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Note

Upper bounds for f-domination number of graphs

Beifang Chen [a,1], Sanming Zhou [b,2,*]

[a] Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

[b] Department of Mathematics, The University of Western Australia, NEDLANDS, Perth, WA 6907, Australia

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Abstract

For an integer-valued function f defined on the vertices of a graph G, the f-domination number $\gamma_f(G)$ of G is the smallest cardinality of a subset $D \subseteq V(G)$ such that each $x \in V(G) - D$ is adjacent to at least $f(x)$ vertices in D. When $f(x) = k$ for all $x \in V(G)$, $\gamma_f(G)$ is the k-domination number $\gamma_k(G)$. In this note, we give a tight upper bound for γ_f and an improvement of the upper bound for a special f-domination number $\mu_{j,k}$ of Stracke and Volkmann (1993). Some upper bounds for γ_k are also obtained. © 1998 Elsevier Science B.V. All rights reserved

Let $G = (V(G), E(G))$ be a finite, undirected, simple graph. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a set $D \subseteq V(G)$ such that each $x \in V(G) - D$ is adjacent to at least one vertex in D. Extensive studies on $\gamma(G)$ and domination-related topics have been done in the last thirty years. In 1985, Fink and Jacobson [4,5] introduced the concept of k-domination. For a positive integer k, a set $D \subseteq V(G)$ is called a *k-dominating set* if each $x \in V(G) - D$ is adjacent to at least k vertices of D. The *k-domination number* $\gamma_k(G)$ is then defined to be the smallest cardinality of a k-dominating set of G (see [4]). The following upper bound for γ_k was proved in [1].

* Corresponding author. E-mail: smzhou@maths.uwa.edu.au.

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Theorem 1 (Caro and Roditty [1]). *Let G be a graph of p vertices and the minimum degree* $\delta(G) \geqslant ((n+1)/n)k - 1$, *where n and k are positive integers. Then*

$$\gamma_k(G) \leqslant \frac{n}{n+1} p. \tag{1}$$

This theorem generalizes the result that $\gamma_k(G) \leqslant kp/(k+1)$ if $\delta(G) \geqslant k$ (see [2]). In [6], a more general domination concept was introduced. For an integer-valued function f defined on $V(G)$, a set $D \subseteq V(G)$ is called an *f-dominating set* of G if each $x \in V(G) - D$ is adjacent to at least $f(x)$ vertices in D. The *f-domination number* $\gamma_f(G)$ is defined to be the smallest cardinality of an f-dominating set of G (see [8]). For integers j, k with $0 \leqslant j \leqslant k$, Stracke and Volkmann [6] defined the function $f_{j,k}(x) = \min\{j, j-k+d(x)\}$, where $d(x)$ is the degree of vertex x in G. Then they studied the $f_{j,k}$-domination number $\mu_{j,k}(G)$ and obtained the following result.

Theorem 2 (Stracke and Volkmann [6]). *If G is a graph of p vertices and* $0 \leqslant j \leqslant k$, *then*

$$\mu_{j,k}(G) \leqslant \begin{cases} \dfrac{2j-k}{2j-k+1} p, & \text{if } j \leqslant k \leqslant 2j-2, \\ p/2, & \text{if } k \geqslant 2j-1. \end{cases} \tag{2}$$

In this note we first generalize Theorem 1 to the case of f-domination number. With this generalization we then give an upper bound for $\mu_{j,k}$ which improves (2) slightly. As consequences, we obtain some upper bounds for γ_k. First we have the following theorem of which a weaker form appeared in [8].

Theorem 3. *Let f be an integer-valued function defined on V(G) and let n be a positive integer. If* $f(x) < (n/(n+1))(d(x)+1+1/n)$ *for each* $x \in V(G)$, *then*

$$\gamma_f(G) \leqslant \frac{n}{n+1} p. \tag{3}$$

Proof. The proof applies a similar idea used in [1]. Set

$$v = \max_{x \in V(G)} ((n+1)f(x) - n(d(x)+1)).$$

Then

$$d(x) \geqslant \frac{n+1}{n} f(x) - 1 - \frac{v}{n}, \tag{4}$$

and the given inequality implies $v < 1$.

Let $V_1, V_2, \ldots, V_{n+1}$ be a partition of $V(G)$ such that $H = G - \bigcup_{i=1}^{n+1} E(G[V_i])$ contains as many edges as possible, where $G[V_i]$ is the subgraph of G induced by V_i. Let $d_H(x)$ denote the degree of x in H. Then $d_H(x) \geqslant \lceil (n/(n+1))d(x) \rceil$ for each $x \in V(G)$

(see [3], an explicit proof can be found in [7, pp. 233]). In fact, suppose to the contrary that $(n+1)d_H(x) < nd(x)$ for a vertex, say, $x \in V_1$. Let $l \geqslant 2$ be such that the number of vertices in V_l which are adjacent to x is as small as possible. Let $W_1 = V_1 - \{x\}$, $W_l = V_l \cup \{x\}$ and $W_i = V_i$, $i \neq 1, l$. Then $G - \bigcup_{i=1}^{n+1} E(G[W_i])$ has more edges than H, a contradiction. From (4) we have

$$d_H(x) \geqslant \left\lceil \frac{n}{n+1}\left(\frac{n+1}{n}f(x) - 1 - \frac{v}{n}\right)\right\rceil$$

$$= \left\lceil f(x) - \frac{n+v}{n+1} \right\rceil$$

$$= \begin{cases} f(x), & \text{if } -n \leqslant v < 1, \\ > f(x), & \text{otherwise.} \end{cases}$$

Without loss of generality we may assume $|V_1| = \max_{1 \leqslant i \leqslant n+1} |V_i|$. From the inequality above we know that $V(G) - V_1$ is an f-dominating set of G. Thus,

$$\gamma_f(G) \leqslant p - |V_1| \leqslant p - \frac{p}{n+1} = \frac{n}{n+1}p. \quad \square$$

Corollary 4. *Let A be a subset of $V(G)$ with $\delta(G[A]) \geqslant 1$. Let*

$$n_0 = \begin{cases} \dfrac{\delta(G[A])}{\delta(G[A]) - k + 1}, & \text{if } (\delta(G[A]) - k + 1) \mid (k-1) \\ \left\lceil \dfrac{k-1}{\delta(G[A]) - k + 1} \right\rceil, & \text{otherwise} \end{cases}$$

for each k with $1 \leqslant k \leqslant \delta(G[A])$. Then $\gamma_k(G) \leqslant p - |A|/(n_0 + 1)$.

Proof. Since $n_0 > (k-1)/(\delta(G[A]) - k + 1)$, we have

$$k < \frac{n_0}{n_0+1}\left(\delta(G[A]) + 1 + \frac{1}{n_0}\right).$$

Hence, $\gamma_k(G[A]) \leqslant (n_0/(n_0+1))|A|$ by Theorem 3. Since a minimum k-dominating set of $G[A]$ together with $V(G) - A$ yields a k-dominating set of G, we get

$$\gamma_k(G) \leqslant p - |A| + \gamma_k(G[A]) \leqslant p - \frac{|A|}{n_0+1}. \quad \square$$

Theorem 3 is a generalization of Theorem 1, and the upper bound in (3) is attainable. For example, let x_0 be a fixed vertex of the complete graph K_p. Let $f(x_0) = p - 2$ and $f(x) = p - 1$ for all $x \in V(K_p) - \{x_0\}$. One can easily check that for $n = p - 1$ the condition in Theorem 3 holds, and it follows from (3) that $\gamma_f(K_p) \leqslant p - 1$. In fact, $\gamma_f(K_p) = p - 1$. Using Theorem 3 we can prove the following:

Theorem 5. *Let j,k be integers such that $0 \leqslant j \leqslant k$. Then*

$$\mu_{j,k}(G) \leqslant \begin{cases} \dfrac{2j-k-1}{2j-k}p, & \text{if } j+1 \leqslant k \leqslant 2j-3, \\ \dfrac{k}{k+1}p, & \text{if } k=j, \\ \frac{2}{3}p, & \text{if } k=2j-2, \\ \frac{1}{2}p, & \text{if } k \geqslant 2j-1. \end{cases} \tag{5}$$

Proof. If $k \geqslant 2j-1$, it was proved in [6] (also implied in Corollary 4 of [8]) that $\mu_{j,k}(G) \leqslant \frac{1}{2}p$. For the case $j+1 \leqslant k \leqslant 2j-3$, we claim that

$$f_{j,k}(x) < \frac{2j-k-1}{2j-k}\left(d(x)+1+\frac{1}{2j-k-1}\right). \tag{6}$$

We divide this into two cases.

Case 1: $d(x) \geqslant k$. Then $f_{j,k}(x)=j$ and (6) becomes $j(2j-k)<(2j-k-1)(d(x)+1)+1$. To prove this, it suffices to show $j(2j-k)<(2j-k-1)(k+1)+1$, or, equivalently, to show

$$\left(k-\frac{3j-2}{2}\right)^2 < \frac{1}{4}(j-2)^2,$$

which is true since $j<k<2j-2$.

Case 2: $d(x) \leqslant k-1$. Then $f_{j,k}(x)=j-k+d(x)$ and (6) is equivalent to $d(x)+1<(k-j+1)(2j-k)+1$. To prove this, it suffices to check $k<(k-j+1)(2j-k)+1$, which is equivalent to

$$\left(k-\frac{3j-2}{2}\right)^2 < \frac{1}{4}(j-2)^2+1.$$

But this is true as we have proved earlier. Thus, (6) is valid provided that $j+1 \leqslant k \leqslant 2j-3$. From Theorem 3 we get

$$\mu_{j,k}(G) \leqslant \frac{2j-k-1}{2j-k}p.$$

By a similar discussion as above it can be easily shown that $f_{j,2j-2}(x) \leqslant \frac{2}{3}(d(x)+1)$ and $f_{k,k}(x) \leqslant (k/(k+1))(d(x)+1)$ for each $x \in V(G)$. Again, we get $\mu_{j,2j-2}(G) \leqslant \frac{2}{3}p$ and $\mu_{k,k}(G) \leqslant (k/(k+1))p$ from Theorem 3. This completes the proof. □

Note that although (5) is just slightly better than (2) when $j+1 \leqslant k \leqslant 2j-3$, the proof is simpler. Theorem 5 implies the following improvement of Theorem 2 of [6].

Corollary 6. *Let k and l be integers with $1 \leqslant k \leqslant l$ and let $A_l = \{x \in V(G)\colon d(x) \geqslant l\}$. Then*

$$\gamma_k(G) \leqslant p - \max\left\{ \max_{k+1 \leqslant l \leqslant 2k-3} \frac{|A_l|}{2k-l}, \frac{|A_k|}{k+1}, \frac{|A_{2k-2}|}{3}, \frac{|A_{2k-1}|}{2} \right\}. \qquad (7)$$

Proof. Similar to the proof of Theorem 2 of [6]. □

If $l \leqslant \delta(G)$, then $|A_l| = p$. So (7) implies

Corollary 7. *For any integer $k \geqslant 1$,*

$$\gamma_k(G) \leqslant \begin{cases} \dfrac{2k-\delta(G)-1}{2k-\delta(G)}p, & \text{if } k+1 \leqslant \delta(G) \leqslant 2k-3, \\ \dfrac{\delta(G)}{\delta(G)+1}p, & \text{if } \delta(G) = k, \\ \frac{2}{3}p, & \text{if } \delta(G) = 2k-2 \geqslant 2, \\ \frac{1}{2}p, & \text{if } \delta(G) \geqslant 2k-1. \end{cases} \qquad (8)$$

This is an improvement of Corollary 2 in [6].

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