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ALMOST COVERS OF 2-ARC TRANSITIVE GRAPHS

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Let Γ be a G-symmetric graph whose vertex set admits a nontrivial G-invariant partition $\mathcal{B}$ with block size v. Let $\Gamma_{\mathcal{B}}$ be the quotient graph of Γ relative to $\mathcal{B}$ and $\Gamma[B,C]$ the bipartite subgraph of Γ induced by adjacent blocks B,C of $\mathcal{B}$. In this paper we study such graphs for which $\Gamma_{\mathcal{B}}$ is connected, $(G,2)$-arc transitive and is almost covered by Γ in the sense that $\Gamma[B,C]$ is a matching of $v-1\geq 2$ edges. Such graphs arose as a natural extremal case in a previous study by the author with Li and Praeger. The case $\Gamma_{\mathcal{B}}\cong K_{v+1}$ is covered by results of Gardiner and Praeger. We consider here the general case where $\Gamma_{\mathcal{B}}\not\cong K_{v+1}$, and prove that, for some even integer $n\geq 4$, $\Gamma_{\mathcal{B}}$ is a near n-gonal graph with respect to a certain G-orbit on n-cycles of $\Gamma_{\mathcal{B}}$. Moreover, we prove that every $(G,2)$-arc transitive near n-gonal graph with respect to a G-orbit on n-cycles arises as a quotient $\Gamma_{\mathcal{B}}$ of a graph with these properties. (A near n-gonal graph is a connected graph Σ of girth at least 4 together with a set $\mathcal{E}$ of n-cycles of Σ such that each 2-arc of Σ is contained in a unique member of $\mathcal{E}$.)

1. Introduction

Let Γ be a finite graph and $s\geq 1$ an integer. An *s-arc* of Γ is a sequence of $s+1$ vertices of Γ, not necessarily all distinct, such that any two consecutive terms are adjacent and any three consecutive terms are distinct. If Γ admits a group G of automorphisms such that G is transitive on the vertex set $V(\Gamma)$ of Γ and, in its induced action, transitive on the set $A_s(\Gamma)$ of s-arcs of Γ, then Γ is said to be *(G,s)-arc transitive*. As usual in the literature, a 1-arc is called an *arc* and a $(G,1)$-arc transitive graph is called a *G-symmetric graph*.

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The study of symmetric graphs and highly arc-transitive graphs has long been one of the main themes in algebraic combinatorics (see e.g. [1]). In a vast number of cases, the vertex set $V(\Gamma)$ of a G-symmetric graph Γ admits a *nontrivial G-invariant partition*, that is, a partition $\mathcal{B}$ of $V(\Gamma)$ such that $1 < |B| < |V(\Gamma)|$ and $B^g \in \mathcal{B}$ for any $B \in \mathcal{B}$ and $g \in G$ (where $B^g := \{\alpha^g : \alpha \in B\}$). If this occurs then Γ is said to be an *imprimitive G-symmetric graph*. From permutation group theory [5], this is the case precisely when the *stabilizer* G_α in G of a vertex $\alpha \in V(\Gamma)$ is not a maximal subgroup of G. In the opposite case, G is primitive on $V(\Gamma)$ and the O'Nan–Scott Theorem [12], which categorizes primitive permutation groups into a number of distinct types, has been proved to be a very useful tool. In this sense the main difficulty in studying symmetric graphs lies in the imprimitive case. For this case it was suggested in [6] that the following three configurations associated with $(\Gamma,\mathcal{B})$ may have a strong influence on the structure of Γ: (i) The *quotient graph* $\Gamma_{\mathcal{B}}$ of Γ with respect to $\mathcal{B}$; (ii) the bipartite subgraph $\Gamma[B,C]$ of Γ induced by adjacent blocks B,C of $\mathcal{B}$; and (iii) a certain 1-design $\mathcal{D}(B)$ with point set B. (These are defined carefully in Section 2, paragraph 2.) In some sense the graph Γ is "decomposed" into the "product" of these configurations, and a natural problem is to characterize Γ in terms of the triple $(\Gamma_{\mathcal{B}},\Gamma[B,C],\mathcal{D}(B))$. In the particular case where $\Gamma[B,C]$ is a perfect matching between B and C, Γ is said to be a *cover* of $\Gamma_{\mathcal{B}}$. The covering graph construction given in [1, pp. 149–154] provides a means for constructing some symmetric graphs with this covering property, and is a standard technique in constructing symmetric graphs.

In [11], Li, Praeger and the author found a very natural and simple method (see Section 2 for details) for constructing larger symmetric graphs from smaller ones which bears some similarity with the covering graph construction mentioned above. The constructed graphs can be characterized ([11, Theorem 1], restated here as Theorem 2.1) as imprimitive G-symmetric graphs Γ such that the block size $v := |B|$ of $\mathcal{B}$ is at least 3 and is one more than the block size of the design $\mathcal{D}(B)$, and $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive, for a certain G-invariant partition $\mathcal{B}$ of $V(\Gamma)$. For such graphs, $\mathcal{D}(B)$ contains no repeated blocks (Theorem 2.1), $\Gamma_{\mathcal{B}}$ has valency v and the actions of G_B on B and on the set of blocks of $\mathcal{B}$ adjacent to B in $\Gamma_{\mathcal{B}}$ are permutationally isomorphic and 2-transitive ([11, Theorem 5(a)(b)]), where G_B is the setwise stabilizer of B in G. In the present paper, we explore this construction in the case where the "inter-block" configuration $\Gamma[B,C]$ is a matching of $v-1$ edges, that is, $\Gamma[B,C] \cong (v-1)\cdot K_2$. In this case we say that Γ is an *almost cover* of $\Gamma_{\mathcal{B}}$, and that $\Gamma_{\mathcal{B}}$ is *almost covered* by Γ. In the special case where $\Gamma_{\mathcal{B}} \cong K_{v+1}$, all possibilities for Γ and G were classified in [8, Theorem 1.1(b)(ii)(iii)(iv)], see also [19, Theorem 3.19] for an explicit list. Here we study the general case where $\Gamma_{\mathcal{B}} \not\cong K_{v+1}$ and $\Gamma_{\mathcal{B}}$ is connected. In this general case we find a very close connection between such graphs Γ and an interesting class of graphs, namely near-polygonal graphs, which are associated with the Buekenhout geometries [2,14] of the following diagram:

For an integer $n \geq 4$, a *near n-gonal graph* [14] is a pair $(\Sigma, \mathcal{E})$ consisting of a connected graph Σ of girth at least 4, together with a set $\mathcal{E}$ of n-cycles of Σ, such that each 2-arc of Σ is contained in a unique member of $\mathcal{E}$. In this case we also say that Σ is a near n-gonal graph with respect to $\mathcal{E}$. (The *girth* of a graph Σ, denoted by girth(Σ), is the length of a shortest cycle of Σ if Σ contains cycles, and is defined to be ∞ otherwise.) Our main result may be stated as follows.

Theorem 1.1. *Suppose Γ is a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ of block size $v \geq 3$ such that $\Gamma_{\mathcal{B}}$ is connected and $\Gamma_{\mathcal{B}} \not\cong K_{v+1}$. Suppose further that Γ almost covers $\Gamma_{\mathcal{B}}$ and that the design $\mathcal{D}(B)$ $(B \in \mathcal{B})$ has no repeated blocks. Then, for some even integer $n \geq 4$, $\Gamma_{\mathcal{B}}$ is a $(G,2)$-arc transitive near n-gonal graph with respect to a certain G-orbit on n-cycles of $\Gamma_{\mathcal{B}}$. Moreover, any $(G,2)$-arc transitive near n-gonal graph (where $n \geq 4$ is even) with respect to a G-orbit on n-cycles can appear as such a quotient $\Gamma_{\mathcal{B}}$.*

We will present and prove this result in terms of the graph construction introduced in [11] (see Theorem 3.1). As a consequence of this result we obtain a sufficient condition for a two-arc transitive graph to be near-polygonal, see Corollary 4.1 for details.

A G-symmetric graph Γ is said to be *G-locally primitive* if, for $\alpha \in V(\Gamma)$, G_α is primitive in its action on the *neighbourhood* $\Gamma(\alpha)$ of α in Γ (that is, the set of vertices adjacent to α in Γ). If Γ is a G-locally primitive graph admitting a G-invariant partiton $\mathcal{B}$ with block size $v \geq 3$ such that v is one more than the block size of $\mathcal{D}(B)$, and if $\Gamma_{\mathcal{B}}$ is connected, then Γ almost covers $\Gamma_{\mathcal{B}}$ ([6, Lemma 3.1(a))]), and $\mathcal{D}(B)$ contains no repeated blocks ([6, Lemma 3.3(c)]) and hence $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive (Theorem 2.1). In this case, we get as a consequence of Theorem 1.1 an amended form of [6, Theorem 5.4] (see Corollary 4.2 in Section 4). Discovering that the proof of the result [6, Theorem 5.4] was incomplete, was one of the motivations for the investigation leading to the results of this paper.

The reader is referred to [18] for a systematic study of the graph construction [11] used in this paper, and to [19,20] for a more general construction of imprimitive symmetric graphs starting from point- and block-transitive 1-designs. In a recent work of the author with Iranmanesh and Praeger we used Theorem 1.1 in the study of a family of symmetric graphs with two-arc transitive quotients, see [10, Theorem 1.4] for details.

2. Definitions and preliminaries

We refer to [5,17] for notation and terminology on permutation groups. If G is a group acting transitively on a finite set Ω, then the *fixed point sets* $\mathrm{fix}_{\Omega}(G_{\alpha}) := \{\beta \in \Omega : \beta^g = \beta \text{ for all } g \in G_{\alpha}\}$, for $\alpha \in \Omega$, form a G-invariant partition $\{(\mathrm{fix}_{\Omega}(G_{\alpha}))^g : g \in G\}$ of Ω ([5, pp. 19]). We write $G_{\alpha\beta} = (G_{\alpha})_{\beta}$, $G_{\alpha\beta\gamma} = (G_{\alpha\beta})_{\gamma}$, etc., for $\alpha, \beta, \gamma \in \Omega$. For a group G acting on two finite sets Ω_1, Ω_2, the actions of G on Ω_1 and Ω_2 are said to be *permutationally equivalent* if there exists a bijection $\lambda : \Omega_1 \to \Omega_2$ such that $\lambda(\alpha^g) = (\lambda(\alpha))^g$ for all $\alpha \in \Omega_1$ and $g \in G$. We use K_n and C_n to denote respectively the complete graph and the cycle on n vertices, and we use $K_{n,n}$ to denote the complete bipartite graph with n vertices in each part of its bipartition. For a finite graph Γ, $n \cdot \Gamma$ denotes the union of n vertex-disjoint copies of Γ. An edge of Γ joining two non-consecutive vertices in a cycle of Γ is said to be a *chord* of the cycle. Instead of $A_1(\Gamma)$ we will use $A(\Gamma)$ to denote the set of arcs of Γ. We will denote an arc (σ, τ) of a graph by $\sigma\tau$ when this is convenient and unlikely to cause confusion.

Let Γ be a finite G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$. The *quotient graph* $\Gamma_{\mathcal{B}}$ of Γ with respect to $\mathcal{B}$ is the graph with vertex set $\mathcal{B}$ in which two blocks $B, C \in \mathcal{B}$ are *adjacent* if and only if there exists at least one edge of Γ joining a vertex of B and a vertex of C. Clearly $\Gamma_{\mathcal{B}}$ is G-symmetric under the induced action (possibly unfaithful) of G on $\mathcal{B}$, and we assume in the following that it has at least one edge. Then each $B \in \mathcal{B}$ is an independent set of Γ [6,15]. Set $\Gamma(B) := \bigcup_{\alpha \in B} \Gamma(\alpha)$. For two adjacent blocks B, C of $\mathcal{B}$, let $\Gamma[B, C]$ denote the induced bipartite subgraph of Γ with bipartition $\{\Gamma(C) \cap B, \Gamma(B) \cap C\}$. Define $\mathcal{D}(B)$ to be the 1-design with point set B and blocks $\Gamma(C) \cap B$ (with possible repetitions) for all blocks $C \in \Gamma_{\mathcal{B}}(B)$, where $\Gamma_{\mathcal{B}}(B)$ is the neighbourhood of B in $\Gamma_{\mathcal{B}}$. Since Γ is G-symmetric, up to isomorphism $\Gamma[B, C]$ and $\mathcal{D}(B)$ are independent of the choice of specific blocks B, C. Moreover, the block size k of $\mathcal{D}(B)$ is $|\Gamma(C) \cap B|$.

Let Σ be a $(G, 2)$-arc transitive graph of valency $v \geq 3$ (where G is a subgroup of the full automorphism group $\mathrm{Aut}(\Sigma)$ of Σ), and let Δ be a G-orbit on $A_3(\Sigma)$. If Δ is *self-paired*, that is, $(\tau, \sigma, \sigma', \tau') \in \Delta$ implies $(\tau', \sigma', \sigma, \tau) \in \Delta$, then the 3-*arc graph* $\mathrm{Arc}_{\Delta}(\Sigma)$ of Σ with respect to Δ is defined [11, Definition 3] to be the graph with vertex set $A(\Sigma)$ in which $\sigma\tau, \sigma'\tau'$ are adjacent if and only if $(\tau,\sigma,\sigma',\tau') \in \Delta$. The requirement that Δ is self-paired guarantees that the adjacency of $\Gamma := \mathrm{Arc}_\Delta(\Sigma)$ is well-defined. One can see that G preserves the adjacency of Γ and hence induces a faithful action as a group of automorphisms of Γ. Moreover, Γ is G-symmetric and admits a G-invariant partition $\mathcal{B}(\Sigma) := \{B(\sigma) : \sigma \in V(\Sigma)\}$ such that $\Sigma \cong \Gamma_{\mathcal{B}(\Sigma)}$ with respect to the bijection $\sigma \mapsto B(\sigma)$ ([11, Theorem 10(b)]), where $B(\sigma) := \{\sigma\tau : \tau \in \Sigma(\sigma)\}$ for $\sigma \in V(\Sigma)$. The 3-arc graphs can be characterized as follows.

Theorem 2.1 **([11, Theorem 1]).** *Let Γ be a finite G-symmetric graph, and $\mathcal{B}$ a nontrivial G-invariant partition of $V(\Gamma)$ with block size $v = k+1 \geq 3$. Then $\mathcal{D}(B)$ contains no repeated blocks if and only if $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive, and in this case $\Gamma \cong \mathrm{Arc}_\Delta(\Gamma_{\mathcal{B}})$ for some self-paired G-orbit Δ of 3-arcs of $\Gamma_{\mathcal{B}}$. Conversely, for any self-paired G-orbit Δ of 3-arcs of a $(G,2)$-arc transitive graph Σ of valency $v \geq 3$, the graph $\Gamma = \mathrm{Arc}_\Delta(\Sigma)$, group G, and partition $\mathcal{B}(\Sigma)$ satisfy all the conditions above.*

Thus, the class of G-symmetric graphs Γ satisfying the conditions of Theorem 1.1 is precisely the class of 3-arc graphs $\Gamma := \mathrm{Arc}_\Delta(\Sigma)$ of connected $(G,2)$-arc transitive graphs Σ such that Γ almost covers Σ (in the sense that it almost covers $\Gamma_{\mathcal{B}(\Sigma)}$). So in the following we may use the language of 3-arc graphs. Parts (a) and (c) of the following lemma are self-evident, and part (b) of it was proved in [11, Theorem 10(a)].

Lemma 2.2. *Let Σ be a finite connected $(G,2)$-arc transitive graph and let $\Delta := (\tau,\sigma,\sigma',\tau')^G$, a G-orbit on $A_3(\Sigma)$. Then*

(a) *Δ is self-paired if and only if $\sigma\tau$ and $\sigma'\tau'$ can be interchanged by an element of G, and in this case*

(b) *for $\varepsilon \in \Sigma(\sigma)$, $\sigma\varepsilon$ is the only vertex of $B(\sigma)$ not adjacent in $\mathrm{Arc}_\Delta(\Sigma)$ to any vertex of $B(\varepsilon)$, and*

(c) *$\mathrm{Arc}_\Delta(\Sigma)$ almost covers Σ if and only if τ' is fixed by $G_{\tau\sigma\sigma'}$ (that is, $G_{\tau\sigma\sigma'} = G_{\tau\sigma\sigma'\tau'}$).*

Let $\Gamma = \mathrm{Arc}_\Delta(\Sigma)$ be a 3-arc graph of the $(G,2)$-arc transitive graph Σ. If Γ almost covers Σ, then for each $\tau \in \Sigma(\sigma) \setminus \{\sigma'\}$ there exists a unique $\tau' \in \Sigma(\sigma') \setminus \{\sigma\}$ such that $(\tau,\sigma,\sigma',\tau') \in \Delta$, and hence $\tau \mapsto \tau'$ defines a bijection from $\Sigma(\sigma) \setminus \{\sigma'\}$ to $\Sigma(\sigma') \setminus \{\sigma\}$. Note that this bijection depends on Δ. Since there will be no danger of confusion, we will denote it just by $\phi_{\sigma\sigma'}$.

Lemma 2.3. *Let Σ be a finite connected $(G,2)$-arc transitive graph, let Δ be a self-paired G-orbit on $A_3(\Sigma)$ and let $\sigma\tau$ be an arc of Σ. Suppose that* *the 3-arc graph* $\Gamma := \mathrm{Arc}_\Delta(\Sigma)$ *almost covers* Σ. *Then the following* (a)–(d) *hold:*

(a) *The actions of* G_σ *on* $B(\sigma)$ *and* $\Sigma(\sigma)$ *are permutationally equivalent, 2-transitive and faithful.*

(b) *The actions of* $G_{\sigma\tau}$ *on* $\Sigma(\sigma)\setminus\{\tau\}$ *and on* $\Gamma(\sigma\tau)$ *are permutationally equivalent, where* $\Gamma(\sigma\tau)$ *is the set of vertices of* Γ *adjacent in* Γ *to the vertex* $\sigma\tau$ *of* Γ. *In particular,* Γ *is G-locally primitive if and only if* G_σ *is 2-primitive on* $\Sigma(\sigma)$; *and* $G_{\sigma\tau}$ *is regular on* $\Gamma(\sigma\tau)$ *if and only if* G_σ *is sharply 2-transitive on* $\Sigma(\sigma)$.

(c) $\phi_{\sigma\tau}^{-1} = \phi_{\tau\sigma}$.

(d) $(\phi_{\sigma\tau}(\varepsilon))^g = \phi_{\sigma^g\tau^g}(\varepsilon^g)$ *for* $\varepsilon \in \Sigma(\sigma)\setminus\{\tau\}$ *and* $g \in G$. *In particular, the actions of* $G_{\sigma\tau}$ *on* $\Sigma(\sigma)\setminus\{\tau\}$ *and* $\Sigma(\tau)\setminus\{\sigma\}$ *are permutationally equivalent with respect to* $\phi_{\sigma\tau}$.

Proof. (a) The actions of G_σ on $B(\sigma)$ and $\Sigma(\sigma)$ are permutationally equivalent with respect to the bijection $B(\sigma) \to \Sigma(\sigma)$ defined by $\sigma\tau \mapsto \tau$ for $\tau \in \Sigma(\sigma)$. Since Σ is $(G,2)$-arc transitive, these actions are 2-transitive. The faithfulness follows from Theorem 2.1 and [11, Lemma 1(a) and Theorem 5(e)].

(b) For each $\varepsilon \in \Sigma(\sigma)\setminus\{\tau\}$, let $\lambda(\varepsilon)$ denote the unique vertex in $B(\varepsilon)$ adjacent to $\sigma\tau$ in Γ. (The existence of $\lambda(\varepsilon)$ follows from Lemma 2.2(b).) Then λ establishes a bijection from $\Sigma(\sigma)\setminus\{\tau\}$ to $\Gamma(\sigma\tau)$, and the actions of $G_{\sigma\tau}$ on $\Sigma(\sigma)\setminus\{\tau\}$ and on $\Gamma(\sigma\tau)$ are permutationally equivalent with respect to λ. From this the last two assertions in (b) follow immediately.

(c) This is obvious from the definition of $\phi_{\sigma\tau}$.

(d) For $(\varepsilon,\sigma,\tau,\eta) \in \Delta$ and $g \in G$, since Δ is G-invariant we have $(\varepsilon^g,\sigma^g,\tau^g,\eta^g) \in \Delta$ and so $(\phi_{\sigma\tau}(\varepsilon))^g = \eta^g = \phi_{\sigma^g\tau^g}(\varepsilon^g)$ (by the definitions of $\phi_{\sigma\tau}$ and $\phi_{\sigma^g\tau^g}$). In particular, $(\phi_{\sigma\tau}(\varepsilon))^g = \phi_{\sigma\tau}(\varepsilon^g)$ for $g \in G_{\sigma\tau}$ and hence the assertion in the last sentence of (d) is true. ∎

The next lemma will be used to prove a corollary of our main result. It shows that, for a $(G,2)$-arc transitive graph Σ, if G_σ is sharply 2-transitive on $\Sigma(\sigma)$ (that is, G_σ holds the "weakest" 2-transitivity on $\Sigma(\sigma)$), then all the 3-arc graphs of Σ are forced to be almost covers of Σ.

Lemma 2.4. *Suppose that* Σ *is a finite* $(G,2)$*-arc transitive graph of valency* $v \geq 3$ *such that* G_σ *is sharply 2-transitive on* $\Sigma(\sigma)$ *for* $\sigma \in V(\Sigma)$. *Then, for every self-paired G-orbit* Δ *on* $A_3(\Sigma)$, *the 3-arc graph* $\Gamma := \mathrm{Arc}_\Delta(\Sigma)$ *is an almost cover of* Σ, *and moreover* $G_{\sigma\tau}$ *is regular on the neighbourhood* $\Gamma(\sigma\tau)$ *of* $\sigma\tau \in V(\Gamma)$ *in* Γ.

Proof. Let $\sigma\tau$ be an arc of Σ. Then the sharp 2-transitivity of G_σ on $\Sigma(\sigma)$ implies that $G_{\sigma\tau}$ is regular on $\Sigma(\sigma)\setminus\{\tau\}$, and hence we have $|G_{\sigma\tau}| =$

$|\Sigma(\sigma)|-1$. Since $\Gamma(\sigma\tau)$ contains exactly s points of each block $B(\delta)$ for $\delta\in\Sigma(\sigma)\setminus\{\tau\}$, where s is the valency of the bipartite graph $\Gamma[B(\sigma),B(\delta)]$, we then have $|\Gamma(\sigma\tau)| = s(|\Sigma(\sigma)|-1) = s|G_{\sigma\tau}|$. On the other hand, since $G_{\sigma\tau}$ is transitive on $\Gamma(\sigma\tau)$, by the well-known orbit-stabilizer property (see e.g. [5, Theorem 1.4A]), $|\Gamma(\sigma\tau)|$ is a divisor of $|G_{\sigma\tau}|$. So we have $s=1$, that is, $\Gamma[B(\sigma),B(\tau)]=(v-1)\cdot K_2$, and hence Γ almost covers Σ. Since $G_{\sigma\tau}$ is regular on $\Sigma(\sigma)\setminus\{\tau\}$, from Lemma 2.3(b) we know that $G_{\sigma\tau}$ is also regular on $\Gamma(\sigma\tau)$. ∎

Lemma 2.5. *Let Σ be a finite connected $(G,2)$-arc transitive graph with valency $v\geq 3$. Then* $\mathrm{girth}(\Sigma)=3$ *if and only if $\Sigma\cong K_{v+1}$, which in turn is true if and only if G is 3-transitive on $V(\Sigma)$.*

Proof. If $\Sigma\cong K_{v+1}$, then $\mathrm{girth}(\Sigma)=3$ and G is 3-transitive on $V(\Sigma)$ since G_σ is 2-transitive on $\Sigma(\sigma)=V(\Sigma)\setminus\{\sigma\}$ and G is transitive on $V(\Sigma)$. Next suppose that G is 3-transitive on $V(\Sigma)$. Then, for each $\sigma\in V(\Sigma)$, G_σ is 2-transitive on $V(\Sigma)\setminus\{\sigma\}$ and hence $V(\Sigma)\setminus\{\sigma\}$ induces a complete graph K_v (note that $V(\Sigma)\setminus\{\sigma\}$ contains adjacent vertices). This implies $\Sigma\cong K_{v+1}$. Finally, if $\mathrm{girth}(\Sigma)=3$, then $\Sigma(\sigma)$ induces a complete graph K_v by the 2-transitivity of G_σ on $\Sigma(\sigma)$. Hence $\Sigma\cong K_{v+1}$ by the connectedness of Σ. ∎

A *circulant* is a Cayley graph $\mathrm{Cay}(\mathbb{Z}_n,S)$ with vertex set the additive group $\mathbb{Z}_n$ of integers modulo n in which $x,y\in\mathbb{Z}_n$ are adjacent if and only if $x-y\in S$, where S is a subset of $\mathbb{Z}_n$ such that $0\notin S$ and $-S:=\{-x:x\in S\}$ is equal to S. For a near n-gonal graph $(\Sigma,\mathcal{E})$, the cycles in $\mathcal{E}$ are called *basic cycles* of $(\Sigma,\mathcal{E})$. We use $C(\sigma,\tau,\varepsilon)$ to denote the unique basic cycle of $(\Sigma,\mathcal{E})$ containing a given 2-arc $(\sigma,\tau,\varepsilon)$ of Σ. We also use $A_3(\Sigma,\mathcal{E})$ to denote the set of all 3-arcs of Σ which are contained in some basic cycle of $(\Sigma,\mathcal{E})$. Any subgroup $G\leq\mathrm{Aut}(\Sigma)$ induces an action on n-cycles of Σ, and if $\mathcal{E}$ is G-invariant, then G induces an action on $\mathcal{E}$.

Lemma 2.6. *Suppose $(\Sigma,\mathcal{E})$ is a finite $(G,2)$-arc transitive near n-gonal graph. Then the following statements* (a)–(c) *are equivalent:*

(a) *$\mathcal{E}$ is G-invariant.*

(b) *$\mathcal{E}$ is a G-orbit on n-cycles of Σ.*

(c) *$A_3(\Sigma,\mathcal{E})$ is a self-paired G-orbit on $A_3(\Sigma)$.*

Moreover, if one of these occurs, then the following (d)–(e) *hold:*

(d) *Any element of G fixing a 2-arc $(\sigma,\tau,\varepsilon)$ of Σ must fix each vertex in $C(\sigma,\tau,\varepsilon)$.*

(e) *The subgraph of Σ induced by the vertex set of a basic cycle of $(\Sigma,\mathcal{E})$ is isomorphic to a circulant graph $\mathrm{Cay}(\mathbb{Z}_n,S)$, for some S with $1\in S$. Moreover, each such basic cycle is chordless (that is, $\mathrm{Cay}(\mathbb{Z}_n,S)\cong C_n$) unless, for adjacent vertices σ,τ of Σ, either*

(i) G_τ *is sharply* 2*-transitive on* $\Sigma(\tau)$ *(and hence* $|\Sigma(\tau)|$ *is a prime power); or*

(ii) $G_{\sigma\tau}$ *is imprimitive on* $\Sigma(\tau)\setminus\{\sigma\}$.

Proof. The equivalence of (a) and (b) is obvious since each 2-arc of Σ lies in a unique cycle of $\mathcal{E}$. If (a) holds, then $A_3(\Sigma,\mathcal{E})$ is a G-orbit on $A_3(\Sigma)$. Moreover, in this case $A_3(\Sigma,\mathcal{E})$ is also self-paired. In fact, for $(\sigma,\tau,\varepsilon,\eta)\in A_3(\Sigma,\mathcal{E})$ there exists $g\in G$ such that $(\sigma,\tau,\varepsilon)^g=(\eta,\varepsilon,\tau)$ as Σ is $(G,2)$-arc transitive. Thus, $(C(\sigma,\tau,\varepsilon))^g=C(\eta,\varepsilon,\tau)$. But $C(\sigma,\tau,\varepsilon)$ is the unique basic cycle containing $(\sigma,\tau,\varepsilon)$, and it is also the unique basic cycle containing (η,ε,τ). So g fixes $C(\sigma,\tau,\varepsilon)$ and $\eta^g=\sigma$, implying $(\eta,\varepsilon,\tau,\sigma)=(\sigma,\tau,\varepsilon,\eta)^g\in A_3(\Sigma,\mathcal{E})$. Hence $A_3(\Sigma,\mathcal{E})$ is self-paired. Thus (a) implies (c). Conversely suppose that (c) holds. Let $C(\sigma_0,\sigma_1,\sigma_2)=(\sigma_0,\sigma_1,\sigma_2,\dots,\sigma_{n-1},\sigma_0)$ be a basic cycle of $(\Sigma,\mathcal{E})$, and let $g\in G$. For each $i=0,1,\dots,n-1$ (subscripts modulo n here and in the remaining part of the proof), it follows from (c) that both $(\sigma_{i-1}^g,\sigma_i^g,\sigma_{i+1}^g,\sigma_{i+2}^g)$ and $(\sigma_i^g,\sigma_{i+1}^g,\sigma_{i+2}^g,\sigma_{i+3}^g)$ lie in basic cycles, and they must lie in the same basic cycle since these two 3-arcs have the 2-arc $(\sigma_i^g,\sigma_{i+1}^g,\sigma_{i+2}^g)$ in common and since each 2-arc of Σ is contained in a unique basic cycle of $(\Sigma,\mathcal{E})$. Since this is true for all i, it follows that $(C(\sigma_0,\sigma_1,\sigma_2))^g$ must be a basic cycle of $(\Sigma,\mathcal{E})$ and hence (c) implies (a).

In the remainder of this proof, we suppose $\mathcal{E}$ is G-invariant, so both (b) and (c) hold. Thus the vertex sets of the basic cycles of $(\Sigma,\mathcal{E})$ induce mutually isomorphic subgraphs. If $g\in G$ fixes the 2-arc $(\sigma_0,\sigma_1,\sigma_2)$, then it fixes the basic cycle $C(\sigma_0,\sigma_1,\sigma_2)$ and, since g fixes each of σ_1,σ_2, it follows that g must fix σ_3. Inductively, one can see that g fixes each vertex in $C(\sigma_0,\sigma_1,\sigma_2)$ and thus (d) is proved.

In proving (e), we set $V:=\{\sigma_0,\sigma_1,\sigma_2,\dots,\sigma_{n-1}\}$, the vertex set of $C(\sigma_0,\sigma_1,\sigma_2)$, and denote by Σ_1 the subgraph of Σ induced by V. Since Σ is $(G,2)$-arc transitive, there exists $h\in G$ such that $(\sigma_{n-1},\sigma_0,\sigma_1)^h=(\sigma_0,\sigma_1,\sigma_2)$. Since $\mathcal{E}$ is G-invariant it follows that h fixes V setwise and leaves $C(\sigma_0,\sigma_1,\sigma_2)$ invariant. The only element of $\mathrm{Aut}(\Sigma_1)$ which leaves $C(\sigma_0,\sigma_1,\sigma_2)$ invariant and maps $(\sigma_{n-1},\sigma_0,\sigma_1)$ to $(\sigma_0,\sigma_1,\sigma_2)$ is the rotation $\rho:\sigma_i\mapsto\sigma_{i+1}$, for all i. Thus the permutation h^V of V induced by h is ρ, and by [1, Lemma 16.3], since $\langle\rho\rangle\cong\mathbb{Z}_n$ is regular on V, Σ_1 is isomorphic to a circulant $\mathrm{Cay}(\mathbb{Z}_n,S)$ for some S. Since σ_i is adjacent to σ_{i+1}, we have $1\in S$ and the first part of (e) is proved. In proving the second part of (e), we assume that $C(\sigma_0,\sigma_1,\sigma_2)$ contains a chord. Since the group induced on $C(\sigma_0,\sigma_1,\sigma_2)$ contains ρ, it follows that σ_1 is adjacent to some vertex σ_i with $i\neq 0,2$, that is to say, $\{\sigma_1,\sigma_i\}$ is a chord; and the set $X:=\mathrm{fix}_{\Sigma(\sigma_1)\setminus\{\sigma_0\}}(G_{\sigma_0\sigma_1\sigma_2})$ contains both σ_2 and σ_i. On the other hand, the $(G,2)$-arc transitivity of Σ implies that $G_{\sigma_0\sigma_1}$ is transitive on $\Sigma(\sigma_1)\setminus\{\sigma_0\}$, and the stabilizer $G_{\sigma_0\sigma_1\sigma_2}$ (which fixes

$C(\sigma_0,\sigma_1,\sigma_2)$ pointwise) fixes $|X|\geq 2$ points of $\Sigma(\sigma_1)\setminus\{\sigma_0\}$. As mentioned at the beginning of this section, X is a block of imprimitivity for $G_{\sigma_0\sigma_1}$ in $\Sigma(\sigma_1)\setminus\{\sigma_0\}$, and hence either $X=\Sigma(\sigma_1)\setminus\{\sigma_0\}$ or X induces a nontrivial $G_{\sigma_0\sigma_1}$-invariant partition of $\Sigma(\sigma_1)\setminus\{\sigma_0\}$. In the former case the possibility (i) in (e) occurs; whilst in the latter case the possibility (ii) in (e) occurs. Note that if (i) occurs then by [17, pp. 23] the valency $|\Sigma(\sigma_1)|$ must be a prime power. ∎

3. Proof of Theorem 1.1

Now we are ready to prove Theorem 1.1. In fact, we will prove the following theorem which, together with Theorems 2.1, yields a proof of Theorem 1.1.

Theorem 3.1. *Suppose that Σ is a finite connected $(G,2)$-arc transitive graph with valency $v\geq 3$ and that $\Sigma\ncong K_{v+1}$. Then Σ is almost covered by a 3-arc graph $\mathrm{Arc}_\Delta(\Sigma)$ of Σ if and only if, for some even integer $n\geq 4$, Σ is a near n-gonal graph with respect to a G-orbit $\mathcal{E}$ of n-cycles of Σ, and in this case we have $\Delta=A_3(\Sigma,\mathcal{E})$, the set of all 3-arcs of Σ contained in the n-cycles in $\mathcal{E}$.*

Proof. Suppose Σ is almost covered by a 3-arc graph $\Gamma:=\mathrm{Arc}_\Delta(\Sigma)$ of Σ, where Δ is a self-paired G-orbit on $A_3(\Sigma)$. Recall that, for adjacent vertices σ,σ' of Σ, we use $\phi_{\sigma\sigma'}$ to denote the the bijection from $\Sigma(\sigma)\setminus\{\sigma'\}$ to $\Sigma(\sigma')\setminus\{\sigma\}$ such that $\phi_{\sigma\sigma'}(\tau)=\tau'$ precisely when $(\tau,\sigma,\sigma',\tau')\in\Delta$. Let $(\sigma_0,\sigma_1,\sigma_2)$ be a 2-arc of Σ. Set $\sigma_3:=\phi_{\sigma_1\sigma_2}(\sigma_0)$, and inductively define $\sigma_{i+2}:=\phi_{\sigma_i\sigma_{i+1}}(\sigma_{i-1})$ for $i\geq 1$. Then we get a sequence $\sigma_0,\sigma_1,\sigma_2,\ldots,\sigma_{i-1},\sigma_i,\sigma_{i+1},\sigma_{i+2},\ldots$ of vertices of Σ such that $(\sigma_{i-1},\sigma_i,\sigma_{i+1},\sigma_{i+2})\in\Delta$ for each $i\geq 1$. Our assumption $\Sigma\ncong K_{v+1}$ implies that $\mathrm{girth}(\Sigma)\geq 4$ (Lemma 2.5) and hence all such 3-arcs $(\sigma_{i-1},\sigma_i,\sigma_{i+1},\sigma_{i+2})$ are proper, that is, any four consecutive vertices in this sequence are pairwise distinct. Since Σ has a finite number of vertices, the sequence must eventually contain repeated vertices. Let σ_n be the first vertex in the sequence that coincides with one of the preceding vertices. We claim that σ_n must coincide with σ_0. Suppose to the contrary that $\sigma_n=\sigma_m$ for some m such that $1\leq m<n$. Then since Σ is $(G,2)$-arc transitive, there exists $g\in G$ such that $(\sigma_m,\sigma_{m+1},\sigma_{m+2})^g=(\sigma_0,\sigma_1,\sigma_2)$. From Lemma 2.3(d), we have $\sigma_{m+3}^g=\phi_{\sigma_{m+1}^g\sigma_{m+2}^g}(\sigma_m^g)=\phi_{\sigma_1\sigma_2}(\sigma_0)=\sigma_3$. Inductively we have that $\sigma_{m+i}^g=\sigma_i$ for each $i\geq 0$. In particular, $\sigma_n^g=\sigma_{m+(n-m)}^g=\sigma_{n-m}$. But since $\sigma_n=\sigma_m$, we have $\sigma_{n-m}=\sigma_n^g=\sigma_m^g=\sigma_0$, contradicting the minimality of n. Therefore we must have $\sigma_n=\sigma_0$. Thus, each 2-arc $(\sigma_0,\sigma_1,\sigma_2)$ of Σ determines a unique (undirected) n-cycle $C(\sigma_0,\sigma_1,\sigma_2):=(\sigma_0,\sigma_1,\sigma_2,\ldots,\sigma_{n-1},\sigma_0)$ of Σ. Note again that $n\geq 4$ since $\mathrm{girth}(\Sigma)\geq 4$.

Set $\tau := \phi_{\sigma_1\sigma_0}(\sigma_2)$, then we have $\sigma_2 = \phi_{\sigma_0\sigma_1}(\tau)$ by Lemma 2.3(c). We claim that τ must coincide with σ_{n-1}. For the 2-arc (τ,σ_0,σ_1), the construction in the previous paragraph will give the sequence $\tau,\sigma_0,\sigma_1,\sigma_2,\ldots,\sigma_{n-1},\sigma_n = \sigma_0$, and since the first repeated vertex is the same as the starting vertex τ, it follows that $\tau = \sigma_{n-1}$. Similarly, one can show that $\sigma_{n-2} = \phi_{\sigma_0\sigma_{n-1}}(\sigma_1)$ and hence $\sigma_1 = \phi_{\sigma_{n-1}\sigma_0}(\sigma_{n-2})$. Therefore, reading the subscripts modulo n (here and in the remainder of this section), we have $\sigma_{i+2} = \phi_{\sigma_i\sigma_{i+1}}(\sigma_{i-1})$ and hence $\sigma_{i-1} = \phi_{\sigma_{i+1}\sigma_i}(\sigma_{i+2})$ for each $i \geq 1$ (Lemma 2.3(c)). This implies that the 2-arcs $(\sigma_{i-1},\sigma_i,\sigma_{i+1})$ and $(\sigma_{i+1},\sigma_i,\sigma_{i-1})$ contained in $C(\sigma_0,\sigma_1,\sigma_2)$ (for $i \geq 1$) also determine the same n-cycle $C(\sigma_0,\sigma_1,\sigma_2)$. By definition of $C(\sigma_0,\sigma_1,\sigma_2)$ and by Lemma 2.3(d), we have $C(\sigma_0^g,\sigma_1^g,\sigma_2^g) = (C(\sigma_0,\sigma_1,\sigma_2))^g$ for $g \in G$ and hence $\mathcal{E} := \{C(\sigma,\tau,\varepsilon) : (\sigma,\tau,\varepsilon) \in A_2(\Sigma)\}$ is G-invariant and each 2-arc lies in a unique cycle of $\mathcal{E}$. By the $(G,2)$-arc transitivity of Σ, the length n of $C(\sigma,\tau,\varepsilon)$ is independent of the choice of $(\sigma,\tau,\varepsilon)$ and G is transitive on $\mathcal{E}$. Thus $\mathcal{E}$ is a G-orbit on n-cycles of Σ and Σ is a near n-gonal graph with respect to $\mathcal{E}$. Moreover, the argument above shows that $\Delta = A_3(\Sigma,\mathcal{E})$. In particular, in the sequence $\sigma_0\sigma_1,\sigma_1\sigma_0,\sigma_2\sigma_3,\sigma_3\sigma_2,\ldots,\sigma_{2i-2}\sigma_{2i-1},\sigma_{2i-1}\sigma_{2i-2},\sigma_{2i}\sigma_{2i+1},\sigma_{2i+1}\sigma_{2i},\ldots$ of vertices of Γ, for each i, the $(2i-1)$-st vertex $\sigma_{2i-2}\sigma_{2i-1}$ and the $2i$-th vertex $\sigma_{2i-1}\sigma_{2i-2}$ are not adjacent, while the $2i$-th vertex and the $(2i+1)$-st vertex $\sigma_{2i}\sigma_{2i+1}$ are adjacent. By the definition of n, the n-th vertex of this sequence is $\sigma_{n-1}\sigma_{n-2}$, and it is adjacent to $\sigma_0\sigma_1$ $(= \sigma_n\sigma_{n+1})$ since $(\sigma_{i-1},\sigma_i,\sigma_{i+1},\sigma_{i+2}) \in \Delta$ for each i (subscripts modulo n). It follows that n must be an even integer.

To prove the "if" part of the theorem, suppose that $(\Sigma,\mathcal{E})$ is a $(G,2)$-arc transitive near n-gonal graph with valency $v \geq 3$ and $\mathcal{E}$ is a G-orbit on n-cycles of Σ, for some even $n \geq 4$. Then by Lemma 2.6, $\Delta := A_3(\Sigma,\mathcal{E})$ is a self-paired G-orbit on $A_3(\Sigma)$. Let $\Gamma := \mathrm{Arc}_\Delta(\Sigma)$ and let $(\tau,\sigma,\sigma',\tau') \in \Delta$. Then $\sigma\tau \in B(\sigma)$ is adjacent to $\sigma'\tau' \in B(\sigma')$ in Γ. If $\sigma\tau$ is adjacent in Γ to a second vertex, say $\sigma'\varepsilon'$, of $B(\sigma')$, then $(\tau,\sigma,\sigma',\tau')$, $(\tau,\sigma,\sigma',\varepsilon')$ are distinct 3-arcs in Δ and hence the 2-arc (τ,σ,σ') is contained in two distinct basic cycles of $(\Sigma,\mathcal{E})$. This contradiction shows that $\Gamma[B(\sigma),B(\sigma')] \cong (v-1)\cdot K_2$ and hence Γ almost covers $\Gamma_{\mathcal{B}(\Sigma)}$. ∎

Remark 3.2. By Lemma 2.6(e), the vertex set of each basic cycle of $(\Sigma,\mathcal{E})$ in Theorem 3.1 induces a circulant subgraph of Σ, and these basic cycles are chordless unless either (e)(i) or (e)(ii) in that lemma occurs. This latter fact is interesting from a combinatorial point of view. The following example shows that the basic cycles of $(\Sigma,\mathcal{E})$ may contain chords. It also provides an example of such a graph Σ with the smallest valency (namely 3) and shows that the near n-gonal graph $(\Sigma,\mathcal{E})$ occurring in Theorem 3.1 is not necessarily an n-gonal graph. (A near n-gonal graph is said to be an *n-gonal graph* [14] if n is equal to the girth of the graph.) Moreover, it shows that the graph $\mathrm{Arc}_{\Delta}(\Sigma)$ may not be connected, even if Σ is connected and $(G,2)$-arc transitive.

Example 3.3. Let Σ be the complete bipartite graph $K_{3,3}$ with vertex set $\{0,1,2,3,4,5\}$ and bipartition $(\{0,2,4\},\{1,3,5\})$. We will show that there exists a unique subgroup $G \le \mathrm{Aut}(\Sigma)$ such that Σ is a $(G,2)$-arc transitive near 6-gonal graph with respect to a G-orbit $\mathcal{E}$ of 6-cycles of Σ. By the definition of near polygonal graphs, one can easily check that

$$\mathcal{E}_1 := \{(0,1,2,3,4,5,0),(0,5,2,1,4,3,0),(0,1,4,5,2,3,0)\}$$

and

$$\mathcal{E}_2 := \{(0,1,2,5,4,3,0),(0,3,2,1,4,5,0),(0,1,4,3,2,5,0)\}$$

are the only possible sets $\mathcal{E}$ of 6-cycles of Σ such that $(\Sigma,\mathcal{E})$ is a near 6-gonal graph. On the other hand, we have $\mathrm{Aut}(\Sigma) = S_3 \,\mathrm{wr}\, S_2 \cong \langle (024),(02),(01)(23)(45)\rangle$ and again it is easily checked that (024) and $(01)(23)(45)$ fix $\mathcal{E}_1$ and $\mathcal{E}_2$ setwise, whilst (02) interchanges $\mathcal{E}_1$ and $\mathcal{E}_2$. Thus $\mathrm{Aut}(\Sigma)$ interchanges $\mathcal{E}_1$ and $\mathcal{E}_2$ and so a subgroup G of $\mathrm{Aut}(\Sigma)$ with index 2 fixes $\mathcal{E}_1$ and $\mathcal{E}_2$ setwise. We have seen that G contains $H = \langle (024),(01)(23)(45)\rangle \cong A_3 \,\mathrm{wr}\, S_2$, but does not contain (02). Thus $|G:H| = 2$. The element (13) is the conjugate of (02) by $(01)(23)(45)$, and hence $(13) \in \mathrm{Aut}(\Sigma)$ and (13) interchanges $\mathcal{E}_1$ and $\mathcal{E}_2$. Therefore $(02)(13)$ fixes $\mathcal{E}_1$ and $\mathcal{E}_2$ setwise and does not lie in H, so $G = \langle H,(02)(13)\rangle$. It is easy to check that G is transitive on the 2-arcs of Σ, and hence $(\Sigma,\mathcal{E}_i)$ is a $(G,2)$-arc transitive near 6-gonal graph for $i=1$ and $i=2$. If Σ is $(K,2)$-arc transitive and K preserves the $\mathcal{E}_i$, then $K \le G$ and $|K|$ is divisible by the number of 2-arcs, that is, by 36. Hence $K = G$. Finally, for $\Delta_i := A_3(\Sigma,\mathcal{E}_i)$, $i=1,2$, we have $\mathrm{Arc}_{\Delta_i}(\Sigma) \cong 3 \cdot C_6$. We show this graph in Figure 1, where the three blocks on the left-hand side are $B(0), B(2)$ and $B(4)$, and that on the right-hand side are $B(1), B(3)$ and $B(5)$.

The following proposition shows further that the graph Σ in Example 3.3 is the only connected trivalent non-complete graph which is $(G,2)$-arc transitive and near n-gonal for an even integer n such that the basic cycles have chords.

Proposition 3.4. *Suppose Σ is a finite, connected, $(G,2)$-arc transitive, trivalent graph and $\Sigma \not\cong K_4$. Suppose Δ is a self-paired G-orbit on $A_3(\Sigma)$ such that $\Gamma := \mathrm{Arc}_{\Delta}(\Sigma)$ almost covers Σ. Then Σ is a near n-gonal graph with respect to some G-orbit $\mathcal{E}$ of n-cycles (and n is even). Moreover the*

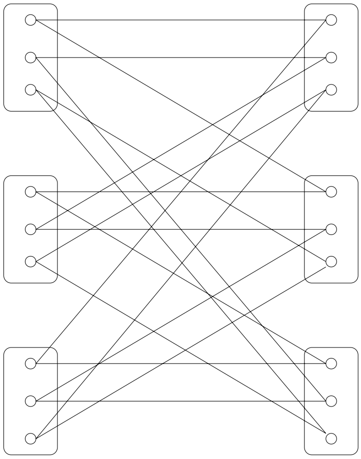

Figure 1. $\Gamma = 3 \cdot C_6, \Sigma = K_{3,3}$

cycles in $\mathcal{E}$ have chords if and only if $\Sigma \cong K_{3,3}$, $\Gamma \cong 3 \cdot C_6$, and $\mathcal{E} \cong \mathcal{E}_1$ or $\mathcal{E}_2$, where G, $\mathcal{E}_1$ and $\mathcal{E}_2$ are as in Example 3.3.

Proof. By Theorem 3.1, Σ is a near n-gonal graph with respect to some G-orbit $\mathcal{E}$ of n-cycles for an even integer $n \geq 4$. So we need only to prove that the cycles in $\mathcal{E}$ have chords if and only if $\Sigma, \Gamma, G, \mathcal{E}$ are as claimed. The "if" part was in fact proved in Example 3.3. We prove the "only if" part in the following.

Suppose $\{\sigma_0, \sigma_m\}$ is a chord of the basic cycle $C(\sigma_0, \sigma_1, \sigma_2) := (\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_0)$. Then $\{\sigma_i, \sigma_{i+m}\}$ is a chord of $C(\sigma_0, \sigma_1, \sigma_2)$ for each i (by Lemma 2.6(e)). Since Σ is trivalent and connected, the only possibility is $m = n/2$ and $\Sigma \cong \mathrm{Cay}(\mathbb{Z}_n, \{1, m, n-1\})$. Since $\Sigma \not\cong K_4$, we have $m \geq 3$. Now the unique n-cycle $C(\sigma_m, \sigma_0, \sigma_1)$ containing $(\sigma_m, \sigma_0, \sigma_1)$ must be the following sequence of vertices: $\sigma_m, \sigma_0, \sigma_1, \sigma_{m+1}, \sigma_{m+2}, \sigma_2, \sigma_3, \sigma_{m+3}, \sigma_{m+4}, \ldots$. If m is even, this sequence does not even form an n-cycle since it never returns to the vertex σ_m. (Once we arrive at σ_{m-1}, the next vertex in the sequence is σ_{n-1} and from σ_{n-1} the sequence returns to σ_0. For example, if $m = 4$, then the sequence is the 7-cycle $(\sigma_0, \sigma_1, \sigma_5, \sigma_6, \sigma_2, \sigma_3, \sigma_7, \sigma_0)$.) So m is odd, and in this case the sequence does give an n-cycle. By the $(G, 2)$-arc transitivity of Σ, there exists $g \in G$ such that $(\sigma_{n-1}, \sigma_0, \sigma_1)^g = (\sigma_m, \sigma_0, \sigma_1)$. From Lemma 2.3(d), we have $(C(\sigma_{n-1}, \sigma_0, \sigma_1))^g = C(\sigma_m, \sigma_0, \sigma_1)$. Therefore, $\sigma_0^g = \sigma_0$, $\sigma_1^g = \sigma_1, \sigma_{n-1}^g = \sigma_m, \sigma_{n-3}^g = \sigma_{n-1}$. Since σ_0, σ_m are adjacent, we know that σ_0^g and σ_m^g are adjacent, and hence the only possibility for σ_m^g is $\sigma_m^g = \sigma_{n-1}$ (note that $\sigma_m^g \neq \sigma_1^g = \sigma_1, \sigma_m^g \neq \sigma_{n-1}^g = \sigma_m$). But $\sigma_{n-3}^g = \sigma_{n-1}$ as mentioned above, so we get $\sigma_m = \sigma_{n-3}$. Therefore,

$n=6$ and hence $\Sigma = \mathrm{Cay}(\mathbb{Z}_6,\{1,3,5\}) \cong K_{3,3}$. From the discussion in Example 3.3, we then have $\Gamma = 3\cdot C_6$, $\mathcal{E}$ is either $\mathcal{E}_1$ or $\mathcal{E}_2$, and G is the group $\langle(024),(02)(13),(01)(23)(45)\rangle$. ∎

4. Corollaries

We conclude the paper by giving two corollaries of our main result. The first one, stated below, might be useful in constructing two-arc transitive near-polygonal graphs.

Corollary 4.1. *Suppose that Σ a finite connected $(G,2)$-arc transitive graph of valency $v \geq 3$ such that $\Sigma \not\cong K_{v+1}$ and G_σ is sharply 2-transitive on $\Sigma(\sigma)$ for $\sigma \in V(\Sigma)$. If one of the G-orbits on $A_3(\Sigma)$ is self-paired (that is, G contains an element reversing a 3-arc of Σ), then, for some even integer $n \geq 4$, Σ is a near n-gonal graph with respect to a G-orbit on n-cycles of Σ.*

This follows immediately from Theorem 3.1 and Lemma 2.4. Moreover, since G_σ is sharply 2-transitive on $\Sigma(\sigma)$, by a well known result (see [5,17]) the valency $v = |\Sigma(\sigma)|$ of Σ must be a prime power. The reader is referred to [16] for information about the group G.

Our second corollary examines an important special case of almost covers which motivated the study in this paper. Recall that if Γ is a G-symmetric, G-locally primitive graph admitting a nontrivial G-invariant partition $\mathcal{B}$ of block size $v = k+1 \geq 3$ such that $\Gamma_{\mathcal{B}}$ is connected, then $\mathcal{D}(B)$ contains no repeated blocks ([6, Lemma 3.3(c)]) and $\Gamma_{\mathcal{B}}$ is almost covered by Γ ([6, Lemma 3.1(a)]). By Theorem 2.1, $\Gamma = \mathrm{Arc}_\Delta(\Sigma)$ for some self-paired G-orbit Δ on 3-arcs of $\Sigma := \Gamma_{\mathcal{B}}$, and $\mathcal{B}$ is identical with $\mathcal{B}(\Sigma)$ (see [11, Section 5]). Note that Lemma 2.3 parts (a) and (b), and the G-local primitivity of Γ, imply that G_B is 2-primitive on B and $\Sigma(B)$. If in addition $\mathrm{girth}(\Sigma) = 3$ (that is, $\Sigma \cong K_{v+1}$, see Lemma 2.5), then G is 3-primitive on $\mathcal{B}$ and the argument in the proof of [6, Theorem 5.4] from (Line, Page) = (25, 534) to (12, 535) is valid, and hence we get the possibilities for (Γ, G) listed in part (a) and the second half of part (b) of [6, Theorem 5.4]. However, in the general case where $\mathrm{girth}(\Sigma) \geq 4$, the argument in [6, lines 33–41, pp. 534] should be modified since the block D therein is not adjacent to C. In this case, as shown in Theorem 1.1, Σ is a near n-gonal graph with $n \geq 4$ and n even. Moreover, $G_B^{\Sigma(B)}$ is 2-primitive. Hence if basic cycles of Σ have chords, then by Lemma 2.6(e), G_B is sharply 2-primitive on $\Sigma(B)$. Hence G_B is also sharply 2-primitive on B, and so v is a prime power and, for $\alpha \in B$, $G_\alpha^{B\setminus\{\alpha\}} = \mathbb{Z}_{v-1}$ with $v-1$ a prime. Hence either $v=3$, or $v = 2^p$ for a prime p with $q=2^p-1$ a Mersenne prime. In the former case Proposition 3.4 implies that $\Sigma=K_{3,3}$, $\Gamma=3\cdot C_6$, and G and $\mathcal{E}$ are as in Example 3.3. In the latter case $G_B^B=(\mathbb{Z}_2)^p.\mathbb{Z}_q$. Theorems 1.1, 2.1 and 3.1 and the argument above imply Corollary 4.2, an amended form of [6, Theorem 5.4].

For a prime power v, and distinct elements u,w,y,z of the projective line $\mathrm{GF}(v)\cup\{\infty\}$, the *cross-ratio* (see e.g. [13, pp. 59]) is defined as $c(u,w;y,z):=(u-y)(w-z)/(u-z)(w-y)$. For each $x\in\mathrm{GF}(v)\setminus\{0\}$, the *cross-ratio graph* $\mathrm{CR}(v,x)$ was defined in [6,9] to be the graph with vertices the ordered pairs of distinct elements of $\mathrm{GF}(v)\cup\{\infty\}$ in which uw and yz are adjacent if and only if $c(u,w;y,z)=x$.

Corollary 4.2. *Suppose that Γ is a finite G-symmetric, G-locally primitive graph admitting a nontrivial G-invariant partition $\mathcal{B}$ of block size $v=k+1\geq 3$ such that $\Gamma_{\mathcal{B}}$ is connected. Then $\Gamma_{\mathcal{B}}$ is a $(G,2)$-arc transitive graph of valency v, the actions of G_B on B and $\Gamma_{\mathcal{B}}(B)$ are permutationally equivalent and 2-primitive, and the following* (a)–(b) *hold.*

(a) *If $\Gamma_{\mathcal{B}}\cong K_{v+1}$, then either* (i) *$\Gamma\cong(v+1)\cdot K_v$ and G is one of the following: S_{v+1} $(v\geq 3)$, A_{v+1} $(v\geq 4)$, M_{v+1} $(v=10,11,22,23)$, M_{11} $(v=11)$,* $\mathrm{PGL}(2,2^p)$ *$(v=2^p$ with 2^p-1 a Mersenne prime$)$, or* (ii) *$\Gamma\cong\mathrm{CR}(3,-1)=3{\cdot}C_4$ and $G=\mathrm{PGL}(2,3)$ $(v=3)$, or* (iii) *$\Gamma\cong\mathrm{CR}(2^p,x)$ and $G=\mathrm{PGL}(2,2^p)$ $(v=2^p)$ for some $x\in\mathrm{GF}(2^p)\setminus\{0,1\}$ with 2^p-1 a Mersenne prime.*

(b) *If $\Gamma_{\mathcal{B}}\not\cong K_{v+1}$, then for some even integer $n\geq 4$, $\Gamma_{\mathcal{B}}$ is a near n-gonal graph with respect to a certain G-orbit $\mathcal{E}$ on n-cycles of $\Gamma_{\mathcal{B}}$ and $\Gamma\cong\mathrm{Arc}_{\Delta}(\Gamma_{\mathcal{B}})$ for $\Delta:=A_3(\Gamma_{\mathcal{B}},\mathcal{E})$. Moreover, each basic cycle of $(\Gamma_{\mathcal{B}},\mathcal{E})$ is chordless unless G_B^B is sharply 2-primitive and either*

(i) *$v=3$, $\Gamma_{\mathcal{B}}\cong K_{3,3}$, $\Gamma\cong 3\cdot C_6$, and G and $\mathcal{E}$ are as in Example 3.3, or*

(ii) *$G_B^B=(\mathbb{Z}_2)^p.\mathbb{Z}_q$ and $v=2^p$ with p a prime and $q=2^p-1$ a Mersenne prime.*

The smallest v in part (b)(ii) above is $v=2^2=4$. In this case we have $G_B^B=(\mathbb{Z}_2)^2.\mathbb{Z}_3$ and a similar argument as in the proof of Proposition 3.4 shows that, if the basic cycles of $(\Gamma_{\mathcal{B}},\mathcal{E})$ have chords, then the subgraph induced by the vertex set of each basic cycle is isomorphic to the circulant $\mathrm{Cay}(\mathbb{Z}_n,S)$ for $S=\{1,n/2,n-1\}$.

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