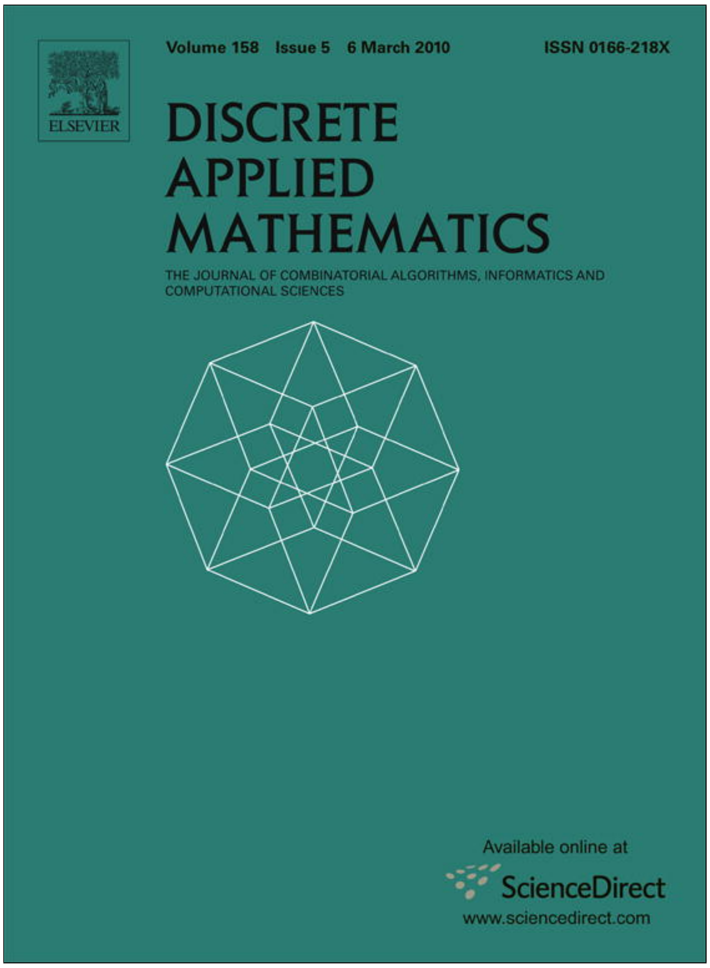
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Optimal radio labellings of complete m-ary trees[☆]

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ABSTRACT

A radio labelling of a connected graph G is a mapping $f : V(G) \to \{0, 1, 2, \ldots\}$ such that $|f(u) - f(v)| \geq \mathrm{diam}(G) - d(u, v) + 1$ for each pair of distinct vertices $u, v \in V(G)$, where $\mathrm{diam}(G)$ is the diameter of G and $d(u, v)$ the distance between u and v. The span of f is defined as $\max_{u,v \in V(G)} |f(u) - f(v)|$, and the radio number of G is the minimum span of a radio labelling of G. A complete m-ary tree ($m \geq 2$) is a rooted tree such that each vertex of degree greater than one has exactly m children and all degree-one vertices are of equal distance (height) to the root. In this paper we determine the radio number of the complete m-ary tree for any $m \geq 2$ with any height and construct explicitly an optimal radio labelling.

1. Introduction

The $L(j, k)$-labelling problem ($j, k \geq 0$) and its variants have been studied extensively (see e.g. [1,2,5–10,17,18]). A major concern of this problem is to seek an assignment of labels (which are nonnegative integers) to the vertices of a graph such that the span (difference) between the largest and smallest labels used is minimized, subject to that adjacent vertices receive labels with separation at least j and vertices at distance two apart receive labels with separation at least k. The minimum span is called [7] the $\lambda_{j,k}$-number of the graph.

Motivated by FM channel assignments, a new model, namely the radio labelling problem was introduced in [3,4] and studied further in [12,15,16]. For a connected graph $G = (V(G), E(G))$, a *radio labelling* of G is a mapping $f : V(G) \to \{0, 1, 2, \ldots\}$ such that, for any two distinct vertices u and v of G,

$$|f(u) - f(v)| \geq \mathrm{diam}(G) - d(u, v) + 1, \tag{1}$$

where $d(u, v)$ is the distance in G between u and v and $\mathrm{diam}(G)$ the diameter of G. Without loss of generality we will always assume $\min_{v \in V(G)} f(v) = 0$, and with this convention the *span* of f is defined to be $\mathrm{span}(f) := \max_{v \in V(G)} f(v)$. The *radio number* of G, $\mathrm{rn}(G)$, is the minimum span of a radio labelling of G, and a radio labelling with span $\mathrm{rn}(G)$ is called an *optimal radio labelling*. We remark that for technical reasons we follow the definitions in [15], and thus the radio number $\mathrm{rn}(G)$ defined here is one less than that defined in [4]. The radio labelling problem can be viewed as an instance of the $L(j_1, j_2, \ldots, j_d)$-labelling problem (see e.g. [7,19]), where $d, j_1, j_2, \ldots, j_d \geq 1$ are given integers, which aims at minimizing

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the span of a labelling $f: V(G) \to \{0, 1, 2, \ldots\}$ subject to $|f(u) - f(v)| \geq j_t$ whenever $d(u, v) = t$, $1 \leq t \leq d$. In the special case where $d = \mathrm{diam}(G)$ and $j_t = d - t + 1$ for each t, the minimum span of such a labelling is exactly the radio number of G. In particular, if $\mathrm{diam}(G) = 2$, then $\mathrm{rn}(G)$ is equal to the $\lambda_{2,1}$-number of G.

Determining the radio number of a graph is an interesting yet difficult combinatorial problem with potential applications to FM channel assignment. So far it has been explored for a few basic families of graphs. For instance, for paths and cycles the problem was studied by Chartrand et al. [3,4,16] and the exact values of the radio number remained open until solved by Liu and Zhu [15]. Recently, the radio number of the square of a path or cycle was studied in [13,14], and the radio number of any hypercube was determined in [11] by using generalized binary Gray codes. The results for paths were generalized [12] to spiders, leading to the exact value of the radio number in certain special cases. (A spider is a tree with at most one vertex of degree greater than two.) Surprisingly, even for paths and cycles finding the radio number was a challenging task. It is envisaged that in general determining the radio number would be difficult even for trees, despite a general lower bound for trees given in [12].

In this paper we solve completely the radio labelling problem for any complete m-ary tree with any height. These trees are chosen due to their many applications in computer science. Given integers $m \geq 2$, $k \geq 1$, the *complete m-ary tree of height k*, denoted by $T_{k,m}$, is a rooted tree such that each vertex other than leaves (degree-one vertices) has m children and all leaves are distance k apart from the root. (In a tree with root r a vertex v is a *child* of another vertex u if u, v are adjacent and $d(r, v) = d(r, u) + 1$.) The unique vertex of degree m is designated as the *root* of $T_{k,m}$, denoted by r. In particular, $T_{k,2}$ is the *complete binary tree of height k*, and we abbreviate it to T_k in the following. It can be easily verified that $\mathrm{rn}(T_{1,m}) = m + 1$ for any $m \geq 2$. Therefore, we assume $k \geq 2$ in the following.

The main results of the paper are the following two theorems, which will be proved in Sections 3 and 4 respectively. Note that the formula in Theorem 2 does not apply to binary trees.

Theorem 1. *Let $k \geq 2$ be an integer. Then*

$$\mathrm{rn}(T_k) = 13 \cdot 2^{k-1} - 4k - 5.$$

Moreover, we give an optimal radio labelling of T_k explicitly.

The optimal radio labelling of T_k will be given in Section 4.2.

Theorem 2. *Let $m \geq 3$ and $k \geq 2$ be integers. Then*

$$\mathrm{rn}(T_{k,m}) = \frac{m^{k+2} + m^{k+1} - 2km^2 + (2k-3)m + 1}{(m-1)^2}.$$

Moreover, we give an optimal radio labelling of $T_{k,m}$ explicitly.

The promised optimal radio labelling of $T_{k,m}$ will be given in Section 4.3.

The paper is structured as follows. After setting up notation and terminology in the next section, in Section 3 we will give lower bounds for $\mathrm{rn}(T_k)$ and $\mathrm{rn}(T_{k,m})$ respectively. In Section 4 we construct radio labellings of T_k and $T_{k,m}$ and prove their optimality by showing that their spans achieve our respective lower bounds. Note that the binary case has to be dealt with separately, and it is more complicated than the general case.

2. Preliminaries

Let T be a tree rooted at a vertex r. A vertex v is called a *descendant* of another vertex u (or u is an *ancestor* of v) if u is on the unique path of T from r to v. Define the *level* of $u \in V(T)$ (with respect to r) by

$$L(u) := d(r, u).$$

A vertex u of T is *in level l* if $L(u) = l$. For distinct $u, v \in V(T)$, define

$$\phi(u, v) := \text{length of the common part of the paths of } T \text{ from } r \text{ to } u \text{ and } v.$$

The subtree of T induced by r, a child u of r, and all descendants of u is referred to as a *branch* of T. (Note that for technical reasons we take the root as in every branch of T.) Obviously, we have the following facts, which has been used in [12].

Lemma 3. *Let T be a tree rooted at r. Then for distinct $u, v \in V(T)$ the following* (a)–(b) *hold.*

(a) $d(u, v) = L(u) + L(v) - 2\phi(u, v)$;
(b) $\phi(u, v) = 0$ *if and only if* $r \in \{u, v\}$ *or u and v belong to different branches.*

Let f be a radio labelling of T. By (1) f is injective, that is, $f(u) \neq f(v)$ for distinct $u, v \in V(T)$. Hence f induces a linear order

$$u_0, u_1, u_2, \ldots, u_{n-1} \tag{2}$$

of the vertices of T, where $n = |V(T)|$, which is defined by

$$0 = f(u_0) < f(u_1) < f(u_2) < \cdots < f(u_{n-1}).$$

Note that the span of f is equal to $f(u_{n-1})$. Note also that, by (1),

$$f(u_{i+1}) - f(u_i) \geq \mathrm{diam}(T) - d(u_i, u_{i+1}) + 1, \quad 0 \leq i \leq n-2.$$

We call

$$J_f(u_i, u_{i+1}) := f(u_{i+1}) - f(u_i) - (\mathrm{diam}(T) - d(u_i, u_{i+1}) + 1), \quad 0 \leq i \leq n-2$$

the *jump* of f from u_i to u_{i+1}. If $J_f(u_i, u_{i+1}) = K$, then f is said to have a K*-jump* from u_i to u_{i+1}. The *total jump* of f is defined as

$$J(f) := \sum_{i=0}^{n-2} J_f(u_i, u_{i+1}).$$

In our subsequent discussion we use the notation and terminology above for $T_{k,m}$ $(m, k \geq 2)$ with the understanding that r is the root of $T_{k,m}$ as specified in its definition. Note that $\mathrm{diam}(T_{k,m}) = 2k$ and level k is the bottom level of $T_{k,m}$. Define $w(T_{k,m}) := \sum_{u \in V(T_{k,m})} L(u)$. Then $w(T_{k,m}) = \sum_{i=1}^{k} m^i i$ and hence $(m-1)w(T_{k,m}) = km^{k+1} - \sum_{i=1}^{k} m^i$. From this we obtain

$$w(T_{k,m}) = \frac{km^{k+2} - (k+1)m^{k+1} + m}{(m-1)^2}. \tag{3}$$

In particular,

$$w(T_k) = (k-1)2^{k+1} + 2. \tag{4}$$

3. Lower bounds

3.1. Jumps in T_k

In this subsection we assume that f is a radio labelling of T_k and that the vertices of T_k are ordered as in (2) with respect to f, where $n = |V(T_k)| = 2^{k+1} - 1$. Let $u_i, u_{i+1}, u_{i+2}, 0 \leq i \leq n-3$, be consecutive vertices in (2), so that $f(u_i) < f(u_{i+1}) < f(u_{i+2})$. To obtain the desired lower bound on $\mathrm{rn}(T_k)$ we first consider jumps from u_i to u_{i+1} and u_{i+1} to u_{i+2} under the following assumptions:

$$u_i, u_{i+2} \text{ are in the same branch of } T_k, \text{ and } u_{i+1} \text{ is in a different branch of } T_k. \tag{5}$$

Lemma 4. *Under the assumption* (5)*, we have*

$$J_f(u_{i+1}, u_{i+2}) \geq \max\{2(\phi(u_i, u_{i+2}) + L(u_{i+1}) - k) - J_f(u_i, u_{i+1}) - 1, 0\}.$$

Proof. Denote $l_i = L(u_i)$, $l_{i+1} = L(u_{i+1})$ and $l_{i+2} = L(u_{i+2})$. From Lemma 3 and the assumption (5), we have $d(u_i, u_{i+1}) = l_i + l_{i+1}$, $d(u_{i+1}, u_{i+2}) = l_{i+1} + l_{i+2}$ and $d(u_i, u_{i+2}) = l_i + l_{i+2} - 2\phi(u_i, u_{i+2})$. Thus $f(u_{i+1}) - f(u_i) = 2k - l_i - l_{i+1} + J_f(u_i, u_{i+1}) + 1$ and $f(u_{i+2}) - f(u_{i+1}) = 2k - l_{i+1} - l_{i+2} + J_f(u_{i+1}, u_{i+2}) + 1$. Summing up we get

$$f(u_{i+2}) - f(u_i) = 4k - l_i - 2l_{i+1} - l_{i+2} + J_f(u_i, u_{i+1}) + J_f(u_{i+1}, u_{i+2}) + 2.$$

On the other hand, since f is a radio labelling, we have

$$\begin{aligned} f(u_{i+2}) - f(u_i) &\geq 2k - d(u_i, u_{i+2}) + 1 \\ &= 2k - l_i - l_{i+2} + 2\phi(u_i, u_{i+2}) + 1. \end{aligned}$$

Combining the two expressions above we get $J_f(u_{i+1}, u_{i+2}) \geq 2(\phi(u_i, u_{i+2}) + l_{i+1} - k) - J_f(u_i, u_{i+1}) - 1$. Since $J_f(u_{i+1}, u_{i+2}) \geq 0$, the result follows immediately. □

In particular, if u_{i+1} is in level k, then Lemma 4 gives the following corollary. Note that if $u_i, u_{i+2} \neq r$ then $\phi(u_i, u_{i+2}) \geq 1$ under the assumption (5).

Corollary 5. *Under the assumption (5), if* u_{i+1} *is in level* k*, then*

$$J_f(u_{i+1}, u_{i+2}) \geq \max\{2\phi(u_i, u_{i+2}) - J_f(u_i, u_{i+1}) - 1, 0\}.$$

In particular, if in addition none of u_i, u_{i+2} *is the root* r*, then*

$$J_f(u_i, u_{i+1}) + J_f(u_{i+1}, u_{i+2}) \geq 1. \tag{6}$$

Note that in this corollary one of u_i, u_{i+2} or both of them can be in level k. This will be used below in deriving a lower bound for $\mathrm{rn}(T_k)$.

3.2. Lower bound for $\mathrm{rn}(T_k)$

We use the notation in Section 3.1 and denote by L_k the set of vertices of T_k in level k. For a segment $D = \{u_i, u_{i+1}, \ldots, u_j\}$ of (2), $0 < i \leq j < n-1$, define

$$J(D) := \sum_{t=i-1}^{j} J_f(u_t, u_{t+1}).$$

Lemma 6. *Let* f *be a radio labelling of* T_k*. Suppose* $D = \{u_i, u_{i+1}, \ldots, u_j\}$*,* $0 < i \leq j < n-1$*, satisfies the following conditions:*

(a) $r \notin D$;
(b) u_i *and* u_j *are both in level* k;
(c) u_t, u_{t+1} *are in different branches, for* $t = i-1, \ldots, j$.

Then

$$J(D) \geq \begin{cases} \dfrac{|D \cap L_k|}{2}, & r \notin \{u_{i-1}, u_{j+1}\} \\ \dfrac{|D \cap L_k| - 1}{2}, & \text{otherwise}. \end{cases} \tag{7}$$

Proof. We distinguish the following two cases.

Case 1: $r \notin \{u_{i-1}, u_{j+1}\}$.

Subcase 1.1: $D \subseteq L_k$. In this case we prove $J(D) \geq |D \cap L_k|/2 \; (= |D|/2 = (j-i+1)/2)$ by induction on $|D|$.

Consider first the case when $|D| = 2d+1 \geq 1$ is odd. If $d = 0$, then since u_i is in level k, Corollary 5 applies to u_{i-1}, u_i, u_{i+1}, and hence $J(D) \geq 1$ by (6). Inductively, suppose, for some $d \geq 0$, that $J(D) \geq (2d+1)/2$ for any $D \subseteq L_k$ with $|D| = 2d+1$. Then, for any $D \subseteq L_k$ with $|D| = 2d+3$, we have $J(\{u_i, u_{i+1}, \ldots, u_{j-2}\}) \geq (2d+1)/2$ by the inductive hypothesis and noting $r \notin D$, and $J_f(u_{j-1}, u_j) + J_f(u_j, u_{j+1}) \geq 1$ by (6); hence $J(D) \geq (2d+3)/2$. Thus the first inequality in (7) holds when $D \subseteq L_k$ and $|D|$ is odd.

Next we consider the case when $|D| = 2d \geq 2$ is even. If $d = 1$, then u_i and $u_{i+1} \; (= u_j)$ are both in level k, and hence $J(D) \geq 1$ by applying (6) to u_{i-1}, u_i, u_{i+1}. Based on this and using (6), by induction as in the previous paragraph one can verify that the first inequality in (7) holds when $D \subseteq L_k$ and $|D|$ is even.

Subcase 1.2: $D \not\subseteq L_k$. Let $D_1, \ldots, D_l$ be maximal segments of consecutive vertices in D that are in level k. Then by the result for Subcase 1.1 we have $J(D_t) \geq |D_t \cap L_k|/2$, $t = 1, \ldots, l$. Hence $J(D) \geq \sum_{t=1}^{l} J(D_t) = |D \cap L_k|/2$.

Case 2: $r \in \{u_{i-1}, u_{j+1}\}$. Consider the case $r = u_{i-1}$ first. If $i = j$, then $|D| = 1$ and the second inequality in (7) becomes $J(D) \geq 0$, which is trivial. Assume then $i < j$. Let i^* be the smallest subscript such that $i^* > i$ and u_{i^*} is in level k. Since $i < j$ and u_j is in level k, i^* is well defined. Let $D' = \{u_{i^*}, \ldots, u_j\}$. Then $u_{i^*-1} \neq r$ and the result for Case 1 can be applied to D'. Thus, $J(D) \geq J(D') \geq |D' \cap L_k|/2 = (|D \cap L_k| - 1)/2$. The case where $r = u_{j+1}$ can be treated similarly. □

Remark 7. We still have $J(D) \geq (|D \cap L_k| - 1)/2$ if we remove the condition (b) in Lemma 6. This is obtained by applying Lemma 6 to the longest possible segment $\{u_{i'}, \ldots, u_{j'}\}$ contained in D such that $u_{i'}$ and $u_{j'}$ are in level k.

Corollary 8. *Let* f *be a radio labelling of* T_k*. Suppose* $D = \{u_i, u_{i+1}, \ldots, u_j\}$*,* $0 < i \leq j < n-1$*, satisfies the following conditions:*

(a) $r \in \{u_{i+1}, \ldots, u_{j-1}\}$;
(b) u_t, u_{t+1} *are in different branches, for* $t = i-1, \ldots, j$.

Then

$$J(D) \geq \frac{|D \cap L_k|}{2} - 1.$$

Proof. By (a) we may assume $u_{i^*} = r$ for some i^* with $i < i^* < j$. Let $D_1 = \{u_i, \dots, u_{i^*-1}\}$ and $D_2 = \{u_{i^*+1}, \dots, u_j\}$. Then both D_1 and D_2 satisfy the conditions (a) and (c) in Lemma 6. Thus, by Lemma 6 and Remark 7, we have $J(D) \geq J(D_1)+J(D_2) \geq (|D_1 \cap L_k| - 1)/2 + (|D_2 \cap L_k| - 1)/2 = (|D \cap L_k|/2) - 1$. □

Equipped with Lemma 6 and Corollary 8 we now prove the following lower bound for $\mathrm{rn}(T_k)$. Let $\mathcal{F}$ denote the set of radio labellings f of T_k such that for each $i = 0, 1, \dots, n-2$ the vertices u_i, u_{i+1} in the linear order (2) induced by f are in different branches unless one of them is r. Recall that $w(T_k) = (k-1)2^{k+1} + 2$ by (4).

Lemma 9. *Let $k \geq 2$ be an integer and $n = |V(T_k)| = 2^{k+1} - 1$. Then*

$$\begin{aligned}\mathrm{rn}(T_k) &\geq (n-1)(2k+1) - 2w(T_k) + 2^{k-1} + 1 \\ &= 13 \cdot 2^{k-1} - 4k - 5.\end{aligned}$$

Proof. Let f be an arbitrary radio labelling of T_k. As in (2) let $u_0, u_1, \dots, u_{n-1}$ be the linear order defined by $0 = f(u_0) < f(u_1) < \cdots < f(u_{n-1})$. Since the diameter of T_k is $2k$, by the definition of a radio labelling,

$$f(u_{i+1}) - f(u_i) = (2k+1) - d(u_i, u_{i+1}) + J_f(u_i, u_{i+1}), \quad 0 \leq i \leq n-2.$$

Thus, using Lemma 3, we have

$$\begin{aligned}\mathrm{span}(f) &= f(u_{n-1}) \\ &= \sum_{i=0}^{n-2} (f(u_{i+1}) - f(u_i)) \\ &= (n-1)(2k+1) - \sum_{i=0}^{n-2} d(u_i, u_{i+1}) + \sum_{i=0}^{n-2} J_f(u_i, u_{i+1}) \\ &= (n-1)(2k+1) - 2w(T_k) + L(u_0) + L(u_{n-1}) + \sigma(f) \qquad (8)\end{aligned}$$

where

$$\sigma(f) := \sum_{i=0}^{n-2} (J_f(u_i, u_{i+1}) + 2\phi(u_i, u_{i+1})).$$

Based on this we now prove

$$\mathrm{span}(f) \geq (n-1)(2k+1) - 2w(T_k) + 2^{k-1} + 1. \qquad (9)$$

Case 1: $f \notin \mathcal{F}$.

A pair u_i, u_{i+1} of vertices is called *bad* if $u_i, u_{i+1} \neq r$ and u_i, u_{i+1} are in the same branch of T_k. Define X to be the subset of $V(T_k)$ such that a vertex of T_k is in X if and only if it is in at least one bad pair. Then $V(T_k) \setminus X$ consists of maximal segments of the sequence $u_0, u_1, \dots, u_{n-1}$, say, $D_0, D_1, \dots, D_l$, such that each pair of consecutive vertices in the same segment are in different branches of T_k. Since there are 2^k vertices in level k, $|V(T_k) \setminus X| \geq |L_k \setminus X| \geq 2^k - |X|$. Exactly one of these segments, say, D_0, contains r. If r is neither the first nor the last vertex of D_0, then Corollary 8 applies directly to D_0; otherwise the second inequality in (7) applies to the subsegment (which may be empty) obtained by deleting r from D_0. (See Remark 7 for the latter case.) For each $t = 1, \dots, l$, the first inequality in (7) applies to the longest possible subsegment $\{u_{i'}, \dots, u_{j'}\}$ of D_t such that $u_{i'}$ and $u_{j'}$ are in level k. Combining all these inequalities, we have

$$\sum_{i=0}^{n-2} J_f(u_i, u_{i+1}) \geq \sum_{t=0}^{l} J(D_t) \geq \sum_{t=0}^{l} \frac{|D_t \cap L_k|}{2} - 1 \geq \frac{2^k - |X|}{2} - 1.$$

Note that there are at least $|X|/2$ bad pairs of vertices and that $\phi(u_i, u_{i+1}) \geq 1$ for each such pair u_i, u_{i+1}. Thus $\sum_{i=0}^{n-2} \phi(u_i, u_{i+1}) \geq |X|/2$ and therefore

$$\sigma(f) \geq \left(\frac{2^k - |X|}{2} - 1\right) + |X| = 2^{k-1} + \frac{|X|}{2} - 1.$$

Since $f \notin \mathcal{F}$, we have $|X| \geq 1$. Hence $\sigma(f) \geq 2^{k-1} - 1/2$, which implies $\sigma(f) \geq 2^{k-1}$ as $\sigma(f)$ is an integer. Therefore, since $L(u_0) + L(u_{n-1}) \geq 1$, (9) follows from (8) immediately.

Case 2: $f \in \mathcal{F}$.

Assume $r \notin \{u_0, u_{n-1}\}$ first. In this case, applying Corollary 8 to $D = \{u_1, \dots, u_{n-2}\}$, we have $\sigma(f) \geq (|D \cap L_k|/2 - 1) \geq ((2^k - 1)/2) - 1$, which implies $\sigma(f) \geq 2^{k-1} - 1$ since $\sigma(f)$ is an integer. Since $r \notin \{u_0, u_{n-1}\}$, $L(u_0) + L(u_{n-1}) \geq 2$ and hence (9) follows from (8).

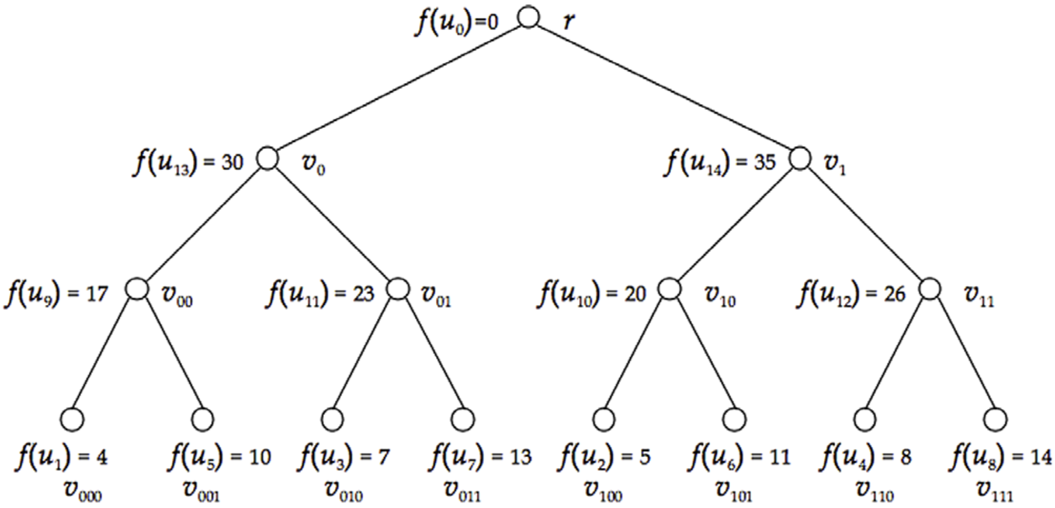

Fig. 1. Vertex-indices and radio labelling of T_3.

Next we assume $r \in \{u_0, u_{n-1}\}$. Then $L(u_0)+L(u_{n-1}) \geq 1$, and the equality holds if and only if the vertex in $\{u_0, u_{n-1}\}\setminus\{r\}$ is a child of r. Setting $D = \{u_1, \ldots, u_{n-2}\}$, we have $|D \cap L_k| = 2^k$ if $L(u_0) + L(u_{n-1}) = 1$ and $|D \cap L_k| \geq 2^k - 1$ if $L(u_0) + L(u_{n-1}) \geq 2$. Thus, applying Lemma 6 to D and taking Remark 7 into account, we have $L(u_0) + L(u_{n-1}) + \sigma(f) \geq L(u_0)+L(u_{n-1})+(|D\cap L_k|-1)/2 \geq 2^{k-1}+1/2$, which implies $L(u_0)+L(u_{n-1})+\sigma(f) \geq 2^{k-1}+1$ since $L(u_0)+L(u_{n-1})+\sigma(f)$ is an integer. In view of (8) we get (9) and hence complete the proof by the arbitrariness of f. □

3.3. *Lower bound for* $\mathrm{rn}(T_{k,m})$, $m \geq 3$

The following bound can be proved by an argument similar to the one that leads to (8), using $L(u_0) + L(u_{n-1}) \geq 1$. It is a special case of the bound given in [12, Theorem 3]. Recall that $w(T_{k,m})$ is given in (3).

Lemma 10. *Let $m \geq 3$, $k \geq 2$ be integers, and let $n = |V(T_{k,m})| = (m^{k+1} - 1)/(m - 1)$. Then*

$$\mathrm{rn}(T_{k,m}) \geq (n-1)(2k+1) - 2w(T_{k,m}) + 1 = \frac{m^{k+2} + m^{k+1} - 2km^2 + (2k-3)m + 1}{(m-1)^2}.$$

4. Optimal radio labellings

In this section we construct radio labellings for T_k and $T_{k,m}$ $(m \geq 3)$ and prove their optimality. The latter is achieved by showing that our radio labellings attain the lower bounds in Lemmas 9 and 10, respectively. Due to jumps in radio labellings of T_k we have to deal with complete binary trees separately. In what follows we first introduce an index scheme that will be used for both T_k and $T_{k,m}$ with $m \geq 3$.

4.1. *Vertex-indices*

As is widely used in the literature we may index the vertices of $T_{k,m}$ in level l by words of length l with alphabet $\{0, 1, \ldots, m-1\}$. More explicitly, the m children of the root r are indexed by $v_0, v_1, v_2, \ldots, v_{m-2}, v_{m-1}$; inductively the m children of a vertex $v_{i_1,i_2,\ldots,i_l}$ $(0 \leq i_1, i_2, \ldots, i_l \leq m-1, l < k)$ in level l are indexed by $v_{i_1,i_2,\ldots,i_l,i_{l+1}}$, where $i_{l+1} = 0, 1, \ldots, m-1$. Thus, for any two vertices $v_{a_1,a_2,\ldots,a_s}$ and $v_{b_1,b_2,\ldots,b_t}$ other than r, we have

$$\phi(v_{a_1,a_2,\ldots,a_s}, v_{b_1,b_2,\ldots,b_t}) = \max\{l : a_1 = b_1, a_2 = b_2, \ldots, a_l = b_l\}. \quad (10)$$

To facilitate our radio labelling we now give another index scheme for $T_{k,m}$ $(m, k \geq 2)$. We first index r by u_0 and then index other vertices bottom-up starting from level k. (See Fig. 1 for an illustration.) More precisely, for any vertex $v_{i_1,i_2,\ldots,i_l} \neq r$, we rename $u_j = v_{i_1,i_2,\ldots,i_l}$, where

$$j = 1 + i_1 + i_2 m + \cdots + i_l m^{l-1} + \sum_{l+1 \leq t \leq k} m^t. \quad (11)$$

For example, a vertex $v_{i_1,i_2,\ldots,i_k}$ in level k is indexed as u_j, where $j = 1 + i_1 + i_2 m + \cdots + i_k m^{k-1}$. Note that the minimum and maximum indices of vertices in level l $(1 \leq l \leq k)$ are $1 + m^{l+1} + \cdots + m^k$ and $m^l + m^{l+1} + \cdots + m^k$, respectively. Thus

(11) together with $u_0 = r$ defines a bijection from $V(T_{k,m})$ to $\{0, 1, \ldots, |V(T_{k,m})| - 1\}$. Note that, by (10) and (11), we have the following observation.

Lemma 11. *$\phi(u_j, u_{j+1}) = 0$, $1 \le j \le n-2$, that is, u_j and u_{j+1} are in different branches of $T_{k,m}$.*

Lemma 12. *Suppose u_a, u_b are in the same level of $T_{k,m}$. Then $|b-a| \ge m^{\zeta}$, where $\zeta = \phi(u_a, u_b)$.*

Proof. Let $u_a = v_{a_1,a_2,\ldots,a_l}$, $u_b = v_{b_1,b_2,\ldots,b_l}$, where $L(u_a) = L(u_b) = l$. Without loss of generality we may assume $b > a$. Let h be the largest subscript such that $a_h \ne b_h$. Then $\zeta + 1 \le h \le l$ and $b - a = (b_{\zeta+1} - a_{\zeta+1})m^{\zeta} + \cdots + (b_h - a_h)m^{h-1}$ by (10) and (11). We must have $b_h > a_h$ for otherwise $b - a \le (m-1)(m^{\zeta} + \cdots + m^{h-2}) - m^{h-1} < 0$, a contradiction. Hence $b - a \ge -(m-1)(m^{\zeta} + \cdots + m^{h-2}) + m^{h-1} = m^{\zeta}$. □

4.2. Optimal radio labelling of T_k

Define $f : V(T_k) \to \{0, 1, 2, \ldots\}$ as follows:

$$f(u_0) = 0;$$
$$f(u_{2t-1}) = k + 3t - 2, \qquad f(u_{2t}) = k + 3t - 1, \quad 1 \le t \le 2^{k-1};$$
$$f(u_{2^k+1}) = k + 3 \cdot 2^{k-1} + 2;$$
$$f(u_{j+1}) = f(u_j) + 2k - (L(u_j) + L(u_{j+1})) + 1, \quad 2^k + 1 \le j \le 2^{k+1} - 2.$$

That is, we label r and the vertices in level k first, and then label other vertices recursively starting from u_{2^k+1}. (Fig. 1 shows this labelling for T_3.) Using the recursive relation above and noting that $u_{2^k+\cdots+2^{l+1}+t}$ is in level l, $1 \le t \le m^l$, we can give explicitly the labels of the vertices in levels 1 to $k-1$, where $1 < l \le k$ and $1 \le t \le 2^{l-1}$:

$$f(u_{2^k+2^{k-1}+\cdots+2^l+t}) = (k + 3 \cdot 2^{k-1}) + \sum_{i=l}^{k-1} (2(k-i)+1)2^i + (2(k-l)+3)t - (k-l+1).$$

Note that the linear order induced by f is $u_0, u_1, \ldots, u_{2^{k+1}-1}$, agreeing with our notation in (2).

Lemma 13. *The mapping f above is an optimal radio labelling of T_k, and moreover $\mathrm{span}(f) = (n-1)(2k+1) - 2w(T_k) + 2^{k-1} + 1$, where $n = |V(T_k)| = 2^{k+1} - 1$ and $w(T_k)$ is as in (4).*

Proof. The major task is to prove that f is indeed a radio labelling of T_k. First, it can be verified by induction that $f(u_j) - f(u_0) = f(u_j) \ge 2k - d(u_0, u_j) + 1$ for $j \ge 1$. Thus in the following we only consider pairs u_a, u_b of vertices with $1 \le a < b \le 2^{k+1} - 2$. Note that, by the definition of f,

$$f(u_{j+1}) - f(u_j) = 2k - (L(u_j) + L(u_{j+1})) + 1 + \epsilon_j, \tag{12}$$

where $\epsilon_j = 1$ if $j = 2, 4, 6, \ldots, 2^k$ and $\epsilon_j = 0$ for all other j. Thus, setting $\zeta = \zeta(a, b) := \phi(u_a, u_b)$,

$$\begin{aligned} f(u_b) - f(u_a) &= (2k+1)(b-a) - \left(L(u_a) + L(u_b) + 2\sum_{j=a+1}^{b-1} L(u_j) \right) + \sum_{j=a}^{b-1} \epsilon_j \\ &= \{2k - (L(u_a) + L(u_b) - 2\zeta) + 1\} + \delta(a, b) \end{aligned}$$

where

$$\delta(a, b) := (2k+1)(b-a-1) + \sum_{j=a}^{b-1} \epsilon_j - 2\sum_{j=a+1}^{b-1} L(u_j) - 2\zeta.$$

To verify that f is a radio labelling of T_k it suffices to show that $\delta(a, b) \ge 0$ for all pairs u_a, u_b. Denote $l_a = L(u_a)$ and $l_b = L(u_b)$. Since $a < b$ by our assumption, we have $l_a \ge l_b \ge \zeta$.

Case 1: $l_a = l_b = l$.

In this case we have $l > \zeta$. Assume first that $l = k$. Then by our index scheme all vertices u_j, $a+1 \le j \le b-1$, are in level k. Thus, $\sum_{j=a+1}^{b-1} L(u_j) = k(b-a-1)$ and $\sum_{j=a}^{b-1} \epsilon_j \ge (b-a-1)/2$. Hence, using Lemma 12, $\delta(a, b) \ge 3(b-a-1)/2 - 2\zeta \ge 3(2^{\zeta} - 1)/2 - 2\zeta$. Note that the right-hand side of this inequality is 0 if $\zeta = 0$; $-1/2$ if $\zeta = 1$; and at least 1 if $\zeta \ge 2$. Since $\delta(a, b)$ is an integer, it follows that $\delta(a, b) \ge 0$.

Next we assume $l < k$. Then $\sum_{j=a+1}^{b-1} L(u_j) = l(b-a-1) \le (k-1)(b-a-1)$. Hence, using Lemma 12, we get $\delta(a, b) \ge 3(b-a-1) - 2\zeta \ge 3(2^{\zeta} - 1) - 2\zeta \ge 0$.

Case 2: $l_a > l_b$.

Assume first that $l_a = k$. If u_{b-1} is in level k, then the same argument as in Case 1 leads to $\delta(a,b) \geq 0$. If u_{b-1} is not in level k, then $L(u_{b-1}) \leq k-1$ and so $\sum_{j=a+1}^{b-1} L(u_j) \leq k(b-a-1)-1$. This together with Lemma 12 implies $\delta(a,b) \geq (b-a+1)-2\zeta \geq 2^{\zeta}+1-2\zeta > 0$.

Now let us assume $l_a < k$. In this case by our index scheme all vertices u_j, $a+1 \leq j \leq b-1$, are in level $k-1$ or above. Hence $\sum_{j=a+1}^{b-1} L(u_j) \leq (k-1)(b-a-1)$. Using Lemma 12 we obtain $\delta(a,b) \geq 3(b-a-1)-2\zeta \geq 3(2^{\zeta}-1)-2\zeta \geq 0$.

In summary, we have proved $\delta(a,b) \geq 0$ in all situations. Therefore, f is a radio labelling of T_k.

By Lemma 11 and (12), f has exactly 2^{k-1} non-zero jumps, all of which are 1-jumps. Hence $J(f) = 2^{k-1}$. Therefore, since $f(u_0) = 0$, $L(u_0) = 0$ and $L(u_{n-1}) = 1$, in view of Lemma 11 we have

$$\begin{aligned}\mathrm{span}(f) &= \sum_{j=0}^{n-2} \big(f(u_{j+1}) - f(u_j)\big) \\ &= (n-1)(2k+1) - \sum_{j=0}^{n-2} (L(u_i) + L(u_{i+1})) + J(f) \\ &= (n-1)(2k+1) - 2w(T_k) + L(u_0) + L(u_{n-1}) + J(f) \\ &= (n-1)(2k+1) - 2w(T_k) + 2^{k-1} + 1.\end{aligned}$$

Comparing this with Lemma 9, we conclude that f is an optimal radio labelling of T_k. □

Proof of Theorem 1. This follows from Lemmas 9 and 13 immediately. □

4.3. *Optimal radio labelling of* $T_{k,m}$, $m \geq 3$

Define $f_m : V(T_{k,m}) \to \{0, 1, 2, \ldots\}$ recursively by

$$f_m(u_j) = \begin{cases} 0, & j = 0 \\ f_m(u_{j-1}) + 2k - (L(u_{j-1}) + L(u_j)) + 1, & 1 \leq j \leq n-1, \end{cases}$$

where $n = |V(T_{k,m})| = (m^{k+1}-1)/(m-1)$. Working recursively, we obtain the following explicit rule, where $1 < l \leq k$ and $1 \leq t \leq m^{l-1}$:

$$f_m(u_{m^k+m^{k-1}+\cdots+m^l+t}) = k + \sum_{i=l}^{k} (2(k-i)+1)\, m^i + (2(k-l)+3)t - (k-l+1).$$

Clearly, the linear order induced by f_m is $u_0, u_1, \ldots, u_{n-1}$, and this agrees with the notation in (2).

Lemma 14. *The mapping f_m above is an optimal radio labelling of $T_{k,m}$, and moreover* $\mathrm{span}(f_m) = (n-1)(2k+1) - 2w(T_{k,m}) + 1$, *where $w(T_{k,m})$ is as in* (3) *and n is as above.*

Proof. First, by induction on j one can easily verify that $f_m(u_j) - f_m(u_0) \geq 2k - d(u_0, u_j) + 1 = 2k - j + 1$, $1 \leq j \leq n-1$. By Lemma 11 and the definition of f_m, it is clear that f_m has no non-zero jumps, that is, $J(f_m) = 0$. Let $1 \leq a < b \leq n-1$ and set $\zeta = \zeta(a,b) := \phi(u_a, u_b)$. Similar to the proof of Lemma 13, using Lemma 12 and noting that $m^{\zeta} \geq 2\zeta + 1$ for $m \geq 3$, $\zeta \geq 0$, we have

$$\begin{aligned}f_m(u_b) - f_m(u_a) &= (2k+1)(b-a) - \left(L(u_a) + L(u_b) + 2\sum_{j=a+1}^{b-1} L(u_j)\right) \\ &\geq (2k+1)(b-a) - (L(u_a) + L(u_b) + 2k(b-a-1)) \\ &= 2k + (b-a) - (L(u_a) + L(u_b)) \\ &\geq 2k + m^{\zeta} - (L(u_a) + L(u_b)) \\ &\geq 2k + 2\zeta + 1 - (L(u_a) + L(u_b)) \\ &= 2k - d(u_a, u_b) + 1.\end{aligned}$$

Hence f_m is a radio labelling of $T_{k,m}$.

Since $f_m(u_0) = 0$, $L(u_0) = 0$, $L(u_{n-1}) = 1$ and $J(f_m) = 0$, by Lemma 11 and the definition of f_m, we have

$$\begin{aligned} \text{span}(f_m) &= \sum_{j=0}^{n-2} \left(f_m(u_{j+1}) - f_m(u_j)\right) \\ &= (n-1)(2k+1) - \sum_{j=0}^{n-2}(L(u_i) + L(u_{i+1})) \\ &= (n-1)(2k+1) - 2w(T_{k,m}) + 1. \end{aligned}$$

Comparing this with Lemma 10, it is clear that f_m is an optimal radio labelling of $T_{k,m}$. □

Proof of Theorem 2. This follows from Lemmas 10 and 14 immediately. □

It may be the case that $T_{k,m}$ has many optimal radio labellings. However, from the proof of Lemmas 10 and 14, any optimal radio labelling of $T_{k,m}$ ($m \geq 3$) has no (non-zero) jump and it always assigns 0 to r and the largest label to a child of r.

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