Trivalent 2-arc transitive graphs of type G_2^1 are near polygonal

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Abstract

A connected graph Σ of girth at least four is called a near n-gonal graph with respect to $\mathcal{E}$, where $n \geq 4$ is an integer, if $\mathcal{E}$ is a set of n-cycles of Σ such that every path of length two is contained in a unique member of $\mathcal{E}$. It is well known that connected trivalent symmetric graphs can be classified into seven types. In this note we prove that every connected trivalent G-symmetric graph $\Sigma \neq K_4$ of type G_2^1 is a near polygonal graph with respect to two G-orbits on cycles of Σ. Moreover, we give an algorithm for constructing the unique cycle in each of these G-orbits containing a given path of length two.

Key words: Symmetric graph; arc-transitive graph; trivalent symmetric graph; near polygonal graph; three-arc graph

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1 Introduction

Let us start with a very simple example – the (3-dimensional) cube Q_3. Obviously, the family of the six 4-cycles of Q_3 has the following property: every path of length two is contained in exactly one cycle in the family. Yet there is another family of cycles of Q_3 possessing the same property, namely, the four 6-cycles $(\alpha, \gamma', \beta, \alpha', \gamma, \beta', \alpha)$, $(\alpha, \gamma', \delta, \alpha', \gamma, \delta', \alpha)$, $(\beta, \delta', \alpha, \beta', \delta, \alpha', \beta)$, $(\beta, \delta', \gamma, \beta', \delta, \gamma', \beta)$ as shown in Figure 1. In this paper we will prove that these observations are not a mere coincidence, and similar results hold for a certain family of trivalent 2-arc transitive graphs.

Let $\Sigma = (V(\Sigma), E(\Sigma))$ be a finite graph and $s \geq 1$ an integer. An *s-arc* of Σ is an $(s+1)$-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of vertices of Σ such that α_i, α_{i+1} are adjacent for $i = 0, \ldots, s-1$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $i = 1, \ldots, s-1$. In the following we will use $\mathtt{Arc}_s(\Sigma)$ to denote the set of s-arcs of Σ, and $\mathtt{Arc}(\Sigma)$ in place of $\mathtt{Arc}_1(\Sigma)$. Σ is said to *admit* a finite group G as a group of automorphisms if G acts on $V(\Sigma)$ such that, for any $\alpha, \beta \in V(\Sigma)$ and $g \in G$, α and β are adjacent in Σ if and only if α^g and β^g are adjacent in Σ. In the case where G is transitive on $V(\Sigma)$ and, under the induced action, transitive on $\mathtt{Arc}_s(\Sigma)$, Σ is said to be (G, s)*-arc transitive*; if in addition the action of G on $\mathtt{Arc}_s(\Sigma)$ is regular, then Σ is said to be (G, s)*-arc regular*. A 1-arc is usually called an *arc*, and a $(G, 1)$-arc transitive graph is called a G*-symmetric graph*.

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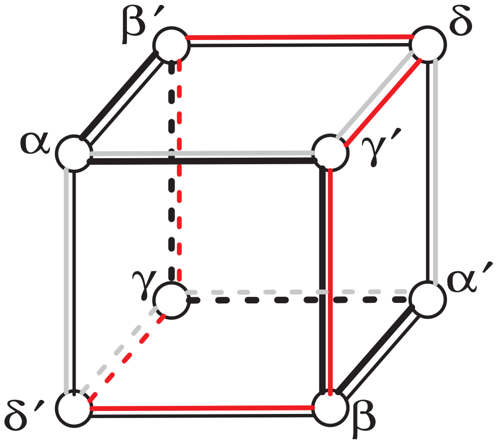

Figure 1: The cube as a near polygonal graph.

A connected graph Σ of girth at least four is called [9] a *near n-gonal graph with respect to $\mathcal{E}$*, where $n \geq 4$ is an integer, if $\mathcal{E}$ is a set of n-cycles of Σ such that each 2-arc of Σ is contained in a unique member of $\mathcal{E}$. In the case when n is the girth of Σ, a near n-gonal graph Σ is called an *n-gonal graph* [8]. Polygonal and near polygonal graphs have attracted considerable attention in recent years. See e.g. [8, 9, 10, 11] for examples, constructions and classifications of some families of such graphs, and [16] for necessary and sufficient conditions for a $(G, 2)$-arc transitive graph to be near polygonal with respect to a G-orbit on cycles.

A well known result of Tutte [13] says that for any trivalent (G, s)-arc transitive graph we must have $1 \leq s \leq 5$. In [3, 5] it was proved further that connected trivalent symmetric graphs can be classified into seven types according to the level of s-arc transitivity and the existence of an involutory automorphism flipping an edge. That is, for a connected trivalent G-symmetric graph Σ, G is a homomorphic image of one of seven finitely-presented groups, G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 or G_5, with subscript s indicating that Σ is (G, s)-arc regular, where

$$G_2^1 := \langle h, a, p \mid h^3 = a^2 = p^2 = 1, apa = p, php = h^{-1} \rangle$$

and the rest groups can be found in [2, 3]. For graphs of type G_2^1, we will prove the following theorem, which is the main result of this paper. (A *double cover* of a graph Σ is a family of cycles of Σ such that each edge of Σ is contained in exactly two cycles in the family.)

Theorem 1 *Let $\Sigma \neq K_4$ be a connected trivalent $(G, 2)$-arc transitive graph. Then Σ is a near polygonal graph with respect to a G-orbit on cycles of Σ if and only if it is of type G_2^1. Moreover, any Σ of type G_2^1 is near polygonal with respect to exactly two G-orbits $\mathcal{E}_1$, $\mathcal{E}_2$ on cycles of Σ, and each of $\mathcal{E}_1$ and $\mathcal{E}_2$ is a double cover of Σ and contains at least three cycles. Furthermore, there is a simple algorithm to construct the unique cycle in $\mathcal{E}_1$ ($\mathcal{E}_2$) containing a given 2-arc of Σ.*

The algorithm will be given during the proof. The 'only if' part of Theorem 1 is easy, and it is a special case of the following observation: any $(G, 3)$-arc transitive graph Σ of valency at least three is not near polygonal with respect to a G-orbit on cycles of Σ. Theorem 1 relies on the main result of [14] and an analysis (Theorem 2) of 3-arc graphs [7, 15] of trivalent symmetric graphs. Such 3-arc graphs appear also in classifying [17] a family of symmetric graphs with 2-arc transitive quotients.

The cube Q_3 is the smallest example of a trivalent symmetric graph of type G_2^1 other than K_4, and for this example Theorem 1 gives exactly the fact mentioned in the beginning of this paper. Theorem 1 is reminiscent of the Petrie polygons [4] of a regular map. The cube Q_3 shows that $\mathcal{E}_1$ and $\mathcal{E}_2$ may or may not consist of Petrie polygons of a regular map with Σ as the underlying graph. In general, it would be interesting to explore possible connections between the result above and the Petrie polygons of some regular maps associated with Σ.

2 Notation and terminology

The reader is referred to [6] for notation and terminology on permutation groups, and to [1, Chapters 17-19] for an introduction to symmetric graphs. Recent developments on symmetric graphs can be found in [12].

To enable us to explain our work we now introduce some notation and terminology for imprimitive symmetric graphs. Let Γ be a G-symmetric graph. A partition $\mathcal{B}$ of $V(\Gamma)$ is said to be *G-invariant* if $B^g \in \mathcal{B}$ for $B \in \mathcal{B}$ and $g \in G$, where $B^g := \{\sigma^g : \sigma \in B\}$. In the case where $V(\Gamma)$ admits a G-invariant partition $\mathcal{B}$ with $1 < |B| < |V(\Gamma)|$, Γ is said to be an *imprimitive G-symmetric graph.* In this case the *quotient graph* of Γ with respect to $\mathcal{B}$, denoted by $\Gamma_{\mathcal{B}}$, is the graph with vertex set $\mathcal{B}$ in which two 'vertices' $B, C \in \mathcal{B}$ are adjacent if and only if there exist $\sigma \in B$ and $\tau \in C$ such that σ, τ are adjacent in Γ. As usual we will assume without mentioning explicitly that $\Gamma_{\mathcal{B}}$ contains at least one edge, so that each block of $\mathcal{B}$ is an independent set of Γ and hence the subgraph of Γ induced by $B \cup C$ is bipartite. Let $\Gamma[B, C]$ denote this bipartite graph without including isolated vertices. Then the bipartition of $\Gamma[B, C]$ is $\{\Gamma(C) \cap B, \Gamma(B) \cap C\}$, where

$$\Gamma(B) := \bigcup_{\sigma \in B} \Gamma(\sigma).$$

In the case where $\Gamma[B, C]$ is a perfect matching between B and C, Γ is a topological cover of the quotient $\Gamma_{\mathcal{B}}$. Similarly, if $\Gamma[B, C]$ is a matching of $|B| - 1$ edges, then Γ is called an *almost cover* [14] of $\Gamma_{\mathcal{B}}$.

Let Σ be a regular graph. A subset Δ of $\mathtt{Arc}_3(\Sigma)$ is called *self-paired* if $(\tau, \sigma, \sigma', \tau') \in \Delta$ implies $(\tau', \sigma', \sigma, \tau) \in \Delta$. For such a Δ the 3-*arc graph* [7, 15] of Σ with respect to Δ, denoted by $\Xi(\Sigma, \Delta)$, is the graph with vertex set $\mathtt{Arc}(\Sigma)$ in which $(\sigma, \tau), (\sigma', \tau')$ are adjacent if and only if $(\tau, \sigma, \sigma', \tau') \in \Delta$. Denote

$$\mathcal{B}(\Sigma) := \{B(\sigma) : \sigma \in V(\Sigma)\}$$

where $B(\sigma) := \{(\sigma, \tau) : \tau \in \Sigma(\sigma)\}$ with $\Sigma(\sigma)$ the *neighbourhood* of σ in Σ. In the case where Σ is G-symmetric and G is transitive on Δ (under the induced action of G on $\mathtt{Arc}_3(\Sigma)$), $\Gamma := \Xi(\Sigma, \Delta)$ is a G-symmetric graph [7, Section 6] which admits $\mathcal{B}(\Sigma)$ as a G-invariant partition such that $\Gamma_{\mathcal{B}(\Sigma)} \cong \Sigma$ with respect to the natural bijection $\sigma \leftrightarrow B(\sigma)$, $\sigma \in V(\Sigma)$.

3 Proof of Theorem 1, and 3-arc graphs of trivalent symmetric graphs

A major step towards the proof of Theorem 1 is the following result (which is also used in the proof of the main result of [17]). Note that a trivalent graph is (G, s)-arc regular if and only if it is (G, s)-arc but not $(G, s+1)$-arc transitive (e.g. [1, Proposition 18.1]).

Theorem 2 *A connected trivalent G-symmetric graph Σ has a self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$ if and only if it is not of type G_2^2. Moreover, the following (a)-(c) hold (where σ, σ' are adjacent vertices, $\Sigma(\sigma) = \{\sigma', \tau, \delta\}$ and $\Sigma(\sigma') = \{\sigma, \tau', \delta'\}$).*

(a) *In the case where Σ is $(G, 1)$-arc regular, there are exactly two self-paired G-orbits on $\mathtt{Arc}_3(\Sigma)$, namely $\Delta_1 := (\tau, \sigma, \sigma', \tau')^G$ and $\Delta_2 := (\delta, \sigma, \sigma', \delta')^G$ where we assume that the unique element of G reversing (σ, σ') maps τ to τ', and $\Xi(\Sigma, \Delta_1) \cong \Xi(\Sigma, \Delta_2) \cong n \cdot K_2$ where $n = |E(\Sigma)|$.*

(b) *In the case where $\Sigma \neq K_4$ is $(G, 2)$-arc regular of type G_2^1, there are exactly two self-paired G-orbits on $\mathtt{Arc}_3(\Sigma)$, namely $\Delta_1 := (\tau, \sigma, \sigma', \tau')^G$ and $\Delta_2 := (\tau, \sigma, \sigma', \delta')^G$, and $\Xi(\Sigma, \Delta_1)$ and $\Xi(\Sigma, \Delta_2)$ are both almost covers of Σ with valency 2.*

(c) *In the case where Σ is (G, s)-arc regular where $3 \leq s \leq 5$, the unique self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$ is $\Delta := \mathtt{Arc}_3(\Sigma)$, and $\Xi(\Sigma, \Delta)$ is a connected G-symmetric but not $(G, 2)$-arc transitive graph of valency 4.*

In the proof of Theorem 2 we will exploit the following known results (where $\Gamma_{\mathcal{B}}(B)$ denotes the neighbourhood of B in $\Gamma_{\mathcal{B}}$):

(A) ([7, Theorem 1]) Let $(\Gamma, \mathcal{B})$ be an imprimitive G-symmetric graph such that $|\Gamma(C) \cap B| = |B| - 1 \geq 2$ for adjacent blocks $B, C \in \mathcal{B}$. Then $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if $\Gamma(C) \cap B \neq \Gamma(D) \cap B$ for distinct $C, D \in \Gamma_{\mathcal{B}}(B)$, and in this case $\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ for a self-paired G-orbit Δ on $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$. Conversely, for any $(G, 2)$-arc transitive graph Σ and any self-paired G-orbit Δ on $\mathtt{Arc}_3(\Sigma)$, the 3-arc graph $\Gamma := \Xi(\Sigma, \Delta)$ together with the G-invariant partition $\mathcal{B} := \mathcal{B}(\Sigma)$ satisfies all the conditions above.

(B) ([7, Theorem 2]) Let $(\Gamma, \mathcal{B})$ be an imprimitive G-symmetric graph such that $|\Gamma(C) \cap B| = |B| - 1 \geq 2$ for adjacent $B, C \in \mathcal{B}$ and $\Gamma(C) \cap B \neq \Gamma(D) \cap B$ for distinct $C, D \in \Gamma_{\mathcal{B}}(B)$. Then $\Gamma_{\mathcal{B}}$ is $(G, 3)$-arc transitive if and only if $\Gamma[B, C]$ is a complete bipartite graph, which in turn is true if and only if $\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ with $\Delta = \mathtt{Arc}_3(\Gamma_{\mathcal{B}})$.

Proof of Theorem 2 Let Σ be a connected trivalent G-symmetric graph. Then by Tutte's theorem [13] Σ must be (G, s)-arc regular for some s with $1 \leq s \leq 5$. In the case where $\Sigma = K_4$, we have $G \cong S_4$ and the eight triangles (with orientation) of Σ form a self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$. Thus in the following we may assume $\Sigma \neq K_4$, so that the girth of Σ is at least 4. Also, we assume that σ, σ' are adjacent vertices of Σ, and $\Sigma(\sigma) = \{\sigma', \tau, \delta\}$ and $\Sigma(\sigma') = \{\sigma, \tau', \delta'\}$. Thus, $B(\sigma) = \{(\sigma, \sigma'), (\sigma, \tau), (\sigma, \delta)\}$ and $B(\sigma') = \{(\sigma', \sigma), (\sigma', \tau'), (\sigma', \delta')\}$. See figure 2(a), where the six vertices involved are pairwise distinct since the girth of Σ is greater than 3.

Suppose first that $3 \leq s \leq 5$. Then, since Σ is $(G, 3)$-arc transitive, $\Delta := \mathtt{Arc}_3(\Sigma)$ is the unique G-orbit on $\mathtt{Arc}_3(\Sigma)$, which is obviously self-paired. From (A)-(B) above it follows that $\Gamma := \Xi(\Sigma, \Delta)$ is G-symmetric and $\Gamma[B(\sigma), B(\sigma')] \cong K_{2,2}$ for adjacent blocks $B(\sigma), B(\sigma')$ of $\mathcal{B}(\Sigma)$. Thus, since Σ is trivalent and $\Gamma(B(\sigma')) \cap B(\sigma)$, $\Gamma(B(\tau)) \cap B(\sigma)$, $\Gamma(B(\delta)) \cap B(\sigma)$ are pairwise distinct by (A), Γ has valency 4. Moreover, since Σ is connected and $\Gamma[B(\sigma), B(\sigma')] \cong K_{2,2}$ for any two adjacent blocks $B(\sigma)$ and $B(\sigma')$, it follows that Γ is connected. Furthermore, since $\Gamma[B(\sigma), B(\sigma')]$ is not a matching, Γ is not $(G, 2)$-arc transitive.

Next we assume $s = 2$, so that Σ is of type G_2^1 or G_2^2. Let us first deal with the case where Σ is of type G_2^1. In this case the unique element g of G such that $(\tau, \sigma, \sigma')^g = (\tau', \sigma', \sigma)$ is an involution

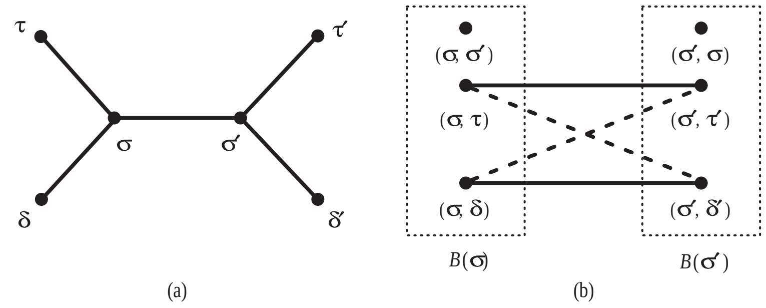

Figure 2: Proof of Theorem 2.

(see [5, Proposition 2(v)]). Thus, $(\tau')^g = \tau^{g^2} = \tau$ and hence $(\tau, \sigma, \sigma', \tau')^g = (\tau', \sigma', \sigma, \tau)$. Therefore, $\Delta_1 := (\tau, \sigma, \sigma', \tau')^G$ is a self-paired G-orbit on the 3-arcs of Σ. Let $\Gamma_1 := \Xi(\Sigma, \Delta_1)$. Since Σ is $(G, 2)$-arc transitive, from (A) above it follows that Γ_1 is a G-symmetric graph admitting $\mathcal{B}(\Sigma)$ as a G-invariant partition of block size three such that $|\Gamma_1(B(\sigma')) \cap B(\sigma)| = |B(\sigma)| - 1 = 2$. Moreover, since Σ is not $(G, 3)$-arc transitive, from (B) above $\Gamma_1[B(\sigma), B(\sigma')]$ cannot be the complete bipartite graph $K_{2,2}$. Thus, we must have $\Gamma_1[B(\sigma), B(\sigma')] \cong 2 \cdot K_2$. The 'vertices' (σ, τ) and (σ', δ') are not adjacent in Γ_1, since otherwise $(\tau, \sigma, \sigma', \tau')^x = (\tau, \sigma, \sigma', \delta')$ for some $x \in G$, which violates the $(G, 2)$-arc regularity of Σ. Similarly, (σ, δ) and (σ', τ') are not adjacent in Γ_1. Thus, the two edges of $\Gamma_1[B(\sigma), B(\sigma')]$ must be $\{(\sigma, \tau), (\sigma', \tau')\}$ and $\{(\sigma, \delta), (\sigma', \delta')\}$. This implies that there exists $h \in G$ such that $(\tau, \sigma, \sigma', \tau')^h = (\delta, \sigma, \sigma', \delta')$. Since h maps (τ, σ, σ') to $(\delta, \sigma, \sigma')$ and Σ is $(G, 2)$-arc regular, δ' is not fixed by h, and hence we must have $(\delta')^h = \tau'$. Noting that g swaps δ and δ', it follows that $(\tau, \sigma, \sigma', \delta')^{hg} = (\delta', \sigma', \sigma, \tau)$ and therefore $\Delta_2 := (\tau, \sigma, \sigma', \delta')^G$ is a self-paired G-orbit on the 3-arcs of Σ. Since $(\tau, \sigma, \sigma', \delta') \notin \Delta_1$ as shown above, we have $\Delta_1 \neq \Delta_2$. Let $\Gamma_2 := \Xi(\Sigma, \Delta_2)$. An argument similar to that used for Γ_1 ensures that the only edges of $\Gamma_2[B(\sigma), B(\sigma')]$ are $\{(\sigma, \tau), (\sigma', \delta')\}$ and $\{(\sigma, \delta), (\sigma', \tau')\}$. Therefore, both Γ_1 and Γ_2 are almost covers of Σ, and obviously they have valency 2. See figure 2(b) where the continuous lines are edges of Γ_1 and the dashed lines are edges of Γ_2. The proof above also ensures that Δ_1, Δ_2 are the only self-paired G-orbits on the 3-arcs of Σ.

Now let us deal with the case where Σ is of type G_2^2. Let g be the unique element of G such that $(\tau, \sigma, \sigma')^g = (\tau', \sigma', \sigma)$. From [5, Proposition 2(v)], in the case of G_2^2 we have $G_{\sigma\sigma'} \cong \mathbb{Z}_2$, $G_{\{\sigma,\sigma'\}} = \langle g \rangle \cong \mathbb{Z}_4$ and g^2 is the non-identity element of $G_{\sigma\sigma'}$, where $G_{\sigma\sigma'}$ and $G_{\{\sigma,\sigma'\}}$ are respectively the stabilisers in G of the arc (σ, σ') and the edge $\{\sigma, \sigma'\}$ of Σ. Since $(\tau, \sigma, \sigma')^g = (\tau', \sigma', \sigma)$, from the $(G, 2)$-arc regularity of Σ the only element of G which could map $(\tau, \sigma, \sigma', \tau')$ to $(\tau', \sigma', \sigma, \tau)$ is g. However, if $(\tau')^g = \tau$, then $(\tau, \sigma, \sigma', \tau')^{g^2} = (\tau, \sigma, \sigma', \tau')$, and this implies $g^2 = 1$ since Σ is $(G, 2)$-arc regular. This contradicts with the fact that $g^2 \neq 1$. Thus, there is no element of G which reverses $(\tau, \sigma, \sigma', \tau')$, and hence there exists no self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$.

Finally, we assume that Σ is $(G, 1)$-arc regular. In this case there exists a unique $g \in G$ such that $(\sigma, \sigma')^g = (\sigma', \sigma)$ and $g^2 = 1$. We may suppose $\tau^g = \tau'$ without loss of generality. Then $(\tau')^g = \tau$, $\delta^g = \delta'$ and $(\delta')^g = \delta$. Hence $(\tau, \sigma, \sigma', \tau')^g = (\tau', \sigma', \sigma, \tau)$ and $\Delta_1 := (\tau, \sigma, \sigma', \tau')^G$ is a self-paired G-orbit on the 3-arcs of Σ. Similarly, $\Delta_2 := (\delta, \sigma, \sigma', \delta')^G$ is a self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$. The $(G, 1)$-arc regularity of Σ implies that there exists no element of G which maps $(\tau, \sigma, \sigma', \tau')$ to $(\delta, \sigma, \sigma', \delta')$, and hence $\Delta_1 \neq \Delta_2$. Thus, (σ, δ) and (σ', δ') are not adjacent in $\Gamma_1 := \Xi(\Sigma, \Delta_1)$, and (σ, τ) and (σ', τ') are not adjacent in $\Gamma_2 := \Xi(\Sigma, \Delta_2)$. Also from the

$(G,1)$-arc regularity of Σ, neither $\{(\sigma,\tau),(\sigma',\delta')\}$ nor $\{(\sigma,\delta),(\sigma',\tau')\}$ is an edge of Γ_1 or Γ_2. Hence $\{(\sigma,\tau),(\sigma',\tau')\}$ is the only edge of $\Gamma_1[B(\sigma),B(\sigma')]$, and $\{(\sigma,\delta),(\sigma',\delta')\}$ is the only edge of $\Gamma_2[B(\sigma),B(\sigma')]$. Therefore, we have $\Gamma_1 \cong \Gamma_2 \cong n\cdot K_2$ where $n = |E(\Sigma)|$. To show that Δ_1, Δ_2 are the only self-paired G-orbits on $\mathtt{Arc}_3(\Sigma)$, it suffices to prove that neither $(\tau,\sigma,\sigma',\delta')^G$ nor $(\delta,\sigma,\sigma',\tau')^G$ is self-paired. Suppose to the contrary that $(\tau,\sigma,\sigma',\delta')^G$ is self-paired. Then there exists $h \in G$ such that $(\tau,\sigma,\sigma',\delta')^h = (\delta',\sigma',\sigma,\tau)$. Thus, hg fixes σ and σ' and moves δ' to τ', violating the $(G,1)$-regularity of Σ. Therefore, $(\tau,\sigma,\sigma',\delta')^G$ is not self-paired, and similarly $(\delta,\sigma,\sigma',\tau')^G$ is not self-paired. This completes the proof. □

The main tool for the proof of Theorem 1 is the following result and its proof.

(C) ([14, Theorem 3.1]) Let $\Sigma \not\cong K_{v+1}$ be a finite connected $(G,2)$-arc transitive graph with valency $v \geq 3$. Then Σ is almost covered by a 3-arc graph $\Xi(\Sigma,\Delta)$ with respect to a self-paired G-orbit Δ on $\mathtt{Arc}_3(\Sigma)$ if and only if, for some integer $n \geq 4$, Σ is a near n-gonal graph with respect to a G-orbit $\mathcal{E}$ on n-cycles of Σ, and in this case Δ is the set of 3-arcs contained in the n-cycles in $\mathcal{E}$.

Moreover, from the proof [14] of this result, for any $(\beta,\alpha,\alpha',\beta') \in \Delta$, the unique n-cycle of $\mathcal{E}$ containing (β,α,α') is the same as the unique n-cycle of $\mathcal{E}$ containing (α,α',β'). Thus, the number of n-cycles of $\mathcal{E}$ containing the edge $\{\alpha,\alpha'\}$ is equal to the number of 3-arcs in Δ which contain $\{\alpha,\alpha'\}$ as 'middle edge'.

Proof of Theorem 1 We will use the same notation as in the proof of Theorem 2. Let $\Sigma \neq K_4$ be a connected trivalent $(G,2)$-arc transitive graph, so that the girth of Σ is at least 4. In the case where Σ is (G,s)-arc regular, $3 \leq s \leq 5$, the only self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$ is $\Delta = \mathtt{Arc}_3(\Sigma)$, and by Theorem 2(c), $\Xi(\Sigma,\Delta)$ is not an almost cover of Σ. Hence by (C) above Σ is not a near polygonal graph with respect to a G-orbit on cycles of Σ. On the other hand, if Σ is of type G_2^1, then by Theorem 2(b) there are exactly two self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$, namely $\Delta_1 = (\tau,\sigma,\sigma',\tau')^G$ and $\Delta_2 = (\tau,\sigma,\sigma',\delta')^G$, and Σ is almost covered by each of $\Xi(\Sigma,\Delta_1)$ and $\Xi(\Sigma,\Delta_2)$. Thus, by (C) there exist integers $n_1, n_2 \geq 4$, and G-orbits $\mathcal{E}_1$ and $\mathcal{E}_2$ on n_1-cycles and n_2-cycles of Σ, respectively, such that $(\Sigma,\mathcal{E}_1)$ and $(\Sigma,\mathcal{E}_2)$ are both near polygonal. Moreover, since Δ_1 and Δ_2 are the only self-paired G-orbit on $\mathtt{Arc}_3(\Sigma)$, by (C) these are the only near polygonal graphs with respect to G-orbits on cycles of Σ. From the proof of Theorem 2, $(\tau,\sigma,\sigma',\tau')$ and $(\delta,\sigma,\sigma',\delta')$ are the only 3-arcs in Δ_1 which contain $\{\sigma,\sigma'\}$ as 'middle edge'. Thus, from the remark after (C), $\{\sigma,\sigma'\}$ is contained in exactly two n_1-cycles of $\mathcal{E}_1$, namely those containing (τ,σ,σ') and (δ,σ,σ') respectively. Therefore, $\mathcal{E}_1$ is a double cover of Σ. Similarly, $\mathcal{E}_2$ is a double cover of Σ. Note that the cycles in $\mathcal{E}_1$ containing the 2-arcs (τ,σ,σ'), (δ,σ,σ'), (τ,σ,δ) respectively must be pairwise distinct. Hence $|\mathcal{E}_1| \geq 3$, and similarly $|\mathcal{E}_2| \geq 3$.

For any arc (α,α') of Σ and any $\beta \in \Sigma(\alpha)\setminus\{\alpha'\}$, there exists a unique vertex $\beta' \in \Sigma(\alpha')\setminus\{\alpha\}$ such that $(\beta,\alpha,\alpha',\beta') \in \Delta_1$. The remaining vertex γ' in $\Sigma(\alpha')\setminus\{\alpha,\beta'\}$ should then satisfy $(\beta,\alpha,\alpha',\gamma') \in \Delta_2$. Thus, $L_{\alpha\alpha'}: \beta \mapsto \beta'$ and $R_{\alpha\alpha'}: \beta \mapsto \gamma'$ define two bijections from $\Sigma(\alpha)\setminus\{\alpha'\}$ to $\Sigma(\alpha')\setminus\{\alpha\}$. For any 2-arc $(\alpha_0,\alpha_1,\alpha_2)$ of Σ, define $\alpha_{i+2} := L_{\alpha_i\alpha_{i+1}}(\alpha_{i-1})$ for $i \geq 1$, and thus obtain a sequence $\alpha_0,\alpha_1,\alpha_2,\ldots,\alpha_{i-1},\alpha_i,\alpha_{i+1},\alpha_{i+2},\ldots$ of vertices of Σ. From the proof of [14, Theorem 3.1], the first vertex α_{n_1} in this sequence that coincides with one of the preceding vertices must coincide with α_0. Moreover, $\mathcal{E}_1$ is given by $\mathcal{E}_1 = (\alpha_0,\alpha_1,\alpha_2,\ldots,\alpha_{n_1-1},\alpha_0)^G$, and $(\alpha_0,\alpha_1,\alpha_2,\ldots,\alpha_{n_1-1},\alpha_0)$ is the unique cycle of $\mathcal{E}_1$ containing the given 2-arc $(\alpha_0,\alpha_1,\alpha_2)$. Similarly, if we use the bijection $R_{\alpha_i\alpha_{i+1}}$ instead of $L_{\alpha_i\alpha_{i+1}}$ in generating the sequence above, then we obtain $\mathcal{E}_2$ and the unique cycle of $\mathcal{E}_2$ containing $(\alpha_0, \alpha_1, \alpha_2)$. $\Box$

Let $\Sigma \neq K_4$ be a connected trivalent G-symmetric graph of type G_2^1. We may imagine that we walk on Σ and regard $L_{\alpha_i\alpha_{i+1}}$ ($R_{\alpha_i\alpha_{i+1}}$) as the rule of walking to the 'left neighbour' ('right neighbour') of α_{i+1}, given that the trail so far is $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}$. Starting from the given 2-arc $(\alpha_0, \alpha_1, \alpha_2)$ and applying the same rule all the time along the walk, we will always return to the initial vertex α_0 and thus obtain a cycle, which is the unique cycle containing $(\alpha_0, \alpha_1, \alpha_2)$ in the corresponding near polygonal graph.

4 Remarks

A complete list of connected trivalent symmetric graphs on up to 768 vertices was given in [2]. In the list there are 122 connected trivalent 2-arc regular graphs of up to 768 vertices, and all but one of them are of type G_2^1. Theorem 1 implies that all these 121 graphs of type G_2^1 except K_4 are near polygonal graphs. From the table in [2] each integer in $\{4, 5, 6, 7, 8, 9, 10, 12, 14\}$ can occur as the girth of such a graph of up to 768 vertices.

Let $\Sigma \neq K_4$ be a connected trivalent G-symmetric graph of type G_2^1. Let Δ_1, Δ_2 be as in Theorem 2, and n_1, n_2 the lengths of the cycles in $\mathcal{E}_1, \mathcal{E}_2$, respectively, as in the proof of Theorem 1. Then $\Xi(\Sigma, \Delta_1) \cong t_1 \cdot C_{n_1}$ and $\Xi(\Sigma, \Delta_2) \cong t_2 \cdot C_{n_2}$, where $t_1 = |\mathcal{E}_1| \geq 3, t_2 = |\mathcal{E}_2| \geq 3$. In fact, by Theorem 2, $\Xi(\Sigma, \Delta_j)$ ($j = 1, 2$) has valency 2 and hence is a vertex-disjoint union of cycles of the same length. Note that (ε, η) and (ε', η') are adjacent in $\Xi(\Sigma, \Delta_j) \Leftrightarrow (\eta, \varepsilon, \varepsilon', \eta') \in \Delta_j$ $\Leftrightarrow (\eta, \varepsilon, \varepsilon', \eta')$ is contained in some n_j-cycle in $\mathcal{E}_j$, where the last statement is from (C). Thus, each n_j-cycle in $\mathcal{E}_j$ gives rise to an n_j-cycle of $\Xi(\Sigma, \Delta_j)$, and vice versa. Therefore, we have $\Xi(\Sigma, \Delta_1) \cong t_1 \cdot C_{n_1}$ and $\Xi(\Sigma, \Delta_2) \cong t_2 \cdot C_{n_2}$ as claimed. Note that $t_1 \geq 3$ and $t_2 \geq 3$ by Theorem 1.

The cube Q_3 is $(S_4 \,\mathrm{wr}\, \mathbb{Z}_2, 2)$-arc regular of type G_2^1 such that $n_1 = 4$ and $n_2 = 6$, and it is a 4-gonal graph with respect to the six 4-cycles as shown in Figure 1. The well known Petersen graph P can be defined to have vertices the unordered pairs ij of distinct elements of $\{1, 2, 3, 4, 5\}$ such that ij and $i'j'$ are adjacent if and only if $\{i, j\} \cap \{i', j'\} = \emptyset$. Thus, P admits A_5 as a 2-arc regular group of automorphisms of type G_2^1, and one can verify that $n_1 = n_2 = 5$ and P is a 5-gonal graph with respect to $\mathcal{E}_1 = \mathcal{E}_2 = (12, 34, 51, 24, 35, 12)^{A_5}$. This example shows that the two near polygonal graphs $(\Sigma, \mathcal{E}_1)$, $(\Sigma, \mathcal{E}_2)$ in Theorem 1 can be identical, and they can be polygonal. However, in general we do not know when $\mathcal{E}_1 = \mathcal{E}_2$ occurs, and Theorem 1 and its proof provide no information about the values of n_1 and n_2. Thus in general we do not know when one of the near polygonal graphs is polygonal.

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