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A NOTE ON GALLAI-TYPE EQUALITY FOR THE TOTAL DOMINATION NUMBER OF A GRAPH

Sanming Zhou*

Department of Mathematics and Statistics
The University of Melbourne
Parkville, VIC 3010, Australia

e-mail: smzhou@ms.unimelb.edu.au

Abstract

We prove the following Gallai-type equality

$$\gamma_t(G) + \varepsilon_t(G) = p$$

for any graph G with no isolated vertex, where p is the number of vertices of G, $\gamma_t(G)$ is the total domination number of G, and $\varepsilon_t(G)$ is the maximum integer s such that there exists a spanning forest F with s the number of pendant edges of F minus the number of star components of F.

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1. Introduction

Let $G = (V(G), E(G))$ be a graph with $p = |V(G)|$ vertices. Let $\alpha(G)$, $\beta(G)$, $\alpha'(G)$ and $\beta'(G)$ be the vertex covering number, the vertex independence number, the edge covering number and the edge independence number of G, respectively. In [3], Gallai established his now classic equalities involving these invariants:

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(I) $\alpha(G) + \beta(G) = p$

(II) $\alpha'(G) + \beta'(G) = p$,

here in (II) G is assumed to have no isolated vertices. Now there are a number of similar Gallai-type equalities for a variety of graphical invariants. The reader is referred to [2] for a comprehensive survey on this topic. The purpose of this paper is to prove a Gallai-type equality for the total domination number of G.

A subset D of $V(G)$ is said to be a *dominating set* of G if each vertex in $V(G) - D$ is adjacent to at least one vertex in D. The minimum cardinality of a dominating set of G is the *domination number* of G, denoted by $\gamma(G)$. A dominating set D is a *total dominating set* of G if the subgraph $G[D]$ induced by D has no isolated vertex. Note that G admits total dominating sets if and only if it contains no isolated vertex. In such a case, the *total domination number* of G, denoted by $\gamma_t(G)$, is defined to be the minimum cardinality of a total dominating set of G. A dominating set D of G is a *connected dominating set* if $G[D]$ is connected. For a connected graph G, the *connected domination number* $\gamma_c(G)$ is the minimum cardinality of a connected dominating set of G. A degree-one vertex of a graph is said to be a *pendant vertex*; and an edge incident with a pendant vertex is a *pendant edge* of the graph. Denote by $\varepsilon(G)$ the maximum number of pendant edges in a spanning forest of G. In [6] Nieminen gave the following Gallai-type equality for domination number $\gamma(G)$.

Theorem 1 ([6]). $\gamma(G) + \varepsilon(G) = p$.

A similar equality holds for connected domination number. Denote by $\varepsilon_T(G)$ the maximum number of pendant edges in a spanning tree of a connected graph G. Hedetniemi and Laskar [5] proved

$$\gamma_c(G) + \varepsilon_T(G) = p \tag{1}$$

for any connected graph G. To the best knowledge of the author, there has been no similar Gallai-type equality so far for total domination number in the literature. In this paper we will provide such an equality, which has the same spirit as above.

For a spanning forest F of G, we denote by $s(F)$ the number of pendant edges of F minus the number of star components of F. (A *star* is a complete bipartite graph $K_{1,n}$ for some $n \geq 1$.) Denote by $\varepsilon_t(G)$ the maximum $s(F)$

taken over all spanning forests F of G. Our main result is the following theorem.

Theorem 2. *Let G be a graph with no isolated vertex. Then*

$$\gamma_t(G) + \varepsilon_t(G) = p.$$

2. Proof of Theorem 2

In order to prove Theorem 2, let us first review some basic ideas (see [1, 2, 4]) involved in the derivation of a lot of known Gallai-type equalities.

Let S be a finite set and Q a property associated with the subsets of S. If a subset X of S possesses Q, then we call X a *Q-set*; otherwise a *$\overline{Q}$-set*. In the following we suppose that Q is *cohereditary* (or *expanding* as used in [2]) in the sense that whenever X is a Q-set and $X \subseteq Y \subseteq S$ then Y is a Q-set. We say that $Y \subseteq S$ is a *Q^*-set* if $X \cup Y \neq S$ holds for each $\overline{Q}$-set X. Let $\beta_Q(S)$ be the minimum cardinality of a Q-set of S, and $\alpha_Q(S)$ the maximum cardinality of a Q^*-set of S. It is not difficult to see [2, Theorem 2′] that $X \subseteq S$ is a Q-set if and only if $\overline{X} = S - X$ is a Q^*-set. This implies the following basic Gallai-type equality:

$$\alpha_Q(S) + \beta_Q(S) = |S|. \tag{2}$$

***Proof of Theorem* 2.** Let $V = V(G)$ be the vertex set of G. Let Q be the property defined on the subsets of V such that $X \subseteq V$ is a Q-set if and only if it is a total dominating set of G. Then obviously Q is cohereditary and $\beta_Q(V) = \gamma_t(G)$. We have the following claim.

Claim. A subset Y of V is a Q^*-set if and only if Y is a set of pendant vertices of a spanning forest F of G such that

(a) F contains no isolated vertex;

(b) each edge of F is incident with at most one vertex in Y; and

(c) the removal of Y from F results in a forest with no isolated vertices.

In fact, if $Y \subseteq V$ is a Q^*-set, then $V - Y$ is a total dominating set according to the discussion above. Thus, for each $y \in Y$ there exists an edge, say e_y, joining y and a vertex in $V - Y$. Also, the subgraph $G[V - Y]$ of G induced by $V - Y$ has no isolated vertex. Let E_Y be a minimal subset of the edge set of $G[V - Y]$ such that it induces a spanning subgraph of $G[V - Y]$ with no isolated vertex. By the minimality, E_Y induces a spanning forest of $G[V - Y]$. Thus, the graph induced by the edges $E_Y \cup \{e_y : y \in Y\}$ is a spanning forest F of G satisfying (a), (b) and (c) above, and Y is a set of pendant vertices of F. Conversely, if $Y \subseteq V$ is a set of pendant vertices of a spanning forest F of G such that (a), (b) and (c) are satisfied, then any $X \subseteq V(G)$ with $X \cup Y = V(G)$ is a total dominating set of G. In other words, in such a case Y is a Q^*-set and hence the Claim is proved.

Now by the Claim above $\alpha_Q(V)$ is equal to the maximum cardinality of a subset Y of V such that Y is a set of pendant vertices of a spanning forest F of G satisfying (a), (b) and (c). Note that for a fixed spanning forest F with no isolated vertices, a set Y of pendant vertices of F satisfying (b) and (c) has the maximum cardinality if and only if Y contains all the pendant vertices of each non-star component and $n - 1$ pendant vertices of each star component $K_{1,n}$ of F. In other words, the maximum cardinality of a set Y of pendant vertices of F satisfying (b) and (c) is precisely $s(F)$. Thus, $\alpha_Q(V)$ is the maximum $s(F)$ taken over all spanning forests F with no isolated vertex. For a spanning forest F of G with isolated vertices, say $x_1, x_2, \ldots, x_n$ $(1 \leq n \leq p)$, since G contains no isolated vertex, each x_i is either adjacent to another x_j or adjacent to a vertex in a nontrivial component of F. (A nontrivial component is a connected component with at least two vertices.) Hence we can add some edges of G to F such that each x_i is incident with exactly one of the added edges. In this way we get a new spanning forest F' of G containing no isolated vertex. It is not difficult to check that $s(F) \leq s(F')$. Thus, $\alpha_Q(V)$ is actually the maximum $s(F)$ taken over all spanning forests F. That is, $\alpha_Q(V) = \varepsilon_t(G)$. Now from (2) we get $\gamma_t(G) + \varepsilon_t(G) = p$ and the proof of Theorem 2 is complete. ∎

We notice that Theorem 1 can be derived from (2) in a similar way. In fact, let Q be the property associated with the subsets of $V = V(G)$ such that $X \subseteq V$ is a Q-set if and only if X is a dominating set of G. Then Q is cohereditary and $\beta_Q(V) = \gamma(G)$. By an argument similar to the proof of Theorem 2 we get $\alpha_Q(V) = \varepsilon(G)$ and hence Theorem 1 follows from (2). Note that (1) cannot be derived from (2) in a similar way since the property of being a connected dominating set is not a cohereditary property.

REFERENCES

[1] B. Bollobás, E.J. Cockayne and C.M. Mynhardt, *On Generalized Minimal Domination Parameters for Paths*, Discrete Math. **86** (1990) 89–97.

[2] E.J. Cockayne, S.T. Hedetniemi and R. Laskar, *Gallai Theorems for Graphs, Hypergraphs and Set Systems*, Discrete Math. **72** (1988) 35–47.

[3] T. Gallai, *Über Extreme Punkt-und Kantenmengen*, Ann. Univ. Sci. Budapest Eötvös Sect. Math. **2** (1959) 133–138.

[4] S.T. Hedetniemi, *Hereditary Properties of Graphs*, J. Combin. Theory **14** (1973) 16–27.

[5] S.T. Hedetniemi and R. Laskar, *Connected Domination in Graphs*, in: B. Bollobás ed., Graph Theory and Combinatorics (Academic Press, 1984) 209–218.

[6] J. Nieminen, *Two Bounds for the Domination Number of a Graph*, J. Inst. Math. Appl. **14** (1974) 183–187.

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