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Note

An inequality between the diameter and the inverse dual degree of a tree

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Abstract

Let T be a nontrivial tree with diameter $D(T)$ and radius $R(T)$. Let $I(T)$ be the inverse dual degree of T which is defined to be $\sum_{u\in V(T)} 1/\bar{d}(u)$, where $\bar{d}(u) = (\sum_{v\in N(u)} d(v))/d(u)$ for $u \in V(T)$. For any longest path P of T, denote by $a(P)$ the number of vertices outside P with degree at least 2, $b(P)$ the number of vertices on P with degree at least 3 and distance at least 2 to each of the end-vertices of P, and $c(P)$ the number of vertices adjacent to one of the end-vertices of P and with degree at least 3. In this note we prove that $I(T) \geqslant D(T)/2 + a(P)/3 + b(P)/10 + c(P)/12 + \frac{5}{6}$. As a corollary we then get

$$I(T) \geqslant \begin{cases} R(T) + 1/3 & \text{if } D(T) \text{ is odd,} \\ R(T) + 5/6 & \text{if } D(T) \text{ is even,} \end{cases}$$

with equality if and only if T is a path of at least four vertices. The latter inequality strengthens a conjecture made by the program Graffiti.

Keywords: Tree; Diameter; Radius; Inverse dual degree; Graffiti conjecture

1. Introduction

Investigation of relations among various graph invariants is one of the most fundamental tasks of graph theory. One can easily find a number of results of this kind in

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any standard textbook on graphs. Well-known examples include the theorem of Brooks which claims that the chromatic number of a connected graph does not exceed its maximum degree unless the graph is complete or is an odd cycle, and Vizing's result which asserts that the edge chromatic number of a simple graph is equal to the maximum degree or the maximum degree plus one. In this short paper we will present an inequality connecting the diameter $D(T)$ of a nontrivial tree $T=(V(T),E(T))$ and the *inverse dual degree* [4] of T, defined by

$$I(T) := \sum_{u\in V(G)} 1/\bar{d}(u)$$

with

$$\bar{d}(u) := \left(\sum_{v\in N(u)} d(v)\right)\Bigg/ d(u),$$

where $N(u):=\{v\in V(T): v$ is adjacent to u in $T\}$ is the *neighbourhood* and $d(u):=|N(u)|$ the *degree* of u in T. The motivation of seeking such an inequality arises from the following conjecture, made by using the program Graffiti [4], about relationship between $I(T)$ and the radius $R(T)$ of T.

Graffiti Conjecture 577. The inequality $I(T)\geqslant R(T)$ holds for any nontrivial tree T.

Our main result, Theorem 1 below, implies that not only is this true but also the difference $I(T)-R(T)$ is large in general. Fajtlowicz has suggested that the difference $I(G)-R(G)$ may be useful as a measure of the "branching" of a graph $G=(V(G),E(G))$ (not necessarily a tree), where $I(G)$ is defined in the same way as above. Other measures of similar flavour include the well-known Wiener index and the Randic index [8], the latter being defined as $\sum_{e\in E(G)} 1/\sqrt{w(e)}$ with $w(e)$ denoting the product of the degrees of the two vertices incident to e. It is reported that Randic index is useful in predicting the boiling point of certain hydrocarbons, see [8] for details.

For a simple and connected graph G, the *distance* $d(u,v)$ in G between two vertices u, $v\in V(G)$ is the minimum length of a path of G joining u and v. The *diameter* $D(G)$ of G is the maximum distance between any two vertices of G. The *radius* $R(G)$ of G is defined to be $\min_{u\in V(G)}\max_{v\in V(G)} d(u,v)$. For $S\subseteq V(G)$ and $u\in V(G)\backslash S$, we define $d(u,S):=\min_{v\in S} d(u,v)$, which can be viewed as the distance in G from u to S. Whenever ambiguity exists we will use subscript G in these notations to emphasize the underlying graph G. So we will write say $d_G(u)$ and $\bar{d}_G(u)$ instead of $d(u)$ and $\bar{d}(u)$ in such cases. It is well-known that, for a tree T, the diameter and radius satisfy $R(T)=\lceil D(T)/2\rceil$, where $\lceil x\rceil$ denotes the smallest integer no less than x. Also, $D(T)$ is equal to the length of a longest path of T. Let $P=v_0v_1\ldots v_D$ be such a path, where

$D = D(T)$. Then $d(v_0) = d(v_D) = 1$ by definition. Define

$$a(P) := |\{v \in V(T)\backslash V(P):\ d(v) \geqslant 2\}|,$$

$$b(P) := \begin{cases} |\{i:\ d(v_i) \geqslant 3,\ 2 \leqslant i \leqslant D-2\}| & \text{if } D \geqslant 4, \\ 0 & \text{otherwise} \end{cases}$$

and

$$c(P) := \begin{cases} |\{i:\ d(v_i) \geqslant 3, i = 1 \text{ or } D-1\}| & \text{if } D \geqslant 2, \\ 0 & \text{if } D = 1. \end{cases}$$

Theorem 1. *For any nontrivial tree T and any longest path P of T, we have*

$$I(T) \geqslant D(T)/2 + a(P)/3 + b(P)/10 + c(P)/12 + \tfrac{5}{6}.$$

Since $R(T) = \lceil D(T)/2 \rceil$, and since $a(P) \geqslant 0$, $b(P) \geqslant 0$ and $c(P) \geqslant 0$ with equality occurring simultaneously if and only if T is a path, this theorem implies the following corollary. (For paths of less than four vertices, the equality in the corollary does not hold, see the beginning of the proof of Theorem 1.)

Corollary 1. *For any nontrivial tree T, we have*

$$I(T) \geqslant \begin{cases} R(T) + 1/3 & \textit{if } D(T) \textit{ is odd}, \\ R(T) + 5/6 & \textit{if } D(T) \textit{ is even}, \end{cases}$$

with equality if and only if T is a path of at least four vertices.

This corollary strengthens the above-mentioned conjecture of Graffiti. The authors were notified by one of the referees that this conjecture was confirmed also by Ronghua Shi who proved that $I(T) \geqslant D(T)/2 + \frac{1}{3}$ (see the on-line form of "Written on the Wall" which extends [4] and is maintained at Fajtlowicz's homepage http://www.math.uh.edu/~siemion/). However, Corollary 1 is slightly stronger, and also it gives a characterization of the extreme graphs. Moreover, Theorem 1 suggests that usually $I(T) - R(T)$ is much larger than $\frac{5}{6}$. In fact, this difference is unbounded above: for the full binary tree T of height $h \geqslant 3$ we have $I(T) - R(T) = 2^{h+2}/5 - h - 1/4$, which can be arbitrarily large as h tends to infinity.

The inequality $I(G) \geqslant R(G)$ is not true in general for graphs containing cycles, and hence so are the inequalities in Theorem 1 and Corollary 1. In view of this, further investigation of the "branching" measure $I(G) - R(G)$ for general graphs G would be necessary. For more results about Graffiti conjectures, the reader is referred to [1–7,9] and the website above.

2. Proof of the main result

In the following we assume T is a nontrivial tree and $P = v_0v_1\dots v_D$ is a longest path of T, where $D = D(T)$. We will simply write a, b and c in place of $a(P)$, $b(P)$ and $c(P)$, respectively. To prove Theorem 1 we will first prune the tree to a caterpillar, and then prune the caterpillar to a path. By definition a tree is called a *caterpillar* if the removal of all degree-one vertices yields a path, called the *spine*. Note that if T is a caterpillar, then $v_1\dots v_{D-1}$ is the spine of T; and if T is not a caterpillar, then $D(T) \geqslant 4$. At each step of the pruning we need to monitor the change of the inverse dual degree, and this is given by the following three lemmas.

Lemma 1. *Suppose T is not a caterpillar, and let u be a vertex not in P such that $d(u) \geqslant 2$ and $d(u, V(P))$ is as large as possible. Let T' be the subtree obtained from T by deleting all degree-one vertices adjacent to u. Then $D(T) = D(T')$ and $I(T) \geqslant I(T') + \frac{1}{3}$.*

Proof. Clearly, $D(T) \geqslant 4$ as mentioned above, and $D(T) = D(T')$ as the specified vertex-deletion does not hurt the path P. We first note that all but one of the neighbours of u have degree one, for otherwise there would be a neighbour w of u not in P with $d(w) \geqslant 2$ and $d(w, V(P)) > d(u, V(P))$, violating the choice of u. Suppose $N(u) = \{u_1, \dots, u_m, v\}$, where $d(u_i) = 1$ for $1 \leqslant i \leqslant m$, and denote $d(v) = r$. Denote $\sigma = \sum_{w \in N(v)\setminus\{u\}} d(w)$. Since $\sigma + 1 \geqslant r$, by the definition of the inverse dual degree we have

$$
\begin{aligned}
I(T) - I(T') &= \sum_{i=1}^{m} \frac{1}{\bar{d}_T(u_i)} + \left(\frac{1}{\bar{d}_T(u)} - \frac{1}{\bar{d}_{T'}(u)}\right) + \left(\frac{1}{\bar{d}_T(v)} - \frac{1}{\bar{d}_{T'}(v)}\right) \\
&= \frac{m}{m+1} + \left(\frac{m+1}{m+r} - \frac{1}{r}\right) + \left(\frac{r}{m+\sigma+1} - \frac{r}{\sigma+1}\right) \\
&= 1 + \frac{1}{m+r} - \frac{1}{r} - \frac{1}{m+1} + \left(\frac{m}{m+r} + \frac{r}{m+\sigma+1} - \frac{r}{\sigma+1}\right) \\
&\geqslant 1 + \frac{1}{m+r} - \frac{1}{r} - \frac{1}{m+1}.
\end{aligned}
$$

Note that $m \geqslant 1$, $r \geqslant 2$, and $1/(m+x) - 1/x$ is an increasing function of x. So furthering the inequality above we get

$$
\begin{aligned}
I(T) - I(T') &\geqslant 1 + \frac{1}{m+2} - \frac{1}{2} - \frac{1}{m+1} \\
&= \frac{1}{2} - \frac{1}{(m+1)(m+2)} \\
&\geqslant \frac{1}{3}
\end{aligned}
$$

as required. □

Lemma 2. *Suppose T is a caterpillar but not a path and $D=D(T)\geqslant 4$. If at least one of $d(v_1)$ and $d(v_{D-1})$ is no less than 3, say $d(v_1)\geqslant 3$, let T' be the subtree obtained from T by deleting all degree-one vertices adjacent to v_1 excepting v_0. Then $D(T)=D(T')$ and $I(T)\geqslant I(T')+1/12$.*

Proof. Clearly, we have $D(T)=D(T')$. Suppose $d(v_1)=m+2\geqslant 3$ and $N(v_1)=\{v_0,v_2,u_1,\ldots,u_m\}$. Denote $d(v_2)=r$ and $d(v_3)=s$. Then $r,s\geqslant 2$ since $D\geqslant 4$. We have

$$
\begin{aligned}
&I(T)-I(T')\\
&=\sum_{i=1}^{m}\frac{1}{\bar{d}_T(u_i)}+\sum_{i=0}^{2}\left(\frac{1}{\bar{d}_T(v_i)}-\frac{1}{\bar{d}_{T'}(v_i)}\right)\\
&=\frac{m}{m+2}+\left(\frac{1}{m+2}-\frac{1}{2}\right)+\left(\frac{m+2}{m+r+1}-\frac{2}{r+1}\right)+\left(\frac{r}{m+r+s}-\frac{r}{r+s}\right)\\
&=m\left\{\left[\frac{1}{2(m+2)}-\frac{1}{(r+1)(m+r+1)}\right]\right.\\
&\qquad\left.+r\left[\frac{1}{(r+1)(m+r+1)}-\frac{1}{(r+s)(m+r+s)}\right]\right\}\\
&\geqslant m\left[\frac{1}{2(m+2)}-\frac{1}{(r+1)(m+r+1)}\right]\\
&\geqslant m\left[\frac{1}{2(m+2)}-\frac{1}{3(m+3)}\right]\\
&=\frac{m(m+5)}{6(m+2)(m+3)}\\
&\geqslant\frac{1}{12}. \qquad \Box
\end{aligned}
$$

Lemma 3. *Suppose T is a caterpillar but not a path and $D=D(T)\geqslant 4$. If $d(v_1)=d(v_{D-1})=2$, let v_α be a vertex on P which is nearest to one end-vertex of P and is such that $d(v_\alpha)\geqslant 3$, and let T' be the subtree obtained by deleting all degree-one neighbours of v_α. Then $D(T)=D(T')$ and $I(T)\geqslant I(T')+\frac{1}{10}$.*

Proof. Again, we have $D(T)=D(T')$. Without loss of generality, we may suppose $\alpha\leqslant D/2$. Let $u_1,\ldots,u_m$ be the degree-one neighbours of v_α, so that $d(v_\alpha)=m+2$. Denote $d(v_{\alpha-2})=r$, $d(v_{\alpha+1})=s$ and $d(v_{\alpha+2})=t$. By the definition of v_α, we have $d(v_{\alpha-1})=2$, and $r=1$ if $\alpha=2$ and $r=2$ otherwise. If $D\geqslant 5$, then $t\geqslant 2$ and hence by the monotonicity of the function $1/(m+x)-1/x$ we have

$$
\begin{aligned}
I(T)-I(T') &= \sum_{i=1}^{m} \frac{1}{\bar{d}_T(u_i)} + \sum_{i=\alpha-1}^{\alpha+1}\left(\frac{1}{\bar{d}_T(v_i)}-\frac{1}{\bar{d}_{T'}(v_i)}\right)\\
&= \frac{m}{m+2}+\left(\frac{2}{m+r+2}-\frac{2}{r+2}\right)+\left(\frac{m+2}{m+s+2}-\frac{2}{s+2}\right)\\
&\quad +\left(\frac{s}{m+s+t}-\frac{s}{s+t}\right)\\
&\geqslant \frac{m}{m+2}+\left(\frac{2}{m+3}-\frac{2}{3}\right)+\left(\frac{m+2}{m+s+2}-\frac{2}{s+2}\right)\\
&\quad +\left(\frac{s}{m+s+2}-\frac{s}{s+2}\right)\\
&= \frac{m(m+5)}{3(m+2)(m+3)}\\
&\geqslant \frac{1}{6}\\
&\geqslant \frac{1}{10}.
\end{aligned}
$$

If $D=4$, then a straightforward calculation shows that

$$
\begin{aligned}
I(T)-I(T') &= \frac{1}{6}+\frac{4}{m+3}-\frac{2}{m+2}-\frac{2}{m+4}\\
&= \frac{1}{6}-\frac{4}{(m+2)(m+3)(m+4)}\\
&\geqslant \frac{1}{10}. \qquad \square
\end{aligned}
$$

Proof of Theorem 1. Let us first deal with paths and the case where $D(T)$ is small. If $T=P_n$, the path with n vertices, then

$$
I(P_n)-D(P_n)/2=\begin{cases} \frac{3}{2}, & n=2,\\ 1, & n=3,\\ \frac{5}{6}, & n\geqslant 4.\end{cases}
$$

If $D(T)=2$, then T is a star with $a=b=0$, $c=1$ and $I(T)-D(T)/2=1\geqslant c/12+\frac{5}{6}$. If $D(T)=3$ and $T\neq P_4$, then $a=b=0$ and T has exactly two vertices (namely v_1

and v_2) with degree $\geqslant 2$. Suppose the degrees of them are $\ell+1$ and $m+1$. Then $\max\{\ell,m\}\geqslant 2$ as $T\neq P_4$, and $c=2$ if both ℓ and m are at least 2 and $c=1$ otherwise. So we have

$$
\begin{aligned}
I(T)-D(T)/2 &= \frac{\ell+m+2}{\ell+m+1}+\frac{\ell}{\ell+1}+\frac{m}{m+1}-\frac{3}{2}\\
&= \frac{1}{\ell+m+1}-\frac{1}{\ell+1}-\frac{1}{m+1}+\frac{3}{2}\\
&\geqslant \frac{c}{12}+\frac{5}{6}.
\end{aligned}
$$

In the following we suppose T is not a path and $D=D(T)\geqslant 4$. If T is not a caterpillar, let u be a vertex not in P such that $d(u)\geqslant 2$ and $d(u,V(P))$ is as large as possible. Then all but one of the neighbours of u have degree one. Removing from T all the degree-one neighbours of u we get a subtree T_1 with $D(T)=D(T_1)$ and $I(T)\geqslant I(T_1)+\frac{1}{3}$, according to Lemma 1. If T_1 is not a caterpillar, then repeat this procedure until a caterpillar is obtained. It is clear that after a steps we get a sequence $T=T_0,T_1,\ldots,T_a$ such that each T_{i+1} is a subtree of T_i, $D(T_i)=D(T_{i+1})$, $I(T_i)\geqslant I(T_{i+1})+\frac{1}{3}$, and T_a is a caterpillar. Thus, we have $D(T)=D(T_a)$ and $I(T)\geqslant I(T_a)+a/3$.

If $d_{T_a}(v_1)=d(v_1)\geqslant 3$, then delete from T_a all the degree-one neighbours of v_1 except v_0. Thus we get a subtree T_{a+1} of T_a with the same diameter as T and with $I(T_a)\geqslant I(T_{a+1})+\frac{1}{12}$, according to Lemma 2. If $d(v_{D-1})\geqslant 3$, then we do the same thing for v_{D-1}. In this way, c subtrees are added to the sequence above and we get $T=T_0,T_1,\ldots,T_a,\ldots,T_{a+c}$ with $D(T)=D(T_{a+c})$ and $I(T)=I(T_{a+c})+a/3+c/12$.

Now we have $d_{T_{a+c}}(v_1)=d_{T_{a+c}}(v_{D-1})=2$ and $d_{T_{a+c}}(v_i)=d(v_i)$ for $i\notin\{1,D-1\}$. If T_{a+c} is not a path, then according to Lemma 3 we can delete all degree-one neighbours of some v_α and obtain a subtree T_{a+c+1} with $I(T_{a+c})\geqslant I(T_{a+c+1})+1/10$. Repeat the procedure until we obtain the path P. When the process stops we get a sequence $T=T_0,T_1,\ldots,T_a,\ldots,T_{a+c},\ldots,T_{a+c+b}=P$ with $I(T)\geqslant I(T_{a+c})+a/3+c/12\geqslant I(P)+a/3+b/10+c/12$. Since $I(P)=D(P)/2+\frac{5}{6}$, as shown at the beginning of the proof, and since $D(P)=D(T)$, we get $I(T)\geqslant D(T)/2+a/3+b/10+c/12+\frac{5}{6}$ as required. □

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