A summation formula for the diagonal symmetry

We can show that in the case of diagonal symmetry the following property holds:

$$\sum_{j=1}^{n} x_{j,j} = \sum_{j=1}^{n} (-1)^{j-1} \lambda_j$$

We have a matrix symmetric about the diagonal, ie $x_{i,j} = x_{j,i}$, which, as found previously, leads to symmetry in the growth models, hence $\lambda_\ell(i,j) = \lambda_\ell(j,i)$. So, using the recurrences (2.4) and (2.7), we get:

$$\begin{aligned} \lambda_1(i,i) &= \lambda_1(i,i-1) + x_{i,i} \\ \lambda_\ell(i,i) &= \lambda_\ell(i,i-1) + \lambda_{\ell-1}(i,i-1) - \lambda_{\ell-1}(i-1,i-1) \end{aligned}$$

Rearranging the latter of these results gives:

$$\lambda_\ell(i,i) + \lambda_{\ell-1}(i-1,i-1) = \lambda_\ell(i,i-1) + \lambda_{\ell-1}(i,i-1) \tag{3.1}$$

We multiply both sides of (3.1) by $(-1)^{\ell-1}$ and sum over ℓ, from $\ell = 2$ to i:

$$\sum_{\ell=2}^{i}(-1)^{\ell-1}\big(\lambda_\ell(i,i)+\lambda_{\ell-1}(i-1,i-1)\big) = \sum_{\ell=2}^{i}(-1)^{\ell-1}\big(\lambda_\ell(i,i-1)+\lambda_{\ell-1}(i,i-1)\big)$$

$$= \sum_{\ell=2}^{i-1}(-1)^{\ell-1}\big(\lambda_\ell(i,i-1)-\lambda_\ell(i,i-1)\big)+(-1)^{i-1}\lambda_i(i,i-1)-\lambda_1(i,i-1)$$

In general, $\lambda_i(i, i-1) = 0$, so we have:

$$\sum_{\ell=2}^{i}(-1)^{\ell-1}\lambda_\ell(i,i) - \sum_{\ell=1}^{i-1}\lambda_\ell(i-1,i-1)\big) = -\lambda_1(i,i-1)$$

$$= -\lambda_1(i,i) + x_{i,i}$$

Hence:

$$x_{i,i} = \sum_{\ell=1}^{i}(-1)^{\ell-1}\lambda_\ell(i,i) - \sum_{\ell=1}^{i-1}(-1)^{\ell-1}\lambda_\ell(i-1,i-1)$$

Now we want to sum $x_{i,i}$ from $i = 1$ to n:

$$\sum_{i=1}^{n} x_{i,i} \;=\; \sum_{i=1}^{n}\Big(\sum_{\ell=1}^{i}(-1)^{\ell-1}\lambda_\ell(i,i) - \sum_{\ell=1}^{i-1}(-1)^{\ell-1}\lambda_\ell(i-1,i-1)\Big)$$

$$= \sum_{i=1}^{n-1}\sum_{\ell=1}^{i}(-1)^{\ell-1}\lambda_\ell(i,i) - \sum_{i=0}^{n-1}\sum_{\ell=1}^{i}(-1)^{\ell-1}\lambda_\ell(i,i) + \sum_{\ell=1}^{n}(-1)^{\ell-1}\lambda_\ell(n,n)$$

The first two terms cancel, so we have:

$$\sum_{i=1}^{n} x_{i,i} \;=\; \sum_{\ell=1}^{n}(-1)^{\ell-1}\lambda_\ell(n,n) \tag{3.2}$$

which is the sought formula since $\lambda_\ell = \lambda_\ell(n, n)$, ie the maximum displacement of the level-ℓ path, from growth models corresponding to the original matrix.

3.1.4 Probabilistic formulae for diagonal symmetry

Similar to the general case, if we choose the right probability to assign to entries of the matrix, we can use Schur polynomials to establish a formula for $\Pr(\lambda_1 \leq h)$. Again we do this by considering the weighting related to entries - but, as the matrix is symmetric about the diagonal, we only need consider the weightings of entries $x_{i,j}$: $i \leq j$, as the value of $x_{i,j}$ for $i > j$ is fixed by symmetry.

Let each entry k in position (i, j) of the matrix, where $i < j$, occur with probability $(1 - q_i q_j)(q_i q_j)^k$, and each entry k in position (i, i) occur with probability $(1 - q_i)q_i^k$. The probability that X has entries $[x_{i,j}]_{i,j=1,\ldots,n}$ is hence:

$$\prod_{i=1}^{n}(1 - q_i)q_i^{x_{i,i}} \prod_{1\leq i<j\leq n} (1 - q_i q_j)(q_i q_j)^{x_{i,j}}$$

We can then give the probability of a matrix X corresponding to a single tableau (because, as found earlier, $P = Q$) of shape λ as:

$$\prod_{i=1}^{n}(1 - q_i) \prod_{1\leq i<j\leq n} (1 - q_i q_j)s_\lambda(q_1, \ldots , q_n)$$

Further, we can generalise the probability of the diagonal entries to $(1-\alpha q_i)(\alpha q_i)^k$, so the above probability becomes:

$$\alpha^{\sum_{\ell=1}^{n} x_{\ell,\ell}} \prod_{i=1}^{n}(1 - \alpha q_i) \prod_{1\leq i<j\leq n} (1 - q_i q_j)s_\lambda(q_1, \ldots , q_n)$$

which, using (3.2), becomes:

$$\alpha^{\sum_{\ell=1}^{n}(-1)^{\ell-1}\lambda_\ell} \prod_{i=1}^{n}(1 - \alpha q_i) \prod_{1\leq i<j\leq n} (1 - q_i q_j)s_\lambda(q_1, \ldots , q_n)$$

And so we have:

$$\Pr(\lambda_1 \leq h) = \prod_{i=1}^{n}(1 - \alpha q_i) \prod_{1\leq i<j\leq n} (1 - q_i q_j) \sum_{\lambda:\lambda_1\leq h} \alpha^{\sum_{\ell=1}^{n}(-1)^{\ell-1}\lambda_\ell} s_\lambda(q_1, \ldots , q_n)$$

Again we require $\Pr(\lambda_1 \leq h) \to 1$ as $h \to \infty$, which gives:

$$\sum_{\lambda} \alpha^{\sum_{\ell=1}^{n}(-1)^{\ell-1}\lambda_\ell} s_\lambda(q_1, \ldots , q_n) = \frac{1}{\prod_{i=1}^{n}(1 - \alpha q_i) \prod_{1\leq i<j\leq n} (1 - q_i q_j)}$$

This is a known analogue to the Cauchy identity, with the case $\alpha = 1$ corresponding to the Littlewood identity [8].

3.2 Symmetry about the anti-diagonal

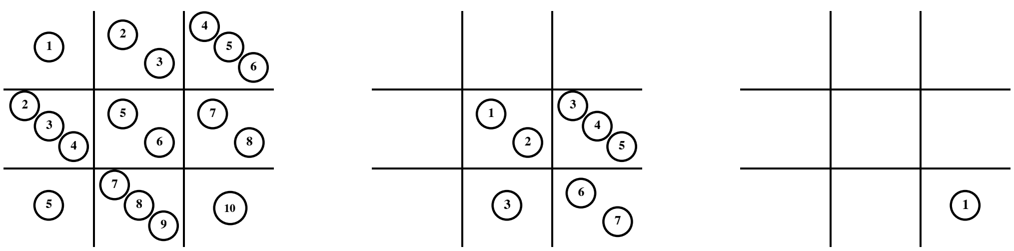
Now we consider the effect of symmetry about the anti-diagonal (for the matrix ball situation, this is from bottom left to top right, and for the growth models it is from top left to bottom right)

3.2.1 Matrix-ball construction

For example, the matrix A:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 1 & 3 & 1 \end{pmatrix}$$

which gives $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$:



We can see that the symmetry about the anti-diagonal carries through to some extent for all $A^{(p)}$, although the line of symmetry changes. As seen previously, each matrix of balls $A^{(p)}$ will not contain balls in the first $(p-1)$ rows or columns. In this case, disregarding the first $(p-1)$ rows and columns, we see symmetry in the anti-diagonal of the remaining $(n-p+1)\times(n-p+1)$ matrix.

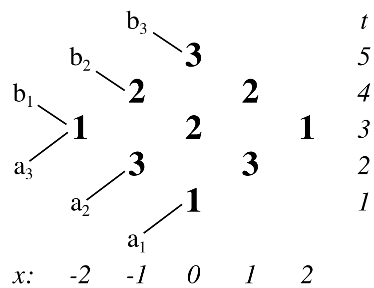
The above matrix-ball construction leads to the tableaux (P, Q):

1	1	1	1	1	2	2	2	2	3
2	2	2	3	3	3	3			
3									

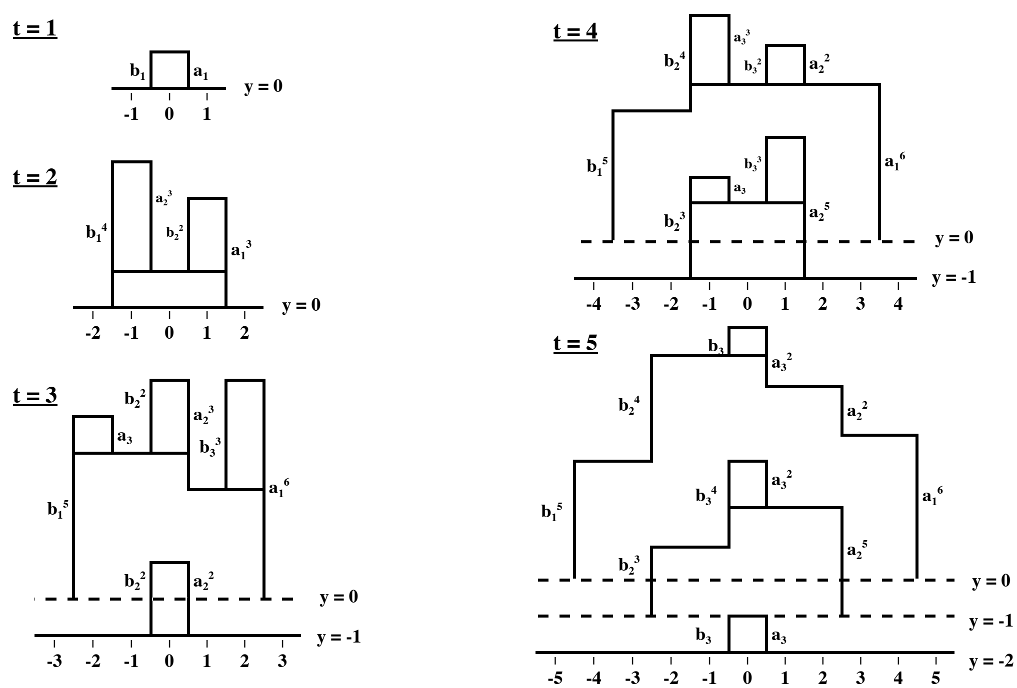
1	1	1	1	1	1	2	2	3	3
2	2	2	2	2	3	3			
3									

3.2.2 Growth models

The equivalent matrix to that used for the matrix-ball construction, rotated, is:



giving the growth models:



Again, as expected, this corresponds to the tableaux (P, Q):

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&2&2&2&2&3\\\hline 2&2&2&3&3&3&3\\\cline{1-7} 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&2&2&3&3\\\hline 2&2&2&2&2&3&3\\\cline{1-7} 3\\\cline{1-1}\end{array}$$

We can observe a relationship between P and Q: the number of 1's in Q is equal to the number of 3's in P, P and Q contain the same number of 2's, and the number of 3's in Q is equal to the number of 1's in P.

We note that in general, for tableaux resulting from a matrix symmetric about the antidiagonal:

$$\#j\text{'s in } P = \#(n+1-j)\text{'s in } Q, \quad j = 1, \ldots, n \tag{3.3}$$

An equivalent relationship can be seen directly from the growth models - the total number of "up-steps" at $x = -2n - \frac{1}{2} + 2j$ is equal to the total number of up-steps at $x = 2j - \frac{3}{2}$, for $j = 1, \ldots, n$. This leads to the sum of the powers of a_j being equal to the sum of the powers of b_{n+1-j}, for $j = 1, \ldots, n$, which, when translated to tableaux, gives (3.3).

Similarly, in the matrix-ball model, as seen in the symmetry found earlier, for each $A^{(p)}$, the number of balls in row j is equal to the number of balls in column $(n+p-j)$, for $j = p, \ldots, n$. When related to tableaux, this also leads to (3.3).

In fact, the tableau Q is the *Schützenberger dual* of P - so the tableaux resulting from a matrix symmetric about the anti-diagonal can be uniquely specified by P, as we can find tableau Q from P by applying the Schützenberger sliding algorithm. For a description of the Schützenberger sliding algorithm, refer to Appendix 2.

3.2.3 Probabilistic formulae for anti-diagonal symmetry

This is similar to the case for symmetry about the diagonal, where we do not consider the weight of the entries which are fixed by symmetry. So we need only consider the weights, and hence the probabilities, of entries $x_{i,j}$: $i \leq n+1-j$.

Let each entry k in position (i,j) of the matrix, where $i < n+1-j$, occur with probability $(1-q_iq_{n+1-j})(q_iq_{n+1-j})^k$, and each entry k in position $(i, n+1-i)$ occur with probability $(1-q_i)q_i^k$. The probability that X has entries $[x_{i,j}]_{i,j-1,\dots,n}$ is hence:

$$\prod_{i=1}^{n}(1-q_i)q_i^{x_{i,i}} \prod_{1\leq i<(n+1-j)\leq n} (1-q_iq_{n+1-j})(q_iq_{n+1-j})^{x_{i,n+1-j}}$$

which can be restated as:

$$\prod_{i=1}^{n}(1-q_i)q_i^{x_{i,i}} \prod_{1\leq i<j\leq n} (1-q_iq_j)(q_iq_j)^{x_{i,j}}$$

We can then give the probability of a matrix X corresponding to a single tableau (as Q is the Schützenberger dual of P, as found earlier) of shape λ as:

$$\prod_{i=1}^{n}(1-q_i) \prod_{1\leq i<j\leq n} (1-q_iq_j)s_\lambda(q_1,\dots,q_n) \tag{3.4}$$

We generalise the probability of the anti-diagonal entries to:

$$\frac{(1-q_i^2)}{1+\beta q_i}\beta^{k \mathrm{mod} 2}q_i^k$$

- we do this to take advantage of the following property [3]:

$$\#\{x_{i,n+1-i} : x_{i,n+1-i} \text{ odd}\} = \#\{\lambda_j : \lambda_j \text{ odd}\} = \sum_{j=1}^{n}(-1)^{j-1}\lambda_j^* \tag{3.5}$$

(where λ_j^* is the length of the j^{th} column in the diagram of λ)

So (3.4) becomes:

$$\beta^{\sum_{\ell=1}^{n} x_{\ell,n+1-\ell} \mathrm{mod} 2} \prod_{i=1}^{n} \frac{(1-q_i^2)}{1+\beta q_i} \prod_{1\le i<j\le n} (1-q_iq_j) s_\lambda(q_1,\ldots,q_n)$$

which, using (3.5), is equivalent to:

$$\beta^{\sum_{\ell=1}^{n} (-1)^{\ell-1}\lambda_\ell^*} \prod_{i=1}^{n} \frac{(1-q_i^2)}{1+\beta q_i} \prod_{1\le i<j\le n} (1-q_iq_j) s_\lambda(q_1,\ldots,q_n)$$

And hence

$$\Pr(\lambda_1 \le h) = \prod_{i=1}^{n} \frac{(1-q_i^2)}{1+\beta q_i} \prod_{1\le i<j\le n} (1-q_iq_j) \sum_{\lambda:\lambda_1\le h} \beta^{\sum_{\ell=1}^{n} (-1)^{\ell-1}\lambda_\ell^*} s_\lambda(q_1,\ldots,q_n)$$

Again we require $\Pr(\lambda_1 \le h) \to 1$ as $h \to \infty$, which gives:

$$\sum_{\lambda} \beta^{\sum_{\ell=1}^{n} (-1)^{\ell-1}\lambda_\ell^*} s_\lambda(q_1,\ldots,q_n) = \frac{1}{\prod_{i=1}^{n} \frac{(1-q_i^2)}{1+\beta q_i} \prod_{1\le i<j\le n} (1-q_iq_j)}$$

which, when $\beta = 1$ is the same as the case of symmetry about the diagonal for $\alpha = 1$, and again is one of the generalisations of the Littlewood identities [8].

Chapter 4

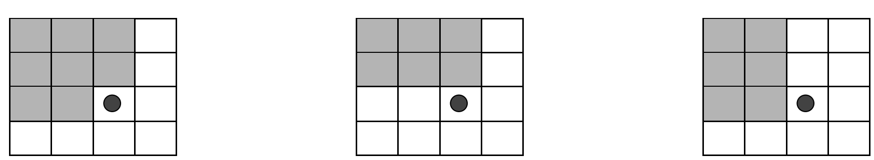
Matrices of 0s and 1s

A modification of the matrix-ball construction of the Robinson-Schensted-Knuth correspondence gives a similar correspondence between zero-one matrices and pairs of tableaux with conjugate shapes. Our task, achieved in this chapter, is then to relate this modified correspondence to a growth model, and consider its probabilistic formulae.

4.1 Modified matrix-ball construction

In place of the northwest ordering used in the general case, we can instead use a Northwest ordering - hence requiring ball numbers to be strictly increasing down columns, but only *weakly* increasing across rows. Similarly we could consider a northWest ordering, amongst others, and this would lead to a different correspondence (the nW ordering is explored in Appendix 2, and an overview given for other cases).

The following figures show, for each ordering, the region to be considered - ie the ball in question must have a greater value than any ball in the shaded region:

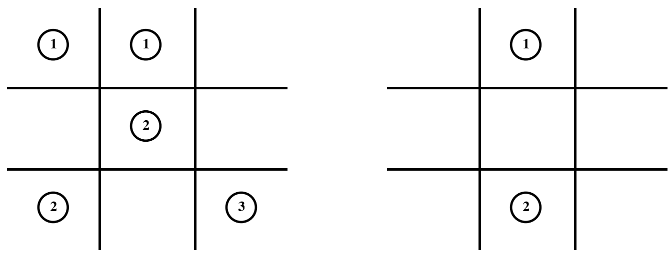


northwest ordering Northwest ordering northWest ordering

For example, taking the matrix A:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Using the Nw ordering for matrix-ball construction, we get $A^{(1)}$ and $A^{(2)}$:



This gives a resultant (P, Q):

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 \\ \cline{1-2} \end{array}$$

But for a Young tableau we require strictly increasing in columns and weakly increasing in rows, whereas Q above is weakly increasing in columns and strictly increasing in rows. However if we take the conjugate of Q, we get a Young tableau. So we end up with a pair of tableaux with conjugate shape:

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}$$

In this way we have set up a one-to-one correspondence between zero-one matrices and pairs of tableaux with conjugate shapes. If P is of shape λ, we call its conjugate shape λ^*, in keeping with our previous notation for the length of the j^{th} column, λ^*_j.

4.2 Growth model for matrices of 0s and 1s

For the zero-one case, similar to the general non-negative integer case, we can develop a growth model - a sequence of non-intersecting lattice paths - and associate these with a pair of tableaux of conjugate shape.

Given an $n \times n$ zero-one matrix, $X = [x_{i,j}]_{i,j=1,\ldots,n}$, we label the columns from the left with a value of x, and the rows from the bottom with a value for t. So, using the same matrix as that used in the Nw matrix ball case (with rows interchanged to account for the different method of numbering), this gives:

$$\begin{array}{cccccc} & & & & t \\ & \mathbf{1} & \mathbf{0} & \mathbf{1} & 3 \\ & \mathbf{0} & \mathbf{1} & \mathbf{0} & 2 \\ & \mathbf{1} & \mathbf{1} & \mathbf{0} & 1 \\ x: & 1 & 2 & 3 & \end{array}$$

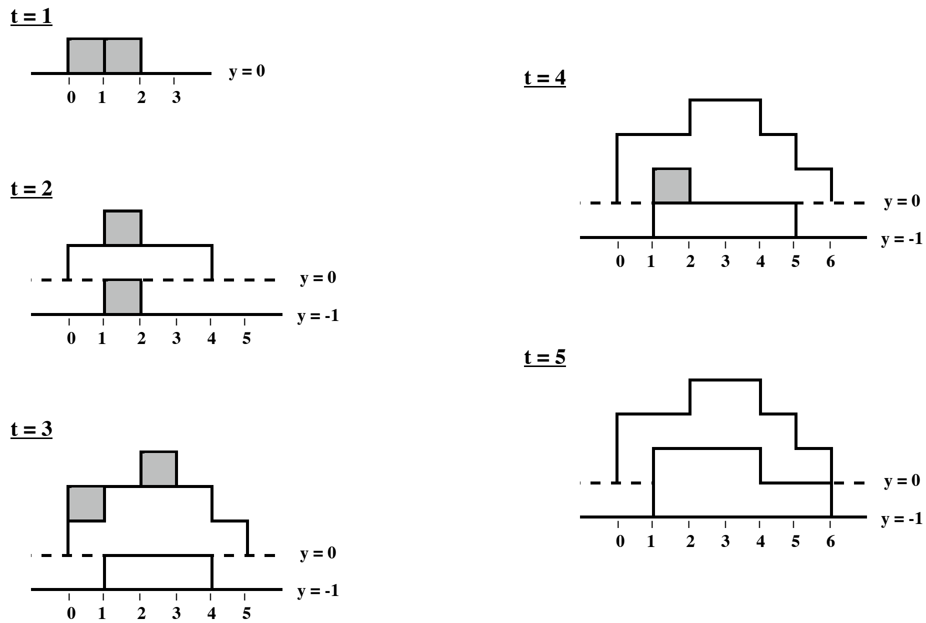
Again, as in the general case, the entries $x_{i,j}$ represent the heights of nucleation events - however in the case of the zero-one matrix it is only possible to have a nucleation event of unit height. For $x_{i,j} = 1$, a nucleation event is added to the growth model at time step $t = i$ from $x = (j-1)$ to $x = j$. So, in our example, at $t = 1$ we have two nucleation events: one from $x = 0$ to $x = 1$, and one from $x = 1$ to $x = 2$.

As $t \mapsto t+1$, the profile of all nucleation events recorded 'grows' one unit in the $+x$ direction (but remains fixed on the left, ie no growth in the $-x$ direction). If there is no nucleation event to the right of a particular event, and it has not already grown, it will grow to $x = n+1$. The overlap of adjacent nucleation events due to this growth (after which time the adjacent nucleation events will be considered as one) is recorded on the line immediately below.

After this growth process occurs at time $(t+1)$, new nucleation events are placed on the growth model at $y = 0$ corresponding to the $(t+1)^{th}$ row of X, and the process will continue at $(t+2)$ with the growth of these events, and so on.

The addition of new nucleation events at $y = 0$ will stop after $t = n$ (as there are n rows in the matrix), and the growth process stops at level ℓ after time $t = n + \ell$ (where level ℓ corresponds to the path at $y = 1 - \ell$) - this is the minimum time required so that all nucleation events resulting from an $n \times n$ matrix have a chance to grow at least once.

In the following growth models (from the above matrix), for greater clarity, shaded squares are used to represent nucleation events, or carried-down overlaps, that have not yet grown.



We can obtain the tableaux from the paths at the final time-step. The height of the ℓth path (the path at $y = 1 - \ell$) tells us the length of the ℓth row in both tableaux. We obtain P from the left path - the number of k's present in the ℓth row of the tableau corresponds to the number of vertical lines of unit length at $x = (k - 1)$, $y = 1 - \ell$.

In the example above, at $t = 5$, the first path has two vertical lines of unit length at $x = 0$ and one at $x = 2$, making the first row of the tableau P $(1, 1, 3)$. The second path has two vertical lines of unit length at $x = 1$, so the second row is $(2, 2)$.

Similarly, we obtain Q from the right path - the number of k's present in the ℓth row of the tableau corresponds to the number of vertical lines of unit length at $x = (2n + 1 - k)$, $t = \ell$.

In the example, the first path has a vertical line of unit length at each of 6, 5, and 4, so the first row of Q is $(1, 2, 3)$. The second path has a vertical line of unit length at 6 and 4, so the second row is $(1, 3)$.

So the resultant tableaux (P, Q) are:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 & 3 \\ \cline{1-2}\end{array}$$

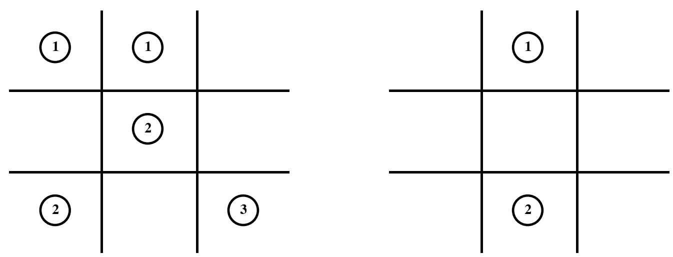
which is the same as those produced using the Nw matrix-ball construction. This will be the case in general, as will become clear from the following discussion.

4.3 Relationship between zero-one matrix-ball and growth models

4.3.1 Matrix-ball to growth model

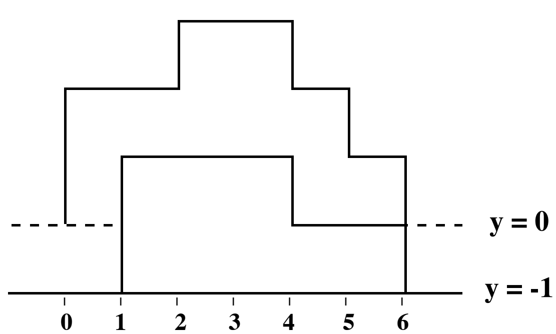
As with the general case, it is quite straightforward in the zero-one case to obtain the set of paths for the final time step from the matrix-ball construction. We relate $A^{(k)}$ to the path at $y = 1 - k$, using the columns of $A^{(k)}$ to determine the left path, and the rows to determine the right path. The maximum number in a ball in the j^{th} column (i^{th} row), minus the maximum number in previous columns (rows), gives the number of up-steps occurring at $x = j - 1$ ($x = 2n + 1 - i$).

Using the previous example to illustrate, we begin with the matrix-ball form:



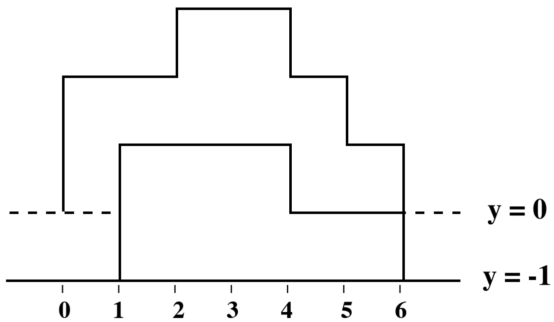
We use $A^{(1)}$ to determine the path at $y = 0$. First the left path: the maximum number in the first column is 2, so we have two up-steps at $x = 0$. The maximum number in the second column is 2, and the third is 3, so we have one up-step at $x = 2$. Now the right path: the maximum number in each row is 1, 2, and 3 respectively, so we have a single up-step at each of $x = 6$, $x = 5$ and $x = 4$.

Similarly, from $A^{(2)}$, we find we have two up-steps at $x = 1$ for the left path, and an up-step at each of $x = 6$ and $x = 4$ for the right path. So we have the following set of paths:



4.3.2 Growth model to matrix-ball

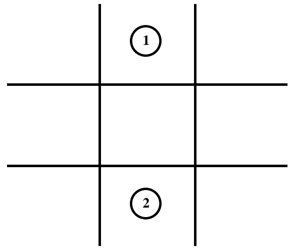
Again, as in the general case, this is less straightforward than going from the matrix-ball to the growth model. Given the set of non-intersecting paths:



we can derive the corresponding matrix-ball form, by relating the path at $y = 1-k$ to the matrix-ball form $A^{(k)}$.

The sum of the heights of all paths at $y \leq 1-k$ gives the total number of balls in $A^{(k)}$, and the height at $y = 1-k$ gives the maximum number permitted on any ball in $A^{(k)}$. The height of the left path at $x = j-1$ (or the right at $x = 2n+1-i$) gives the maximum number permitted in a ball at or to the left of the j^{th} column (or at or above the i^{th} row). From this we can deduce the matrix-ball form, beginning with the lowest path, at $y = 1-\ell$, as we did with the general case.

For the example above, we use the path at $y = -1$ to find the matrix-ball form $A^{(2)}$. The sum of all paths at $y \leq -1$ is 2, and the height at $y = -1$ is 2, so there are two balls in $A^{(2)}$, numbered up to 2. The height at $x = 1$ is 2, so the balls in the second column will be numbered up to 2. The height at $x = 6$ and $x = 4$ is 1 and 2 respectively, so the first row will be numbered up to 1, and the third up to 2. This leads to the matrix-ball form $A^{(2)}$:



The matrix-ball form $A^{(\ell)}$ helps in deducing $A^{(\ell-1)}$, as for every ball present in $A^{(\ell)}$ there must be two balls in $A^{(\ell-1)}$ with the same number, one strictly to the left, and the other weakly above (as the numbers will be weakly increasing across rows and strictly increasing down columns).

So for the example we use the path at $y = 0$ combined with $A^{(2)}$ to find the matrix-ball form $A^{(1)}$. The sum of the heights of all paths at $y \leq 0$ is 5, and the height at $y = 0$ is 3, so there are five balls in $A^{(1)}$, numbered up to 3. The height at $x = 0$ is 2, and at $x = 2$ is 3, so the numbers 1 and 2 will be present in the first column, but 3 will not appear until the third. Similarly, from the right path, we can see that the numbers 1, 2 and 3 will first appear in the first, second, and third row respectively.

Now we use the above information, combined with $A^{(2)}$, to deduce $A^{(1)}$. As there is a ball at $A^{(2)}_{1,2}$ we know that there must be a ball at each of $A^{(1)}_{1,1}$ and $A^{(1)}_{1,2}$, with the same number (which must be 1, as this is only number permitted on the first row). The ball at $A^{(2)}_{3,2}$ tells us there must be one at $A^{(1)}_{3,1}$, and one in either $A^{(1)}_{2,2}$ or $A^{(1)}_{3,2}$, with the same number (which must be 2, as this is the maximum number in the first column). As the number 3 first appears in the third row and third column, there must be a ball at $A^{(1)}_{3,3}$. So four of the balls are fixed, and we can use the fact that the number 2 will first appear in the second row to decide that the fifth (the one mentionned earlier with two possible positions) must be at $A^{(1)}_{2,2}$. So we have $A^{(1)}$:

1	1	
	2	
2		3

4.4 Probabilistic formulae for zero-one matrices

Again, with an appropriate choice of probabilities, and making use of the Schur polynomials, we can derive some probabilistic formulae, in the zero-one case. The significant difference is, since entries (and hence the value of 'k' used in previous probabilistic calculations) can only take on the values 0 or 1, we must change the normalisation accordingly.

Let each entry k in position (i, j) of the matrix occur with probability:

$$\frac{(a_i b_j)^k}{(1 + a_i b_j)}$$

Hence the probability of the matrix X having entries $[x_{i,j}]_{i,j=1,\ldots,n}$ is:

$$\prod_{i,j=1}^{n} \frac{(a_i b_j)^{x_{i,j}}}{(1 + a_i b_j)}$$

And the probability that X corresponds to a pair of tableaux with shape λ is:

$$\prod_{i,j=1}^{n} \frac{s_\lambda(a_1, \ldots, a_n) s_{\lambda^*}(b_1, \ldots, b_n)}{(1 + a_i b_j)}$$

Hence $\Pr(\lambda_1 \leq h) = \prod_{i,j=1}^{n} \frac{1}{(1 + a_i b_j)} \sum_{\lambda:\lambda_1 \leq h} s_\lambda(a_1, \ldots, a_n) s_{\lambda^*}(b_1, \ldots, b_n)$

We require $\Pr(\lambda_1 \leq h) \to 1$ as $h \to \infty$, which gives:

$$\sum_{\lambda} s_\lambda(a_1, \ldots, a_n) s_{\lambda^*}(b_1, \ldots, b_n) = \prod_{i,j=1}^{n} (1 + a_i b_j)$$

This is the conjugate Cauchy identity, which is known [12] in the theory of Schur polynomials.

Chapter 5

Conclusion and possible extensions

The RSK correspondence, in the form proposed by Knuth, came into being in 1970, and many fundamental properties were derived at that time. The matrix-ball interpretation, and the closely related growth models, were not realised until the late 1990's. This viewpoint has led to the discovery of additional fundamental properties of the RSK correspondence, and has furthermore suggested applications which could not have been seen from the original formulation. A comprehensive account of aspects of these developments has been the subject of this thesis, although it is also clear that more fundamental properties remain to be fully explored. Let us then make mention of a few which have become apparent in the course of this study.

We have considered thoroughly the case of one of the matrix-ball orderings, the nw case, along with its respective growth model, but, as mentioned in Appendix 1, similar correspondences are possible with all other orderings. A natural extension from this would be to consider possible growth models that could be fit to each case, which could result in further study of the probabilistic formulae.

Similarly for the zero-one case, as touched on in Appendix 3, we have variations of the RSK correspondence for different orderings of the matrix, and these correspondences could be further developed with the aid of growth models. Also, in the zero-one case, it would most likely prove fruitful to investigate recurrences for the lengths of the resultant tableaux of conjugate shape.

Further investigation of the property (3.5), which could be understood in terms of the matrix-balls and the growth models, seems to indicate a possible theory associated with the column lengths, λ_j^*, similar to the way in which (3.2) gives one associated with row lengths, λ_ℓ.

Another direction that could be pursued comes from the idea that we can relate λ_1 to the *Last passage time*, $L^{(1)}_{X_{n,n}}$, of an $n \times n$ matrix. If we consider 'travelling' from the bottom left corner to the top right corner of a matrix (where rows are counted from the bottom), where we liken each entry to a time, the last passage time is the maximum time this process can take for a given matrix, with the restriction that we can only step up or right. Explicitly:

$$L^{(1)}_{X_{n,n}} = \max \sum_{(1,1) \text{ U/R to } (n,n)} x_{i,j} \tag{5.1}$$

Considering a truncated matrix X_{n_1,n_2}, the last passage time satisfies the recurrence:

$$L^{(1)}_{X_{n_1,n_2}} = \max\big(\lambda_1(n_1, n_2 - 1), \lambda_1(n_1 - 1, n_2)\big) + x_{n_1,n_2},$$

$$\text{with } \; L^{(1)}_{X_{0,n_2}} = L^{(1)}_{X_{n_1,0}} = 0,$$

so we can conclude from the recurrence found for $\lambda_1(n_1, n_2)$ that $L^{(1)}_{X_{n_1,n_2}} = \lambda_1(n_1, n_2)$.

Hence our study of the probabilities related to λ_1 also apply to $L^{(1)}_X$. We could continue this study to consider more generally $L^{(k)}_X$, which we expect to be able to relate to the level-ℓ heights λ_ℓ for each $\ell = 1, \ldots, k$. We denote by $(\text{U/R})^k$ the set of k disjoint paths, restricted to consist of only up- or right-steps, from some $x_{i',j'}$ to some $x_{i,j}$, where $1 \le i' \le i \le n_1$, $1 \le j' \le j \le n_2$. We generalise (5.1) to get:

$$L^{(k)}_{X_{n,n}} = \max \sum_{(\text{U/R})^k} x_{i,j}$$

Results of Greene [7] suggest that:

$$L^{(k)}_{X_{n_1,n_2}} = \sum_{\ell=1}^{k} \lambda_\ell(n_1, n_2)$$

To derive this result in the context of recurrence relations is another problem for future study.

Appendices

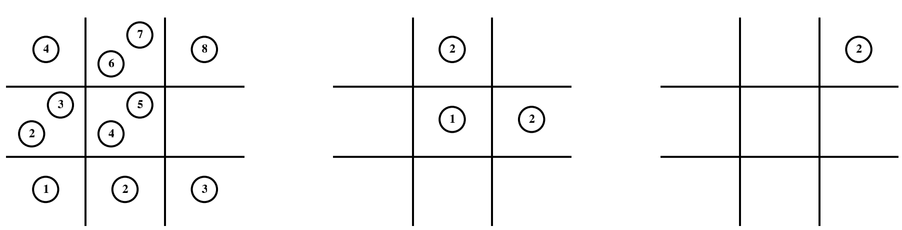
Appendix 1: Variations on the RSK correspondence

We can obtain different correspondences, related to the RSK correspondence, by using different matrix-ball orderings. For example, we can use a **southwest** ordering, numbering balls from southwest to northeast. Similar to the nw case, we place $(\ell-1)$ balls in $A^{(k+1)}$ for every ℓ balls in $A^{(k)}$ with the same number, such that the new ball is to the right of each of these (except the last), and directly above the next ball.

For example, using a southwest ordering on the matrix A:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

gives $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$:



The system for numbering the tableaux is similar to the nw case. The ℓ^{th} row of P is still determined by listing the left-most (ie, "west-most") columns where each number first occurs, so P will be a semi-standard tableau. However, the ℓ^{th} row of Q is determined by listing the lowest (ie, "south-most") rows where each number first occurs. When we are measuring from the bottom (as in this case) or from the right (in an "east" case), we insert a *dual number* in the diagram, which we denote by k^*.

So, from $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ we can read off the tableaux corresponding to A:

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&2&3\\ \hline 2&2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|c|c|c|}\hline 3^*&3^*&3^*&2^*&2^*&1^*&1^*&1^*\\ \hline 2^*&2^* \\ \cline{1-2} 1^* \\ \cline{1-1}\end{array}$$

We note that Q is not in this case, and will not in general be, a semi-standard tableau, as the lowest row in which a value s first appears will be greater than or equal to the lowest row in which a value $(s+1)$ first appears. To obtain a semi-standard tableau from Q we can substitute $(n+1-k)$ for each k^* to get the tableau S:

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3&3\\ \hline 2&2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

We say that Q is the *dual* of S, which we write as $Q = S^*$. Another significant point that can be seen is that S would be the tableau related to the rows that we would get from using an ordering in the opposite direction, ie northeast. This is in fact the case in general.

Fulton, in [5], gives a summary of the relationship the tableaux from different orderings have with each other.

For weak orderings, we have:

$$\begin{array}{rcl@{\qquad}rcl} \text{nw ordering} & \rightarrow & (P, Q) & \text{sw ordering} & \rightarrow & (R, S^*) \\ \text{se ordering} & \rightarrow & (P^*, Q^*) & \text{ne ordering} & \rightarrow & (R^*, S) \end{array}$$

and for strong orderings, we have:

$$\begin{array}{rcl@{\qquad}rcl} \text{NW ordering} & \rightarrow & (R, S) & \text{SW ordering} & \rightarrow & (P, Q^*) \\ \text{SE ordering} & \rightarrow & (R^*, S^*) & \text{NE ordering} & \rightarrow & (P^*, Q) \end{array}$$

So we can note that the weak orderings and the strong orderings are related by interchanging P and R, and interchanging Q and S.

Appendix 2: Schützenberger sliding algorithm

Note: The Schützenberger sliding algorithm is often used to form a **product tableau** *of two tableaux, but we wish to use it to find the Schützenberger dual of a tableau, and as such the description that follows is particular to this case.*

For a given tableau P, of shape λ, with n rows, we can form its Schützenberger dual, which will also be of shape λ, as follows:

Begin with the tableau P and an "empty" (ie not numbered) tableau of the same shape λ, which we will call Q. The sliding algorithm proceeds as follows:

(1) Remove the number, of value x, from the top left corner of the tableau. The resultant box can now be considered as a "hole".

(2) Of the box to the right and the box below this hole, choose the one with lesser value. If they are of equal value, choose the box below. "Slide" this value into the hole, thus leaving a hole in the position vacated by this value.

(3) With respect to the new hole created each time, repeat (2) until there is no value to the right or below the hole.

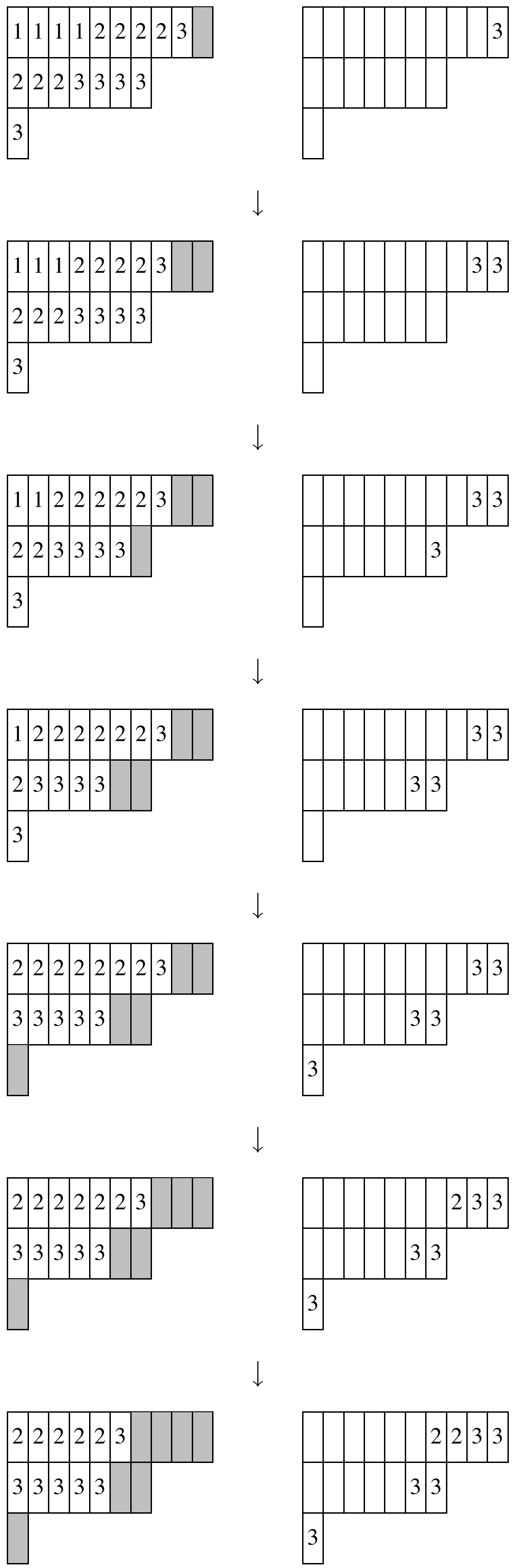
(4) Assign the value $(n - x)$ to the box in Q corresponding to the position of the hole in P, then delete the hole from P (ie remove this box from the tableau).

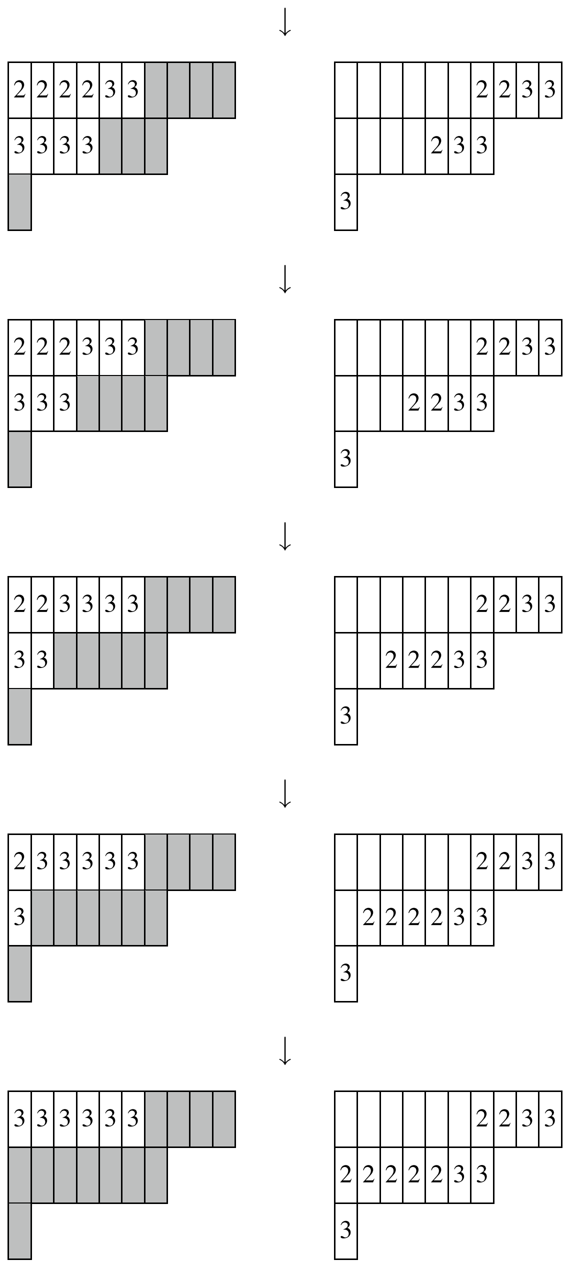
(5) Repeat steps $(1) - (4)$ until no boxes remain in P, and hence all boxes in Q are labelled. The resultant Q is the Schützenberger dual of (the original) P.

This is best shown through example, so we will take the tableau P found earlier in the example of the matrix symmetric about the anti-diagonal, and an empty tableau of the same shape:

1	1	1	1	1	2	2	2	2	3
2	2	2	3	3	3	3			
3									

and proceed with the algorithm, as shown in the following pages.





So the resultant Q is:

1	1	1	1	1	1	2	2	3	3
2	2	2	2	2	3	3			
3									

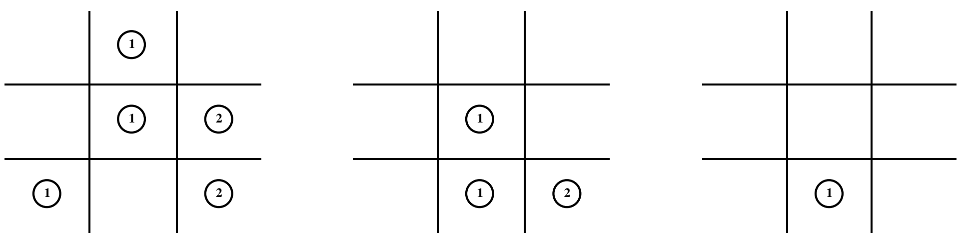
which, as expected, is the same as the tableau Q found earlier for the matrix symmetric about the anti-diagonal.

Appendix 3: Matrix-ball variations - zero-one case

We can obtain different correspondences, related to the RSK correspondence, by using different matrix-ball orderings. For example, we can use a **southwest** ordering, numbering balls from southwest to northeast. We apply the northWest ordering for matrix-ball construction to the matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

which gives us $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$:



And so the resultant (P, Q) are:

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}$$

But, similar to the Northwest case, P is weakly increasing in columns and strictly increasing in rows, so to get a tableau from this we take the conjugate, and hence we end up with a pair of tableaux with conjugate shape:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 3 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}$$

As in the general case, Fulton [5] has summarised the possibilities with different orderings. We use $\{\tilde{P}, \tilde{Q}\}$ to denote a pair of conjugate tableaux.

For strong column and weak row orderings, we have:

$$\begin{array}{rcl@{\qquad}rcl} \text{nW ordering} & \rightarrow & \{\tilde{P}, \tilde{Q}\} & \text{sW ordering} & \rightarrow & \{\tilde{R}, \tilde{S^*}\} \\ \text{sE ordering} & \rightarrow & \{\tilde{P^*}, \tilde{Q^*}\} & \text{nE ordering} & \rightarrow & \{\tilde{R^*}, \tilde{S}\} \end{array}$$

For weak column and strong row orderings, we have:

$$\begin{array}{rcl@{\qquad}rcl} \text{Nw ordering} & \rightarrow & \{\tilde{R}, \tilde{S}\} & \text{Sw ordering} & \rightarrow & \{\tilde{P}, \tilde{Q^*}\} \\ \text{Se ordering} & \rightarrow & \{\tilde{R^*}, \tilde{S^*}\} & \text{Ne ordering} & \rightarrow & \{\tilde{P^*}, \tilde{Q}\} \end{array}$$

These are the exact same relationships as found for the general case, with the substitution of the pairs of conjugate tableau $\{\tilde{P}, \tilde{Q}\}$ and $\{\tilde{R}, \tilde{S}\}$ for (P,Q) and (R,S).

We can also note that, similar to the general case, the strong column, weak row orderings and the weak column, strong row orderings are related by interchanging P and R, and interchanging Q and S.

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