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Weight Distribution of the Bases of a Binary Matroid

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Abstract—Let M be a weighted binary matroid and $w_1 < \cdots < w_m$ be the increasing sequence of all possible distinct weights of bases of M. We give a sufficient condition for the property that $w_1, \ldots, w_m$ is an arithmetical progression of common difference d. We also give conditions which guarantee that $w_{i+1} - w_i \leq d$, $1 \leq i \leq m-1$. Dual forms for these results are given also. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Matroid, Weight, Arithmetical progression.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a connected graph and $\mathcal{F}(G)$ the set of spanning trees of G. Let $w : E(G) \to \mathbb{R}$ be a weight function which associates a real number weight $w(e)$ with each edge $e \in E(G)$. For each $T \in \mathcal{F}(G)$, the weight of T is $w(T) = \sum_{e \in E(T)} w(e)$. Denote all distinct weights of spanning trees of G by $w_1 > \cdots > w_m$. The spanning trees with weight w_i are called the i^{th} maximal spanning trees. For each $T \in \mathcal{F}(G)$ and integer k, $0 \leq k \leq |V(G)|$, let $\mathcal{L}_k(T) = \{T' \in \mathcal{F}(G) : |T' \setminus T| \leq k\}$. Kano [1], conjectured that for any maximum weight spanning tree A and each i with $1 \leq i \leq k$, $\mathcal{L}_{k-1}(A)$ contains an i^{th} maximal spanning tree of G. He proved [1] that the conjecture is true when $w_1, \ldots, w_m$ is an arithmetical progression. Although the conjecture has been fully proved [2,3], we feel that the problem of when $w_1, \ldots, w_m$ is an arithmetical progression is of interest for its own reason. In this direction, an early result of Hakimi and Maeda [4] says that if the weight $w(e)$ of each edge e is c, $c + d$, or $c + 2d$ for some constants c and $d > 0$, then $w_1, \ldots, w_m$ is an arithmetical progression. On the other hand, it seems that we do not know much about the distribution of the weights of spanning trees of a graph, although a lot of combinatorial optimization problems, such as the minimum spanning tree problem, relate closely to the weights of spanning trees. In general, it is difficult to have a detailed understanding of the distribution of the weights of bases of a weighted matroid.

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In this paper, we tentatively give a condition which guarantees that the weights of bases of a weighted binary matroid consist of an arithmetical progression. Also we give a sufficient condition for the property that for each i, the difference of the $(i+1)^{\text{th}}$ minimal and the i^{th} minimal weights does not exceed a constant d. The dual versions of these results are provided.

2. MAIN RESULTS AND THE PROOF

The reader is referred to [5] for terminologies on matroids. Let M be a *matroid* on a finite set S and $\mathcal{B}(M)$ the set of *bases* of M. For any $B \in \mathcal{B}(M)$ and $x \in S \setminus B$, $B \cup \{x\}$ contains a unique circuit $C(x, B)$, called the *fundamental circuit* of x in the base B. Note that $x \in C(x, B)$.

LEMMA 1. *(See [5].) Suppose $B \in \mathcal{B}(M), x \in S \setminus B, y \in B$. Then $(B \setminus \{y\}) \cup \{x\} \in \mathcal{B}(M)$ if and only if $y \in C(x, B)$ or $y = x$.*

If for any two distinct circuits C_1, C_2 of M, the symmetric difference $C_1 \triangle C_2$ contains a circuit, then M is said to be a *binary matroid* [5]. Note that there are alternative ways to define a binary matroid. We present the following equivalent condition which will be used later.

LEMMA 2. *(See [5].) M is a binary matroid if and only if the symmetric difference of any collection of distinct circuits is the union of disjoint circuits of M.*

In the following, we always suppose M is a binary matroid on finite S. For a subset X of S, the incidence vector of X is the vector $(i_x)_{x \in S}$ with entries indexed by the elements of S, where i_x is 1 or 0 depending on whether x is or is not in X. The *circuit space* of M, denoted by $V(M)$, is the vector space over the field $GF(2)$ generated by the incidence vectors of the circuits of M. We can view the vectors of $V(M)$ as symmetric differences of some circuits of M (or equivalently as disjoint union of some circuits). The sum of $X, Y \in V(M)$ is the symmetric difference $X \triangle Y$. We call a base of $V(M)$ a *circuit base* if each vector in this base is a circuit of M. Note that the dimension of $V(M)$ is $\rho = |S| - r$, where $r = \text{rank}(M)$ is the rank of M.

Let $w : S \to \mathbb{R}$ be a weight function, where $\mathbb{R}$ is the set of real numbers. Thus, M is a weighted matroid with *weight* $w(x)$ for each $x \in S$. The *weight of a base* $B \in \mathcal{B}(M)$ is $w(B) = \sum_{x \in B} w(x)$. A base with maximum weight is said to be a *maximum base*. Suppose $w_1 < \cdots < w_m$ is the sequence of all distinct weights of bases of M. In this section, we always suppose that the following condition is satisfied.

CONDITION. There exists a circuit base $\mathcal{C} = \{C_1, \ldots, C_\rho\}$ of $V(M)$ such that for each C_i there exists at most one C_j with $C_i \cap C_j \neq \emptyset$, $j \neq i$.

For the case of a cycle matroid of a graph G, this condition is satisfied when, for example, the cycles of G are pairwise edge disjoint. We have the following.

LEMMA 3.

(i) *If $x_1, \ldots, x_\rho \in S$ satisfy*

$$x_i \in C_i \Big\backslash \bigcup_{j \neq i} C_j, \qquad 1 \leq i \leq \rho, \tag{1}$$

then $B = S \setminus \{x_1, \ldots, x_\rho\} \in \mathcal{B}(M)$ and $C_i = C(x_i, B)$.

(ii) *Conversely, for any $B \in \mathcal{B}(M)$ there exists an order $x_1, \ldots, x_\rho$ of the elements of $S \setminus B$ such that (1) is satisfied.*

PROOF.

(i) Since $|B| = |S \setminus \{x_1, \ldots, x_\rho\}| = r$, it suffices to show that B is an independent set. Suppose otherwise, then there exists a circuit C which is contained in B. Since $\mathcal{C}$ is a base for the vector space $V(M)$, C can be expressed as $C_{i_1} \triangle \cdots \triangle C_{i_k}, 1 \leq i_1 < \cdots < i_k \leq \rho$. From (1) we have $x_{i_1} \in C \subseteq B$, a contradiction. So B is an independent set and hence $B \in \mathcal{B}(M)$. By $C_i \setminus \{x_i\} \subseteq B$, we know $C_i = C(x_i, B)$.

(ii) We need to prove that there exists a bijection $f : S \setminus B \to \mathcal{C}$ such that $x \in f(x)$, $x \notin f(y)$ for any distinct $x, y \in S \setminus B$.

For any $x \in S \setminus B$, let $C(x,B) = C_{i_1} \triangle \cdots \triangle C_{i_k}, 1 \le i_1 < \cdots < i_k \le \rho$. From the above-mentioned condition and $x \in C(x,B)$, we know x belongs to exactly one C_{i_t}. Without loss of generality, we suppose $x \in C_{i_1} \setminus \bigcup_{t=2}^{k} C_{i_t}$. Set $f(x) = C_{i_1}$. In this way, we define a mapping f from $S \setminus B$ to $\mathcal{C}$. For $y \in S \setminus B, y \neq x$, let $C(y,B) = C_{j_1} \triangle \cdots \triangle C_{j_l}, 1 \le j_1 < \cdots < j_l \le \rho$. Also, we may suppose $y \in C_{j_1} \setminus \bigcup_{t=2}^{l} C_{j_t}$. Then $f(y) = C_{j_1}$. Now we prove

$$f(x) \neq f(y) \tag{2}$$

and

$$x \notin f(y). \tag{3}$$

If these are achieved, then from (2), we know f is injective and hence bijective since $|S \setminus B| = |\mathcal{C}|$, and from (3), we get (1).

Let us prove (2) first. Suppose to the contrary that $f(x) = f(y)$, i.e., $C_{i_1} = C_{j_1}$. Then $k, l \ge 2$. In fact, if $k = 1$, then from $y \in C_{j_1} = C_{i_1} = C(x,B)$ we know $y \in C(x,B) \setminus \{x\} \subseteq B$, a contradiction. Similarly, $l \ge 2$. Since $y \notin C(x,B)$ but $y \in C_{i_1}$, there exists, say, C_{i_2} which contains y. From the above-mentioned condition, we have $y \notin \bigcup_{t=3}^{k} C_{i_t}$. Similarly, we can suppose $x \in C_{j_2}$ and $x \notin \bigcup_{t=3}^{l} C_{j_t}$. Note that $C_{i_1} \neq C_{i_2}, C_{j_2}$, but $C_{i_1} \cap C_{i_2} \neq \emptyset$, $C_{i_1} \cap C_{j_2} \neq \emptyset$. This contradicts the hypothesis of the condition, and hence, (2) follows.

Now, we prove (3). If $x \in f(y) = C_{j_1}$, then there exists exactly one C_{j_t} such that $x \in C_{j_t}$, $t \ge 2$. Without loss of generality, we suppose $x \in C_{j_2}$. Then we must have $C_{i_1} = C_{j_2}$, since otherwise, the pairwise distinct $C_{i_1}, C_{j_1}, C_{j_2}$ will have a common element x, violating the hypothesis in the condition. We claim that there exists no z with $z \in C_{j_1} \setminus B, z \neq x, y$. Suppose otherwise, then by $z \notin C(y,B)$ and by the condition, we know there exists a unique C_{j_t} with $z \in C_{j_t}$, $t \ge 2$. If $t > 2$, then C_{j_1} has nonempty intersection with both C_{j_2} and C_{j_t}, a contradiction. So we must have $t = 2$. That is, $z \in C_{j_2} = C_{i_1}$. But $x \notin C(x,B)$, so there exists a unique C_{i_s} with $z \in C_{i_s}$, $s \ge 2$. Note that $C_{i_s} \neq C_{j_1}$, for otherwise x will be in C_{i_s}. Thus, C_{j_1} has nonempty intersection with C_{i_1} and C_{i_s}, which contradicts the condition. So there exists no z with $z \in C_{j_1} \setminus B, z \neq x, y$, and hence, $C(x,B) \triangle C(y,B) \triangle C_{j_1} \subseteq B$. But M is binary implies that $C(x,B) \triangle C(y,B) \triangle C_{j_1}$ is the union of disjoint circuits. So the base B must contain circuits. This contradiction completes the proof of (3) and hence of Lemma 3.

LEMMA 4. *Suppose $B \in \mathcal{B}(M)$ and $S \setminus B = \{x_1, \ldots, x_\rho\}$ satisfies (1). Then B is a maximum base if and only if x_i is a minimum weight element in C_i, $1 \le i \le \rho$.*

PROOF. Suppose x_i is not a minimum weight element of C_i for some i. Then there exists $y_i \in C_i \setminus \{x_i\}$ with $w(y_i) < w(x_i)$. By Lemma 3, we have $C_i = C(x_i, B)$, and hence, $(B \setminus \{y_i\}) \cup \{x_i\} \in \mathcal{B}(M)$. B is not a maximum weight base since $w((B \setminus \{y_i\}) \cup \{x_i\}) = w(B) - w(y_i) + w(x_i) > w(B)$.

Conversely suppose each x_i is a minimum weight element in C_i. By Lemma 3, for any $B' \in \mathcal{B}(M)$, the elements of $S \setminus B'$ can be ordered as $x'_1, \ldots, x'_\rho$ such that $x'_i \in C_i \setminus \bigcup_{j \neq i} C_j$. Since $w(x_i) \le w(x'_i), 1 \le i \le \rho$, we have $w(B') = w(B) + \sum_{i=1}^{\rho} (w(x_i) - w(x'_i)) \le w(B)$, and hence, B is a maximum base. This completes the proof of Lemma 4.

For a circuit C of M, let $c_1 < \cdots < c_n$ be all distinct weights of elements of C. If $c_1, \ldots, c_n$ is an arithmetical progression with common difference d, for some real number $d > 0$, then C is said to satisfy the *d-condition*. If $c_{i+1} - c_i \le d$, $1 \le i \le n-1$, then we say C satisfies the $d^{\le}$-condition. We have the following lemma.

LEMMA 5. *Suppose $B \in \mathcal{B}(M)$ is not a maximum base. Then*

(i) *if each C_i satisfies the d-condition, $1 \le i \le \rho$, then there exists $B' \in \mathcal{B}(M)$ such that $w(B') = w(B) + d$;*

(ii) *if each C_i satisfies the $d^{\le}$-condition, $1 \le i \le \rho$, then there exists $B' \in \mathcal{B}(M)$ such that $w(B) < w(B') \le w(B) + d$.*

PROOF. By Lemma 3, we can suppose $S \setminus B = \{x_1, \dots, x_\rho\}$ satisfies (1) and $C_i = C(x_i, B), 1 \leq i \leq \rho$. If each C_i satisfies the d-condition, then by Lemma 4 and the assumption that B is not a maximum base, we know there exist C_i and $x_i' \in C_i \setminus \{x_i\}$ such that $w(x_i') = w(x_i) - d$. By Lemma 1, $B' = (B \setminus \{x_i'\}) \cup \{x_i\} \in \mathcal{B}(M)$. The weight of B' is $w(B') = w(B) - w(x_i') + w(x_i) = w(B) + d$. In a similar way, one can prove (ii).

From Lemma 5, we get our main result.

THEOREM 1. *Suppose S, M, w, w_i are as before and d is a positive number. Suppose there exists a circuit base $\mathcal{C} = \{C_1, \dots, C_\rho\}$ of $V(M)$ which satisfies the condition.*

(i) *If each C_i satisfies the d-condition, then $w_1, \dots, w_m$ is an arithmetical progression with common difference d.*
(ii) *If each C_i satisfies the $d^{\leq}$-condition, then $0 < w_{i+1} - w_i \leq d,\ 1 \leq i \leq m-1$.*

An *integer interval* is a set of consecutive integers. From Theorem 1, we have the following.

COROLLARY 1. *Suppose M is a binary matroid on S and there exists a circuit base $\mathcal{C}$ of $V(M)$ which satisfies the condition. If w is an integer-valued weight function defined on S such that the weights of the elements in each C_i consist of an integer interval, then the weights of the bases of M also consist of an integer interval.*

3. DUAL THEOREM

The *cocircuit space* $V^*(M)$ of M is the vector space over $GF(2)$ generated by the incidence vectors of the cocircuits of M. The dimension of $V^*(M)$ is r. A base $C_1^*, \dots, C_r^*$ of $V^*(M)$ is said to be a *cocircuit base* if each C_i^* is a cocircuit of M. Let $\mathcal{B}^*(M)$ be the set of cobases of M. The weight of a cobase B^* is $w(B^*) = \sum_{x \in B^*} w(x)$. Let $w_1^* < \cdots < w_m^*$ be all the possible distinct weights of cobases of M. From Theorem 1 and the duality principle [5] for matroids, we get the following.

THEOREM 2. *Suppose S, M, w, w_i^* are as before and d is a positive number. Suppose there exists a cocircuit base $\mathcal{C}^* = \{C_1^*, \dots, C_r^*\}$ of $V^*(M)$ such that each C_i^* has nonempty intersection with at most one $C_j^*, j \neq i$.*

(i) *If each C_i^* satisfies the d-condition, then $w_1^*, \dots, w_m^*$ is an arithmetical progression with common difference d.*
(ii) *If each C_i^* satisfies the $d^{\leq}$-condition, then $0 < w_{i+1}^* - w_i^* \leq d,\ 1 \leq i \leq m-1$.*

COROLLARY 2. *Suppose M is a binary matroid on S and there exists a cocircuit base $\mathcal{C}^* = \{C_1^*, \dots, C_r^*\}$ of $V^*(M)$ such that each C_i^* intersects at most one other C_j^*. If w is an integer-valued weight function for M such that the weights of the elements in each C_i^* consist of an integer interval, then the weights of the cobases of M also consist of an integer interval.*

In particular, the corollaries of Theorems 1 and 2 are valid for the cycle and cocycle matroids of a graph since they are both binary.

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