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Distance Labelling Problems for Hypercubes and Hamming Graphs – A Survey

Sanming Zhou [1,2]

Department of Mathematics and Statistics
The University of Melbourne
Parkville, VIC 3010, Australia

Abstract

We survey recent results on a few distance labelling problems for hypercubes and Hamming graphs.

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1 Introduction

Let $\Gamma = (V, E)$ be a graph and $i_1, i_2, \ldots, i_k \geq 0$ integers. An $L(i_1, i_2, \ldots, i_k)$-*labelling* [7] of Γ is a mapping $\phi : V \to \{0, 1, 2, \ldots\}$ such that $|\phi(u) - \phi(v)| \geq i_t$ for any $u, v \in V$ with distance t apart, $t = 1, 2, \ldots, k$. Call $\phi(u)$ the *label* of u under ϕ. Assuming $\min_{v \in V} \phi(v) = 0$ w.l.o.g, we call $\mathrm{sp}(\Gamma; \phi) := \max_{v \in V} \phi(v)$ the *span* of ϕ and $\lambda_{i_1, i_2, \ldots, i_k}(\Gamma) := \min_\phi \mathrm{sp}(\Gamma; \phi)$ the $\lambda_{i_1, i_2, \ldots, i_k}$-*number* of Γ. An unused label between 0 and the largest label used is called a

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[2] Email: smzhou@ms.unimelb.edu.au

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hole, and the meaning of a *no-hole* $L(i_1, i_2, \ldots, i_k)$-labelling is self-evident. Define $\overline{\lambda}_{i_1,i_2,\ldots,i_k}(\Gamma)$ to be the minimum span of a no-hole $L(i_1, i_2, \ldots, i_k)$-labelling of Γ if it exists and ∞ otherwise. A labelling $\phi : V \to \{0, 1, 2, \ldots\}$ such that $|\phi(u) - \phi(v)|_\ell \geq i_t$ for any $u, v \in V$ with distance t apart, $t = 1, 2, \ldots, k$, is called an *ℓ-cyclic $L(i_1, i_2, \ldots, i_k)$-labelling* [9][13], where $|x|_\ell := \min\{|x|, \ell - |x|\}$. (The term 'circular' was used in [9][13].) Let $\sigma_{i_1,i_2,\ldots,i_k}(\Gamma)$ be the minimum integer $\ell - 1$ such that Γ admits such a labelling; and let $\overline{\sigma}_{i_1,i_2,\ldots,i_k}(\Gamma)$ be the minimum $\ell - 1$ such that Γ admits a no-hole ℓ-cyclic $L(i_1, i_2, \ldots, i_k)$-labelling, and ∞ if no such a labelling exists.

The notions above were originated from radio channel assignment [8]. A related problem [17] from optical networking seeks to colour the vertices of Γ such that any two vertices of distance at most k receive different colours. Let $\chi_{\bar{k}}(\Gamma)$ be the minimum number of colours required in such a colouring. One can check that, for any given $i_1, i_2, \ldots, i_k \geq 1$, $\chi_{\bar{k}}(\Gamma)$ is [21] the minimum number of labels needed in an $L(i_1, i_2, \ldots, i_k)$-labelling of Γ. Thus, for example, $\chi_{\bar{2}}(\Gamma)$ is the chromatic number of the square graph of Γ. Another invariant [17] arising from optical networking is the minimum number $\chi_k(\Gamma)$ of labels in an $L(0, 0, \ldots, 0, 1)$-labelling of Γ.

There has been an extensive literature on the invariants above, and so far most studies have been focused on the case where $k = 2$ or 3. See [1] for a recent survey on $L(j, k)$-labellings. The present paper is intended to be a complementary survey with focus on hypercubes and Hamming graphs. Given $q_1, q_2, \ldots, q_d \geq 2$, the *Hamming graph* $H_{q_1,q_2,\ldots,q_d}$ is defined to have vertex set $\mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \cdots \times \mathbb{Z}_{q_d}$ such that two vertices are adjacent if and only if they differ in exactly one coordinate. If $q_t = q$ for all t then we write $H(d, q)$ in place of $H_{q_1,q_2,\ldots,q_d}$. Thus, $H(d, 2)$ is the *hypercube* Q_d of dimension d. A labelling/colouring of a graph is said to be *homogeneous* (or *balanced*) if each label/colour is used by the same number of vertices.

2 Hypercubes and beyond

Let G be a group and X a subset of $G \setminus \{1_G\}$ with $X^{-1} := \{x^{-1} : x \in X\} = X$. The *Cayley graph* $\Gamma(G, X)$ on G relative to X is defined to have vertex set G in which $x, y \in G$ are adjacent if and only if $xy^{-1} \in X$. In [21] the author introduced a group-theoretic approach to $L(j, k)$-labelling Cayley graphs over abelian groups. Using this approach we obtained the following result, where $n := 1 + \lfloor \log_2 d \rfloor, t := \min\{2^n - d - 1, n\}$ for given $d \geq 1$.

Theorem 2.1 *(Zhou [21]) Let Γ be a connected graph whose automorphism* *group contains a vertex-transitive abelian subgroup. Let d be the degree of vertices of* Γ*, and* n, t *be as above. Then, for any* $j \geq k \geq 1$*, we have* $\lambda_{j,k}(\Gamma) \leq 2^n \max\{k, \lceil j/2 \rceil\} + 2^{n-t} \min\{j-k, \lfloor j/2 \rfloor\} - j$ *and* $d+1 \leq \chi_{\bar{2}}(\Gamma) \leq 2^n$*.*

Thus, if $2k \geq j$, then $\lambda_{j,k}(\Gamma) \leq 2^n k + 2^{n-t}(j-k) - j$. In particular, for $L(2,1)$-labellings, Theorem 2.1 implies

Corollary 2.2 *(Zhou [21]) Let* Γ *and* d *be the same as in Theorem 2.1. Then* $\lambda_{2,1}(\Gamma) \leq 2^n + 2^{n-t} - 2$ *and* $d + 1 \leq \chi_{\bar{2}}(\Gamma) \leq 2^n$*.*

Since $Q_d \cong \Gamma(\mathbb{Z}_2^d, X)$ and it admits $\mathbb{Z}_2^d$ as a vertex-transitive group of automorphisms, where X is the set of elements of $\mathbb{Z}_2^d$ with exactly one non-zero coordinate, Theorem 2.1 and Corollary 2.2 imply the following two corollaries for Q_d.

Corollary 2.3 *(Zhou [21]) Let* d*,* j *and* k *be integers with* $d \geq 1$ *and* $j \geq k \geq 1$*. Then* $\lambda_{j,k}(Q_d) \leq 2^n \max\{k, \lceil j/2 \rceil\} + 2^{n-t} \min\{j-k, \lfloor j/2 \rfloor\} - j$ *and* $d + 1 \leq \chi_{\bar{2}}(Q_d) \leq 2^n$*.*

The proof of Theorem 2.1 gives a systematic way of generating $L(j,k)$-labellings of Q_d which use 2^n labels and have span the bound above. A specific $L(2,1)$-labelling of Q_d was given in [17, Theorem 2] under different terminology. Again, if $2k \geq j$, then we get $\lambda_{j,k}(Q_d) \leq 2^n k + 2^{n-t}(j-k) - j$. In particular, Corollary 2.3 implies the following upper bounds, which were the first results on $\lambda_{2,1}$ and $\chi_{\bar{2}}$ for hypercubes.

Corollary 2.4 *We have* $\lambda_{2,1}(Q_d) \leq 2^n + 2^{n-t} - 2$ *(Whittlesey, Georges and Mauro [18, Theorem 3.7]) and* $d + 1 \leq \chi_{\bar{2}}(Q_d) \leq 2^n$ *(Wan [17, line 12, pp.185]).*

In [20] the author studied χ_2 for Cayley graphs Γ over abelian groups. Among the findings are a connection [20, Theorem 1] between $\chi_2(\Gamma)$ and $\chi_2(\hat{\Gamma})$ for certain quotient graphs $\hat{\Gamma}$ of Γ, and an upper bound [20, Corollary 1] on $\chi_2(\Gamma)$. Using this bound together with techniques from linear algebra we established the following result.

Theorem 2.5 *Let* Γ *be a connected triangle-free graph of degree* d*. Suppose the automorphism group of* Γ *contains a vertex-transitive abelian subgroup. Then* $d \leq \chi_2(\Gamma) \leq 2^{\lceil \log_2 d \rceil}$*. Moreover, we give explicitly homogeneous* $L_{0,1}$*-labellings (not unique) of* Γ *using* $2^{\lceil \log_2 d \rceil}$ *labels.*

Theorem 2.5 implies the following known bounds for hypercubes.

Corollary 2.6 *For $d \geq 2$, we have $d \leq \chi_2(Q_d) \leq 2^{\lceil \log_2 d \rceil}$ ([17]). Moreover, from any $d \times \lceil \log_2 d \rceil$ matrix over* GF(2) *with rank $\lceil \log_2 d \rceil$ and pairwise distinct rows we can construct explicitly a homogeneous $L(0,1)$-labelling of Q_d which uses $2^{\lceil \log_2 d \rceil}$ labels.*

Now let us move on to $\chi_{\bar{3}}$ and $L(i_1, i_2, i_3)$-labellings for Q_d.

Theorem 2.7 *(Kim, Du and Pardalos [11])* $2d \leq \chi_{\bar{3}}(Q_d) \leq 2^{\lceil \log_2 d \rceil + 1}$.

In [11] the same authors also obtained lower and upper bounds for $\chi_{\bar{k}}(Q_d)$, which were improved as follows by Ngo, Du and Graham.

Theorem 2.8 *(Ngo, Du and Graham [15]) Let $t = \lfloor k/2 \rfloor$ and $\left(\binom{d}{m}\right)$ denote $\sum_{i=0}^{m} \binom{d}{i}$. Then, when k is even, we have*

$$\left(\binom{d}{t}\right) + \frac{1}{\lfloor \frac{d}{t+1} \rfloor} \binom{d}{t} \left(\frac{d-t}{t+1} - \left\lfloor \frac{d-t}{t+1} \right\rfloor \right) \leq \chi_{\bar{k}}(Q_d) \leq 2^{\left\lfloor \log_2 \left(\binom{d-1}{k-1}\right) \right\rfloor + 1};$$

and when k is odd, we have

$$2\left(\left(\binom{d-1}{t}\right) + \frac{\binom{d-1}{t}\left(\frac{d-1-t}{t+1} - \left\lfloor \frac{d-1-t}{t+1} \right\rfloor\right)}{\lfloor \frac{d-1}{t+1} \rfloor} \right) \leq \chi_{\bar{k}}(Q_d) \leq 2^{\left\lfloor \log_2 \left(\binom{d-2}{k-2}\right) \right\rfloor + 2}.$$

Note that, if $d = 2^n - 1$, then $\chi_{\bar{2}}(Q_d) = 2^n$ by Corollary 2.4. Wan conjectured [17] that $\chi_{\bar{2}}(Q_d) = 2^n$ for any d. This is disproved by $13 \leq \chi_{\bar{2}}(Q_8) \leq 14$, obtained independently by Hougardy [19] and Royle [10, Section 9.7]. In general, we have the following interesting asymptotic result.

Theorem 2.9 *(Östergård [16])* $\lim_{d \to \infty} \chi_{\bar{2}}(Q_d)/d = 1$, $\lim_{d \to \infty} \chi_{\bar{3}}(Q_d)/d = 2$.

Let $n = 1 + \lfloor \log_2 d \rfloor$ and $q = \max\{d + 1 + \lceil \log_2(d+1) \rceil - 2^{\lceil \log_2(d+1) \rceil}, 0\}$.

Theorem 2.10 *(Zhou [22]) Let $d \geq 3$ and n, q be as above. Then for any integers $i_1 \geq i_2 \geq i_3 \geq 1$ we have*

$$i_2(d-1) + i_1 \leq \lambda_{i_1,i_2,i_3}(Q_d) \leq \begin{cases} 2^n(i_3 + r) + 2^q(i_1 - r) - i_1, & 2^{n-1} < d \leq 2^n - 1 \\ (2^n - 2)r + i_1, & d = 2^{n-1} \end{cases}$$

where $r := \max\{i_2, \lceil i_1/2 \rceil\}$, and we can give explicitly homogeneous $L(i_1, i_2, i_3)$-labellings of Q_d which use $2^{\lceil \log_2 d \rceil + 1}$ labels and have span the upper bound above. In addition, if $i_1 \leq 2$, then $\lambda_{i_1,i_2,i_3}(Q_d) \geq 2(d-1) + i_1$.

Theorem 2.10 gives the same upper bound $\chi_{\bar{3}}(Q_d) \leq 2^{\lceil \log_2 d \rceil + 1}$ as in Theorem 2.7, and it implies the following result in the special case where $(i_1, i_2, i_3) = (2, 1, 1)$.

Corollary 2.11 *(Zhou [22]) Let $d \geq 3$ and n, q be as above. If $2^{n-1} < d \leq 2^n - 1$, then $2d \leq \lambda_{2,1,1}(Q_d) \leq 2^{n+1} + 2^q - 2$; if $d = 2^{n-1}$, then $\lambda_{2,1,1}(Q_d) = 2d$ and Q_d admits a homogeneous $L(2,1,1)$-labelling with span $2d$ and exactly one hole.*

3 Hamming graphs

We assume $q_1 \geq q_2 \geq \cdots \geq q_d \geq 2$ throughout this section. Again by using the group-theoretic approach developed in [21], the author obtained the following results.

Theorem 3.1 *(Zhou [21]) Suppose $q_1 > d \geq 2$, q_2 divides q_1 and each prime factor of q_1 is no less than d. Then for any $j \geq k \geq 1$ and $q_3, \ldots, q_d$ we have $\lambda_{j,k}(H_{q_1,q_2,\ldots,q_d}) \leq (q_1 q_2 - 1)\max\{k, \lceil j/2 \rceil\}$, $\chi_{\bar{2}}(H_{q_1,q_2,\ldots,q_d}) = q_1 q_2$, and we can give an $L(j,k)$-labelling of $H_{q_1,q_2,\ldots,q_d}$ which is optimal for $\chi_{\bar{2}}$ and has span $(q_1 q_2 - 1)\max\{k, \lceil j/2 \rceil\}$. If in addition $2k \geq j$, then $\lambda_{j,k}(H_{q_1,q_2,\ldots,q_d}) = (q_1 q_2 - 1)k$ and this $L(j,k)$-labelling is optimal for $\lambda_{j,k}$ and $\chi_{\bar{2}}$ simultaneously.*

Thus, if $2k \geq j$, then the trivial lower bounds $\lambda_{j,k}(H_{q_1,q_2,\ldots,q_d}) \geq (q_1 q_2 - 1)k$ and $\chi_{\bar{2}}(H_{q_1,q_2,\ldots,q_d}) \geq q_1 q_2$ are attained simultaneously. Interestingly, both $\lambda_{j,k}$ and $\chi_{\bar{2}}$ are irrelevant to j in this case.

Corollary 3.2 *(Zhou [21]) Let $q_1, q_2, \ldots, q_d$ and $d \geq 2$ be as in Theorem 3.1. Then $\lambda_{2,1}(H_{q_1,q_2,\ldots,q_d}) = q_1 q_2 - 1$ and $\chi_{\bar{2}}(H_{q_1,q_2,\ldots,q_d}) = q_1 q_2$. Moreover, we can give a no-hole $L(2,1)$-labelling of $H_{q_1,q_2,\ldots,q_d}$ which is optimal for $\lambda_{2,1}$ and $\chi_{\bar{2}}$ simultaneously.*

Corollary 3.3 *(Zhou [21]) Let $q = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$, where p_i is a prime and $r_i \geq 1$ for each i. Let d be such that $2 \leq d \leq p_i$ for all i and $\sum_{i=1}^{t}(p_i - d + r_i) \geq 2$. Then for any $j \geq k \geq 1$ we have $\lambda_{j,k}(H(d,q)) \leq (q^2 - 1)\max\{k, \lceil j/2 \rceil\}$ and $\chi_{\bar{2}}(H(d,q)) = q^2$. If in addition $2k \geq j$, then $\lambda_{j,k}(H(d,q)) = (q^2 - 1)k$.*

Corollaries 3.2 and 3.3 imply the following result of Georges, Mauro and Stein [6]: for any prime p and integers $d, r \geq 1$ such that $3 \leq d \leq p$ and $(p-d)+r \geq 2$, we have $\lambda_{2,1}(H(d,p^r)) = p^{2r} - 1$. The following theorem of the same authors suggests that the bound $(q_1 q_2 - 1)\max\{k, \lceil j/2 \rceil\}$ in Theorem 3.1 may be far away from the actually value of $\lambda_{j,k}(H_{q_1,q_2})$ when j/k is large.

Theorem 3.4 *(Georges, Mauro and Stein [6]) Let* $j \geq k$, $q_1 > q_2 \geq 2$ *and* $q \geq 2$. *Then* $\lambda_{j,k}(H_{q_1,q_2}) = (q_1-1)j + (q_2-1)k$ *if* $j/k > q_2$ *and* $(q_1q_2-1)k$ *if* $j/k \leq q_2$, *and* $\lambda_{j,k}(H(2,q)) = (q-1)j + (2q-2)k$ *if* $j/k > q-1$ *and* $(q^2-1)k$ *if* $j/k \leq q-1$.

Theorem 3.5 *(Erwin, Georges and Mauro [4]) Let* $q_1 > q_2 > \cdots > q_d$ *be relatively prime and* $d \geq 3$. *Then* $\lambda_{j,k}(H_{q_1,q_2,\ldots,q_d}) = (q_1q_2-1)k$ *if* $j/k \leq q_2$ *and* $(q_1-1)j + (q_2-1)k$ *if* $j/k > q_2$.

Theorem 3.1 inspired the following questions. (Note that in [21] the necessary condition $j/k \leq q_1q_2 - \sum_{i=1}^{d} q_i + d$ was missed.)

Problem 3.6 *(Zhou [21]) Let* $2k \geq j \geq k \geq 1$. *Is* $\lambda_{j,k}(H_{q_1,q_2,\ldots,q_d}) = (q_1q_2-1)k$ *true for any* $H_{q_1,q_2,\ldots,q_d} \neq Q_d$ *such that* $j/k \leq q_1q_2 - \sum_{i=1}^{d} q_i + d$?

Theorem 3.7 *(Chang, Lu and Zhou [2])* $H_{q_1,q_2,\ldots,q_d}$ *admits a no-hole* $L(2,1)$-*labelling* $\Leftrightarrow$ $H_{q_1,q_2,\ldots,q_d}$ *admits a no-hole cyclic* $L(2,1)$-*labelling* $\Leftrightarrow$ $H_{q_1,q_2,\ldots,q_d} \neq Q_2$. *Moreover, if* $q_1 \geq d + n - 1 + \sum_{2\leq l<m\leq d} \max\{0, q_l + q_m - q_2 - 1\}$ *(where* n *is the largest integer such that* $q_2 = q_n$*), then* $\lambda_{2,1}(H_{q_1,q_2,\ldots,q_d}) = \overline{\lambda}_{2,1}(H_{q_1,q_2,\ldots,q_d}) = \overline{\sigma}_{2,1}(H_{q_1,q_2,\ldots,q_d}) = \sigma_{2,1}(H_{q_1,q_2,\ldots,q_d}) = q_1q_2 - 1$ *and we give a labelling of* $H_{q_1,q_2,\ldots,q_d}$ *which is optimal for* $\lambda_{2,1}, \overline{\lambda}_{2,1}, \overline{\sigma}_{2,1}, \sigma_{2,1}$ *simultaneously.*

In particular, this answers Problem 3.6 for $(j,k) = (2,1)$ and sufficiently large q_1. Theorem 3.7 implies [2] $\lambda_{1,1} = \overline{\lambda}_{1,1} = \overline{\sigma}_{1,1} = \sigma_{1,1} = q_1q_2 - 1$ for $H_{q_1,q_2,\ldots,q_d}$ under the same conditions.

Problem 3.8 *(Chang, Lu and Zhou [2]) Is it true that* $\lambda_{2,1}(H_{q_1,q_2,\ldots,q_d}) = \overline{\lambda}_{2,1}(H_{q_1,q_2,\ldots,q_d}) = \overline{\sigma}_{2,1}(H_{q_1,q_2,\ldots,q_d}) = \sigma_{2,1}(H_{q_1,q_2,\ldots,q_d}) = q_1q_2-1$ *for any* $H_{q_1,q_2,\ldots,q_d} \neq Q_d$ *with* $\sum_{t=1}^{d} q_t \leq q_1q_2 + d - 2$?

An affirmative answer follows [2] if we can prove $\overline{\sigma}_{2,1}(H_{q_1,q_2,\ldots,q_d}) \leq q_1q_2 - 1$.

Theorem 3.9 *(Chang, Lu and Zhou [2]) We have* $\lambda_{2,0}(H_{q_1,q_2,\ldots,q_d}) = 2q_1 - 2$ *and* $\sigma_{2,0}(H_{q_1,q_2,\ldots,q_d}) = 2q_1 - 1$. *Moreover,* $H_{q_1,q_2,\ldots,q_d}$ *admits a no-hole* $L(2,0)$-*labelling* $\Leftrightarrow$ $H_{q_1,q_2,\ldots,q_d}$ *admits a no-hole cyclic* $L(2,0)$-*labelling* $\Leftrightarrow$ $H_{q_1,q_2,\ldots,q_d} \neq Q_2$, *and in this case the following* (a)-(c) *hold:*

(a) *if* $q_1, q_2, \ldots, q_d$ *are not all the same, then* $\overline{\lambda}_{2,0}(H_{q_1,q_2,\ldots,q_d}) = \overline{\sigma}_{2,0}(H_{q_1,q_2,\ldots,q_d}) = 2q_1 - 1$;

(b) *if* $d \geq 3$ *and* $q \geq 2$, *then* $\overline{\lambda}_{2,0}(H(d,q)) = 2q - 1$ *and* $\overline{\sigma}_{2,0}(H(d,q)) = 2q$;

(c) *if* $d = 2$ *and* $q \geq 3$, *then* $\overline{\lambda}_{2,0}(H(2,q)) = 2q, \overline{\sigma}_{2,0}(H(2,3)) = 8$ *and* $\overline{\sigma}_{2,0}(H(2,q)) = 2q$ or $2q+1$ $(q \geq 4)$.

Furthermore, we construct explicitly an optimal labelling in each case with the exception of $\overline{\sigma}_{2,0}(H(2,q))$, $q \geq 4$.

In [2] Chang, Lu and the author conjectured that $\overline{\sigma}_{2,0}(H(2,q)) = 2q+1$ if $q \geq 4$, and they confirmed this for $q = 4, 5, 6$.

4 A brief discussion on Cayley graphs

Theorems 2.1 and 3.1, and Corollaries 2.2, 2.3, 3.2 and 3.3 are all based on a general approach [21] to $L(j,k)$-labelling Cayley graphs $\Gamma(G,X)$ over abelian groups G. Among other results it was proved [21] that, for any subgroup H of G such that $H \cap X = \emptyset$ and $H \cap X^2 = \{1\}$ (where $X^2 := \{xx' : x, x' \in X\}$), we have $\lambda_{j,k}(\Gamma(G,X)) \leq |G:H| \max\{k, \lceil j/2 \rceil\} + |G : \langle G - HX \rangle| \min\{j - k, \lfloor j/2 \rfloor\} - j$ and $\chi_{\overline{2}}(\Gamma(G,X)) \leq |G:H|$. The key and the most difficult part in using this generic approach is to find a suitable subgroup H so that the bound above for $\lambda_{j,k}$ is as small as possible.

In [3], Chang, Lu and the author studied the no-hole $L(2,0)$-labelling problem for Cayley graphs $\Gamma(G,X)$ over finitely generated abelian groups G, where $|X|$ is finite. Sufficient conditions for the existence of a no-hole $L(2,0)$-labelling and upper bounds on $\overline{\lambda}_{2,0}(\Gamma(G,X))$ were obtained in [3]. In particular, for a finite abelian group G, by using the hamiltonicity [14] of $\Gamma(G,X)$ it was shown [3] that $\Gamma(G,X)$ admits a no-hole $L(2,0)$-labelling if and only if $\langle G - X \rangle = G$. Finally, a forthcoming paper [12] investigates the $L(2,1)$-labelling problem for cubic Cayley graphs on dihedral groups.

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