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OPTIMAL LABELLING OF UNIT INTERVAL GRAPHS

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Abstract. This paper shows that, for every unit interval graph, there is a labelling which is simultaneously optimal for the following seven graph labelling problems: bandwidth, cyclic bandwidth, profile, fill-in, cutwidth, modified cutwidth, and bandwidth sum(linear arrangement).

1. Introduction

Given a finite simple graph $G(V, E)$ with $|V| = n$. A labelling of G is a bijection $f : V \rightarrow \{1, 2, \dots, n\}$. When a labelling f of a graph $G(V, E)$ is given, we set $S_k = \{v \in V \mid f(v) \leq k\}$ and $T_k = \{v \in V \mid f(v) > k\}$, $1 \leq k < n$.

We consider the following seven graph labelling problems.

(1) *Bandwidth problem*[8,1]:

For a given graph $G(V, E)$ and a labelling f of G , set

$$B(G, f) = \max\{|f(u) - f(v)| \mid uv \in E(G)\}.$$

The bandwidth of graph G is defined as

$$B(G) = \min\{B(G, f) \mid f \text{ is a labelling of } G\}.$$

A labelling f of G with $B(G, f) = B(G)$ is called an optimal bandwidth labelling of G .

(2) *Cyclic bandwidth problem*[3]:

For a given graph $G(V, E)$ with $|V| = n$ and a labelling f of G , set

$$B_c(G, f) = \max_{uv \in E} \min\{|f(u) - f(v)|, n - |f(u) - f(v)|\}.$$

The cyclic bandwidth of G is defined as

$$B_c(G) = \min\{B_c(G, f) \mid f \text{ is a labelling of } G\}.$$

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A labelling f of G with $B_c(G, f) = B_c(G)$ is called an optimal cyclic bandwidth labelling of G .

(3) *Profile problem*[8,5]:

For a given graph $G(V, E)$ with $|V| = n$, and a labelling f of G , set

$$P(G, f) = \sum_{v \in V} (f(v) - \min\{f(x) \mid x = v \text{ or } xv \in E\}).$$

The profile of G is defined as

$$P(G) = \min\{P(G, f) \mid f \text{ is a labelling of } G\}.$$

A labelling f of G with $P(G, f) = P(G)$ is called an optimal bandwidth labelling of G .

(4) *Fill-in problem*[9,7,4]:

Given a graph $G(V, E)$ with $|V| = n$ and a labelling f of G . Set

$$v_i = f^{-1}(i), \quad i = 1, 2, \dots, n;$$

$$G_0 = G;$$

$$E_i(f) = \{xy \notin E(G_{i-1}) \mid v_i x, v_i y \in E(G_{i-1}), x \neq y\};$$

$$G_i = G_{i-1} - v_i + E_i(f), \quad i = 1, 2, \dots, n-1;$$

$$F(G, f) = \sum_{i=1}^{n-1} |E_i(f)|.$$

The fill-in of G is defined as

$$F(G) = \min\{F(G, f) \mid f \text{ is a labelling of } G\}.$$

A labelling f of G with $F(G, f) = F(G)$ is called an optimal fill-in labelling of G .

(5) *Cutwidth problem*[1,6]:

Given a graph $G(V, E)$ with $|V| = n$ and a labelling f of G . Set

$$C(G, f) = \max_{1 \leq k \leq n-1} |(S_k, T_k)|,$$

where $(S, T) = \{uv \in E \mid u \in S, v \in T\}$ for $S, T \subseteq V, S \cap T = \emptyset$. The cutwidth of G is defined as

$$C(G) = \min\{C(G, f) \mid f \text{ is a labelling of } G\}.$$

A labelling f of G with $C(G, f) = C(G)$ is called an optimal cutwidth labelling of G .

(6) *Modified cutwidth problem*[6]:

Substitute T_{k+1} for T_k , $1 \leq k \leq n-2$, and MC for C in the cutwidth problem.

(7) *Bandwidth sum (linear arrangement) problem*[1]:

Given a graph $G(V, E)$ and a labelling f of G . Set

$$BS(G, f) = \sum_{uv \in E} |f(u) - f(v)|.$$

The bandwidth sum of G is defined as

$$BS(G) = \min\{BS(G, f) \mid f \text{ is a labelling of } G\}.$$

A labelling f of G with $BS(G, f) = BS(G)$ is called an optimal bandwidth sum labelling of G .

All of the above seven problems are strongly motivated by practical considerations. For example, the bandwidth problem and the profile problem originate from the storage of symmetric matrices[8,5]; the cyclic bandwidth problem and the (modified) cutwidth problem arise from the circuit layout of VLSI designs[3,6]; the fill-in problem comes from the Gaussian elimination of symmetric positive definite matrix[7]; and the bandwidth sum problem is considered in finite element analysis. Unfortunately they are all NP-complete[1,3,9,6,2].

In this paper, we focus our interests on unit interval graphs.

Suppose that J_1, J_2, \dots, J_n are some unit closed intervals in the real line. A graph G with

$$\begin{aligned} V(G) &= \{J_1, J_2, \dots, J_n\}, \\ E(G) &= \{J_i J_j \mid i \neq j, J_i \cap J_j \neq \emptyset\} \end{aligned}$$

is called a unit interval graph.

A labelling f of a graph G is called a regular labelling of G if for every edge $uv \in E(G)$, say with $f(u) < f(v)$, $V(u, v) = \{x \in V(G) \mid f(u) \leq f(x) \leq f(v)\}$ is a clique of G , i.e., every two distinct vertices of $V(u, v)$ are adjacent in G .

Not every graph has regular labelling. The following result is prove in [11].

Theorem 1.1[11]. *A graph G is a unit interval graph if and only if G has a regular labelling.*

In section 2 we shall prove that the regular labelling of a unit interval graph is also the optimal labelling for the seven graph labelling problems.

2. Optimal Labelling

Denote by $Q(G)$ the clique number of a graph G . For the bandwidth and cyclic bandwidth, we have from the definitions

$$\begin{aligned} B(G) &\geq Q(G) - 1, \\ B_c(G) &\geq \min\{\lfloor |V(G)|/2 \rfloor, Q(G) - 1\}. \end{aligned}$$

For a regular labelling f of a unit interval graph G , from Theorem 1.1, we can check that

$$\begin{aligned} B(G, f) &= Q(G) - 1, \\ B_c(G, f) &= \min\{\lfloor |V(G)|/2 \rfloor, Q(G) - 1\}. \end{aligned}$$

These imply the following two results in [11] and [10], respectively:

$$\begin{aligned} B(G) &= Q(G) - 1, \\ B_c(G) &= \min\{\lfloor |V(G)|/2 \rfloor, Q(G) - 1\}, \end{aligned}$$

for every unit interval graph G . Hence we have

Theorem 2.1. *A regular labelling of a unit interval graph is an optimal bandwidth labelling.*

Theorem 2.2. *A regular labelling of a unit interval graph is an optimal cyclic bandwidth labelling.*

For the profile of a graph G , we have $P(G) \geq |E(G)|$. This follows from the definitions, or see [5] for the more general result: $P(G) = |E(G)|$ if and only if G is an interval graph. For a regular labelling of a unit interval graph G , from Theorem 1.1, we can easily check that $P(G, f) = |E(G)|$. Hence we have

Theorem 2.3. *A regular labelling of a unit interval graph is an optimal profile labelling.*

For the fill-in of a graph G , a result in [4] shows that $F(G) \leq P(G) - |E(G)|$. Hence for a unit interval graph G , we have $F(G) = 0$. This also follows from another result in [4]: $F(G) = 0$ if and only if G is a chordal graph. A unit interval graph is certainly a chordal graph. For a regular labelling f of a unit interval graph G , from Theorem 1.1, we can easily see that $E_i(f)$ is empty for each i , and hence $F(G, f) = 0$. Then we have

Theorem 2.4. *A regular labelling of a unit interval graph is an optimal fill-in labelling.*

The above discussions are in fact trivial. Now we consider the other three problems.

Lemma 2.5. Suppose that G is a graph with $|V(G)| = n$. Let $S, T \subseteq V(G)$ such that $S \cup T = V(G)$ and $S \cap T = \emptyset$. If S and T are two cliques of G , then $C(G) \geq |(S, T)|$, where $(S, T) = \{uv \in E(G) \mid u \in S, v \in T\}$.

Proof. Suppose $|S| = s$, $|T| = t$. Then $s + t = n$ and

$$|E(G)| = |(S, T)| + s(s-1)/2 + t(t-1)/2$$

since S and T are two cliques of G .

For an optimal cutwidth labelling f of G , we have

$$\begin{aligned} C(G) = C(G, f) &= \max_{1 \leq k \leq n-1} |(S_k, T_k)| \geq |(S_s, T_s)| \\ &= |E(G)| - |\{uv \in E(G) \mid u, v \in S_s\}| - |\{uv \in E(G) \mid u, v \in T_s\}| \\ &\geq |E(G)| - s(s-1)/2 - (n-s)(n-s-1)/2 = |(S, T)|. \quad \square \end{aligned}$$

Lemma 2.6. Suppose that G is a graph with $|V(G)| = n$. Let $w \in V(G)$ and $S, T \subseteq V(G)$ such that $S \cup T = V(G) \setminus \{w\}$ and $S \cap T = \emptyset$. If S and T are two cliques of G and w is adjacent to every vertex in $S \cup T$, then $MC(G) \geq |(S, T)|$.

Proof. Suppose $|S| = s$, $|T| = t$. Then $s + t = n - 1$ and

$$|E(G)| = |(S, T)| + s(s-1)/2 + t(t-1)/2 + (n-1)$$

since S and T are two cliques of G .

For an optimal modified cutwidth labelling f of G , we have

$$\begin{aligned} MC(G) = MC(G, f) &= \max_{1 \leq k \leq n-1} |(S_k, T_{k+1})| \geq |(S_s, T_{s+1})| \\ &\geq |E(G)| - s(s-1)/2 - t(t-1)/2 - (n-1) = |(S, T)|. \quad \square \end{aligned}$$

Theorem 2.7. A regular labelling of a unit interval graph is an optimal cutwidth labelling.

Proof. Suppose that G is a unit interval graph and f is a regular labelling of G . From the definition of $C(G, f)$, there is an integer k , $1 \leq k \leq |V(G)| - 1$, such that $C(G, f) = |(S_k, T_k)|$. Set

$$\begin{aligned} S &= \{u \in S_k \mid \text{there is } v \in T_k \text{ such that } uv \in E(G)\}, \\ T &= \{v \in T_k \mid \text{there is } u \in S_k \text{ such that } uv \in E(G)\}. \end{aligned}$$

Then we can easily see that $(S, T) = (S_k, T_k)$. Hence $C(G, f) = |(S, T)|$. From Theorem 1.1, we see that S and T are two cliques of G . Let $H = G[S \cup T]$ be a subgraph of G with vertex set $S \cup T$ and edge set $\{uv \in E(G) \mid u, v \in S \cup T\}$. Then $H \subseteq G$, $S \cup T = V(H)$, $S \cap T = \emptyset$, and S and T are two cliques in H . From Lemma 2.5, we have

$$C(G) \geq C(H) \geq |(S, T)_H| = |(S, T)| = C(G, f).$$

This implies $C(G, f) = C(G)$. \square

Theorem 2.8. *A regular labelling of a unit interval graph is an optimal modified cutwidth labelling.*

Proof. Suppose that G is a unit interval graph and f is a regular labelling of G . Let k be an integer, $1 \leq k \leq |V(G)| - 2$, such that $MC(G, f) = |(S_k, T_{k+1})|$. Set

$$w = f^{-1}(k+1),$$

$$S = \{u \in S_k \mid \text{there is } v \in T_{k+1} \text{ such that } uv \in E(G)\},$$

$$T = \{v \in T_{k+1} \mid \text{there is } u \in S_k \text{ such that } uv \in E(G)\}.$$

Then $(S, T) = (S_k, T_{k+1})$, $MC(G, f) = |(S, T)|$, S and T are two cliques in G , and w is adjacent to every vertex in $S \cup T$. Set $H = G[S \cup T \cup \{w\}]$. Then $H \subseteq G$ and $S \cup T \cup w = V(H)$. From Lemma 2.6 we have

$$MC(G) \geq MC(H) \geq |(S, T)_H| = |(S, T)| = MC(G, f). \quad \square$$

Lemma 2.9. *Suppose that G is a graph, v is a vertex of G , and $d_G(v) = k$. If $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ is a clique of G , then*

$$BS(G) \geq BS(G - v) + k(k+1)/2.$$

Proof. Suppose that f is an optimal bandwidth sum labelling of graph G , i.e., $BS(G) = BS(G, f)$. Define a labelling f' of the graph $G - v$ by

$$f'(u) = f(u), \quad \text{for } u \in V(G), f(u) < f(v);$$

$$f'(u) = f(u) - 1, \quad \text{for } u \in V(G), f(u) > f(v).$$

Set

$$E_0 = \{xy \in E(G) \mid x, y \in N(v)\}; \quad E_1 = \{xv \mid x \in N(v)\}.$$

Then it is easy to see that

$$|f'(x) - f'(y)| \leq |f(x) - f(y)|, \quad \text{for } xy \in E(G) \setminus (E_0 \cup E_1).$$

Let

$$S = \{u \in N(v), f(u) < f(v)\}; \quad T = \{u \in N(v), f(u) > f(v)\}.$$

Let $|S| = s$ and $|T| = t = k - s$. Since $N(v)$ is a clique of G , it is easy to see that

$$\begin{aligned} \sum_{xy \in E_1} |f(x) - f(y)| &= \sum_{x \in S} (f(v) - f(x)) + \sum_{x \in T} (f(x) - f(v)) \\ &\geq \sum_{i=1}^s i + \sum_{i=1}^t i = s(s+1)/2 + t(t+1)/2; \\ \sum_{xy \in E_0} |f(x) - f(y)| &= \sum_{xy \in (S, T)} (f(x) - f(y)) + \sum_{xy \in E_0 \setminus (S, T)} |f(x) - f(y)| \\ &= st + \sum_{xy \in (S, T)} (f'(x) - f'(y)) + \sum_{xy \in E_0 \setminus (S, T)} |f'(x) - f'(y)| \\ &= st + \sum_{xy \in E_0} |f'(x) - f'(y)|. \end{aligned}$$

Hence we have

$$\begin{aligned}
 BS(G) &= BS(G, f) \\
 &= \sum_{xy \in E(G) \setminus (E_0 \cup E_1)} |f(x) - f(y)| + \sum_{xy \in E_1} |f(x) - f(y)| + \sum_{xy \in E_0} |f(x) - f(y)| \\
 &\geq \sum_{xy \in E(G) \setminus (E_0 \cup E_1)} |f'(x) - f'(y)| + s(s+1)/2 + t(t+1)/2 + st + \sum_{xy \in E_0} |f'(x) - f'(y)| \\
 &= BS(G - v, f') + (s(s+1) + t(t+1) + 2st)/2 \\
 &\geq BS(G - v) + k(k+1)/2. \quad \square
 \end{aligned}$$

Theorem 2.10. *A regular labelling of a unit interval graph is an optimal bandwidth sum labelling.*

Proof. Suppose that G is a unit interval graph. We prove this theorem by induction on $|V(G)|$. When $|V(G)| = 1$, the result is obviously true. Now suppose that for every unit interval graph H with $|V(H)| < |V(G)|$ (≥ 2) the result is true.

For a regular labelling f of G , let $v = f^{-1}(1)$ and $k = |N(v)|$. Then

$$N(v) = \{u \in V(G) \mid 2 \leq f(u) \leq k+1\}$$

by Theorem 1.1. Define a labelling f' of $G - v$ by $f'(u) = f(u) - 1$ for $u \in V(G) \setminus \{v\}$. From Theorem 1.1, we can see that f' is a regular labelling of $G - v$. Hence by the induction assumption, $BS(G - v, f') = BS(G - v)$. Furthermore we can check that

$$BS(G) \leq BS(G, f) = k(k+1)/2 + BS(G - v, f') = BS(G - v) + k(k+1)/2.$$

Combining with Lemma 2.9, we can deduce that $BS(G, f) = BS(G)$. \square

3. Conclusion

The results in this paper show that a regular labelling of a unit interval graph is also an optimal labelling for the seven problems considered. This implies that the seven graph labelling problems are linearly solvable for unit interval graphs.

Another interesting fact can be stated as follow.

Theorem 3.1. *Let G^k denote the k -th power graph of a graph G . If G is a unit interval graph, then G^k is also a unit interval graph. If f is a regular labelling of G , then f is also a regular labelling of G^k .*

This result follows readily from Theorem 1.1. A by-product of Theorem 3.1 is

Theorem 3.2. *If G is a unit interval graph, then there is a labelling f of G such that f is an optimal labelling of G^k ($k \geq 1$) for the seven graph labelling problems considered in this paper.*

It should be noted that Theorem 3.2 is not true for graphs in general. It is suggested by Professor Zhang Fuji to find some other interesting graphs such that Theorem 3.2 is true for some graph labelling problems. We believe that his suggestion is worthy of further research.

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