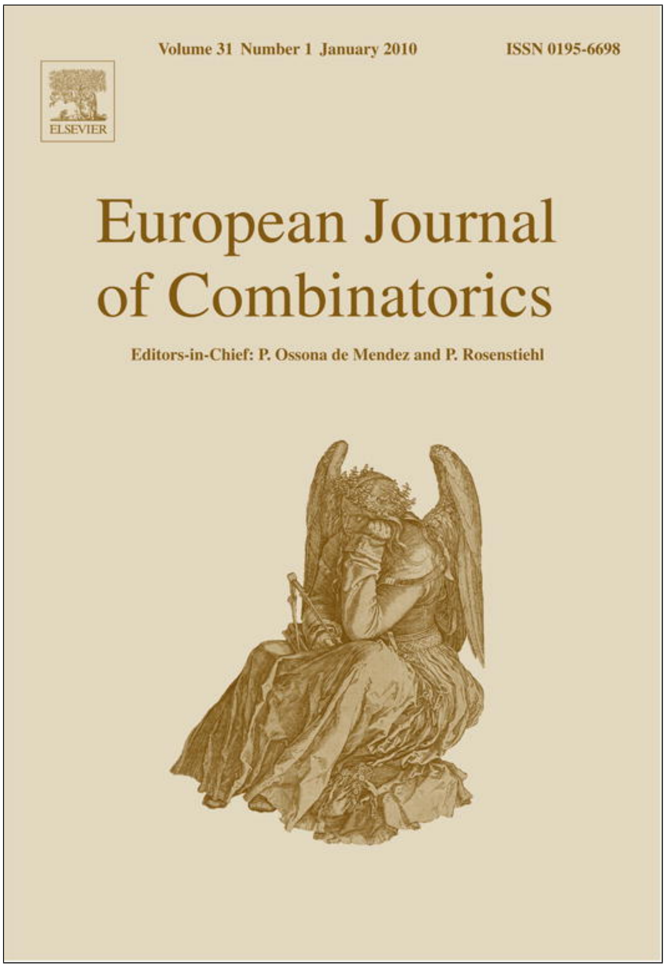
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# Imprimitive symmetric graphs with cyclic blocks

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ABSTRACT

Let $\Gamma$ be a graph admitting an arc-transitive subgroup $G$ of automorphisms that leaves invariant a vertex partition $\mathcal{B}$ with parts of size $v \geq 3$. In this paper we study such graphs where: for $B, C \in \mathcal{B}$ connected by some edge of $\Gamma$, exactly two vertices of $B$ lie on no edge with a vertex of $C$; and as $C$ runs over all parts of $\mathcal{B}$ connected to $B$ these vertex pairs (ignoring multiplicities) form a cycle. We prove that this occurs if and only if $v = 3$ or 4, and moreover we give three geometric or group theoretic constructions of infinite families of such graphs.



## 1. Introduction

A graph $\Gamma = (V, E)$ is $G$-*symmetric* if $G \leq \mathrm{Aut}(\Gamma)$ is transitive on the set $\mathtt{Arc}(\Gamma)$ of arcs of $\Gamma$, where an *arc* is an ordered pair of adjacent vertices. For a $G$-symmetric graph $\Gamma$, a partition $\mathcal{B}$ of $V$ is $G$-*invariant* if $B \in \mathcal{B}$ implies $B^g \in \mathcal{B}$ for all $g \in G$, where $B^g = \{\alpha^g : \alpha \in B\}$, and $\mathcal{B}$ is *nontrivial* if $1 < |B| < |V|$. Such a vertex partition gives rise to a *quotient graph* $\Gamma_{\mathcal{B}}$, namely the graph with vertex set $\mathcal{B}$ in which $B, C \in \mathcal{B}$ are adjacent if and only if there exists an edge of $\Gamma$ joining a vertex of $B$ to a vertex of $C$. Since $\Gamma$ is $G$-symmetric and $\mathcal{B}$ is $G$-invariant, $\Gamma_{\mathcal{B}}$ is $G$-symmetric under the induced (not necessarily faithful) action of $G$ on $\mathcal{B}$. Moreover, if $\Gamma$ is connected, then $\Gamma_{\mathcal{B}}$ is connected and in particular all arcs join distinct parts of $\mathcal{B}$. For an arc $(B, C)$ of $\Gamma_{\mathcal{B}}$, the subgraph $\Gamma[B, C]$ of $\Gamma$ induced on $B \cup C$ with isolated vertices deleted is bipartite and, up to isomorphism, is independent of $(B, C)$. In some examples, such as the case where $\Gamma$ is a cover of $\Gamma_{\mathcal{B}}$, all vertices of $B$ and $C$ occur in $\Gamma[B, C]$, but many other possibilities also arise.

For an arc $(B, C)$ of $\Gamma_{\mathcal{B}}$, let $\Gamma(C) = \bigcup_{\alpha \in C} \Gamma(\alpha)$, where $\Gamma(\alpha)$ denotes the set of vertices adjacent to $\alpha$ in $\Gamma$, and set

$$v := |B|, \qquad k := |\Gamma(C) \cap B|. \tag{1}$$

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An approach to understanding general $G$-symmetric graphs $\Gamma$ in terms of $\Gamma_{\mathcal{B}}$, $\Gamma[B, C]$ and a 1-design induced on $B$ was suggested in [3], and developed further in [5,8,9] in the case $k = v - 1$, where special additional structure on the parts $B$ can be defined and exploited.

If $k = v - 2$ it turns out that we may also define additional structure on the parts. Since $\Gamma[B, C]$ consists of $k$ vertices from each of $B$ and $C$, in particular $v = k + 2 \geq 3$, and the set $B \setminus \Gamma(C)$ contains exactly two vertices. Thus we may define a multigraph $\Gamma^B$ with vertex set $B$ and an edge joining the two vertices of $B \setminus \Gamma(C)$ for each $C$ in the set $\Gamma_{\mathcal{B}}(B)$ of parts of $\mathcal{B}$ adjacent to $B$ in $\Gamma_{\mathcal{B}}$. Denote by $\mathtt{Simple}(\Gamma^B)$ the underlying simple graph of $\Gamma^B$. It was proved [4, Theorem 2.1] that $\mathtt{Simple}(\Gamma^B)$ is $G_B$-vertex-transitive and $G_B$-edge-transitive, and either $\Gamma^B$ is connected or $\mathtt{Simple}(\Gamma^B)$ is a perfect matching $(v/2) \cdot K_2$, where $G_B$ is the setwise stabiliser of $B$ in $G$. In the latter case detailed information about $\Gamma$ was obtained in [4, Theorem 1.3] when $\Gamma^B$ is simple. However, no information about $\Gamma$ was obtained in the case where $\Gamma^B$ is connected. Here we considered the simplest possibility, namely $\mathtt{Simple}(\Gamma^B)$ has valency two. We find with surprise that the parts of $\mathcal{B}$ must have size 3 or 4 in this case. Our main result is Theorem 1.1 below. It involves the multiplicity $m$ of the edges of the multigraph $\Gamma^B$, that is, for a pair $\{\alpha, \beta\}$ of adjacent vertices of $\Gamma^B$,

$$m = |\{C \in \Gamma_{\mathcal{B}}(B) : B \setminus \Gamma(C) = \{\alpha, \beta\}\}|.$$

**Theorem 1.1.** *Suppose $\Gamma$ is a $G$-symmetric graph (where $G \leq \mathrm{Aut}(\Gamma)$) whose vertex set admits a nontrivial $G$-invariant partition $\mathcal{B}$ such that $k = v - 2 \geq 1$ with $k, v$ as in (1), $\Gamma_{\mathcal{B}}$ is connected, and $\mathtt{Simple}(\Gamma^B)$ has valency two. Then $\mathtt{Simple}(\Gamma^B) = C_v$, $\Gamma_{\mathcal{B}}$ has valency $mv$, and one of the following* (a)–(c) *occurs for an arc $(B, C)$ of $\Gamma_{\mathcal{B}}$.*

(a) *$v = 3$ and $\Gamma$ has valency $m$;*
(b) *$v = 4$, $\Gamma[B, C] = K_{2,2}$, and $\Gamma$ is connected of valency $4m$;*
(c) *$v = 4$, $\Gamma[B, C] = 2 \cdot K_2$, and $\Gamma$ has valency $2m$.*

**Remark 1.2.** (1) In particular, if $\Gamma^B$ is simple, then in case (a) we have $\Gamma = (|V(\Gamma)|/2) \cdot K_2$, and, in case (c), $\Gamma$ has valency two and hence is a vertex-disjoint union of cycles of the same length. In Section 3 we construct an infinite family of graphs for each of these cases, and an infinite family of graphs for case (b) with $\Gamma^B$ simple by using the coset graph construction.

(2) In cases (b) and (c) we prove that $G_B^B \cong D_8$, and for an arc $(B, C)$ of $\Gamma_{\mathcal{B}}$, we prove that $G_{BC}^{B \cup C} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ in case (b), and $\mathbb{Z}_2$ in case (c).

(3) In case (a), $\Gamma$ can be $(G, 2)$-arc-transitive even when $m > 1$; see [4, Example 4.6] for an infinite family of such graphs. In case (b) it is clear that $\Gamma$ is not $(G, 2)$-arc-transitive. In case (c), if $m > 1$, then the stabiliser $G_\alpha$ of $\alpha$ in $G$ is imprimitive on $\Gamma(\alpha)$ and hence $\Gamma$ is not $(G, 2)$-arc-transitive. An example for case (c) such that $\Gamma$ is $(G, 2)$-arc-transitive (hence $m = 1$) can be found in [4, Example 4.7].

Our construction for case (b) leads to an infinite family of connected 4-valent symmetric graphs $\Gamma$ which have a 4-valent quotient not covered by $\Gamma$. To the best of our knowledge this is the first infinite family of symmetric graphs with these properties.

**Corollary 1.3.** *There exists an infinite family of connected symmetric graphs $\Gamma$ of valency 4 which have a quotient graph $\Gamma_{\mathcal{B}}$ of valency 4 such that $\Gamma$ is not a cover of $\Gamma_{\mathcal{B}}$.*

In the light of Theorem 1.1 we ask, for other connected graphs $\mathtt{Simple}(\Gamma^B)$:

**Question 1.4.** *In the case where $k = v - 2$ and $\Gamma^B$ is connected, is $v$ bounded by some function of the valency of $\mathtt{Simple}(\Gamma^B)$?*

We may also ask the following question.

**Question 1.5.** *Can $\Gamma$ in Theorem 1.1 be determined for small values of $m$?*

The proof of Theorem 1.1 is given in Section 2 and the examples are constructed in Section 3. The reader is referred to [1] for group theoretic terminology used in the paper.

## 2. Proof of Theorem 1.1

Two parts $B, C \in \mathcal{B}$ are called *adjacent* if they are adjacent in the quotient graph $\Gamma_{\mathcal{B}}$, and if $B, C$ are adjacent we write $G_{BC} = (G_B)_C$ and let $e(B, C)$ be the edge of $\mathtt{Simple}(\Gamma^B)$ joining the two vertices of $B \setminus \Gamma(C)$.

**Proof of Theorem 1.1.** Let $(\Gamma, G, \mathcal{B})$ satisfy the conditions of Theorem 1.1, and let $B, C$ be adjacent parts. Then $\mathtt{Simple}(\Gamma^B) \neq (v/2) \cdot K_2$ since it is of valency two by our assumption. Thus $\Gamma^B$, and hence also $\mathtt{Simple}(\Gamma^B)$, is connected [4, Theorem 2.1] and so $\mathtt{Simple}(\Gamma^B) = C_v$. Thus, by the definition of $\Gamma^B$, the valency of $\Gamma_{\mathcal{B}}$ is $mv$.

**Case 1**: $v$ odd. Since $v$ is odd, there exists a unique vertex $\alpha \in B$ which is 'antipodal' to the edge $e(B, C)$ of $\mathtt{Simple}(\Gamma^B)$, that is, $\alpha$ is the unique vertex equi-distant in $\mathtt{Simple}(\Gamma^B)$ from the two vertices of $e(B, C)$. Now each element of $G_{BC}$ fixes $B \setminus \Gamma(C)$ setwise and hence fixes $\alpha$. (In the case where $m > 1$, an element of $G_{BC}$ may permute the $m$ edges of $\Gamma^B$ joining the two vertices of $B \setminus \Gamma(C)$.) Thus, $G_{BC} \leq G_\alpha$. Since $\alpha \notin B \setminus \Gamma(C)$, there exists $\beta \in C$ adjacent to $\alpha$ in $\Gamma$. Suppose $v \geq 5$. Then there exists a vertex $\gamma \in B$ such that $\gamma \notin \{\alpha\} \cup (B \setminus \Gamma(C))$ and so $\gamma$ is adjacent to a vertex $\delta \in C$. Since $\Gamma$ is $G$-symmetric, there exists $g \in G$ such that $(\alpha, \beta)^g = (\gamma, \delta)$. Since $g$ maps $\alpha \in B$ to $\gamma \in B$ and $\beta \in C$ to $\delta \in C$, it fixes $B$ and $C$ setwise. Thus, $g \in G_{BC} \leq G_\alpha$, which is a contradiction since $\alpha^g = \gamma \neq \alpha$. Therefore, $v = 3$ and consequently $\Gamma$ has valency $m$.

**Case 2**: $v$ even. Since $v$ is even, there exists a unique edge of $\mathtt{Simple}(\Gamma^B)$, say, $e = \{\alpha, \beta\}$, which is 'antipodal' to $e(B, C)$ in $\mathtt{Simple}(\Gamma^B)$, that is, $\alpha$ and $\beta$ are both at maximum distance $v/2$ from some vertex of $e(B, C)$. Note that $\alpha, \beta \in B \cap \Gamma(C)$. Each vertex $\gamma \in B \cap \Gamma(C)$ is adjacent to some vertex $\delta_\gamma \in C$. Since $\Gamma$ is $G$-symmetric, for each such $\gamma$ there exists $g_\gamma \in G$ such that $(\alpha, \delta_\alpha)^{g_\gamma} = (\gamma, \delta_\gamma)$. Since $g_\gamma$ maps $\alpha \in B$ to $\gamma \in B$ and $\delta_\alpha \in C$ to $\delta_\gamma \in C$, we have $g_\gamma \in G_{BC}$. Thus for each $\gamma \in B \cap \Gamma(C)$, $g_\gamma$ fixes $e(B, C)$ setwise and hence fixes $e = \{\alpha, \beta\}$ setwise also. Thus $\alpha^{g_\gamma} = \gamma \in \{\alpha, \beta\}$ and in particular $v = 4$ and $B \cap \Gamma(C) = \{\alpha, \beta\}$. Since $g_\beta$ fixes $e$ setwise, it interchanges $\alpha$ and $\beta$. Since $G_B^B$ is transitive on $B$, it follows that $G_B^B \cong D_8$. Therefore, $1 \neq G_{BC}^{B \cup C} \leq \langle x^B \rangle \times \langle x^C \rangle$, where $x^B$ is the reflection of $\Gamma^B$ in $e(B, C)$ and $x^C$ is the reflection of $\Gamma^C$ in $e(C, B)$. Note that $x^B$ interchanges $\alpha$ and $\beta$ since it interchanges the two vertices of $e(B, C)$. Thus $g_\beta^B = x^B$. Similarly, $x^C$ interchanges the two vertices of $C \setminus e(C, B) = \{\eta, \zeta\}$, say, and $G_{BC}$ contains an element $h$ such that $h^C = x^C$. Thus either $G_{BC}^{B \cup C} = \langle x^B \rangle \times \langle x^C \rangle$ or $G_{BC}^{B \cup C} = \langle x^B x^C \rangle \cong \mathbb{Z}_2$. Since $G_{BC}$ preserves the adjacency of $\Gamma$, the first possibility occurs if and only if $\alpha$ is adjacent to both of the vertices of $C \setminus e(C, B)$ and hence $\Gamma[B, C] = K_{2,2}$ is the 4-cycle $(\alpha, \eta, \beta, \zeta, \alpha)$. Since $\mathtt{Simple}(\Gamma^B) = C_4$, in this case $\alpha$ is at distance 2 in $\Gamma$ from $\beta$ and from one of the vertices of $e(B, C)$, and at distance 3 or 4 from the other vertex of $e(B, C)$. Since $\Gamma_{\mathcal{B}}$ is connected, it follows that $\Gamma$ is connected of valency $4m$ in this case. Suppose now that $G_{BC}^{B \cup C} = \langle x^B x^C \rangle \cong \mathbb{Z}_2$. Then the bipartite graph $\Gamma[B, C]$ consists of two edges only, namely, $\{\alpha, \delta_\alpha\}$ and $\{\beta, \delta_\beta\}$. Hence $\Gamma[B, C] = 2 \cdot K_2$ and $\Gamma$ has valency $2m$. □

## 3. Constructions

In this section we present several constructions of infinite families of graphs that satisfy the conditions of Theorem 1.1 in the case where the multigraph $\Gamma^B$ is simple, that is $\Gamma^B = \mathtt{Simple}(\Gamma^B)$ or equivalently $m = 1$. The first two constructions involve regular maps on surfaces. Here and in what follows our use of the term 'regular map' agrees with that of [2], that is, a regular map is a 2-cell embedding of a connected (multi)graph on a closed surface such that its automorphism group is regular on incident vertex–edge–face triples.

### 3.1. *Truncations of trivalent symmetric graphs*

The construction below produces all graphs that arise in case (a) of Theorem 1.1 with $m = 1$.

**Construction 3.1.** Let $\Sigma$ be a trivalent $G$-symmetric graph with $n$ edges. Define $\Gamma(\Sigma)$ to be the graph with vertex set $\mathtt{Arc}(\Sigma)$ and edges $\{(\sigma, \tau), (\tau, \sigma)\}$ for $(\sigma, \tau) \in \mathtt{Arc}(\Sigma)$ [4, Example 2.4]. Then $\Gamma(\Sigma) = n \cdot K_2$, $\Gamma(\Sigma)$ is $G$-symmetric, and its vertex set admits the $G$-invariant partition

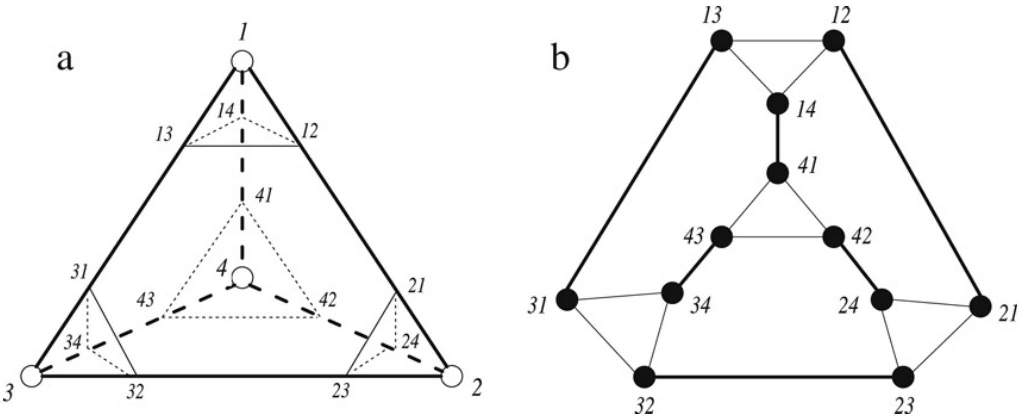


**Fig. 1.** Obtaining $\Gamma = 6 \cdot K_2$ (heavy edges in (b)) by truncating the tetrahedron as in (a).

$\mathcal{B}(\Sigma) = \{B(\sigma) : \sigma \in V(\Sigma)\}$ with parts of size $v = 3$, where $B(\sigma)$ is the set of arcs of $\Sigma$ with first vertex $\sigma$. For this partition we have $k = v - 2 = 1$, $\Gamma(\Sigma)^{B(\sigma)}$ is the simple graph $C_3$, and $\Sigma$ is isomorphic to the quotient graph $\Gamma(\Sigma)_{\mathcal{B}(\Sigma)}$ via the bijection $\sigma \mapsto B(\sigma)$.

As explained in [4, Example 2.4] this construction produces all imprimitive $G$-symmetric graphs $(\Gamma, \mathcal{B})$ such that $k = v - 2 = 1$ and $\Gamma^B = C_3$ is simple.

In case (a) of Theorem 1.1, if $m = 1$, then $G_B^B \cong \mathbb{Z}_3$ or $D_6$. From [2, Theorem 1.1] the former occurs if and only if $\Gamma_{\mathcal{B}}$ admits an embedding as an orientably-regular (rotary) map $M$ on a closed orientable surface. In fact, $\Gamma_{\mathcal{B}}$ admits[1] two such embeddings which are mirror images of each other such that their automorphism groups are isomorphic to $G$. In this case we may view $\Gamma$ as obtained from $M$ by truncation: cutting off each corner and then removing the edges in the triangles thus produced. In particular, let $M$ be the tetrahedron and let $G = A_4$ act on the vertices of $M$ in its natural action. Then Construction 3.1 applied with $\Sigma$ the underlying graph of $M$ gives rise to $\Gamma(\Sigma) = 6 \cdot K_2$ as shown in Fig. 1.

### 3.2. *Flag graphs of* 4*-valent regular maps*

Next we construct four infinite families of graphs that arise in case (c) of Theorem 1.1 with $m = 1$. The constructions take as input a 4-valent regular map $M$ with automorphism group $G = \mathrm{Aut}(M)$ so that the underlying graph $\Sigma$ of $M$ is $G$-symmetric and, for $\sigma \in V(\Sigma)$, $G_\sigma \cong G_\sigma^{\Sigma(\sigma)} = D_8$. The output of Construction 3.2 involves incident vertex–face pairs of $M$ of the form $(\sigma, h)$ where $\sigma$ is a vertex and $h$ is a face incident with $\sigma$.

**Construction 3.2.** Let $M$ be a regular map on a closed surface such that its underlying graph $\Sigma$ has valency four, and let $G = \mathrm{Aut}(M)$. For each edge $\{\sigma, \sigma'\}$ of $\Sigma$, let $f, f'$ denote the faces of $M$ such that $\{\sigma, \sigma'\}$ is on the boundary of both $f$ and $f'$. Let $\mathrm{opp}_\sigma(f)$ and $\mathrm{opp}_\sigma(f')$ be the other two faces of $M$ incident with $\sigma$ and opposite to $f$ and $f'$ respectively, and define $\mathrm{opp}_{\sigma'}(f)$ and $\mathrm{opp}_{\sigma'}(f')$ similarly. Define four graphs $\Gamma_1(M), \Gamma_2(M), \Gamma_3(M), \Gamma_4(M)$ with vertices the incident vertex–face pairs of $M$ and adjacency defined as follows (where $\sim$ means adjacency): for each edge $\{\sigma, \sigma'\}$ of $\Sigma$, $(\sigma, f) \sim (\sigma', f)$ and $(\sigma, f') \sim (\sigma', f')$ in $\Gamma_1(M)$; $(\sigma, f) \sim (\sigma', f')$ and $(\sigma, f') \sim (\sigma', f)$ in $\Gamma_2(M)$; $(\sigma, \mathrm{opp}_\sigma(f)) \sim (\sigma', \mathrm{opp}_{\sigma'}(f))$ and $(\sigma, \mathrm{opp}_\sigma(f')) \sim (\sigma', \mathrm{opp}_{\sigma'}(f'))$ in $\Gamma_3(M)$; $(\sigma, \mathrm{opp}_\sigma(f)) \sim (\sigma', \mathrm{opp}_{\sigma'}(f'))$ and $(\sigma, \mathrm{opp}_\sigma(f')) \sim (\sigma', \mathrm{opp}_{\sigma'}(f))$ in $\Gamma_4(M)$.

Let $\mathcal{B}(M) = \{B(\sigma) : \sigma \in V(\Sigma)\}$, where $B(\sigma) = \{(\sigma, f) : \sigma$ incident with $f\}$. The following lemma shows that the graphs produced by Construction 3.2 have the required properties.

**Lemma 3.3.** *Let $M$, $\Sigma$, $G$ be as in Construction 3.2 and let $\Gamma = \Gamma_i(M)$ be as defined there, where $1 \le i \le 4$. Then $\Gamma$ is a $G$-symmetric graph of valency two whose vertex set admits $\mathcal{B}(M)$ as a $G$-invariant*

[1] Details may be obtained from the authors.

*partition such that* $k = v - 2 = 2$, $\Gamma_{\mathcal{B}} \cong \Sigma$, *and* $\Gamma^{B(\sigma)} = C_4$ *is simple. Moreover, for adjacent blocks* $B(\sigma), B(\tau) \in \mathcal{B}(M)$, $\Gamma[B(\sigma), B(\tau)] = 2 \cdot K_2$.

**Proof.** Since $M$ is a regular map, $G = \mathrm{Aut}(M)$ is transitive on the vertices of $\Gamma$ and $\mathcal{B}(M)$ is a $G$-invariant partition of the vertex set of $\Gamma$. Since the underlying graph $\Sigma$ of $M$ is of valency four, the parts of $\mathcal{B}(M)$ have size $v = 4$ and a typical part is of the form $B(\sigma) = \{(\sigma, f_i) : 1 \leq i \leq 4\}$, where $f_1, f_2, f_3, f_4$ are the faces of $M$ surrounding $\sigma$. Let $\tau_i$, $1 \leq i \leq 4$ be the vertices of $M$ adjacent to $\sigma$ such that $\tau_{i-1}$ and $\tau_i$ are incident with the face $f_i$, where subscripts are taken modulo 4. If $\Gamma = \Gamma_1(M)$ then $(\sigma, f_i)$ is adjacent to $(\tau_{i-1}, f_i)$ and $(\tau_i, f_i)$ only, and hence $\Gamma$ has valency two. Similarly if $\Gamma = \Gamma_2(M)$ then $(\sigma, f_i)$ is adjacent to $(\tau_{i-1}, f_{i-1})$ and $(\tau_i, f_{i+1})$ only, and again $\Gamma$ has valency two. In either case $\Gamma[B(\sigma), B(\tau_1)]$ consists of two edges, namely $\{(\sigma, f_1), (\tau_1, f_1)\}$ and $\{(\sigma, f_2), (\tau_1, f_2)\}$ for $\Gamma_1(M)$, and $\{(\sigma, f_2), (\tau_1, f_1)\}$ and $\{(\sigma, f_1), (\tau_1, f_2)\}$ for $\Gamma_2(M)$, and hence $k = 2$ and $\Gamma[B(\sigma), B(\tau_1)] = 2 \cdot K_2$. Moreover, $\Gamma^{B(\sigma)}$ is a cycle $C_4$, namely $((\sigma, f_1), (\sigma, f_2), (\sigma, f_3), (\sigma, f_4), (\sigma, f_1))$ in both cases, and $\Gamma_{\mathcal{B}} \cong \Sigma$ via the mapping $B(\sigma) \mapsto \sigma$. Since $M$ is a regular map, there exists $g \in G_\sigma$ which fixes $f_1$, interchanges $\tau_1$ and $\tau_4$, and interchanges $f_2$ and $f_4$. Thus $g$ interchanges the two vertices adjacent to $(\sigma, f_1)$ in both cases, so $\Gamma$ is $G$-symmetric.

Similarly one can verify that all statements hold for $\Gamma = \Gamma_3(M)$ or $\Gamma_4(M)$. $\square$

Each of $\Gamma_1(M)$, $\Gamma_2(M)$, $\Gamma_3(M)$ and $\Gamma_4(M)$ in Construction 3.2 is a union of cycles since it has valency two. For example, $\Gamma_1(M) \cong s \cdot C_t$ and each face of $M$ gives rise to a cycle of $\Gamma_1(M)$, where $t$ is the face length and $s$ the number of faces of $M$. For the octahedron $M$ one can check that $\Gamma_1(M) \cong 8 \cdot C_3$, $\Gamma_2(M) \cong 4 \cdot C_6$, $\Gamma_3(M) \cong 6 \cdot C_4$ and $\Gamma_4(M) \cong 4 \cdot C_6$.

### 3.3. *An explicit group theoretic construction*

Finally, we give a Sabidussi coset graph construction (see e.g. [6]) for an infinite family of graphs that satisfy part (b) of Theorem 1.1 with $m = 1$. Given a group $G$, a core-free subgroup $H$ of $G$ and a 2-element $g$ such that $g \notin \mathbf{N}_G(H)$ and $g^2 \in H \cap H^g$, the coset graph $\mathrm{Cos}(G, H, HgH)$ is defined to have vertex set $[G : H] = \{Hx : x \in G\}$ such that $Hx, Hy$ are adjacent if and only if $xy^{-1} \in HgH$. It is known, see for example [6], that $\mathrm{Cos}(G, H, HgH)$ is $G$-symmetric and is connected if and only if $\langle H, g\rangle = G$. For a subgroup $L < H$, let $B = [H : L] = \{Lh \mid h \in H\}$. For $x \in G$, let $B^x = \{Lhx \mid h \in H\}$, and let $\mathcal{B} = \{B^x \mid x \in G\}$. Then $\mathcal{B}$ is a $G$-invariant partition of $[G : L]$. Further, we have the following link between the two coset graphs.

**Lemma 3.4.** *Let* $\Gamma = \mathrm{Cos}(G, L, LgL)$ *and* $\Sigma = \mathrm{Cos}(G, H, HgH)$. *Then* $\Sigma \cong \Gamma_{\mathcal{B}}$.

**Proof.** Define a one-to-one correspondence between $[G : H]$ and $\mathcal{B}$ by:

$$\varphi : Hx \mapsto B^x, \quad x \in G.$$

We claim that $\varphi$ induces an isomorphism between $\Sigma$ and $\Gamma_{\mathcal{B}}$. For any $x, y \in G$, we have (where $\sim$ means adjacency):

$$\begin{aligned} Hx \sim Hy \quad \text{in } \Sigma &\Longrightarrow yx^{-1} \in HgH \\ &\Longrightarrow yx^{-1} = h_1 g h_2 \quad \text{for some } h_1, h_2 \in H \\ &\Longrightarrow h_1^{-1} y (h_2 x)^{-1} = g \in LgL \\ &\Longrightarrow Lh_2 x \sim Lh_1^{-1} y \quad \text{in } \Gamma \\ &\Longrightarrow B^x \sim B^y \quad \text{in } \Gamma_{\mathcal{B}}. \\ B^x \sim B^y \quad \text{in } \Gamma_{\mathcal{B}} &\Longrightarrow Lh_1 x \sim Lh_2 y \quad \text{in } \Gamma, \text{for some } h_1, h_2 \in H \\ &\Longrightarrow h_2 y (h_1 x)^{-1} \in LgL \\ &\Longrightarrow yx^{-1} \in h_2^{-1} LgL h_1 \subset HgH \\ &\Longrightarrow Hx \sim Hy \quad \text{in } \Sigma. \end{aligned}$$

Thus $\Sigma \cong \Gamma_{\mathcal{B}}$, as claimed. $\square$

Now we construct examples satisfying part (b) of Theorem 1.1.

**Construction 3.5.** Let $p$ be a prime such that $p \equiv 1 \pmod{16}$, and let $G = \mathrm{PSL}(2, p)$. Let $H$ be a Sylow 2-subgroup of $G$. Then $H = \langle a \rangle : \langle b \rangle \cong D_{16}$, $\langle a^4, b \rangle \cong \mathbb{Z}_2^2$, and $\mathbf{N}_G(\langle a^4, b \rangle) = S_4$. There exists an involution $g \in \mathbf{N}_G(\langle a^4, b \rangle) \setminus \langle a^2, b \rangle$ such that $g$ interchanges $a^4$ and $b$. Let $L = \langle a^4, ba \rangle \cong \mathbb{Z}_2^2$, and define

$$\Sigma = \mathrm{Cos}(G, H, HgH), \qquad \Gamma = \mathrm{Cos}(G, L, LgL).$$

**Lemma 3.6.** *Using the notation defined above, the following all hold:*

(a) *both $\Gamma$ and $\Sigma$ are $G$-symmetric, connected and of valency* 4*;*
(b) *$\mathcal{B}$ is a $G$-invariant partition of $V(\Gamma)$ such that $k = v - 2 = 2$, $\Gamma^B = C_4$ and $\Sigma \cong \Gamma_{\mathcal{B}}$;*
(c) *for $B = [H : L]$ and $C = B^g \in \Gamma_{\mathcal{B}}(B)$, the induced subgraph $\Gamma[B, C] = K_{2,2}$.*

**Proof.** It follows from the classification of the subgroups of $G$, see for example [7, pp. 417], that $\langle H, g \rangle$ is contained in no maximal subgroup of $G$. Thus $\langle H, g \rangle = G$, and so $\Sigma$ is connected. Moreover, since $(a^4)^g = b$, it follows that $b, a \in \langle a^4, ba, g \rangle$. Thus $\langle L, g \rangle = G$, and so $\Gamma$ is connected.

By the definition, $\langle a^4, b, g \rangle \cong D_8$, and $H \cap H^g = \langle a^4, b \rangle \cong \mathbb{Z}_2^2$. Hence $\Sigma$ has valency 4. Since $L$ is abelian, $L \cap L^g \lhd L$. Also $L \cap L^g$ is normalised by the involution $g$, and hence $L \cap L^g$ is normal in $\langle L, g \rangle = G$. As $G$ is simple, $L \cap L^g = 1$, and so $\Gamma$ is of valency 4. Part (a) now follows by Lemma 3.4.

As above $\mathcal{B}$ is a $G$-invariant partition of $V(\Gamma)$ with parts of size $v = |H : L| = 4$. The stabiliser $G_B = H$, and for $C = B^g$, we have $G_{BC} = G_B \cap G_C = H \cap H^g = \langle a^4, b \rangle$. Label the vertex $L$ of $\Gamma$ as $\alpha$. Then $\alpha \in B$, $G_\alpha = L = \langle a^4, ba \rangle$, and $G_\alpha \cap G_{BC} = \langle a^4 \rangle$. The vertex $\beta = \alpha^g = Lg$ lies in $C \cap \Gamma(\alpha)$, and so $\beta^{a^4} \in C \cap \Gamma(\alpha)$ and $\{\beta, \beta^{a^4}\} \subseteq C \cap \Gamma(\alpha)$. Also, since $G_{\alpha\beta} = L \cap L^g = 1$, $a^4$ does not fix $\beta$ and hence $\beta \neq \beta^{a^4}$. Counting the numbers of edge of $\Sigma$ and $\Gamma$, we conclude that there are exactly 4 edges of $\Gamma$ between $B$ and $C$. It follows that $\Gamma[B, C] = K_{2,2}$ and $k = 2$. This together with the fact that both $\Gamma$ and $\Sigma$ have valency 4 forces $\Gamma^B$ to be simple and isomorphic to $C_4$. Finally by Lemma 3.4, $\Sigma \cong \Gamma_{\mathcal{B}}$. This completes the proof of parts (b) and (c). □

Corollary 1.3 follows from Lemma 3.6 immediately.

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