



Domination number and neighbourhood conditions

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Abstract

The domination number γ of a graph G is the minimum cardinality of a subset D of vertices of G such that each vertex outside D is adjacent to at least one vertex in D . For any subset A of the vertex set of G , let $\partial^+(A)$ be the set of vertices not in A which are adjacent to at least one vertex in A . Let $\bar{N}(A)$ be the union of A and $\partial^+(A)$, and $d(A)$ be the sum of degrees of all the vertices of A . In this paper we prove the inequality $2q \leq (p - \gamma)(p - \gamma + 2) - |\partial^+(A)| (p - \gamma + 1) + d(\bar{N}(A))$, and characterize the extremal graphs for which the equality holds, where p and q are the numbers of vertices and edges of G , respectively. From this we then get an upper bound for γ which generalizes the known upper bound $\gamma \leq p + 1 - \sqrt{2q + 1}$. Let $I(A)$ be the set of vertices adjacent to all vertices of A , and $\bar{I}(A)$ be the union of A and $I(A)$. We prove that $2q \leq (p - \gamma - |\bar{I}(A)| + 2)(p - \gamma + 4) + d(\bar{I}(A)) - \min\{p - \gamma - |\bar{I}(A)| + 2, |A|, |I(A)|, 3\}$, which implies an upper bound for γ as well. © 1999 Elsevier Science B.V. All rights reserved

1. Introduction

Let G be a finite, undirected and simple graph with vertex set $V(G)$. A subset D of $V(G)$ is a *dominating set* if each vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D . The *domination number* of G , denoted by $\gamma(G)$, is defined to be the minimum cardinality of a dominating set of G . Topics on domination number and related parameters have long attracted graph theorists for their strongly practical background and theoretical interesting. It has been proved [7] that the decision problem corresponding to the domination number for arbitrary graphs is NP-complete. Thus, the exploration of lower and upper bounds for the domination number as sharp as possible is of great significance. In this direction there are now a number of estimations for the domination number of a graph in terms of some basic parameters such as the numbers of vertices and edges, the minimum and maximum degrees, and so on. For example,

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an early result of Ore [10] states that the domination number $\gamma \leq p/2$ if G contains no isolated vertices, where p is the number of vertices of G . This result was improved as $\gamma \leq 2p/5$ in [9] for connected graph G which has minimum degree at least two and is not one of seven exceptional graphs. In 1965, Vizing [14] proved the following inequality involving the domination number and the numbers of vertices and edges.

Theorem 1 (Vizing [14]). *For any graph G with p vertices and q edges, the domination number γ satisfies*

$$2q \leq (p - \gamma)(p - \gamma + 2). \quad (1)$$

Moreover, the equality holds if and only if G is the vertex disjoint union of $\gamma - 2$ isolated vertices and a $(p - \gamma + 2)$ -clique with the removal of a minimum edge covering.

Theorem 1 implies the following upper bound.

Corollary 1 (Vizing [14]). $\gamma \leq p + 1 - \sqrt{2q + 1}$. (2)

As an improvement of Vizing's inequality, Fulmann [6] proved the following

Theorem 2 (Fulman [6]). *Let Δ be the maximum degree of G . Then*

$$2q \leq (p - \gamma)(p - \gamma + 2) - \Delta(p - \gamma - \Delta). \quad (3)$$

By using this, Fulmann [6] gave a short proof for a result of Sanchis [13] which states that

$$2q \leq (p - \gamma)(p - \gamma + 1)$$

if $\gamma \geq 3$ and G has no isolated vertices (the same result was proved in [15] for connected graph with $\gamma \geq 3$). The equality is unattainable when $\Delta \leq p - \gamma - 1$, as showed in [6]. In [11] Payan proved that

$$\gamma \leq \frac{1}{2}(p + 2 - \delta)$$

and

$$\gamma \leq \frac{(p - 1 - \Delta)(p - 2 - \delta)}{p - 1} + 2,$$

where Δ and δ are, respectively, the maximum and minimum degrees of G . Further, Payan [11] stated without proof the inequality

$$\gamma \leq \frac{1}{2}(p + 1 - \delta)$$

for the graphs without isolated vertices not isomorphic to the complement of a 1-regular graph or with at least one component not isomorphic to a square. This inequality was proved in [5]. In the same paper, Flach and Volkmann also gave several upper bounds

for γ in terms of the neighbourhood union $N(A)$ of a subset A of $V(G)$. They proved among others that

$$\gamma \leq \frac{1}{2} \left(p + |A| - (\delta - 1) \frac{|N(A) \setminus A|}{\delta} \right), \quad (4)$$

where $N(A)$ is the set of vertices adjacent in G to at least one vertex of A . From this they obtained that

$$\gamma \leq \frac{1}{2} \left(p + 1 - \frac{\Delta(\delta - 1)}{\delta} \right)$$

and

$$\gamma \leq \frac{1}{2}(p - (\delta - 2)\alpha),$$

where α is the maximum cardinality of an independent set of G such that no vertex of G is adjacent to two distinct vertices in the set. Relationships between the domination number and the neighbourhood union can be found in [2, Lemma 2] also. For other estimations of the domination number, the reader can consult, for example, [2–5, 11, 12].

The main purpose of this paper is to investigate further the relationships between the domination number and the neighbourhood union and intersection. More precisely, we will give inequalities involving γ and either the neighbourhood union or the neighbourhood intersection of a subset of $V(G)$, and then derive sharp upper bounds for γ . It is expected that this can supplement the existing results mentioned above. The work was mainly motivated by the desire of giving a more general form (see Theorem 3 in the next section) for Theorems 1 and 2. It was also inspired by the recent year's work on the characterization of the hamiltonicity by using neighbourhood conditions (see e.g. the survey paper [8]).

Throughout the paper we assume G is a finite, undirected and simple graph with vertex set $V(G)$. As above, we use γ , p , q , Δ and δ to denote the domination number, the number of vertices, the number of edges, the maximum degree and the minimum degree of G , respectively. Dominating sets of G with the minimum cardinality are called the *minimum dominating sets*. The neighbourhood $N(x)$ of a vertex x of G is the set of vertices adjacent to x in G . For a subset A of $V(G)$, the *neighbourhood union* and the *neighbourhood intersection* of A are defined to be $N(A) = \bigcup_{x \in A} N(x)$ and $I(A) = \bigcap_{x \in A} N(x)$, respectively. Denote $\bar{N}(A) = A \cup N(A)$ and $\bar{I}(A) = A \cup I(A)$. We call $\partial^+(A) = N(A) \setminus A$ and $\partial^-(A) = \partial^+(V(G) \setminus A)$ the *outer* and *inner boundaries* of A , respectively. For any vertex $x \in V(G)$, let $d_A(x) = |N(x) \cap A|$. Thus, $d_{V(G)}(x) = d(x)$ is the degree of x in G , and if $x \in A$ then $d_A(x)$ is the degree of x in $G[A]$, the subgraph of G induced by A . We use $d(A)$ to denote the sum of degrees $d(x)$ of all the vertices $x \in A$. A *minimum edge covering* of a graph is a smallest set of edges such that each vertex of the graph is incident with at least one edge in the set. For other undefined notations and terminology, the reader is referred to [1].

2. Domination number and neighborhood unions

Let $A \subseteq V(G)$ and $S = V(G) \setminus \overline{N}(A)$. We call A a *type 1 set* of G if

- (a) $\partial^+(A) \neq \emptyset, S \neq \emptyset$;
 - (b) S is an independent set of G ;
 - (c) $A \cup S$ is a minimum dominating set of G ; and
 - (d) $d_S(y) = 1$ for all $y \in \partial^+(A)$, and $d_{\partial^+(A)}(z) \geq 1$ for all $z \in S$.
- A is said to be a *type 2 set* if it satisfies (a) and the following (e)–(g):
- (e) $G[S]$ is a complete graph with the removal of a perfect matching;
 - (f) A together with any two nonadjacent vertices in S is a minimum dominating set of G ;
 - (g) $d_S(y) = |S| - 1$ for all $y \in \partial^+(A)$, and if $|S| = 2$ then $d_{\partial^+(A)}(z) \geq 1$ for each $z \in S$.

Note that in both cases, A is the unique minimum dominating set of $G[\overline{N}(A)]$. The main result in this section is the following theorem.

Theorem 3. *For any subset A of $V(G)$, we have*

$$2q \leq (p - \gamma)(p - \gamma + 2) - |\partial^+(A)| (p - \gamma + 1) + d(\overline{N}(A)). \quad (5)$$

Furthermore, if G contains no isolated vertices, then the equality holds if and only if A is either a type 1 set or a type 2 set.

Proof. Let $S = V(G) \setminus \overline{N}(A)$. Then

$$|S| = p - |A| - |\partial^+(A)|. \quad (6)$$

Since the union of A and a minimum dominating set of $G[S]$ is a dominating set of G , we have

$$\gamma(G[S]) \geq \gamma - |A|. \quad (7)$$

Clearly, $(S \setminus N(y)) \cup A \cup \{y\}$ is a dominating set of G for any $y \in \partial^+(A)$. This implies

$$|N(y) \cap S| \leq |S| + |A| + 1 - \gamma. \quad (8)$$

Note that $2q = d(V(G))$ and $2q(G[S]) \leq (|S| - \gamma(G[S]))(|S| - \gamma(G[S]) + 2)$ by Theorem 1. From (6)–(8) we get

$$\begin{aligned} 2q &= 2q(G[S]) + \sum_{y \in \partial^+(A)} |N(y) \cap S| + d(\overline{N}(A)) \\ &\leq (|S| - \gamma(G[S]))(|S| - \gamma(G[S]) + 2) + |\partial^+(A)|(|S| + |A| + 1 - \gamma) + d(\overline{N}(A)) \\ &\leq (|S| + |A| - \gamma)(|S| + |A| + 2 - \gamma) + |\partial^+(A)|(|S| + |A| + 1 - \gamma) + d(\overline{N}(A)) \\ &= (|S| + |A| - \gamma)(p - \gamma + 2) + |\partial^+(A)| + d(\overline{N}(A)) \\ &= (p - \gamma)(p - \gamma + 2) - |\partial^+(A)| (p - \gamma + 1) + d(\overline{N}(A)), \end{aligned}$$

which is just (5).

It is not difficult to check that if A is a type 1 set or a type 2 set, then the equality in (5) occurs. Conversely, suppose that the equality in (5) is attained. Then from the proof above we have

- (i) $\gamma = |A| + \gamma(G[S])$;
- (ii) for any $y \in \partial^+(A)$, $(S \setminus N(y)) \cup A \cup \{y\}$ is a minimum dominating set of G , thus
 - (ii.1) $|N(y) \cap S| = |S| + |A| + 1 - \gamma$,
 - (ii.2) $S \setminus N(y)$ is an independent set,
 - (ii.3) if $y' \in \partial^+(A)$, $y' \neq y$, then $d_{S \setminus N(y)}(y') \leq 1$,
 - (ii.4) for each $z \in N(y) \cap S$, $d_{S \setminus N(y)}(z) \leq 1$; and
- (iii) S can be partitioned into S_1 and S_2 such that $G[S_1]$ is an $(|S| - k + 2)$ -clique with the removal of a minimum edge covering and S_2 is a set of $k - 2$ isolated vertices in $G[S]$, where $k = \gamma(G[S])$.

From (i) and (ii) we know $S \neq \emptyset$. If $|S| = 1$, then $\gamma = |A| + 1$ by (i) and $|N(y) \cap S| = 1$ by (ii.1). So clearly A is a type 1 set in such case. In the following we suppose $|S| \geq 2$. From (i) and (ii) we have $k = |S \setminus N(y)| + 1$ for each $y \in \partial^+(A)$. Thus,

$$|S \setminus N(y)| = |S_2| + 1, \tag{9}$$

$$|N(y) \cap S| = |S_1| - 1. \tag{10}$$

We distinguish two cases.

Case 1: $|S| - k$ is even.

Then $G[S_1]$ is an $(|S| - k + 2)$ -clique with the removal of a perfect matching. So by (ii.2) there are for each $y \in \partial^+(A)$ at most two vertices of S_1 which are not in $N(y) \cap S$.

Subcase 1.1: There exists a vertex $y_0 \in \partial^+(A)$ such that there are exactly two vertices $z, z' \in S_1$ which are not in $N(y_0) \cap S$. Then $S_1 \setminus \{z, z'\} \subseteq N(y_0) \cap S$ and $z, z' \in S \setminus N(y_0)$ are not adjacent. In fact, we must have $S_1 = \{z, z'\}$ since otherwise a vertex of $S_1 \setminus \{z, z'\}$ is adjacent to both z and z' , contradicting (ii.4). So for any $y \in \partial^+(A)$, $|N(y) \cap S| = 1$ by (10). Note that S is in fact an independent set and $\gamma = |A| + |S|$ by (i). So A is a type 1 set.

Subcase 1.2: For each $y \in \partial^+(A)$, there is exactly one vertex $z_y \in S_1$ which is not in $N(y) \cap S$. From (10) we know $N(y) \cap S = S_1 \setminus \{z_y\}$. Since G has no isolated vertices, we have $S_2 = \emptyset$. Hence $k = 2$ and $\gamma = |A| + 2$ by (i). Clearly, A is a type 2 set.

Case 2: $|S| - k$ is odd.

In such case $|S_1| \geq 3$ and there is a vertex $z_0 \in S_1$ with $d_{S_1}(z_0) = |S_1| - 3$ and $d_S(z) = |S_1| - 2$ for all $z \in S_1 \setminus \{z_0\}$. Suppose $z_1, z_2 \in S_1$ are the vertices not adjacent to z_0 . From (ii.2) and (10) we know $|S_1| - 2 \leq d_{S_1}(y) \leq |S_1| - 1$ for each $y \in \partial^+(A)$.

Subcase 2.1: There exists $y_0 \in \partial^+(A)$ with $d_{S_1}(y_0) = |S_1| - 2$. Then $N(y_0) \cap S = (S_1 \setminus \{z, z'\}) \cup \{w\}$ and $S \setminus N(y_0) = (S_2 \setminus \{w\}) \cup \{z, z'\}$, where z, z' are nonadjacent vertices of $G[S_1]$ and $w \in S_2$. Note that $S_1 \setminus \{z, z'\} \neq \emptyset$ and by (ii.4) any vertex of $S_1 \setminus \{z, z'\}$ cannot be adjacent to z and z' simultaneously. We must have $|S_1| = 3$ since otherwise there exists a vertex of $S_1 \setminus \{z, z'\}$ which is adjacent to both z and z' . Thus $S_1 = \{z_0, z_1, z_2\}$ and $\{z, z'\} = \{z_0, z_1\}$ or $\{z_0, z_2\}$. From (10), $d_S(y) = 2$ for all $y \in \partial^+(A)$.

Since $G[S]$ contains only one edge z_1z_2 , we have $2q = d(\overline{N}(A)) + 2|\partial^+(A)| + 2$. On the other hand, we have $\gamma = |A| + |S| - 1$ by (i) and hence the right-hand side of (5) is $d(\overline{N}(A)) + 2|\partial^+(A)| + 3$, contradicting our assumption.

Subcase 2.2: For all $y \in \partial^+(A)$, we have $d_{S_1}(y) = |S_1| - 1$. Then $N(y) \cap S = N(y) \cap S_1 = S \setminus \{z_y\}$ for a vertex $z_y \in S_1$. Since G contains no isolated vertices, we have $S_2 = \emptyset$. Thus $\gamma(G[S]) = 2$ and $\gamma = |A| + 2$. Summing up the degrees of the vertices of G , we get $2q = d(\overline{N}(A)) + |\partial^+(A)|(|S| - 1) + |S|(|S| - 2) - 1$. But the right-hand side of (5) can be simplified as $d(\overline{N}(A)) + |\partial^+(A)|(|S| - 1) + |S|(|S| - 2)$, a contradiction as well.

In summary, we have proved that if G contains no isolated vertices and if the equality in (5) occurs, then A is either a type 1 set or a type 2 set. This completes the proof of Theorem 3. \square

Theorem 2 can be deduced from Theorem 3 by setting A to be the singleton of a maximum degree vertex. Moreover, we are now able to characterize the extremal graphs for (3).

Corollary 2. *For any graph G , we have*

$$2q \leq (p - \gamma)(p - \gamma + 2) - \Delta(p - \gamma - \Delta). \quad (3)$$

Furthermore, if G contains no isolated vertices, then the equality holds if and only if G is a complete graph with the removal of a perfect matching or there exists an even number $k \geq 4$ such that $p = k^2 - k + 1$ and G is a complete $(k^2 - 2k)$ -regular graph with the properties that for any vertex $x \in V(G)$,

- (a) $S_x = V(G) \setminus \overline{N}(x)$ induces a complete graph with the removal of a perfect matching; and
- (b) any vertex in $N(x)$ is adjacent to $k - 1$ vertices in S_x .

Proof. Let x be a vertex with the maximum degree Δ . By setting $A = \{x\}$ and noting that $d(\overline{N}(A)) \leq \Delta(\Delta + 1)$ we get (3) immediately from (5). One can check that if G is one of the graphs described in the corollary, then the equality in (3) occurs. Conversely, suppose that the equality in (3) holds, then by Theorem 3 for any maximum degree vertex x , $\{x\}$ is either a type 1 set or a type 2 set and $d(y) = \Delta$ for all $y \in N(x)$. Let $S_x = V(G) \setminus \overline{N}(x)$. Then $|S_x| = p - \Delta - 1$. We distinguish two cases.

Case 1: There exists a maximum degree vertex x such that $\{x\}$ is a type 1 set.

Then $d_{N(x)}(y) = d(y) - 2 = \Delta - 2$ for each $y \in N(x)$. Thus $G[N(x)]$ is a complete graph with the removal of a perfect matching (which implies that Δ is even). For any $y \in N(x)$, let z_y be the unique vertex in S_x adjacent to y . If $y, y' \in N(x)$ are not adjacent, then we have $z_y = z_{y'}$ since otherwise $(S_x \setminus \{z_y\}) \cup \{y\}$ would be a dominating set of G , contradicting the fact that $\gamma = |S_x| + 1$. For the case when $|S_x| = 1$, it can be seen easily that G is a complete graph with the removal of a perfect matching. Suppose now $|S_x| \geq 2$. Then $\Delta \geq 4$ and for $y \in N(x)$ we have $N(y) = (N(x) \setminus \{y, y'\}) \cup \{x, z_y\}$, $S_y = (S_x \setminus \{z_y\}) \cup \{y'\}$, where y' is the unique vertex in $N(x)$ not adjacent to y . If $|S_x| \geq 3$, then by noting that S_y is an independent set we know $\{y\}$ cannot be a type 2

set. So it must be of type 1. Thus any $y'' \in N(x) \setminus \{y, y'\} \subseteq N(y)$ is adjacent to exactly one vertex in S_y . Since y'' is adjacent to $y' \in S_y$, we conclude that y'' is not adjacent to any vertex in $S_x \setminus \{z_y\}$. But $y'' \in N(x)$, so it is adjacent to exactly one vertex in S_x . This implies that y'' is adjacent to z_y and hence the vertices in $S_x \setminus \{z_y\}$ are isolated, contradicting our assumption. If $|S_x|=2$, then $\{x\}$ and all $\{y\}$ are both type 1 and type 2 sets. Thus, the vertices in $N(y)$ must all have degree Δ . Since $z_y \in N(y)$, z_y is adjacent to each vertex in $N(x)$. Therefore, the vertex in $S_x \setminus \{z_y\}$ is isolated, a contradiction as well.

Case 2: For all maximum degree vertex x , $\{x\}$ is a type 2 set.

Then $|S_x| \geq 2$ is even. If $|S_x|=2$, then a contradiction can be made by a similar discussion as above. In the following, we suppose $|S_x| \geq 4$. Note that G is connected and the vertices adjacent to a maximum degree vertex are also of maximum degree. So G is Δ -regular. Since each $y \in N(x)$ is adjacent to $|S_x| - 1$ vertices in S_x and $G[S_x]$ is a complete graph with the removal of a perfect matching, we have $\Delta|S_x| = d(S_x) = \Delta(|S_x| - 1) + |S_x|(|S_x| - 2)$, which implies

$$|S_x| = 1 + \sqrt{\Delta + 1}. \tag{11}$$

Thus $\Delta + 1$ is a square number (the square of a positive integer). Since $|S_x| = p - \Delta - 1$, (11) gives $p - \Delta - 1 = 1 + \sqrt{\Delta + 1}$. So we get $\Delta = (2p - 3 - \sqrt{4p - 3})/2$, which implies that $4p - 3$ is also a square number. Suppose $4p - 3 = (2k - 1)^2$. Then $p = k^2 - k + 1$, $\Delta = k^2 - 2k$ and $|S_x| = k$. Hence k must be even. Since $\{x\}$ is a type 2 set, each vertex in $N(x)$ is adjacent to exactly $k - 1$ vertices in S_x . Thus, G satisfies (a) and (b). The proof is complete. \square

Note that in Theorem 3 and Corollary 2 the condition that G contains no isolated vertices is non-essential since $p - \gamma$ and q remain unchanged when isolated vertices are deleted from a graph. The extremal graphs described in Corollary 2 have domination number 2 or 3. So we have the following consequence which shows that (3) can be slightly improved in some cases.

Corollary 3. *If the graph resulted from G by deleting all the isolated vertices has domination number ≥ 4 or is not a regular graph of degree $4l(l - 1)$ for some $l \geq 2$, then*

$$2q \leq \begin{cases} (p - \gamma)(p - \gamma + 2) - \Delta(p - \gamma - \Delta) - 2 & \text{if both } p - \gamma \text{ and } \Delta \text{ are even,} \\ (p - \gamma)(p - \gamma + 2) - \Delta(p - \gamma - \Delta) - 1 & \text{otherwise.} \end{cases} \tag{12}$$

By choosing A to be a 2-subset of $V(G)$ in (5), we get the following:

Corollary 4. *For any two distinct vertices $x, y \in V(G)$, we have*

$$2q \leq (p - \gamma)(p - \gamma + 2) - (p - \gamma - \Delta)(|N(x) \cup N(y)| - 2\delta_{x,y}) + |N(x) \cap N(y)| + 2\delta_{x,y}, \tag{13}$$

where $\delta_{x,y} = 1$ if x, y are adjacent in G , and 0 otherwise.

If $G[A]$ contains no isolated vertices, then we can do even better than (5). In fact, in such case $G[A \setminus \partial^-(A)]$ also contains no isolated vertices and hence $\gamma(G[A \setminus \partial^-(A)]) \leq |A \setminus \partial^-(A)|/2$ by Ore [10]. Note that the union of $\partial^-(A)$, a minimum dominating set of $G[S]$ and a minimum dominating set of $G[A \setminus \partial^-(A)]$ is a dominating set of G . Thus, $\gamma \leq \gamma(G[S]) + |\partial^-(A)| + |A \setminus \partial^-(A)|/2$ and (7) can be improved as $\gamma(G[S]) \geq \gamma - (|A| + |\partial^-(A)|)/2$. Similarly, for any $y \in \partial^+(A)$, $(S \setminus N(y)) \cup \partial^-(A) \cup \{y\}$ together with a minimum dominating set of $G[A \setminus \partial^-(A)]$ gives a dominating set of G . So (8) can be improved as $|N(y) \cap S| \leq |S| + (|A| + |\partial^-(A)|)/2 + 1 - \gamma$. Similar to the proof of (5) we can prove the following

Theorem 3'. *If $A \subseteq V(G)$ and $G[A]$ contains no isolated vertices, then*

$$2q \leq (p - \gamma)(p - \gamma + 2) - (|A \setminus \partial^-(A)| + |\partial^+(A)|)(p - \gamma + 1) + \frac{1}{4}|A \setminus \partial^-(A)|(|A \setminus \partial^-(A)| + 2|\partial^+(A)|) + d(\bar{N}(A)). \quad (5')$$

This is better than (5) when $\delta(G[A]) > 0$. Note that (3) can give an upper bound for γ only if $A \leq 2(\sqrt{6q+4}-1)/3$. However, (5) and (5') always imply upper bounds for the domination number.

Corollary 5. (i) *For any $A \subseteq V(G)$, we have*

$$\gamma \leq p + 1 - \frac{1}{2}(|\partial^+(A)| + \sqrt{|\partial^+(A)|^2 + 8q + 4 - 4d(\bar{N}(A))}), \quad (14)$$

with equality if and only if A is a type 1 set or a type 2 set.

(ii) *If in addition $\delta(G[A]) > 0$, then*

$$\gamma \leq p + 1 - \frac{1}{2}(|A \setminus \partial^-(A)| + |\partial^+(A)| + \sqrt{|\partial^+(A)|^2 + 8q + 4 - 4d(\bar{N}(A))}). \quad (14')$$

Since A is arbitrary, we can specify (14) and (14') and thus get interesting upper bounds for γ by taking A to be special subsets of $V(G)$. For example, in the degenerate case where $A = \emptyset$, (14) becomes Vizing's upper bound (2). Taking $A = \{x\}$, we get the following corollary.

Corollary 6. *For any $x \in V(G)$,*

$$\gamma \leq p + 1 - \frac{1}{2}(d(x) + \sqrt{(d(x))^2 + 8q + 4 - 4d(\bar{N}(x))}). \quad (15)$$

In particular, we have

$$\gamma \leq p + 1 - \frac{1}{2}(A + \sqrt{A^2 + 8q + 4 - 4d_A}), \quad (16)$$

where $d_A = \min_{d(x)=A} d(\bar{N}(x))$.

The upper bound (15) ((16), respectively) is better than the known bound (2) if $d(\bar{N}(x)) \leq d(x)\sqrt{2q+1}$ ($d_A \leq A\sqrt{2q+1}$, respectively). Inequalities (5) and (13) are

sometimes better than (3), and (14) and (15) are better than (2) in some cases when A and x are chosen appropriately. Also, neither (14) nor (4) is implied by the other.

3. Domination number and neighbourhood intersections

In this section we discuss the relationships between the domination number and the neighbourhood intersection. Since $I(A) = N(A)$ when $|A| = 1$ and the case has been studied in Section 2, we assume $|A| \geq 2$ in the following discussion.

Theorem 4. *Suppose $A \subseteq V(G)$ with $|A| \geq 2$ and $|I(A)| \geq 1$. Then*

$$2q \leq (p - \gamma - |\bar{I}(A)| + 2)(p - \gamma + 4) + d(\bar{I}(A)) - \min\{p - \gamma - |\bar{I}(A)| + 2, |A|, |I(A)|, 3\}. \tag{17}$$

Proof. Let $S = V(G) \setminus \bar{I}(A)$. Then $\gamma(G[S]) \geq \gamma - 2$. For any $x \in A, y \in I(A), (S \setminus N(x)) \cup \{x, y\}$ and $(S \setminus N(y)) \cup \{x, y\}$ are both dominating sets of G . So we have $|N(x) \cap S| \leq |S| - \gamma + 2$ and $|N(y) \cap S| \leq |S| - \gamma + 2$. By using Vizing's theorem and summing up the degrees of vertices of G , we get

$$\begin{aligned} 2q &\leq 2q(G[S]) + \sum_{x \in A} (|N(x) \cap S| + |N(x)|) + \sum_{y \in I(A)} (|N(y) \cap S| + |N(y)|) \\ &\leq (|S| - \gamma(G[S]))(|S| - \gamma(G[S]) + 2) + |\bar{I}(A)| (|S| - \gamma + 2) + d(\bar{I}(A)) \\ &\leq (|S| - \gamma + 2)(|S| - \gamma + 4) + |\bar{I}(A)| (|S| - \gamma + 2) + d(\bar{I}(A)) \\ &= (p - \gamma - |\bar{I}(A)| + 2)(p - \gamma + 4) + d(\bar{I}(A)). \end{aligned} \tag{18}$$

We will show that this upper bound can be further improved.

Case 1: There exists $w \in \bar{I}(A)$ with $N(w) \cap S = \emptyset$.

Then from the proof above we know the right-hand side of (18) can be decreased by $|S| - \gamma + 2$.

Case 2: For all $w \in \bar{I}(A), N(w) \cap S \neq \emptyset$.

Let

$$\begin{aligned} A_1 &= \{x \in A: |N(x) \cap S| = |S| - \gamma + 2\}, \\ I_1 &= \{y \in I(A): |N(y) \cap S| = |S| - \gamma + 2\}, \\ A_2 &= A \setminus A_1 \text{ and } I_2 = I(A) \setminus I_1. \end{aligned}$$

Subcase 2.1: $A_1 = \emptyset$ or $I_1 = \emptyset$. If $A_1 = \emptyset$, then $|N(x) \cap S| \leq |S| - \gamma + 1$ for all $x \in A$ and the right-hand side of (18) decreases by $|A|$. If $I_1 = \emptyset$, it decreases by $|I(A)|$.

Subcase 2.2: $A_1 \neq \emptyset$ and $I_1 \neq \emptyset$. Then we have the following:

Claim 1. *If $x \in A_1$, then $N(y) \cap S \subseteq N(x) \cap S$ for all $y \in I(A)$.*

In fact, $(S \setminus N(x)) \cup \{x, y\}$ is a minimum dominating set since $x \in A_1$. If $N(y) \cap S \not\subseteq N(x) \cap S$, then there exists $z \in S \setminus N(x)$ which is adjacent to y . So $((S \setminus N(x)) \setminus \{z\}) \cup \{x, y\}$ is a dominating set, a contradiction. Similarly, we have

Claim 2. *If $y \in I_1$, then $N(x) \cap S \subseteq N(y) \cap S$ for all $x \in A$.*

Claims 1 and 2 imply that $N(w) \cap S$ are all identical for $w \in A_1 \cup I_1$, i.e., there exists $\emptyset \neq S^* \subseteq S$ with $|S^*| = |S| - \gamma + 2$ such that $N(w) \cap S = S^*$ for all $w \in A_1 \cup I_1$. Thus, we have $A_2 \neq \emptyset$ since otherwise the vertices in S^* are adjacent to all vertices in A , a contradiction. Also, from Claims 1 and 2 we know $N(w) \cap S \subseteq S^*$ for all $w \in A_2 \cup I_2$.

Now, we prove that $2q(G[S]) < (|S| - \gamma + 2)(|S| - \gamma + 4)$. Suppose otherwise, then $\gamma = \gamma(G[S]) + 2$ and by Vizing's theorem S can be partitioned into two parts S_1 and S_2 such that $G[S_1]$ is a complete graph with the removal of a minimum edge covering and S_2 is a set of isolated vertices of $G[S]$. Thus $|S_1| = |S| - \gamma + 4$ and $|S_2| = \gamma(G[S]) - 2 = \gamma - 4$. Since $|S^*| = |S| - \gamma + 2$, there are at least two vertices in S_1 which are not in S^* . But $S \setminus S^*$ is an independent set, so there are exactly two nonadjacent vertices z_1, z_2 of S_1 which are not in S^* . Therefore, we have $S^* = S_1 \setminus \{z_1, z_2\}, S \setminus S^* = S_2 \cup \{z_1, z_2\}$. If $|S^*| \geq 2$, then by the structure of $G[S_1]$ there exists a vertex $z \in S^*$ which is adjacent to both z_1 and z_2 . Thus, for $x \in A, y \in I(A)$, $S_2 \cup \{x, y, z\}$ is a dominating set of $\gamma - 1$ vertices, a contradiction. So we must have $|S^*| = 1$. But $\emptyset \neq N(w) \cap S \subseteq S^*$ for all $w \in A_2$, so we get $N(w) \cap S = S^*$, a contradiction as well. Thus, we have proved that

$$2q(G[S]) \leq (|S| - \gamma + 2)(|S| - \gamma + 4) - 1. \quad (19)$$

If $I_2 = \emptyset$, take a vertex $z_0 \in S^*$ and put $A' = A \cup \{z_0\}$. Then $I(A') = I(A)$. Note that $A_2 \neq \emptyset$ and hence $d(z_0) \leq (|\bar{I}(A)| - 1) + (|S_1| - 2) = p - \gamma + 1$, we get from (18) that

$$\begin{aligned} 2q &\leq (p - \gamma - |\bar{I}(A')| + 2)(p - \gamma + 4) + d(\bar{I}(A')) \\ &= (p - \gamma - |\bar{I}(A)| + 1)(p - \gamma + 4) + d(\bar{I}(A)) + d(z_0) \\ &\leq (p - \gamma - |\bar{I}(A)| + 2)(p - \gamma + 4) + d(\bar{I}(A)) - 3. \end{aligned}$$

If $I_2 \neq \emptyset$, then from (19), the proof of (18) and the fact $A_2 \neq \emptyset$ we know that the above inequality is also true. This completes the proof of Theorem 4. \square

Corollary 7. *Suppose $A \subseteq V(G)$ with $|A| \geq 2$ and $|I(A)| \geq 1$. Let*

$$\lambda(A) = \begin{cases} 0 & \text{if } d(x) = 0 \text{ for all } x \in V(G) \setminus \bar{I}(A), \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$\gamma \leq p + 3 - \frac{1}{2}(|\bar{I}(A)|) + \sqrt{(|\bar{I}(A)| + 2)^2 + 8q + 4\lambda(A) - 4d(\bar{I}(A))}. \quad (20)$$

Proof. Let $\mu(A) = \min\{p - \gamma - |\bar{I}(A)| + 2, |A|, |I(A)|, 3\}$. Then $\mu(A) \geq \lambda(A)$ and (20) follows from (17) immediately. \square

Both (17) and (20) are attainable. As a simple example, we take G to be a complete bipartite graph with the bipartition $X \cup Y$, where $2 \leq |X| \leq |Y|$, and set $A = X$. Then the equalities in both (17) and (20) appear. Unfortunately, at the moment we are not able to characterize the extremal graphs for (17).

It is expected that for some special families of graphs the results in this paper can be further improved.

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References

- [1] J.A. Bondy, U.S.R. Murty, *Graph Theory with Applications*, Macmillan Press, London and Basingstoke, 1976.
- [2] R.C. Brigham, R.D. Dutton, A compilation of relations between graph invariants, *Networks* 15 (1985) 73–107.
- [3] R.C. Brigham, R.D. Dutton, Bounds on the domination number of a graph, *Quart. J. Math. Oxford* 41 (1989) 269–275.
- [4] R.C. Brigham, R.D. Dutton, A compilation of relations between graph invariants — Suppl. I, *Networks* 21 (1991) 421–455.
- [5] P. Flach, L. Volkmann, Estimations for the domination number of a graph, *Discrete Math.* 80 (1990) 145–151.
- [6] J. Fulman, A generalization of Vizing's theorem on domination, *Discrete Math.* 126 (1994) 403–406.
- [7] M.R. Garey, D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman, San Francisco, 1979.
- [8] R.J. Gould, Updating the Hamiltonian problem — a survey, *J. Graph Theory* 15 (1991) 121–157.
- [9] W. McCuaig, B. Shepherd, Domination in graphs with minimum degree two, *J. Graph Theory* 13 (1989) 749–762.
- [10] O. Ore, *Theory of Graphs*, Am. Math. Soc. Colloq. Publ. 38, Providence, RI, 1962.
- [11] C. Payan, Sur le nombre d'absorption d'un graphe simple, *Cahiers Centre Etudes Rech. Oper.* 17 (1975) 307–317.
- [12] C. Payan, *Sur quelques problèmes de couverture et de couplage en combinatoire*, Thèse Grenoble, 1977.
- [13] L.A. Sanchis, Maximum number of edges in connected graphs with a given domination number, *Discrete Math.* 87 (1991) 65–72.
- [14] V.G. Vizing, A bound on the external stability number of a graph, *Dokl. Akad. Nauk.* 164 (1965) 729–731.
- [15] V.G. Vizing, Estimate the number of external stability of a graph, *Doklady* (1965) 1275–1278.