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Note

Matroid tree graphs and interpolation theorems

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Abstract

Using the Hamiltonicity of matroid tree graphs we give a new proof for an interpolation theorem of Barefoot (1984) and other related results. From the proof we refine a general approach for dealing with interpolation problems of graphs.

Let G be a simple, connected graph of order p and size q. For each integer m, $p-1\leqslant m\leqslant q$, denote by $C_m(G)$ the set of connected spanning subgraphs of G with m edges. Whenever $H\in C_m(G)$, let $\varphi(H)$ be the number of pendant vertices of H. Thus, φ is a mapping from $C_m(G)$ to $\mathbb{Z}^+$, the set of nonnegative integers. If $m=p-1$, $C_m(G)$ is exactly the set of spanning trees of G. It was first proved in [6] that $\varphi(C_{p-1}(G))$ is an integer interval. An elegant proof for this result was given in [4]. In general, Barefoot [1] proved the following theorem.

Theorem. *The image set $\varphi(C_m(G))$ is an integer interval.*

By using the same idea as in [4] we will give this theorem a short proof and then refine a general approach for interpolation problems. Briefly, we use the same symbol for a graph and its edge set. Note first that if $H_1, H_2\in C_m(G)$ and $e_1\in H_1\setminus H_2$, then there exists $e_2\in H_2\setminus H_1$ such that $H_1-e_1+e_2\in C_m(G)$. Hence, $C_m(G)$ can be taken as the base set of a matroid on G [5]. Let $T_m(G)$ be the tree graph of this matroid. Thus, the vertex set of $T_m(G)$ is $C_m(G)$ and H_1, H_2 adjacent iff $|H_1\setminus H_2|=1$. If $m=q$, $T_m(G)$ is the trival graph. If $m<q$, then the minimum degree of $T_m(G)$ is at least 2. Therefore, $T_m(G)$ contains cycles and, by the Hamiltonicity theorem of matroid tree graphs [3], it is Hamiltonian.

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Now we can prove the theorem. The result is true for $m=q$. If $m=q-1$, we have $C_{q-1}(G)=\{G-e \mid e\in G$ is not a bridge$\}$ and $\varphi(G-e)=\varphi(G)$, $\varphi(G)+1$ or $\varphi(G)+2$. When $\varphi(G-e)=\varphi(G)$ and $\varphi(G-e'')=\varphi(G)+2$ appear for some e, e'', it is easy to see that there is an edge $e'=uv$ which is not a bridge of G such that $d(u)=2$ and $d(v)\geqslant 3$. So $\varphi(G-e')=\varphi(G)+1$. Consequently, the result is valid for any connected graph G and $m=|E(G)|-1$.

For the general case $p-1\leqslant m\leqslant q-1$, denote by $N(H)$ the subset of $C_m(G)$ consisting of $H\in C_m(G)$ and all neighbors of H in $T_m(G)$. Then

$$N(H)=\bigcup_{e\in G\setminus H} C_m(H+e) \quad \text{and} \quad \varphi(N(H))=\bigcup_{e\in G\setminus H} \varphi(C_m(H+e)).$$

As we have just proved, all $\varphi(C_m(H+e))$ are integer intervals which share a common element $\varphi(H)$. Hence, $\varphi(N(H))$ is an integer interval.

Since $T_m(G)$ is Hamiltonian, there is a Hamilton path $H_1, H_2, \ldots, H_t$ in $T_m(G)$. We have

$$\varphi(C_m(G))=\bigcup_{1\leqslant i\leqslant t} \varphi(N(H_i)).$$

Note that $H_{i+1}\in N(H_i)$, two intervals $\varphi(N(H_i))$ and $\varphi(N(H_{i+1}))$ have a common element $\varphi(H_{i+1})$ $(1\leqslant i\leqslant t-1)$. Thus, the union of t intervals above is also an interval. This ends the proof.

The proof above tells us more. Suppose ψ is an integral graphical invariant and p and m are positive integers with $p-1\leqslant m$. If, for each connected graph H of order p and size $m+1$, $\psi(C_m(H))$ is an interval, then, for any connected graph G of order p and size $\geqslant m$, $\psi(C_m(G))$ is also an interval. Using this observation we get again (see [2, 7]) that for any connected graph G with order p and size q and each m, $p-1\leqslant m\leqslant q$, $\psi(C_m(G))$ is an integer interval if ψ is one of the following invariants: connectivity, edge connectivity, independence number, edge independence number, covering number, edge covering number, chromatic number, edge chromatic number, maximum degree, minimum degree, clique number and domination number.

Note furthermore that the Hamiltonicity of $T_m(G)$ is not really needed; the connectedness of it is used only in the proof. So the proof hints the following general approach for dealing with interpolation problems. Let $\mathscr{F}$ be any family of objects under consideration and $\psi:\mathscr{F}\to\mathbb{Z}$ an integral function defined on $\mathscr{F}$. Then $\psi(\mathscr{F})$ is an integer interval if and only if there exists a connected graph $T(\mathscr{F})$ with vertex set $\mathscr{F}$ such that for each $F\in\mathscr{F}$, $\psi(N(F))$ is an integer interval, where $N(F)$ is the subset of $\mathscr{F}$ consisting of F and all neighbors of it in $T(\mathscr{F})$. This idea indicates the connection of local and total interpolation, and generalizes some basic principles used in [2, 7]. The author believes it will be useful in future studies of interpolation properties of graphs.

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