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Distance-two labellings of Hamming graphs

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ABSTRACT

Let $j \geq k \geq 0$ be integers. An ℓ-$L(j, k)$-labelling of a graph $G = (V, E)$ is a mapping $\phi : V \to \{0, 1, 2, \ldots, \ell\}$ such that $|\phi(u) - \phi(v)| \geq j$ if u, v are adjacent and $|\phi(u) - \phi(v)| \geq k$ if they are distance two apart. Let $\lambda_{j,k}(G)$ be the smallest integer ℓ such that G admits an ℓ-$L(j, k)$-labelling. Define $\overline{\lambda}_{j,k}(G)$ to be the smallest ℓ if G admits an ℓ-$L(j, k)$-labelling with $\phi(V) = \{0, 1, 2, \ldots, \ell\}$ and ∞ otherwise. An ℓ-cyclic $L(j, k)$-labelling is a mapping $\phi : V \to \mathbb{Z}_\ell$ such that $|\phi(u) - \phi(v)|_\ell \geq j$ if u, v are adjacent and $|\phi(u) - \phi(v)|_\ell \geq k$ if they are distance two apart, where $|x|_\ell = \min\{x, \ell - x\}$ for x between 0 and ℓ. Let $\sigma_{j,k}(G)$ be the smallest $\ell - 1$ of such a labelling, and define $\overline{\sigma}_{j,k}(G)$ similarly to $\overline{\lambda}_{j,k}(G)$. We determine $\lambda_{2,0}$, $\overline{\lambda}_{2,0}$, $\sigma_{2,0}$ and $\overline{\sigma}_{2,0}$ for all Hamming graphs $K_{q_1} \Box K_{q_2} \Box \cdots \Box K_{q_d}$ ($d \geq 2$, $q_1 \geq q_2 \geq \cdots \geq q_d \geq 2$) and give optimal labellings, with the only exception being $2q \leq \overline{\sigma}_{2,0}(K_q \Box K_q) \leq 2q + 1$ for $q \geq 4$. We also prove the following "sandwich theorem": If q_1 is sufficiently large then $\lambda_{2,1}(G) = \overline{\lambda}_{2,1}(G) = \overline{\sigma}_{2,1}(G) = \sigma_{2,1}(G) = \lambda_{1,1}(G) = \overline{\lambda}_{1,1}(G) = \overline{\sigma}_{1,1}(G) = \sigma_{1,1}(G) = q_1 q_2 - 1$ for any graph G between $K_{q_1} \Box K_{q_2}$ and $K_{q_1} \Box K_{q_2} \Box \cdots \Box K_{q_d}$, and moreover we give a labelling which is optimal for these eight invariants simultaneously.

1. Introduction

We investigate four versions of the well-known $L(j, k)$-labelling problem (Table 1) which originated from channel assignment in communication networks. The reader is referred to [1] for a survey and [11,12,15,22,23] for background information on this problem. In the present paper we concentrate on Hamming graphs, namely Cartesian products of complete graphs, and the case where $(j, k) = (2, 0)$, $(2, 1)$ or $(1, 1)$. In recent years considerable efforts have been made toward the $L(j, k)$-labelling problem for Hamming graphs; see [7,9,23] for related results and [24] for a short survey of related results. Due to close connection between Hamming graphs and coding theory, the results obtained in this paper can be easily interpreted in coding-theoretic language.

Let $G = (V, E)$ be a graph and $j \geq k \geq 0$ integers. A mapping $\phi : V \to \{0, 1, 2, \ldots\}$ is an *$L(j, k)$-labelling* [8,11] of G if, for $u, v \in V$,

$$|\phi(u) - \phi(v)| \geq \begin{cases} j, & d_G(u, v) = 1; \\ k, & d_G(u, v) = 2, \end{cases}$$

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Table 1
Four versions of the $L(j,k)$-labelling problem.

	$L(j,k)$-labelling	No-hole $L(j,k)$-labelling
Euclidean metric	$\lambda_{j,k}$ $\lambda := \lambda_{2,1}$	$\overline{\lambda}_{j,k}$ $\overline{\lambda} := \overline{\lambda}_{2,1}$
Cyclic metric	$\sigma_{j,k}$ $\sigma := \sigma_{2,1}$	$\overline{\sigma}_{j,k}$ $\overline{\sigma} := \overline{\sigma}_{2,1}$

where $d_G(u,v)$ is the distance in G between u and v. We will always assume w.l.o.g that $\min_{v\in V}\phi(v)=0$. Call $\phi(u)$ the *label* of u under ϕ, and $\mathrm{sp}(G;\phi)=\max_{v\in V}\phi(v)$ the *span* of ϕ. The $\lambda_{j,k}$*-number* [8,11] of G, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labellings of G. An $L(j,k)$-labelling ϕ is *no-hole* if $\{\phi(v): v\in V\}$ is a set of consecutive integers. Define $\overline{\lambda}_{j,k}(G)$ to be the minimum span over all no-hole $L(j,k)$-labellings of G if such a labelling exists and ∞ otherwise. In the literature $\lambda(G) := \lambda_{2,1}(G)$ is widely known as the λ-*number* [11] and an $L(2,0)$-labelling is called a *2-distant colouring*. Denote $\overline{\lambda}(G) := \overline{\lambda}_{2,1}(G)$.

The cyclic version of the $L(j,k)$-labelling problem was first studied in [13,22] for $(j,k)=(d,0),(2,1)$ respectively. An *ℓ-cyclic $L(j,k)$-labelling* of G is a mapping $\phi: V\to\mathbb{Z}_\ell$ such that

$$|\phi(u)-\phi(v)|_\ell \geq \begin{cases} j, & d_G(u,v)=1;\\ k, & d_G(u,v)=2\end{cases}$$

for $u,v\in V$, where $|x-y|_\ell := \min\{|x-y|, \ell-|x-y|\}$ is the *ℓ-cyclic distance*. We may assume w.l.o.g that $\min_{v\in V}\phi(v)=0$. An ℓ-cyclic $L(j,k)$-labelling of G exists for sufficiently large ℓ. Define $\sigma_{j,k}(G)$ to be the minimum integer $\ell-1$ such that G admits an ℓ-cyclic $L(j,k)$-labelling. A cyclic $L(j,k)$-labelling ϕ is *no-hole* if $\{\phi(v): v\in V\}$ is a set of consecutive integers. Let $\overline{\sigma}_{j,k}(G)$ be the minimum $\ell-1$ such that G admits a no-hole ℓ-cyclic $L(j,k)$-labelling, and ∞ if no such a labelling exists. Denote $\sigma(G) := \sigma_{2,1}(G)$ and $\overline{\sigma}(G) := \overline{\sigma}_{2,1}(G)$. *Note that $\sigma(G)$ thus defined is one smaller than the σ-number defined in* [13]. (It seems more convenient to define $\sigma_{j,k}(G)$ as above but not the minimum ℓ such that G admits an ℓ-cyclic $L(j,k)$-labelling.)

In general, it is hard to determine $\lambda_{j,k}$, $\overline{\lambda}_{j,k}$, $\sigma_{j,k}$ and/or $\overline{\sigma}_{j,k}$ even for small values of j and k. The reader may consult [2–7, 14,16–18,20,21], respectively, for known results on λ and $\overline{\lambda}_{2,0}$. In this paper we focus on *Hamming graphs* $H_{q_1,q_2,\ldots,q_d} := K_{q_1}\square K_{q_2}\square\cdots\square K_{q_d}$ (where $d\geq 2$ and we always assume $q_1\geq q_2\geq\cdots\geq q_d\geq 2$) and the case where $(j,k)=(2,0),(2,1)$ or $(1,1)$. The vertex set of $H_{q_1,q_2,\ldots,q_d}$ is $\mathbb{Z}_{q_1}\times\mathbb{Z}_{q_2}\times\cdots\times\mathbb{Z}_{q_d}$ and two vertices $(i_1,i_2,\ldots,i_d)$ and $(j_1,j_2,\ldots,j_d)$ are adjacent in $H_{q_1,q_2,\ldots,q_d}$ if and only if they differ at exactly one coordinate. In the case $q_1=q_2=\cdots=q_d=q$, we write $H(d,q)$ in place of $H_{q_1,q_2,\ldots,q_d}$. In particular, $H(d,2)$ is the *hypercube* Q_d of dimension d, and $H(2,2)=Q_2\cong C_4$ is the cycle of length 4.

Our first main result is the following theorem. (We include the trivial result $\lambda_{2,0}(H_{q_1,q_2,\ldots,q_d})=2(\chi(H_{q_1,q_2,\ldots,q_d})-1)=2q_1-2$ for completeness of the theorem.)

Theorem 1.1. *Let $d\geq 2$ and $q_1\geq q_2\geq\cdots\geq q_d\geq 2$ be integers. Then*
(a) $\lambda_{2,0}(H_{q_1,q_2,\ldots,q_d})=2q_1-2$ *and* $\sigma_{2,0}(H_{q_1,q_2,\ldots,q_d})=2q_1-1$.
(b) $H_{q_1,q_2,\ldots,q_d}$ *admits a no-hole $L(2,0)$-labelling* $\Leftrightarrow$ $H_{q_1,q_2,\ldots,q_d}$ *admits a no-hole cyclic $L(2,0)$-labelling* $\Leftrightarrow$ $H_{q_1,q_2,\ldots,q_d}\neq Q_2$, *and in this case the following* (i)–(iii) *hold:*
 (i) *if $q_1,q_2,\ldots,q_d$ are not all the same, then*
$$\overline{\lambda}_{2,0}(H_{q_1,q_2,\ldots,q_d})=\overline{\sigma}_{2,0}(H_{q_1,q_2,\ldots,q_d})=2q_1-1;$$
 (ii) *if $d\geq 3$ and $q\geq 2$, then*
$$\overline{\lambda}_{2,0}(H(d,q))=2q-1,\qquad \overline{\sigma}_{2,0}(H(d,q))=2q;$$
 (iii) *if $d=2$ and $q\geq 3$, then*
$$\overline{\lambda}_{2,0}(H(2,q))=2q,\qquad \overline{\sigma}_{2,0}(H(2,3))=8,\qquad 2q\leq\overline{\sigma}_{2,0}(H(2,q))\leq 2q+1\quad (q\geq 4).$$
Moreover, we construct explicitly an optimal labelling in each case except $\overline{\sigma}_{2,0}(H(2,q))$ with $q\geq 4$; for this exceptional case we give a no-hole $(2q+2)$-cyclic $L(2,0)$-labelling of $H(2,q)$.

We conjecture that $\overline{\sigma}_{2,0}(H(2,q))$ is always equal to $2q+1$ for any integer $q\geq 4$. We have proved this for $q=4,5,6$, but the proof requires significant deviation and hence is not included in this paper. Theorem 1.1(a) together with the monotonicity of $\sigma_{2,0}$ (Lemma 2.8) implies the following corollary (similar result for $\lambda_{2,0}$ is obvious).

Corollary 1.2. *We have $\sigma_{2,0}(G)=2q_1-1$ for any subgraph G of $H_{q_1,q_2,\ldots,q_d}$ with clique number $\omega(G)\geq q_1$. Moreover, the restriction to G of any optimal cyclic $L(2,0)$-labelling of $H_{q_1,q_2,\ldots,q_d}$ is an optimal cyclic $L(2,0)$-labelling of G.*

The problem of determining the λ-number of an arbitrary Hamming graph seems to be a difficult task [9,23]. In [23, Question 6.1(b)] it was asked whether $\lambda(H_{q_1,q_2,\ldots,q_d})=q_1q_2-1$ for any $q_1\geq q_2\geq\cdots\geq q_d\ (\geq 2)$ not all equal to 2. Theorem 1.3 gives a partial solution to this problem. Let $n=n(q_2,q_3,\ldots,q_d)$ be the largest integer such that $q_2=q_n$, and define

$$N(q_2,q_3,\ldots,q_d) := d+n-1+\sum_{2\leq k\leq d}(k-2)(q_k-1). \tag{1}$$

The *square* G^2 of a graph G is defined to have the same vertex set as G such that two vertices are adjacent if and only if their distance in G is at most two.

Theorem 1.3. *Let $d \geq 2$ and $q_1 \geq q_2 \geq \cdots \geq q_d \geq 2$ be integers. Then, $H_{q_1,q_2,\ldots,q_d}$ admits a no-hole $L(2,1)$-labelling $\Leftrightarrow$ $H_{q_1,q_2,\ldots,q_d}$ admits a no-hole cyclic $L(2,1)$-labelling $\Leftrightarrow$ $H_{q_1,q_2,\ldots,q_d} \neq Q_2$. Moreover, if $q_1 \geq N(q_2, q_3, \ldots, q_d)$, then*

$$\lambda(H_{q_1,q_2,\ldots,q_d}) = \overline{\lambda}(H_{q_1,q_2,\ldots,q_d}) = \overline{\sigma}(H_{q_1,q_2,\ldots,q_d}) = \sigma(H_{q_1,q_2,\ldots,q_d}) = q_1q_2 - 1,$$
$$\lambda_{1,1}(H_{q_1,q_2,\ldots,q_d}) = \overline{\lambda}_{1,1}(H_{q_1,q_2,\ldots,q_d}) = \overline{\sigma}_{1,1}(H_{q_1,q_2,\ldots,q_d}) = \sigma_{1,1}(H_{q_1,q_2,\ldots,q_d}) = q_1q_2 - 1,$$

and we give a labelling of $H_{q_1,q_2,\ldots,q_d}$ which is optimal for all these invariants simultaneously. Furthermore, in this case we have $\chi(H^2_{q_1,q_2,\ldots,q_d}) = q_1q_2$ and the same labelling gives rise to a minimum (proper) vertex-colouring of $H^2_{q_1,q_2,\ldots,q_d}$ as well.

In the two-dimensional case $H_{q_1,q_2} \neq Q_2$, we have $d = 2$, $n = 2$ and $q_1 \geq 3 = N(q_1, q_2)$. Thus, we obtain the following corollary of Theorem 1.3 which is partly known in the literature. (In [13, Theorem 3.2] it was proved that $\lambda(H_{q_1,q_2}) = \sigma(H_{q_1,q_2}) = q_1q_2 - 1$ for $H_{q_1,q_2} \neq Q_2$.)

Corollary 1.4. *Let $q_1 \geq q_2 \geq 2$ be integers such that $(q_1, q_2) \neq (2, 2)$. Then*

$$\lambda(H_{q_1,q_2}) = \overline{\lambda}(H_{q_1,q_2}) = \overline{\sigma}(H_{q_1,q_2}) = \sigma(H_{q_1,q_2}) = q_1q_2 - 1,$$
$$\lambda_{1,1}(H_{q_1,q_2}) = \overline{\lambda}_{1,1}(H_{q_1,q_2}) = \overline{\sigma}_{1,1}(H_{q_1,q_2}) = \sigma_{1,1}(H_{q_1,q_2}) = q_1q_2 - 1,$$

and we give a labelling of H_{q_1,q_2} which is optimal for the eight invariants simultaneously.

Theorem 1.3 implies, and is equivalent to, the following "sandwich theorem".

Corollary 1.5. *Let $d \geq 2$ and $q_1 \geq q_2 \geq \cdots \geq q_d \geq 2$ be such that $q_1 \geq N(q_2, q_3, \ldots, q_d)$. Then, for any subgraph G of $H_{q_1,q_2,\ldots,q_d}$ which contains H_{q_1,q_2} as a subgraph, we have*

$$\lambda(G) = \overline{\lambda}(G) = \overline{\sigma}(G) = \sigma(G) = q_1q_2 - 1,$$
$$\lambda_{1,1}(G) = \overline{\lambda}_{1,1}(G) = \overline{\sigma}_{1,1}(G) = \sigma_{1,1}(G) = q_1q_2 - 1.$$

Moreover, the restriction to G of the optimal labelling of $H_{q_1,q_2,\ldots,q_d}$ guaranteed in Theorem 1.3 is optimal for these eight invariants simultaneously. Furthermore, $\chi(G^2) = q_1q_2$ and the same labelling is a minimum (proper) vertex-colouring of G^2 as well.

All results above can be translated into coding-theoretic language due to close connections between Hamming graphs and coding theory.

The rest of this paper is organized as follows. In the next section we list preliminary results that will be used in subsequent discussions. In Section 3 we prove Theorem 1.1 and Corollary 1.2 and construct the corresponding optimal labellings. In Section 4 we prove Theorem 1.3 and Corollary 1.5. The paper concludes with remarks and an open problem related to these results.

2. Preliminaries

Let G^c denote the complement of G. The equivalence of the second and the third statements in the following lemma is known in [18], and that of the third and the fourth statements is given in [10]. Other equivalences can be easily established and hence we omit their proofs.

Lemma 2.1. *Let G be a graph with n vertices. Then, G admits a no-hole $L(2,1)$-labelling $\Leftrightarrow$ G admits a no-hole $L(2,0)$-labelling $\Leftrightarrow$ G^c contains a Hamiltonian path $\Leftrightarrow$ $\lambda(G) \leq n - 1$.*

Similarly, one can prove the following lemma (the equivalence of the last two statements was proved in [13, Theorem 2.2]).

Lemma 2.2. *Let G be a graph with n vertices. Then, G admits a no-hole cyclic $L(2,1)$-labelling $\Leftrightarrow$ G admits a no-hole cyclic $L(2,0)$-labelling $\Leftrightarrow$ G^c is Hamiltonian $\Leftrightarrow$ $\sigma(G) \leq n - 1$.*

By Lemma 2.1, if G^c contains a Hamiltonian path, then $\overline{\lambda}(G), \overline{\lambda}_{2,0}(G)$ are finite and moreover $\lambda(G) \leq \overline{\lambda}(G), \overline{\lambda}_{2,0}(G) \leq \overline{\lambda}(G)$. Similarly, by Lemma 2.2 if G^c is Hamiltonian then $\overline{\sigma}(G), \overline{\sigma}_{2,0}(G)$ are finite and $\sigma(G) \leq \overline{\sigma}(G), \overline{\sigma}_{2,0}(G) \leq \overline{\sigma}(G)$. The following inequalities can be easily established.

Lemma 2.3. *The following (2) and (3) hold for any graph G, and (4) and (5) hold for any graph G such that G^c is Hamiltonian.*

$$\lambda(G) \leq \sigma(G) \leq \lambda(G) + 1, \qquad [13,22] \tag{2}$$
$$\lambda_{2,0}(G) \leq \sigma_{2,0}(G) \leq \lambda_{2,0}(G) + 1, \tag{3}$$
$$\overline{\lambda}(G) \leq \overline{\sigma}(G), \tag{4}$$
$$\overline{\lambda}_{2,0}(G) \leq \overline{\sigma}_{2,0}(G). \tag{5}$$

Lemma 2.4. *Let G be a graph with n vertices. Then the following inequalities hold, where we assume that G^c contains a Hamiltonian path in (6) and G^c is Hamiltonian in (7):*

$$\max\{\lambda(G), \overline{\lambda}_{2,0}(G)\} \leq \overline{\lambda}(G) \leq n - 1, \tag{6}$$

$$\max\{\sigma(G), \overline{\sigma}_{2,0}(G)\} \leq \overline{\sigma}(G) \leq n - 1. \tag{7}$$

Hence we have the following results immediately (that $\lambda(G) = \sigma(G) = n - 1$ was proved in [13, Theorem 3.1]).

Lemma 2.5. *Let G be a graph with order n and diameter 2.*

(a) *If G^c contains a Hamiltonian path, then $\lambda(G) = \overline{\lambda}(G) = n - 1$;*
(b) *if G^c is Hamiltonian, then $\sigma(G) = \overline{\sigma}(G) = n - 1$.*

The following result will be used in the proof of Lemma 2.7. Since we have been unable to locate it in the literature, we include its proof for completeness of this paper.

Lemma 2.6. *Let $d \geq 2$ and $q_1 \geq q_2 \geq \cdots \geq q_d \geq 2$ be integers. Then, $H^c_{q_1,q_2,\ldots,q_d}$ is Hamiltonian $\Leftrightarrow$ $H^c_{q_1,q_2,\ldots,q_d}$ contains a Hamiltonian path $\Leftrightarrow$ $H_{q_1,q_2,\ldots,q_d} \neq Q_2$.*

Proof. First, $H^c_{q_1,q_2,\ldots,q_d}$ is Hamiltonian $\Rightarrow$ $H^c_{q_1,q_2,\ldots,q_d}$ contains a Hamiltonian path $\Rightarrow$ $H_{q_1,q_2,\ldots,q_d} \neq Q_2$. It suffices to show that $H^c_{q_1,q_2,\ldots,q_d}$ is Hamiltonian if $H_{q_1,q_2,\ldots,q_d} \neq Q_2$. Note that $H^c_{q_1,q_2,\ldots,q_d}$ has degree $\prod_{t=1}^{d} q_t - 1 - \sum_{t=1}^{d}(q_t - 1)$. One can verify that, unless $d = 2$, $q_1 \geq 3$ and $q_2 = 2$, or $d = 2$ and $(q_1, q_2) = (3, 3)$, we have

$$\prod_{t=1}^{d} q_t - 1 - \sum_{t=1}^{d}(q_t - 1) \geq \frac{1}{2}\prod_{t=1}^{d} q_t$$

and so $H^c_{q_1,q_2,\ldots,q_d}$ is Hamiltonian by Dirac's condition for Hamiltonicity. In the two exceptional cases it is straightforward to check that $H^c_{q_1,q_2}$ contains a Hamiltonian cycle. □

Lemmas 2.1, 2.2 and 2.6 together imply the following result.

Lemma 2.7. *Let $d \geq 2$ and $q_1 \geq q_2 \geq \cdots \geq q_d \geq 2$. Then, $H_{q_1,q_2,\ldots,q_d}$ admits a no-hole $L(2,1)$-labelling $\Leftrightarrow$ $H_{q_1,q_2,\ldots,q_d}$ admits a no-hole cyclic $L(2,1)$-labelling $\Leftrightarrow$ $H_{q_1,q_2,\ldots,q_d}$ admits a no-hole $L(2,0)$-labelling $\Leftrightarrow$ $H_{q_1,q_2,\ldots,q_d}$ admits a no-hole cyclic $L(2,0)$-labelling $\Leftrightarrow$ $H_{q_1,q_2,\ldots,q_d} \neq Q_2$.*

Thus, the statements in Theorems 1.1 and 1.3 about the existence of the four types of labellings have been established.

A graphical invariant η is *monotonically increasing* (see e.g. [25]) if $\eta(G) \leq \eta(H)$ whenever G is a subgraph of H. The following observation is obvious.

Lemma 2.8. *$\lambda_{j,k}$ and $\sigma_{j,k}$ are both monotonically increasing.*

3. Proof of Theorem 1.1

The proof of Theorem 1.1 consists of a series of lemmas. For any fixed vertex $(i_1, i_2, \ldots, i_d)$ of $H_{q_1,q_2,\ldots,q_d}$, the set $\{(j, i_2, \ldots, i_d) : j \in \mathbb{Z}_{q_1}\}$ induces a subgraph of $H_{q_1,q_2,\ldots,q_d}$ isomorphic to K_{q_1}, which we call the *K_{q_1}-copy* of $H_{q_1,q_2,\ldots,q_d}$ containing $(i_1, i_2, \ldots, i_d)$.

Lemma 3.1. *Let $d \geq 2$ and $q_1 \geq q_2 \geq \cdots \geq q_d \geq 2$ be integers. Then*

$$\lambda_{2,0}(H_{q_1,q_2,\ldots,q_d}) = 2q_1 - 2 \quad \textit{and} \quad \sigma_{2,0}(H_{q_1,q_2,\ldots,q_d}) = 2q_1 - 1.$$

Proof. We have $\lambda_{2,0}(H_{q_1,q_2,\ldots,q_d}) = 2(\chi(H_{q_1,q_2,\ldots,q_d}) - 1) = 2q_1 - 2$. Under any cyclic $L(2,0)$-labelling of $H_{q_1,q_2,\ldots,q_d}$ the labels of any two vertices in the same K_{q_1}-copy must differ by at least 2 with respect to the cyclic metric. Thus, $\sigma_{2,0}(H_{q_1,q_2,\ldots,q_d}) \geq 2q_1 - 1$. The labelling defined by

$$\phi(i_1, i_2, \ldots, i_d) = (2i_1 + 2i_2 + \cdots + 2i_d) \mod 2q_1 \tag{8}$$

is a $2q_1$-cyclic $L(2,0)$-labelling of $H_{q_1,q_2,\ldots,q_d}$. Therefore, $\sigma_{2,0}(H_{q_1,q_2,\ldots,q_d}) = 2q_1 - 1$ and ϕ is an optimal cyclic $L(2,0)$-labelling of $H_{q_1,q_2,\ldots,q_d}$. □

Lemma 3.1 proves part (a) of Theorem 1.1. As we will see in the following the labelling ϕ defined in (8) induces an optimal cyclic $L(2,0)$-labelling for any subgraph G of $H_{q_1,q_2,\ldots,q_d}$ containing K_{q_1}.

Proof of Corollary 1.2. Suppose G is a subgraph of $H_{q_1,q_2,\ldots,q_d}$ containing a copy of K_{q_1}. Since $\sigma_{2,0}$ is monotonically increasing by Lemma 2.8, using Lemma 3.1 we have $2q_1 - 1 = \sigma_{2,0}(K_{q_1}) \le \sigma_{2,0}(G) \le \sigma_{2,0}(H_{q_1,q_2,\ldots,q_d}) = 2q_1 - 1$. Hence $\sigma_{2,0}(G) = 2q_1 - 1$ and the restriction to G of any optimal cyclic $L(2, 0)$-labelling of $H_{q_1,q_2,\ldots,q_d}$ is an optimal cyclic $L(2, 0)$-labelling of G. □

Lemma 3.2. *Let $d \ge 2$ and $q_1 \ge q_2 \ge \cdots \ge q_d \ge 2$ be integers such that $H_{q_1,q_2,\ldots,q_d} \ne Q_2$. Then*

$$2q_1 - 1 \le \overline{\lambda}_{2,0}(H_{q_1,q_2,\ldots,q_d}) \le \overline{\sigma}_{2,0}(H_{q_1,q_2,\ldots,q_d}). \tag{9}$$

Proof. The second inequality follows from (5). For any no-hole $L(2, 0)$-labelling of $H_{q_1,q_2,\ldots,q_d}$, choose a vertex u of label 1 and a K_{q_1}-copy containing u. Then the labels of any two vertices in this K_{q_1}-copy must differ by at least 2. Thus, the maximum label used is at least $2q_1 - 1$ and so $\overline{\lambda}_{2,0}(H_{q_1,q_2,\ldots,q_d}) \ge 2q_1 - 1$. □

That $\overline{\sigma}_{2,0}(H_{q_1,q_2,\ldots,q_d}) \ge 2q_1 - 1$ (which is implied by (9) can be also obtained from Lemma 3.1 and the fact that $\sigma_{2,0}(H_{q_1,q_2,\ldots,q_d}) \le \overline{\sigma}_{2,0}(H_{q_1,q_2,\ldots,q_d})$.

Lemma 3.3. *Let $d \ge 2$ and $q_1 \ge q_2 \ge \cdots \ge q_d \ge 2$ be integers such that $q_1, q_2, \ldots, q_d$ are not all the same. Then*

$$\overline{\sigma}_{2,0}(H_{q_1,q_2,\ldots,q_d}) \le 2q_1 - 1.$$

Proof. Since $q_1, q_2, \ldots, q_d$ are not all the same, we have $q_1 > q_d$. Define

$$\phi(i_1, i_2, \ldots, i_d) = \begin{cases} (2i_1 + 2i_2 + \cdots + 2i_d) \mod 2q_1, & i_d \ne q_d - 1; \\ (2i_1 + 2i_2 + \cdots + 2i_d + 1) \mod 2q_1, & i_d = q_d - 1 \end{cases} \tag{10}$$

for $0 \le i_t \le q_t - 1$ and $1 \le t \le d$. Let u and v be two adjacent vertices of $H_{q_1,q_2,\ldots,q_d}$, and suppose that they differ at the kth position only. Let $i_k \ne j_k$ be the k th coordinates of u and v, respectively. If $k < d$ or $k = d$ but neither i_d nor j_d is equal to $q_d - 1$, then $|\phi(u) - \phi(v)| = 2|i_k - j_k| \mod 2q_1$ and hence $2 \le |\phi(u) - \phi(v)| \le 2q_1 - 2$. If $k = d$ and exactly one of i_d and j_d is equal to $q_d - 1$, say $i_d = q_d - 1$ and $j_d \ne q_d - 1$ (hence $0 \le j_d \le q_d - 2$), then $|\phi(u) - \phi(v)| = |2(q_d - 1) + 1 - 2j_d| \mod 2q_1 = 2(q_d - j_d) - 1$. Noting that $0 \le j_d \le q_d - 2$ and $q_d < q_1$, in this case we have $3 \le |\phi(u) - \phi(v)| \le 2q_d - 1 \le 2(q_1 - 1) - 1 = 2q_1 - 3$. Thus, we have proved $|\phi(u) - \phi(v)|_{2q_1} \ge 2$ in all possibilities, and hence ϕ is a $2q_1$-cyclic $L(2, 0)$-labelling of $H_{q_1,q_2,\ldots,q_d}$. Note that $\phi(i_1, 0, \ldots, 0) = 2i_1$ takes values $0, 2, \ldots, 2q_1 - 2$ when i_1 runs from 0 to $q_1 - 1$. Also, $\phi(i_1, 0, \ldots, 0, q_d - 1) = (2i_1 + 2q_d - 1) \mod 2q_1$, which takes values $2q_d - 1, 2q_d + 1, \ldots, 2q_1 - 1, 1, 3, \ldots, 2q_d - 3$ when i_1 runs from 0 to $q_1 - 1$. Thus, ϕ is a no-hole $2q_1$-cyclic $L(2, 0)$-labelling of $H_{q_1,q_2,\ldots,q_d}$ with span $2q_1 - 1$. Therefore, $\overline{\sigma}_{2,0}(H_{q_1,q_2,\ldots,q_d}) \le 2q_1 - 1$ and the proof is complete. □

By Lemmas 3.2 and 3.3, if $q_1, q_2, \ldots, q_d$ are not all the same, then $\overline{\lambda}_{2,0}(H_{q_1,q_2,\ldots,q_d}) = \overline{\sigma}_{2,0}(H_{q_1,q_2,\ldots,q_d}) = 2q_1 - 1$, and this proves (b)(i) of Theorem 1.1. Moreover, ϕ given by (10) is an optimal no-hole $L(2, 0)$-labelling as well as an optimal no-hole cyclic $L(2, 0)$-labelling of $H_{q_1,q_2,\ldots,q_d}$.

In the case where all $q_1, q_2, \ldots, q_d$ are the same, ϕ defined in (10) is not an $L(2, 0)$-labelling of $H_{q_1,q_2,\ldots,q_d}$. (For instance, if $d = 2$ and $q_1 = q_2$, then $\phi(1, 0) = 2$ and $\phi(1, q_1 - 1) = 1$, violating the 2-distant condition.) In fact, this special case is relatively harder to handle than the general case, and this is the task of the remainder of this section.

Lemma 3.4. *Let $d \ge 3$ and $q \ge 2$ be integers. Then*

$$\overline{\lambda}_{2,0}(H(d, q)) = 2q - 1.$$

Proof. By Lemma 3.2 it suffices to show that $\overline{\lambda}_{2,0}(H(d, q)) \le 2q - 1$. This is achieved by constructing a no-hole $L(2, 0)$-labelling ϕ of $H(d, q)$ with span $2q - 1$ as follows. For any vertex $(i_1, i_2, \ldots, i_d)$ in $H(d, q)$, define

$$\psi(i_1, i_2, \ldots, i_d) = (2i_1 + 1) + ((2i_2 + 2i_3 + \cdots + 2i_d) \mod (2q + 2)), \tag{11}$$

$$\phi(i_1, i_2, \ldots, i_d) = \psi(i_1, i_2, \ldots, i_d) \mod (2q + 1). \tag{12}$$

For any two adjacent vertices $(i_1, i_2, \ldots, i_d)$ and $(j_1, j_2, \ldots, j_d)$, there is exactly one subscript t, $1 \le t \le d$, with $i_t \ne j_t$. By the definition of ψ, the difference (in absolute value) of $\psi(i_1, i_2, \ldots, i_d)$ and $\psi(j_1, j_2, \ldots, j_d)$ is between 2 and $2q - 2$. Thus, $|\phi(i_1, i_2, \ldots, i_d) - \phi(j_1, j_2, \ldots, j_d)| \ge 2$ and so ϕ is an $L(2, 0)$-labelling of $H(d, q)$.

Next we argue that ϕ uses all labels from 0 to $2q - 1$. In fact, while $(i_1, i_2, \ldots, i_d)$ runs over all vertices in $H(d, q)$, $2i_1 + 1$ runs over all odd integers from 1 to $2q - 1$ and, since $d \ge 3$, $(2i_2 + 2i_3 + \cdots + 2i_d) \mod (2q + 2)$ runs over all even integers from 0 to $2q$. Hence $\psi(i_1, i_2, \ldots, i_d)$ runs over all odd integers from 1 to $4q - 1$. After taking modulo $2q + 1$, $\phi(i_1, i_2, \ldots, i_d)$ runs over all integers from 1 to $2q - 1$. Note that $2q + 1, 2q + 3, \ldots, 4q - 1$ respectively become $0, 2, \ldots, 2q - 2$ after taken modulo $2q + 1$. □

Lemma 3.5. *Let $d \geq 3$ and $q \geq 2$ be integers. Then*

$$\overline{\sigma}_{2,0}(H(d,q)) = 2q.$$

Proof. We first show that $2q$ is a lower bound for $\overline{\sigma}_{2,0}(H(d,q))$. Suppose otherwise. Then $\overline{\sigma}_{2,0}(H(d,q)) = 2q-1$ by (9). Let ϕ be a no-hole $2q$-cyclic $L(2,0)$-labelling of $H(d,q)$. Since ϕ is an $L(2,0)$-labelling, under ϕ the vertices of any K_q-copy must receive labels with pairwise cyclic difference (in absolute value) at least 2. Hence each K_q-copy of $H(d,q)$ uses either $\{0,2,\ldots,2q-2\}$ or $\{1,3,\ldots,2q-1\}$ as the label set. Since $H(d,q)$ is connected and every vertex of $H(d,q)$ is contained in d K_q-copies, it follows that either all vertices of $H(d,q)$ use even labels $0,2,\ldots,2q-2$, or all vertices of $H(d,q)$ use odd labels $1,3,\ldots,2q-1$. This contradicts the no-hole condition, and hence $\overline{\sigma}_{2,0}(H(d,q)) \geq 2q$.

Define

$$\phi(i_1,i_2,\ldots,i_d) = (2i_1+2i_2+\cdots+2i_d+1) \mod (2q+1) \tag{13}$$

for each $(i_1,i_2,\ldots,i_d)$. Then, for any two adjacent vertices u and v of $H(d,q)$, we have $2 \leq |\phi(u)-\phi(v)| \leq 2q-2$ and hence $|\phi(u)-\phi(v)|_{2q+1} \geq 2$. Since $d \geq 3$, $\sum_{t=1}^{d} i_t$ can take integers $0,1,2,\ldots,q-1,q,q+1,\ldots,2q-2,2q-1,2q,\ldots,$ and hence $\phi(i_1,i_2,\ldots,i_d)$ can take $1,3,5,\ldots,2q-1,0,2,\ldots,2q-4,2q-2,2q,\ldots$ correspondingly. Thus, ϕ is a no-hole $(2q+1)$-cyclic $L(2,0)$-labelling of $H(d,q)$ and the proof is complete. □

Part (b)(ii) of Theorem 1.1 follows from Lemmas 3.4 and 3.5 immediately. Moreover, as shown in the proofs above, (11) and (12) define an optimal no-hole $L(2,0)$-labelling and (13) an optimal no-hole cyclic $L(2,0)$-labelling of $H(d,q)$ when $d \geq 3$ and $q \geq 2$.

Lemma 3.6. *Let $q \geq 3$ be an integer. Then*

$$\overline{\lambda}_{2,0}(H(2,q)) = 2q.$$

Proof. Recall that $H(2,q)$ has vertex set $\mathbb{Z}_q \times \mathbb{Z}_q$. We think of $H(2,q)$ as a drawing on the plane in the usual way, so we can talk about its rows and columns: the $(i+1)$th row consists of those vertices with the first coordinate i, and the $(j+1)$th column consists of vertices with the second coordinate j, for $0 \leq i,j \leq q-1$. The vertices in the same row/column induce a complete subgraph K_q of $H(2,q)$, and hence they must receive labels with mutual difference at least 2 under any $L(2,0)$-labelling.

Let us prove first that $\overline{\lambda}_{2,0}(H(2,q)) \geq 2q$. Suppose otherwise. Then $\overline{\lambda}_{2,0}(H(2,q)) = 2q-1$ by Lemma 3.2, and $H(2,q)$ has a no-hole $L(2,0)$-labelling ϕ with span $2q-1$. Since ϕ is no-hole, $2q-2$ must appear in some row of $H(2,q)$, say, row R, and hence both $2q-3$ and $2q-1$ do not appear in R. Since $\{0,2,\ldots,2q-2\}$ is the unique q-subset of $[0,2q-2]$ of which any two members differ by at least 2, the vertices in R must receive labels $0,2,4,\ldots,2q-2$. Also, 1 must appear in some column of $H(2,q)$, say, column C. This implies that both 0 and 2 do not appear in column C. Again, since $\{1,3,\ldots,2q-1\}$ is the unique q-subset of $[1,2q-1]$ of which any two members differ by at least 2, the labels used in column C are $1,3,5,\ldots,2q-1$. Since $d=2$, there is a unique common vertex of row R and column C. From the discussion above this vertex must be labelled by an odd integer, as well as an even integer. This is a contradiction and hence we have $\overline{\lambda}_{2,0}(H(2,q)) \geq 2q$.

It remains to prove that $2q$ is an upper bound for $\overline{\lambda}_{2,0}(H(2,q))$. Define

$$\phi(i,j) = \begin{cases} 0, & (i,j) = (0,q-1),(1,q-2); \\ 2, & (i,j) = (1,q-1); \\ (2i+2j+4) \mod (2q+1), & (i,j) \neq (0,q-1),(1,q-2),(1,q-1). \end{cases} \tag{14}$$

Under this labelling ϕ, the vertices in the first row are labelled $4,6,8,\ldots,2q-2,2q,0$, and hence the mutual differences of these labels are at least 2. Similarly, the labels of the vertices in the second row are $6,8,10,\ldots,2q,0,2$, which differ pairwise by at least 2. The vertices in the last and second last columns receive labels $0,2,5,\ldots,2q-5,2q-3,2q-1$ and $2q,0,3,\ldots,2q-7,2q-5,2q-3$, respectively, and hence they satisfy the 2-distant condition as well. For all other vertices (i,j), where $2 \leq i \leq q-1$ and $0 \leq j \leq q-3$, we have $\phi(i,j) = (2i+2j+4) \bmod (2q+1)$, and hence two such vertices in the same row or column receive labels with difference at least 2. Thus, ϕ is an $L(2,0)$-labelling of $H(2,q)$. Since $q \geq 3$, $\phi(q-1,j) = 2j+1$, which takes values $1,3,5,\ldots,2q-1$ when j runs from 0 to $q-1$. Also, $\phi(i,0) = 2i+4 = 4,6,\ldots,2q$ when i runs from 0 to $q-2$. In addition, $\phi(0,q-1) = 0$ and $\phi(1,q-1) = 2$ by definition. So ϕ is a no-hole $L(2,0)$-labelling with span $2q$, and the proof is complete. □

Lemma 3.6 contributes to part (b)(iii) of Theorem 1.1, and (14) gives an optimal no-hole $L(2,0)$-labelling of $H(2,q)$ for any $q \geq 3$.

Lemma 3.7. $\overline{\sigma}_{2,0}(H(2,3)) = 8$ *and* $2q \leq \overline{\sigma}_{2,0}(H(2,q)) \leq 2q+1$ *for* $q \geq 4$.

Proof. From (5) and Lemma 3.6 it follows that $\overline{\sigma}_{2,0}(H(2,q)) \geq 2q$. (This can be proved also by using the method in the first paragraph of the proof of Lemma 3.5.)

We first prove $\overline{\sigma}_{2,0}(H(2,3)) = 8$. Suppose otherwise. Then since $\overline{\sigma}_{2,0}(H(2,3)) \geq 6$, $H(2,3)$ admits a no-hole ℓ-cyclic $L(2,0)$-labelling ϕ, for $\ell = 7$ or 8. Since $H(2,3)$ has 9 vertices, there is at least one label $a \in \mathbb{Z}_\ell$ which is used twice by ϕ.

By adding $\ell - a$ to every label (mod ℓ), we may assume w.l.o.g that $a = 0$. The two vertices labelled 0 must be in different row and different column, and by permuting rows and columns when necessary we may assume $\phi(0, 0) = \phi(1, 1) = 0$. Then neither 1 nor $\ell - 1$ can appear in the first two rows or the first two columns. Thus, (2, 2) is the only position for both 1 and $\ell - 1$. This contradiction shows that $\overline{\sigma}_{2,0}(H(2, 3)) \geq 8$. On the other hand, one can easily find a no-hole 9-cyclic $L(2, 0)$-labelling for $H(2, 3)$. Hence $\overline{\sigma}_{2,0}(H(2, 3)) = 8$.

Let $q \geq 4$ and define ϕ in the same way as in (14) except $\phi(q - 2, 0) = 2q + 1$. Similar to the proof of Lemma 3.6, one can verify that ϕ is a no-hole $(2q + 2)$-cyclic $L(2, 0)$-labelling of $H(2, q)$. Hence $\overline{\sigma}_{2,0}(H(2, q)) \leq 2q + 1$ for $q \geq 4$. □

Part (b)(iii) of Theorem 1.1 follows from (5) and Lemmas 3.6 and 3.7, and this completes the proof of Theorem 1.1.

Note that the labellings (11)–(13) for $H(d, q)$ $(d \geq 3)$ do not work for $H(2, q)$, and the labelling (14) for $H(2, q)$ does not apply to $H(d, q)$ $(d \geq 3)$.

4. Proofs of Theorem 1.3 and Corollary 1.5

Since H_{q_1,q_2} is a subgraph of $H_{q_1,q_2,\ldots,q_d}$ with diameter 2, its vertices must receive distinct labels in any no-hole cyclic $L(2, 1)$-labelling. Hence $\overline{\sigma}(H_{q_1,q_2,\ldots,q_d}) \geq q_1q_2 - 1$. The following lemma is crucial for the proof of Theorem 1.3.

Lemma 4.1. *Let $d \geq 2$ and $q_1 \geq q_2 \geq \cdots \geq q_d \geq 2$ be integers such that $H_{q_1,q_2,\ldots,q_d} \neq Q_2$. If $\overline{\sigma}(H_{q_1,q_2,\ldots,q_d}) \leq q_1q_2 - 1$, then*

$$\lambda(H_{q_1,q_2,\ldots,q_d}) = \overline{\lambda}(H_{q_1,q_2,\ldots,q_d}) = \overline{\sigma}(H_{q_1,q_2,\ldots,q_d}) = \sigma(H_{q_1,q_2,\ldots,q_d}) = q_1q_2 - 1,$$
$$\lambda_{1,1}(H_{q_1,q_2,\ldots,q_d}) = \overline{\lambda}_{1,1}(H_{q_1,q_2,\ldots,q_d}) = \overline{\sigma}_{1,1}(H_{q_1,q_2,\ldots,q_d}) = \sigma_{1,1}(H_{q_1,q_2,\ldots,q_d}) = q_1q_2 - 1.$$

Moreover, any optimal no-hole cyclic $L(2, 1)$-labelling of $H_{q_1,q_2,\ldots,q_d}$ is optimal for $\lambda, \overline{\lambda}, \overline{\sigma}, \sigma, \lambda_{1,1}, \overline{\lambda}_{1,1}, \sigma_{1,1}$ and $\overline{\sigma}_{1,1}$ simultaneously. Furthermore, $\chi(H^2_{q_1,q_2,\ldots,q_d}) = q_1q_2$ and the same labelling is a minimum (proper) vertex-colouring of $H^2_{q_1,q_2,\ldots,q_d}$.

Proof. Since by Lemma 2.6 $H^c_{q_1,q_2,\ldots,q_d}$ is Hamiltonian, Lemmas 2.3 and 2.4 apply. From (4) and (6) we have $\lambda(H_{q_1,q_2,\ldots,q_d}) \leq \overline{\lambda}(H_{q_1,q_2,\ldots,q_d}) \leq \overline{\sigma}(H_{q_1,q_2,\ldots,q_d})$. However, as noticed in [23], $q_1q_2 - 1 \leq \lambda(H_{q_1,q_2}) \leq \lambda(H_{q_1,q_2,\ldots,q_d})$ since H_{q_1,q_2} is a diameter-two subgraph of $H_{q_1,q_2,\ldots,q_d}$. Thus, since $\overline{\sigma} \leq q_1q_2 - 1$ by our assumption, we must have $\lambda = \overline{\lambda} = \overline{\sigma} = q_1q_2 - 1$. (Here and in the rest of the proof parameters refer to that of $H_{q_1,q_2,\ldots,q_d}$ unless specified otherwise.) Combining this with (2) and (7) we get $q_1q_2 - 1 = \lambda \leq \sigma \leq \overline{\sigma} = q_1q_2 - 1$, and hence $\sigma = q_1q_2 - 1$.

It is clear that any (cyclic, no-hole, no-hole cyclic) $L(2, 1)$-labelling is also an $L(1, 1)$-labelling of the same type. Thus, since $H_{q_1,q_2,\ldots,q_d}$ admits no-hole $L(2, 1)$- and no-hole cyclic $L(2, 1)$-labellings by Lemma 2.7, it also admits $L(1, 1)$-labellings of the same types. Moreover, $\lambda_{1,1} \leq \lambda$, $\overline{\lambda}_{1,1} \leq \overline{\lambda}$, $\overline{\sigma}_{1,1} \leq \overline{\sigma}$, $\sigma_{1,1} \leq \sigma$, and the right-hand sides of these inequalities are all equal to $q_1q_2 - 1$ as shown above. Similar to (2) and (7), one can see that $\lambda_{1,1} \leq \sigma_{1,1} \leq \overline{\sigma}_{1,1} \leq \overline{\sigma} = q_1q_2 - 1$. However, under any $L(1, 1)$-labelling the vertices in H_{q_1,q_2} must all receive distinct labels. Thus, $q_1q_2 - 1 \leq \lambda_{1,1}$ and consequently $\lambda_{1,1} = \sigma_{1,1} = \overline{\sigma}_{1,1} = q_1q_2 - 1$. Similar to (6) and (4), we have $\lambda_{1,1} \leq \overline{\lambda}_{1,1} \leq \overline{\sigma}_{1,1}$ and this forces $\overline{\lambda}_{1,1} = q_1q_2 - 1$. Clearly, $\lambda_{1,1} + 1 \geq \chi(H^2_{q_1,q_2,\ldots,q_d})$, and $\chi(H^2_{q_1,q_2,\ldots,q_d}) \geq q_1q_2$ due to the subgraph $H^2_{q_1,q_2} \cong K_{q_1q_2}$ of $H^2_{q_1,q_2,\ldots,q_d}$. Since $\lambda_{1,1} + 1 = q_1q_2$, it follows that $\chi(H^2_{q_1,q_2,\ldots,q_d}) = q_1q_2$.

From the arguments above one can see that any optimal no-hole cyclic $L(2, 1)$-labelling of $H_{q_1,q_2,\ldots,q_d}$ is also optimal for the eight spans and $\chi(H^2_{q_1,q_2,\ldots,q_d})$ simultaneously. □

Proof of Theorem 1.3. By Lemma 4.1, it suffices to prove that $H_{q_1,q_2,\ldots,q_d}$ admits a no-hole q_1q_2-cyclic $L(2, 1)$-labelling under the condition $q_1 \geq N(q_2, q_3, \ldots, q_d)$. We will define such a labelling recursively as follows. Denote by $\langle i_2, i_3, \ldots, i_d\rangle$ the K_{q_1}-copy induced by $\{(i_1, i_2, i_3, \ldots, i_d) : i_1 \in \mathbb{Z}_{q_1}\}$. Define a linear order $\prec$ on the set of all K_{q_1}-copies of $H_{q_1,q_2,\ldots,q_d}$ by:

$$\langle i'_2, i'_3, \ldots, i'_d\rangle \prec \langle i_2, i_3, \ldots, i_d\rangle \Leftrightarrow \text{ there is some } j \text{ such that } i'_j < i_j \text{ and } i'_p = i_p \text{ for } p < j.$$

Under this order, the first K_{q_1}-copy is $\langle 0, 0, \ldots, 0\rangle$ and the last copy is $\langle q_2 - 1, q_3 - 1, \ldots, q_d - 1\rangle$.

For $i = 0, 1, \ldots, q_2 - 1$, denote

$$[i] = \{i + i_1q_2 : i_1 = 0, 1, \ldots, q_1 - 1\}.$$

Then, for any fixed i, we have $q_2 \leq |j - k| \leq q_1q_2 - q_2$ for any two distinct $j, k \in [i]$, and consequently

$$|j - k|_{q_1q_2} \geq q_2 \geq 2. \tag{15}$$

In the following we will label the vertices in the K_{q_1}-copies sequentially in accordance with $\prec$. Suppose $\langle i_2, i_3, \ldots, i_d\rangle$ is the first K_{q_1}-copy that has not been labelled. We will label vertices in $\langle i_2, i_3, \ldots, i_d\rangle$ using integers in $[(\sum_{t=2}^{d} i_t) \mod q_2]$ as follows. First, we label $(0, i_2, i_3, \ldots, i_d)$ with an integer in $[(\sum_{t=2}^{d} i_t) \mod q_2]$ which does not violate the conditions of a q_1q_2-cyclic $L(2, 1)$-labelling with the previously labelled vertices. (In the following we will justify the existence of such an integer.) Then define

$$\phi(i_1, i_2, i_3, \ldots, i_d) = (\phi(0, i_2, i_3, \ldots, i_d) + i_1q_2) \mod q_1q_2, \quad i_1 = 0, 1, \ldots, q_1 - 1. \tag{16}$$

From (15), any two vertices in this K_{q_1}-copy receive labels that differ by at least 2 under the q_1q_2-cyclic metric. Moreover, $\langle i_2, i_3, \ldots, i_d\rangle$ uses up all integers in $[(\sum_{t=2}^{d} i_t) \mod q_2]$. Clearly, $(\sum_{t=2}^{d} i_t) \mod q_2$ takes all values in $[0, q_2-1]$ when i_t runs over $0, 1, \ldots, q_t-1, 2 \le t \le d$. Since the remainder classes $[i]$, $i = 0, 1, 2, \ldots, q_2-1$, form a partition of $[0, q_1q_2-1]$, it follows that ϕ is a no-hole labelling with span q_1q_2-1. The remaining part of the proof is to show that this labelling is a well-defined q_1q_2-cyclic $L(2, 1)$-labelling of $H_{q_1,q_2,\ldots,q_d}$.

We first verify that $(0, i_2, i_3, \ldots, i_d)$ can be labelled by an integer in $[(\sum_{t=2}^{d} i_t) \mod q_2]$ which does not violate the conditions of a q_1q_2-cyclic $L(2, 1)$-labelling with the previously labelled vertices.

Suppose $(i'_1, i'_2, i'_3, \ldots, i'_d)$ is a previously labelled vertex adjacent to $(0, i_2, i_3, \ldots, i_d)$. Then, they only differ at one coordinate, say $0 \le i'_j < i_j$ for some $j \ge 2$. In this case,

$$\phi(0, i_2, i_3, \ldots, i_d) - \phi(i'_1, i'_2, i'_3, \ldots, i'_d) \equiv i_j - i'_j \pmod{q_2}.$$

The only possibilities for a violation are when $i'_j = i_j - 1$, or $i'_j = 0$ with $i_j = q_j - 1 = q_2 - 1$. There are at most $d-1$ possibilities for the former case and at most $n-1$ possibilities for the latter. Hence there are at most $(d-1)+(n-1)$ colors in $[\Sigma_{i=2}^{d}(q_i-1) \bmod q_2]$ that are forbidden for $(0, i_2, i_3, \ldots, i_d)$.

Suppose $(i'_1, i'_2, i'_3, \ldots, i'_d)$ is a labelled vertex with distance two from $(0, i_2, i_3, \ldots, i_d)$ such that $\phi(0, i_2, i_3, \ldots, i_d) = \phi(i'_1, i'_2, i'_3, \ldots, i'_d)$. Then, they only differ at exactly two coordinates, say $i'_j < i_j$ and $i'_k \ne i_k$ for some $1 \le j < k \le d$. In fact, $j \ge 2$ for otherwise

$$0 = \phi(0, i_2, i_3, \ldots, i_d) - \phi(i'_1, i'_2, i'_3, \ldots, i'_d) \equiv i_k - i'_k \pmod{q_2}$$

contradicting $0 \le i'_k \ne i_k < q_k \le q_2$. Now $2 \le j < k \le d$ gives that

$$0 = \phi(0, i_2, i_3, \ldots, i_d) - \phi(i'_1, i'_2, i'_3, \ldots, i'_d) \equiv (i_j - i'_j) + (i_k - i'_k) \pmod{q_2},$$

where $0 \le i'_j < i_j < q_j \le q_2$ and $0 \le i'_k \ne i_k < q_k \le q_2$. There are at most $q_k - 1$ such pairs (i'_j, i'_k). Hence at most

$$\sum_{2\le k\le d}(k-2)(q_k-1)$$

integers violate in total. From this and the violations for distance-one vertices, it follows that if $q_1 > N(q_2, q_3, \ldots, q_d)$ then we can always choose a proper label for $(0, i_2, i_3, \ldots, i_d)$.

Next we claim that if we have labelled the vertex $x' = (0, i_2, i_3, \ldots, i_d)$ properly, then the label defined in (16) for $x = (i_1, i_2, i_3, \ldots, i_d)$ is also proper. To see this, for any previously labelled vertex y, consider $y' = y - (i_1, 0, 0, \ldots, 0)$. Notice that $x - x' = y - y' = (i_1, 0, 0, \ldots, 0)$. So, $d_G(x, y) = d_G(x', y')$ and $\phi(x) - \phi(y) \equiv \phi(x') - \phi(y') \pmod{q_1q_2}$. The fact that the label for x' is proper then implies that the label for x is proper. This completes the proof of the theorem. □

Proof of Corollary 1.5. Suppose $q_1 \ge N(q_2, q_3, \ldots, q_d)$ and G is a subgraph of $H_{q_1,q_2,\ldots,q_d}$ containing H_{q_1,q_2}. Since by Lemma 2.8 the invariants $\eta = \lambda, \sigma, \lambda_{1,1}, \sigma_{1,1}$ are all monotonically increasing, using Theorem 1.3 and Corollary 1.4 we obtain $q_1q_2 - 1 = \eta(H_{q_1,q_2}) \le \eta(G) \le \eta(H_{q_1,q_2,\ldots,q_d}) = q_1q_2 - 1$ and hence $\eta(G) = q_1q_2 - 1$ for $\eta = \lambda, \sigma, \lambda_{1,1}, \sigma_{1,1}$.

Since H_{q_1,q_2} is a diameter-two subgraph of $H_{q_1,q_2,\ldots,q_d}$, for $(j, k) = (2, 1), (1, 1)$ and any optimal no-hole (cyclic) $L(j, k)$-labelling ϕ of $H_{q_1,q_2,\ldots,q_d}$ (which has span $q_1q_2 - 1$), all labels must be present in $H_{q_1,q_2} \subseteq G$ and hence $\phi|_G$ is a no-hole (cyclic) $L(j, k)$-labelling of G. Thus, $\eta(G) \le \eta(H_{q_1,q_2,\ldots,q_d}) = q_1q_2 - 1$ for $\eta = \overline{\lambda}, \overline{\sigma}, \overline{\lambda}_{1,1}, \overline{\sigma}_{1,1}$. Similarly, $\eta(H_{q_1,q_2}) \le \eta(G)$ since H_{q_1,q_2} is a subgraph of G. Now that $\eta(H_{q_1,q_2}) = q_1q_2 - 1$ by Corollary 1.4, it follows that $\eta(G) = q_1q_2 - 1$ for $\eta = \overline{\lambda}, \overline{\sigma}, \overline{\lambda}_{1,1}, \overline{\sigma}_{1,1}$. The truth of $\chi(G^2) = q_1q_2$ follows from $\chi(H^2_{q_1,q_2,\ldots,q_d}) = q_1q_2$ (Theorem 1.3) and the inclusions $K_{q_1q_2} \cong H^2_{q_1,q_2} \subseteq G^2 \subseteq H^2_{q_1,q_2,\ldots,q_d}$.

From the arguments above one can see that, for any optimal labelling ϕ guaranteed in Theorem 1.3, $\phi|_G$ is optimal for $\lambda(G), \overline{\lambda}(G), \overline{\sigma}(G), \sigma(G), \lambda_{1,1}(G), \overline{\lambda}_{1,1}(G), \overline{\sigma}_{1,1}(G), \sigma_{1,1}(G)$ and $\chi(G^2)$ simultaneously. □

5. Remarks

Since $H_{q_1,q_2,\ldots,q_d}$ has degree $\sum_{t=1}^{d}(q_t-1)$, a necessary condition for $\lambda(H_{q_1,q_2,\ldots,q_d}) = q_1q_2 - 1$ is $\sum_{t=1}^{d} q_t \le q_1q_2 + d - 2$. However, this condition is not sufficient since, for example, $\lambda(H_{3,2,2}) = \lambda(C_3 \square C_4) = 8$ [19]. Rewriting this necessary condition, the following question arises naturally from Theorem 1.3.

Question 5.1. *Let $q_2 \ge \cdots \ge q_d \ge 2$ be integers. Determine the smallest integer $N \ge (\sum_{t=2}^{d} q_t - d + 2)/(q_2 - 1)$ such that if $q_1 \ge N$ then $\lambda_{j,k}(H_{q_1,q_2,\ldots,q_d}) = \overline{\lambda}_{j,k}(H_{q_1,q_2,\ldots,q_d}) = \overline{\sigma}_{j,k}(H_{q_1,q_2,\ldots,q_d}) = \sigma_{j,k}(H_{q_1,q_2,\ldots,q_d}) = q_1q_2 - 1$ for $(j, k) = (2, 1), (1, 1)$.*

The existence of this integer N is guaranteed by Theorem 1.3. As in Corollary 1.5 the same condition would ensure that all these invariants are equal to $q_1q_2 - 1$ for any graph between H_{q_1,q_2} and $H_{q_1,q_2,\ldots,q_d}$. The proof of Theorem 1.3 suggests that if we can find a "better" linear order $\prec$ then we can reduce the threshold $N(q_2, q_3, \ldots, q_d)$. In view of Lemma 4.1, Question 5.1 is equivalent to determining the smallest $N \ge (\sum_{t=2}^{d} q_t - d + 2)/(q_2 - 1)$ such that $\overline{\sigma}(H_{q_1,q_2,\ldots,q_d}) \le q_1q_2 - 1$ for any $q_1 \ge N$.

Question 5.1 is related to [23, Question 6.1], where a similar question was asked for $\lambda_{j,k}$ with $2k \geq j \geq k \geq 1$ and $j/k \leq q_1 q_2 - \sum_{i=1}^{d} q_i + d$. (The latter condition, which is necessary, was neglected in [23, Question 6.1].)

As is widely known we may identify $H(d, q)$ with the d-dimensional Hamming space over an alphabet of size q. In this way we may view $H_{q_1,q_2,\ldots,q_d}$ as a subset of $H(d, q_1)$, that is, a q_1-ary block code. Thus, labelling the vertices of $H_{q_1,q_2,\ldots,q_d}$ is meant labelling the codewords in $H_{q_1,q_2,\ldots,q_d}$, and all results in this paper can be stated in terms of codes and Hamming distance in an obvious manner.

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