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UNIFYING APPROACHES FOR CONSTRUCTING LABELED GRAPHS FROM KNOWN ONES

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In this paper we give some unifying methods for constructing five kinds of additively labeled graphs from known ones.

1. INTRODUCTION

In the last more than twenty-five years, much attention has been paid to the graph labeling problems. There is now extensive literature dealing with various kinds of labelings of graphs. The increasing interest on labeling problems results from two reasons. One is the motivations of various problems from natural, social and technological sciences, the other is the requirements of internal development of graph theory. In general, it seems hopeless to get simple but nontrivial criterions for judging which graphs admit labelings of a particular kind. Nevertheless, much work has been done for some special classes of graphs. In this paper we will consider the problem of constructing labeled graphs from known ones. In particular, several known results will be generalized.

Unless stated otherwise, terms and notations used here are consistent with [1]. Graphs we consider are finite, undirected and simple (i.e., without loops and multiple edges). An edge e incident with vertices u and v is denoted by $e = uv$. In such a case call u, v the end-vertices of e, or u, v joined by e. A graph G of p vertices and q edges is indicated by $G = G(p, q)$. By K_p and $\overline{K}_p$ we mean the complete graph and the empty graph on p vertices, respectively. For two graphs G and H, the union $G \cup H$ of them is the graph $(V(G) \cup V(H), E(G) \cup E(H))$. When G and H are vertex disjoint we use $G + H$ in place of $G \cup H$. If E is a subset of the edge set of the complement graph $\overline{G}$ of G, $G \cup E$ means the graph $V(G)$, $E(G) \cup E)$. If G and H are vertex disjoint, let $G \vee H$ be the join of the two graphs. That is $G \vee H = (G + H) \cup E$, where E is the set of all edges with one end-vertex in $V(G)$ and the other in $V(H)$. In particular, $G \cup K_0 = G + K_0 = G \vee K_0 = G$. Let a, b be two integers call $[a, b] = \{x : a \le x \le b$ and x is an integer$\}$ an interval. When $a > b$, let $[a, b] = \emptyset$. For two intervals $[a, b]$ and $[c, d]$, let $[a, b] + [c, d] = \{x + y : x \in [a, b], y \in [c, d]\}$. Particularly,

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$c + [a, b] = [c + a, c + b]$. In general, let $A + B = \{x + y : x \in A, y \in B\}$ and $c + A = \{c\} + A$ for any number sets A, B and number c. Denote $\lfloor c \rfloor$ and $\lceil c \rceil$ the largest integer no more than c and the smallest integer no less than c, respectively.

A (vertex) labeling of a given graph G by integers is an assignment of distinct nonnegative integers to the vertices of G. This is to say that a labeling of G is an injection $f: V(G) \to N$ from $V(G)$ to the set N of nonnegative integers. For each $v \in V(G)$, $f(v)$ is called the label of v or v is f-labeled. In the paper, we use $f_{\min}$ and $f_{\max}$ to denote the minimum and maximum labels of the vertices of G, respectively. Naturally, f induces an edge labeling $f^+ : E(G) \to N$ defined by $f^+(uv) = f(u) + f(v)$ for each $uv \in E(G)$. Often, some additional constraints are attached to a labeling f and its induced edge labeling f^+. Different constraints result in different kinds of labelings. For the purpose of this paper we list definitions of the following five kinds of labelings.

DEFINITION. Let $G = G(p, q)$ and f be a labeling of G.

(1) If $f(V(G) \subseteq [0, q' - 1]$ and $f^+(E(G)) = [k, k + q - 1]$ for a positive integer k, then f is called a k-sequential labeling of G, where $q' = q + 1$ if G is a tree and $q' = q$ otherwise;

(2) If $f(V(G)) \subseteq [0, q - 1]$ and $f^+(E(G)) = [k, k + q - 1]$, for an integer k, then f is said to be a strong k-harmonious labeling of G;

(3) If $f(V(G)) \subseteq [0, q]$ and $f^+(E(G)) = [k, k + q - 1]$, call f a strong k-elegant labeling of G;

(4) If $f(V(G)) = [0, p - 1]$ and $f^+(E(G)) = [k, k + q - 1]$, then f is called a strong k-indexer of G;

(5) If $f^+(E(G)) = \{k, k + d, \ldots, k + (q - 1)d\}$ is an arithmetic progression with first term k and common difference d, f is said to be a (k, d)-arithmetic labeling of G.

Correspondingly, when a graph G admits a k-sequential labeling, strong k-harmonious labeling, *etc.*, then G is said to be a k-sequential graph, strongly k-harmonious graph, etc., respectively.

The five kinds of labeled graphs mentioned above have been studied by many authors and a number of results concerning some special graphs belonging to some of the five kinds stated above have been obtained ([2, 3, 4, 5]). Due to limited space, we cannot list these results. But we refer to the following observations from the aforecited literature.

OBSERVATION 1. Let $G = G(p, q)$ be a graph.

(1) If G is a k-sequential tree, then $k \le q = p - 1$; if G is a k-sequential graph but not a tree, then $k \le q - 2$.

(2) If G is strongly k-harmonious, then $k \le q - 2$.

(3) If G is strongly k-elegant, then $k \le q$.

(4) If G is strongly k-indexable, then $k \le 2p - q - 2$ and $q \le 2p - 3$.

Proof. We prove (4) only, the others can be proved similarly. Let $f: V(G) \to [0, p - 1]$ be a strong k-indexer of G, then $f^+(E(G)) = [k, k+q-1]$. Since for any edge $e = uv \in E(G)$, $f^+(e) = f(u) + f(v) \leq (p-1) + (p-2)$, we must have $k+q-1 \leq 2p-3$. This gives $k \leq 2p-q-2$. For the same reason $f^+(E(G)) \subseteq [1, 2p-3]$; so we have $q \leq 2p-3$ (this has been observed in [4] independently).

OBSERVATION 2. (1) For a graph G other than a tree, the k-sequential labelings coincide with the strong k-harmonious labelings of G. Hence the class of k-sequential graphs other than trees is exactly the class of strongly k-harmonious graphs.

(2) For a tree, the k-sequential labelings, the strong k-elegant labelings and the strong k-indexers all agree. So the class of k-sequential trees, of strongly k-elegant trees and of strongly k-indexable trees are identical.

(3) The terms of k-sequential labelings, strong k-harmonious labelings and strong k-indexers agree on the class of connected unicyclic graphs.

The aim of this paper is to give some unifying methods for constructing new labeled graphs from old ones. The results obtained in this paper will be handy to produce a wide variety of additively labeled graphs of any of the considered types. Therefore, they may be useful to check or illustrate structural properties of graphs that admit a given type of labeling. The results may be also helpful to solve some extremal problems concerning labeled graphs discussed in this paper. The methods used fall into three categories: methods by adding or deleting edges, by joining graphs and by adding new edges connecting many labeled graphs.

2. ADDING AND DELETING EDGES

Perhaps the simplest and most natural idea for constructing labeled graphs is to use elementary edge transformation for a given labeled graph. The following theorem is self-evident.

THEOREM 1. *Let G be a k-sequential (strongly k-harmonious, strongly k-elegant, strongly k-indexable or (k, d)-arithmetic) graph with a k-sequential labeling (strong k-harmonious labeling, strong k-elegant labeling, strong k-indexer or (k, d)-arithmetic labeling, respectively) f. For each $i \in f^+(E(G))$, let e_i be the edge of G with $f^+(e_i) = i$. Let $E_i = \{uv \in E(\overline{G}) : f(u) + f(v) = i\}$. If $E_i \neq \emptyset$, then for each $e_i' \in E_i$, the graph $G + e_i' - e_i$ is k-sequential graph (strongly k-harmonious graph, strongly k-elegant graph, strongly k-indexable graph or (k, d)-arithmetic graph, respectively).*

The following five theorems indicate how we can construct one kind of labeled graphs from another. The proofs are fairly easy, so we prove Theorem 3 only.

THEOREM 2. *Let G be a k-sequential tree of p vertices with a k-sequential labeling f.*

(1) *If $k = 1$, then the graph obtained from G by adding an edge the end-vertices of which are labeled i and $p-i$ for some i $(1 \leq i \leq p-1)$ is 1-sequential. It is also strongly 1-harmonious.*

(2) *If* $k = p - 1$, *then for every* i, $0 \le i \le \lfloor \frac{p-3}{3} \rfloor$, *the graph obtained by adding an edge, which connects two vertices labeled* i *and* $p - 2 - i$, *to* G *is* $(p-2)$-*sequential. It is also strongly* $(p-2)$-*harmonious.*

(3) *If* $1 < k < p - 1$, *then for each* i, $0 \le i \le \lfloor \frac{k-2}{2} \rfloor$, *the graph obtained by adding an edge, which connects the vertices labeled* i *and* $k - i - 1$ *to* G, *is* $(k-1)$-*sequential, it is also strongly* $(k-1)$-*harmonious. Furthermore, for each* i, $k \le i \le \lfloor \frac{k+p-2}{2} \rfloor$, *the graph obtained from* G *by adding the edge the end-vertices of which are labeled* i *and* $k + p - i - 1$ *is* k-*sequential. It is also strongly* k-*harmonious.*

THEOREM 3. *Let* $G = G(p, q)$ *be a* k-*sequential graph other than a tree. Then deleting the* k-*labeled edge from* G *results in a strongly* $(k+1)$-*elegant graph and deleting the* $(k+q-1)$-*labeled edge results in a strongly* k-*elegant graph. Conversely for any strongly* k-*elegant graph* $G = G(p, q)$, *we have*

(1) *If* $k = 1$ *and* $p > \lceil \frac{q+2}{2} \rceil$, *then there exist two nonadjacent vertices of* G, *the sum of whose labels is* $q + 1$. *And, the graph obtained from* G *by adding the edge connecting these two vertices is* 1-*sequential, as well as strongly* 1-*harmonious.*

(2) *If* $k = q$ *and* $p > \lceil \frac{q+2}{2} \rceil$, *then there are two nonadjacent vertices of* G *the sum of whose labels is* $q - 1$. *Connecting these two vertices gives a* $(q-1)$-*sequential graph. This graph is strongly* $(q-1)$-*harmonious as well.*

(3) *If* $1 < k < q$, *we have two subcases:*

(i) *If* $k > 2(q-p) + 3$, *then there exist two vertices not adjacent in* G *such that the sum of their labels is* $k - 1$. *And, connecting these two vertices in* G *results in a* $(k-1)$-*sequential (also strongly* $(k-1)$-*harmonious) graph.*

(ii) *If* $k < 2p - q - 2$, *then there exist two nonadjacent vertices of* G *the sum of whose labels is* $k + q$. *Adding the edge connecting these two vertices results in a* k-*sequential (also strongly* k-*harmonious) graph.*

Proof. The first part of the theorem is obvious. Let's prove the second part. Let $f: V \to [0, q]$ be a strong k-elegant labeling of G.

(1) If $k = 1$ and there exist no two vertices the label sum of which is $q + 1$, then

$$p = |f(V(G))| = |\{0\} \cap f(V(G))| + |[1, q] \cap f(V(G))|$$
$$\le 1 + \lceil \frac{q}{2} \rceil = \lceil \frac{q+2}{2} \rceil.$$

This contradicts our hypothesis. So there are two vertices u and v such that $f(u)+f(v)=q+1$. Note $q+1\notin f^{+}(E(G))$, $uv\notin E(G)$. Since $G+uv$ has $q+1$ edges, f is a 1-sequential labeling (also strong 1-harmonious labeling) of $G+uv$.

(2) If $k=q, p>\lceil\frac{q+2}{2}\rceil$, by analogous argument as in (1) we know that there are two nonadjacent vertices u and v such that $f(u)+f(v)=q-1$ and $G+uv$ is a $(q-1)$-sequential (also strongly $(q-1)$-harmonious) graph.

(3) Suppose $1<k<q$, $k>2(q-p)+3$ and there are no two vertices with label sum $k-1$; then

$$p=|f(V(G))|=|[0,k-1]\cap f(V(G))|+|[k,q]\cap f(V(G))|$$
$$\leq\lceil\frac{k}{2}\rceil+q-k+1$$
$$\leq\frac{k+1}{2}+q-k+1$$

i.e., $$k\leq 2(q-p)+3,$$

a contradiction. Hence there exist $u, v\in V(G)$ with $f(u)+f(v)=k-1$. u, v must be nonadjacent since $k-1\notin f^{+}(E(G))$. It is clear that f is a $(k-1)$-sequential (strong $(k-1)$-harmonious) labeling of $G+uv$.

The proof for the case $k<2p-q-2$ is similar to that above. We omit it.

THEOREM 4. *If G is a strongly k-harmonious graph, then the graph obtained from G by deleting the k-labeled edge is strongly $(k+1)$-elegant, and the graph obtained from G by deleting the $(k+q-1)$-labeled edge is strongly k-elegant.*

THEOREM 5. *Let $G=G(p,q)$ be strongly k-harmonious, then it is also strongly k-indexable if $p\geq q$. On the other hand if G is strongly k-indexable and $p\leq q$, then it is also strongly k-harmonious. Moreover, a strongly k-harmonious graph $G=G(p,q)$ plus $m(\geq q-p)$ isolated vertices is strongly k-indexable.*

THEOREM 6. *If $G=G(p,q)$ is strongly k-elegant and $p\geq q+1$, then it is strongly k-indexable. Conversely if G is strongly k-indexable and $p\leq q+1$, then it is strongly k-elegant. A strongly k-elegant graph with $p\leq q+1$ plus $q-p+1$ isolated vertices results in a strongly k-indexable graph.*

3. JOINING GRAPHS

In [2], T. Grace proved that if T is a k-sequential tree, then $T\vee\overline{K_n}$ is also k-sequential for any nonnegative integer n. This result was generalized in [6] as follows.

THEOREM 7 ([6]). *Let $G = G(p, q)$ be any k-sequential graph and n a positive integer. Then $(G + \overline{K}_m) \vee \overline{K}_n$ is also k-sequential for each integer m with $q - p + 1 \leq m \leq q - p + k$.*

We find this method is applicable to other kinds of labeling graphs. For harmonious graphs we have

THEOREM 8. *Let $G = G(p, q)$ be a strongly k-harmonious graph. Then for every positive integer n and integer m with $q - p \leq m \leq q - p + k$, $(G + \overline{K}_m) \vee \overline{K}_n$ is also strongly k-harmonious.*

Proof. Let $H = (G + \overline{K}_m) \vee \overline{K}_n$, $V(\overline{K}_n) = \{v_1, \ldots, v_n\}$. Then $H = H(p + m + n, q + (p + m)n)$. Suppose $f: V(G) \to [0, q - 1]$ is a strong k-harmonious labeling of G. Define $g : V(H) \to N$ as follows. Let $g(v) = f(v)$ for each $v \in V(G)$. Then $g(V(G)) \subseteq [0, q - 1] \subseteq [0, p + m - 1]$ and $[0, p + m - 1]\backslash g(V(G))$ contains exactly m integers. Let's label the m vertices of $\overline{K}_m$ with these m integers in any way. Thus $g(V(G + \overline{K}_m)) = [0, p + m - 1]$. Let

$$g(v_i) = k + q + (i - 1)(p + m), \quad 1 \leq i \leq n.$$

The smallest label of $v_1, \ldots, v_n$ is $g(v_1) = k + q \geq p + m$, and the largest one is $g(v_n) = k + q + (n - 1)(p + m) \leq 2q - 2 + (n - 1)(p + m) < q + (p + m)n - 1$, So $g : V(H) \to [0, |E(H)| - 1]$ is injective.

Note that $g^+(E(G)) = [k, k + q - 1]$ and $g(v_i) + g(V(G + \overline{K}_m)) = [k + q(i - 1)(p + m),\ k + q + i(p + m) - 1]$ for $1 \leq i \leq n$. We have $g^+(E(H)) = [k, k + q + n(p + m) - 1]$. Thus g is a strong k-harmonious labeling of H. This completes the proof.

By using similar methods as used in the above proof, we can prove the following

THEOREM 9. *Let $G = G(p, q)$ be a strongly k-elegant graph. Then for any integers $n \geq 0$ and m satisfying $q - p + 1 \leq m \leq q - p + k$, $(G + \overline{K}_m) \vee \overline{K}_n$ is also strongly k-elegant.*

THEOREM 10. *Let $G = G(p, q)$ be a strongly k-indexable graph and $m = q - p + k$, then $(G + \overline{K}_m) \vee K_1$ is strongly k-indexable.*

Proof. Let $H = (G + \overline{K}_m) \vee K_1 = H(p + m + 1, q + p + m)$ and $f: V(G) \to [0, p - 1]$ be a strong k-indexer of G. Let $G(v) = f(v)$ for each $v \in V(G)$ and the m vertices of $\overline{K}_m$ be labeled with m integers in $[0, p + m - 1]\backslash g(V(G))$. Label the only vertex of K_1 with $p + m$. Then $g : V(H) \to [0, p + m]$ is a bijection. It is easy to check that $g^+(E(H)) = [k, k + q + p + m - 1]$. This completes the proof.

The method of Theorem 7 was used in [5] for (k, d)-arithmetic graph. But in Theorem 10 of [5], a vital condition "Q is an arithmetic progression with common difference d" was neglected. Also in the proof of that theorem the label of $x_i \in X$ seems to be $k + qd - a + d(p + r)(i - 1)$ instead of $k + qd - f_{\min}(G) + d(p + r)(i - 1)$, where a is the first term of Q. Here we give the revised versions of the theorem and the proof, and then generalize the theorem. Note first that if G is a (k, d)-arithmetic graph, then the graphs obtained from G by adding or deleting any number of isolated vertices are also (k, d)-arithmetic graphs. So without loss of generality we may suppose a (k, d)-arithmetic graph has no isolated vertices. We have

THEOREM 11. *Let $G = G(p, q)$ be a (k, d)-arithmetic graph without isolated vertices, and f be a (k, d)-arithmetic labeling of G. If $f(V(G))$ can be arranged as a subsequence P of an arithmetic progression Q with common difference d such that the last term of Q is $f_{\max}$, then $(G + \overline{K}_m) \vee \overline{K}_n)$ is a (k, d)-arithmetic graph, where $m = |Q \backslash P|$ and $n \geq 1$ is any integer.*

Actually Theorem 11 is an immediate corollary of the following:

THEOREM 12. *Let $G = G(p, q)$ be a (k, d)-arithmetic graph without isolated vertices, and f be a (k, d)-arithmetic labeling of G. Let W be a subset of $V(G)$ such that (1) $f_{\text{mix}} \in f(W)$; (2) $f(W)$ can be arranged as a subsequence of an arithmetic progression Q with common difference d; (3) the last term of Q is that of P, and (4) $Q \cap f(V(G) \backslash W) = \varnothing$. Then $((G[W] + \overline{K}_m) \vee \overline{K}_n) \cup G$ is a (k, d)-arithmetic graph, where $m = |Q \backslash f(W)|$ and $n \geq 1$.*

Putting $W = V(G)$ in Theorem 12, we get Theorem 11.

Proof. Let a be the first term of Q; then $a \leq \min\{f(v) : v \in W\} = f_{\min}$ by condition (1). Let $s = |W|$, then $Q = \{a, a + d, \ldots, a + (m + s - 1)d\}$. Let $V(\overline{K}_n) = \{v_1, \ldots, v_n\}$. Define a labeling g of $H = ((G[W] + \overline{K}_m) \vee \overline{K}_n) \cup G$ as follows. Let $g(v) = f(v)$ for every $v \in V(G)$, then $Q \backslash g(W) = Q \backslash f(W)$ contains exactly m integers. Label the vertices of $\overline{K}_m$ with these m integers in any way. Then $g(W \cup V(\overline{K}_m)) = Q$.

Let

$$g(v_i) = k + qd - a + (i - 1)(m + s)d, \quad 1 \leq i \leq n.$$

We will prove that g is an injection. Because of the condition (4), $g(V(G) \cap g(V(\overline{K}_m)) = \varnothing$. So to prove g is injective it suffices to show that for each $i = 1, \ldots, n$, $g(v_i) \notin Q \cup f(V(G) \backslash W) = Q \backslash f(W)) \cup f(V(G))$. Suppose to the contrary that $g(v_i) \in (Q \backslash f(W)) \cup f(V(G))$ for some i.

CASE 1. $g(v_i) \in f(V(G))$

Let $g(v_i) = f(v)$ for $v \in V(G)$, i.e., $a + f(v) = k + qd + (i - 1)(m + s)d \geq k + qd$. Since G contains no isolated vertices we may suppose u is adjacent to v in G. Then $f(u) \geq f_{\min} \geq a$ by (1). So $f(u) + f(v) \geq k + qd$. This contradicts the fact $f(u) + f(v) + f^+(uv) \in f^+(E(G)) = \{k, k + d, \ldots, k + (q - 1)d\}$.

CASE 2. $g(v_i) \in Q \backslash f(W)$

Because of (3) we have $g(v_i) < \max\{f(v) : v \in W\} = f(v_i)$. Then

$$a + f(v_i) > k + qd + (i - 1)(m + s)d.$$

We can reach a contradiction by the same argument as in case 1. Thus we have proved g is injective.

Since $g(v_i) + g(W \cup V(\overline{K}_m)) = \{k + qd + (i-1)(m+s)d,\ k + qd + (i-1)(m+s)d + d, \ldots, k + qd + i(m+s)d - d\}$, $1 \le i \le n$, we conclude that $g^+(E(H))$ is an arithmetic progression with the first term k and common difference d. This completes the proof.

T. Grace's result ([2]) can be generalized in another way. This is what we have just done for (k, d)-arithmetic graphs in Theorem 12. For sequential graphs we have

THEOREM 13. *Let $G = G(p, q)$ be a k-sequential graph with a k-sequential labeling f and $a = f_{\min}$. If $m' = \max\{x : [a, a+x] \subseteq f(V(G))\} \ge k - a$, then for every m with $k - a \le m \le m'$, the graph H obtained from G by joining all vertices of $V_m = f^{-1}([a, a+m]) = \{v \in V(G) : a \le f(v) \le a + m\}$ with each of other $n \ge 0$ additional isolated vertices is also k-sequential.*

Proof. Let $v_1, \ldots, v_n$ be the n additional vertices as stated in the theorem.

If G is not a tree. Define $g : V(H) \to N$ in the following way. First let $g(v) = f(v)$ for every $v \in V(G)$, then $g(V(G)) \subseteq [a, q-1]$. For each v_i, let $g(v_i) = k + q - a + (i-1)(m+1)$. The smallest label of these n additional vertices is $g(v_1) = k + q - a \ge q$ (since $a < k$), and the largest one is $g(v_s) = k + q - a + (n-1)(m+1) = (q + n(m+1) - 1) + (k - a - m) \le q + n(m+1) - 1 = |E(H)| - 1$. So g is an injection from $V(H)$ to $[0, |E(H)| - 1]$. Since $g(V_m) = [a, a+m]$ we have

$$g(v_i) + g(V_m) = [k + q + (i-1)(m+1),\ k + q + i(m+1) - 1], \quad 1 \le i \le n.$$

It is easy to see that $g^+(E(H)) = [k, k + q + n(m+1) - 1]$. Thus g is a k-sequential labeling of H.

If G is a tree, then $a = 0$, $f(V(G)) = [0, q]$ and $m' = q \ge k$. Let $g(v) = f(v)$ for each $v \in V(G)$ and $g(v_i) = k + q + (i-1)(m+1)$, $1 \le i \le n$. A routine calculation shows that g is a k-sequential labeling of H. This completes the proof.

COROLLARY 1. *If $T = T(p, p-1)$ is a k-sequential tree with a k-sequential labeling, then for every integer m with $0 \le m \le p-1$, the graph obtained by adding any number of isolated vertices to T and connecting every one of them to each vertex of T with label no more than m is also k-sequential. In particular, $T \vee \overline{K}_n$ is k-sequential for each $n \ge 0$.*

Because of Observation 2(1) we have

THEOREM 13′. *Let $G = G(p, q)$ be a strongly k-harmonious graph with a strong k-harmonious labeling f and $a = f_{\mathrm{mix}}$. If $m' = \max\{x : [a, a+x] \subseteq f(V(G))\} \ge k - a$, then for each m, $k - a \le m \le m'$, the graph obtained from G by connecting all vertices of $V_m = f^{-1}([a, a+m])$ to each of $n \ge 0$ additional isolated vertices is also strongly k-harmonious.*

THEOREM 14. *Let $G = G(p, q)$ be a strongly k-elegant graph with a strong k-elegant labeling f, $a = f_{\min}$. Let $\overline{K}_n$ $(n \ge 0)$ be the empty graph without common vertices with G. If $m' = \max\{x : [a, a+x] \subseteq f(V(G))\} \ge k - a - 1$, then for each m, $k - a - 1 \le m \le m'$, $H = (G[V_m] \vee \overline{K}_n \cup G$ is strongly k-elegant, where $G[V_m]$ is the subgraph of G induced by $V_m = f^{-1}([a, a+m])$.*

Proof. Suppose $V(\overline{K}_g) = \{v_1, \ldots, v_n\}$. Let

$$g(v) = f(v), \quad v \in V(G),$$
$$g(v_i) = k + q - a + (i-1)(m+1), \quad 1 \le i \le n.$$

Then g is the required strong k-elegant labeling of H.

The similarity of Theorem 13 for strongly k-indexable graphs is the following

THEOREM 15. *Let $G = G(p, q)$ be a strongly k-indexable graph with a strong k-indexer f. Suppose $p \le q + 1$. Then there exists a vertex u of G with $f(u) = k + q - p$, and the graph obtained by attaching any number of pendant edges to G at u is also strongly k-indexable.*

Proof. Since $k \le 2p - q - 2$ by observation 1 (4), we have $k + q - p \in [0, p-1] = f(V(G))$. So there exists $u \in V(G)$ such that $f(u) = k + q - p$.

Let $uv_1, \ldots, uv_n$ be the additional pendant edges attached at u and H be the resultant graph. Put

$$g(v) = f(v), \quad v \in V(G),$$
$$g(v_i) = p + i - 1, \quad 1 \le i \le n.$$

Then $g : V(H) \to [0, p + n - 1]$ is a bijection and $g^+(E(H)) = [k, k + q + n - 1]$. So g is a strong k-indexer of H.

It is notable that in Theorem 7 we can join $G + \overline{K}_m$ to a star instead of an empty graph. We have

THEOREM 16. *Let $G = G(p, q)$ be a k-sequential graph (strongly k-harmonious graph, strongly k-elegant graph) and S_{n+1} a star of $n + 1$ vertices. If $m = q - p + k$ then $H = (G + \overline{K}_m) \vee S_{n+1}$ is a k-sequential graph (strongly k-harmonious graph, strongly k-elegant graph, respectively) as well.*

Proof. Suppose G is k-sequential and f is a k-sequential labeling of G. Let v_0 be the center and $v_1, \ldots, v_n$ the pendant vertices of S_{n+1}. Define a labeling g of H as follows. Let $g(v) = f(v)$ for every $v \in V(G)$. Then $[0, k + q - 1] \backslash g(V(G))$ contains exactly m integers. Label the m vertices of $\overline{K}_m$ with these m integers. Then $g(V(G + \overline{K}_m)) = [0, k + q - 1]$. Now let $g(v_0) = k + q$ and $g(v_i) = (i+1)(k+q) + i - 1$ $(1 \le i \le n)$. It is easy to check that $g^+(E(H)) = [k, (n+2)(k+q) + n - 1]$ and the maximum vertex label of H under g is $(n+1)(k+q) + n - 1$, which is less than the number of edges of H. So g is a k-sequential labeling of H.

The above proof is applicable also to the case when G is strongly k-harmonious or k-elegant graph. This completes the proof.

The method given in Theorem 16 gives exactly the same result as Theorem 10 when applied to strongly k-indexable graphs.

4. Adding Edges over Different Graphs

This section is devoted to methods of adding edges over a number of labeled graphs to get new labeled graphs. First we have the following rather general result.

THEOREM 17. *Let $T_i = T_i(p_i, p_i - 1)$ be a k_i-sequential tree, $1 \le i \le n$, $k_1 \le \ldots \le k_n$. Suppose these n trees are pairwise vertex disjoint and l_i is such that $p_i \le l_i \le \min\{k_i + p_i - 1, p_i + p_{i+1} - k_{i+1} - 1\}$, $1 \le i \le n$. Then we can add to $T_1 + \ldots + T_n$ $k_{i+1} - k_i - p_i + 2l_i + 1$ edges, which connects vertices of T_i and T_{i+1} for each i, $1 \le i \le n-1$, such that the resultant graph is k_1-sequential.*

Proof. Suppose f_i is a k_i-sequential labeling of T_i. Let $l_0 = 0$ and $f: \bigcup_{i=1}^{n} V(T_i) \to N$ be such that $f(v) = f_i(v) + (l_1 + \ldots + l_{i-1})$, $v \in V(T_i)$, $1 \le i \le n$. Then $f(V(T_i)) = [l_1 + \ldots + l_{i-1}, l_1 + \ldots + l_{i-1} + p_i - 1]$. Since $p_i \le l_i$ for each i, we know f is an injection. Note that

$$f^+(E(T_i)) = [2(l_1 + \ldots + l_{i-1}) + k_i,\ 2(l_1 + \ldots + l_{i-1}) + k_i + p_i - 2],\ 1 \le i \le n.$$

We have

$$\begin{aligned} f^+(E(T_1 + \ldots + T_n)) &= [k_1, k_n + 2(l_1 + \ldots + l_{n-1}) + p_n - 2] \setminus \\ &\quad \bigcup_{i=1}^{n-1} [k_i + 2(l_1 + \ldots + l_{i-1}) + p_i - 1, k_{i+1} + 2(l_1 + \ldots + l_i) - 1] \\ &= [k_1, k_n + 2(l_1 + \ldots + l_{n-1}) + p_n - 2) \setminus \\ &\quad \bigcup_{i=1}^{n} \{2(l_1 + \ldots + l_{i-1}) + l_i + (k_i + p_i - l_i - 1 + j) : \\ &\qquad\qquad j = 0, 1, \ldots, k_{i+1} - k_i - p_i + 2l_i\}. \end{aligned}$$

Since

$$\begin{aligned} 0 &\le k_i + p_i - l_i - 1 \\ &\le k_i + p_i - l_i - 1 + j \\ &\le (k_i + p_i - l_i - 1) + (k_{i+1} - k_i - p_i + 2l_i) \\ &= k_{i+1} + l_i - 1 \\ &\le p_i + p_{i+1} - 2, \end{aligned}$$

and $f(V(T_i)) = [0, p_i - 1]$, $f(V(T_{i+1})) = [0, p_{i+1} - 1]$, we have $k_i + p_i - l_i - 1 + j \in f(V(T_i)) + f(V(T_{i+1}))$. Hence

$$\begin{aligned} E_{ij} &= \{uv : u \in V(T_i), v \in V(T_{i+1}), f_i(u) + f_{i+1}(v) = k_i + p_i - l_i - 1 + j\} \\ &\ne \varnothing,\ 1 \le i \le n-1, 0 \le j \le k_{i+1} - k_i - p_i + 2l_i. \end{aligned}$$

Let e_{ij} be any edge in E_{ij}; then

$$f^+(e_{ij}) = 2(l_1 + \ldots + l_{i-1}) + l_i + (k_i + p_i - l_i - 1 + j).$$

Put $E = \{e_{ij} : 1 \le i \le n-1, 0 \le j \le k_{i+1} - k_i - p_i + 2l_i\}$ and $H = (T_1 + \ldots + T_n) \cup E$. Then H has

$\sum_{i=1}^{n} p_i$ vertices and $p_n + k_n - k_1 - 1 + 2\sum_{i=1}^{n-1} l_i$ edges. Note that $f_{\max} = p_n - 1 + \sum_{i=1}^{n-1} l_i < |E(H)|$ and the above procedure shows that $f^+(E(H)) = [k_1, p_n + k_n - 2 + 2\sum_{i=1}^{n-1} l_i]$ is an integer interval containing $|E(H)|$ integers. Thus f is a k_1-sequential labeling of H. This ends the proof.

Taking $l_i = p_i$, $1 \le i \le n$, in Theorem 17, we get

COROLLARY 2. *Let $T_i = T_i(p_i, p_i - 1)$ be a k_i-sequential tree, $1 \le i \le n$, $k_1 \le \ldots \le k_n$. Then we can add to $T_1 + \ldots T_n$ $k_{i+1} - k_i + p_i + 1$ edges, which connect vertices of T_i and T_{i+1} for each i, $1 \le i \le n-1$, such that the resultant graph is k_1-sequential.*

If $T_1, \ldots, T_n$ are pairwise isomprphic and $l_1 = \ldots = l_{n-1}$ in Theorem 17 we get

COROLLARY 3. *Let $T = T(p, p-1)$ be a k-sequential tree and $T_1, \ldots, T_n$ be n pairwise vertex disjoint copies of T. If $p \le l \le \min\{k + p - 1, 2p - k + 1\}$, then we can add to $T_1 + \ldots + T_n$, $2l - p + 1$ edges connecting over T_i and T_{i+1}, $1 \le i \le n-1$, such that the new graph is also k-sequential.*

THEOREM 18. *Let $G = G(p, q)$ be a k-sequential (strongly k-harmonious, strongly k-elegant) graph other than a tree with a k-sequential (strong k-harmonious, strong k-elegant) labeling f, and $G_1, \ldots, G_n$ be n pairwise vertex disjoint copies of G. If $\max\left\{\lceil \frac{q}{2} \rceil, \beta + 1\right\} \le l \le q (\beta = f_{\max})$, then we can add to $G_1 + \ldots + G_n$ $2l - q$ edges connecting over G_i and G_{i+1}, $1 \le i \le n-1$, such that the graph obtained is k-sequential (strongly k-harmonious, strongly k-elegant, respectively).*

Proof. Let $g(v) = f(v) + (i-1)l$, $v \in V(G_i)$. Then $g(V(G_i)) \subseteq [(i-1)l, (i-1)l + \beta]$, $l \le i \le n$. Since $\beta + 1 \le l$, the n intervals $[(i-1)l, (i-1)l + \beta]$ $(1 \le i \le n)$ are disjoint from one another. Hence g is an injection from $V(G_1 + \ldots + G_n)$ to $[0, (n-1)l + \beta]$. Because $l \ge \lceil \frac{q}{2} \rceil \ge \frac{q}{2}$, the n intervals $g^+(E(G_i)) = [k + 2(i-1)l, k + 2(i-1)l + q - 1]$ $(1 \le i \le n)$ are also pairwise disjoint. We have

$$g^+(E(G_1 + \ldots + G_n)) = [k, k + 2(n-1)l + q - 1] \setminus \bigcup_{i=1}^{n-1} [k + 2(i-1)l + q, k + 2il - 1]$$

$$= [k, k + 2(n-1)l + q - 1] \setminus \bigcup_{i=1}^{n-1} \{2(i-1)l + k + q - l + j : j = 0, 1, \ldots, 2l - q - 1\}.$$

Since

$$k \le k + q - l$$
$$\le k + q - l + j$$

$$\leq k + q - l + (2l - q - 1)$$
$$= k + l - 1$$
$$\leq k + q - 1,$$

$k + q - l + j \in f(E(G))$ for each j and hence $E_{ij} = \{uv : u \in V(G_i), v \in V(G_{i+1}), f(u) + f(v) = k + q - l + j\} \neq \emptyset$, $1 \leq i \leq n-1, 0 \leq j \leq 2l - q - 1$. Let E be any subset of $\bigcup_{i=1}^{n-1} \bigcup_{j=0}^{2l-q-1} E_{ij}$ such that E intersects each E_{ij} at exactly one element. Then $H = (G_1 + \ldots + G_n) \cup E = H(np, q + 2(n-1)l)$. Note that $g_{\max} = \beta + (n-1)l < q + (n-1)l \leq q + 2(n-1)l$ and $g^+(E(H)) = [k, k + 2(n-1)l + q - 1]$. We conclude that g is a k-sequential labeling of H.

By taking $l = q$ in the theorem above we get

COROLLARY 4. *Let $G = G(p, q)$ be a k-sequential (strongly k-harmonious, strongly k-elegant) graph which is not a tree, and $G_1, \ldots, G_n$ be n copies of G. Then in $G_1 + \ldots + G_n$ we can add q edges connecting over G_i and G_{i+1} for each i, $1 \leq i \leq n-1$, such that the resultant graph is k-sequential (strongly k-harmonious, strongly k-elegant, respectively).*

The following theorem is a partial generalization of Theorem 17.

THEOREM 19. *Let $G_i = G_i(p_i, q_i)$ be k_i-sequential with $q_i = p_i - 1$ or p_i, $1 \leq i \leq n$, $k_1 \leq \ldots \leq k_n$. If $q_i' \leq l_i \leq \min\{k_i + q_i, q_i + q_{i+1} - k_{i+1} - k_{i+1} - 1\}$, $1 \leq i \leq n-1$, where $q_i' = q_i + 1$ if G_i is a tree and q_i otherwise, then we can add to $G_1 + \ldots + G_n$ $k_{i+1} - k_i - q_i + 2l_i$ edges connecting over G_i and G_{i+1}, $1 \leq i \leq n-1$, such that the graph obtained is k_1-sequential.*

Proof. Let f_i be a k_i-sequential labeling of G_i and $f(v) = f_i(v) + \sum_{t=1}^{i-1} l_t$, $v \in V(G_i)$, $1 \leq i \leq n$. Since $\frac{q_i}{2} \leq q_i' \leq l_i$ for each i, f is an injection and the n intervals $f(E(G_i)) = [2\sum_{t=1}^{i-1} l_t + k_i, 2\sum_{t=1}^{i-1} l_t + k_i + q_i - 1]$ $(1 \leq i \leq n)$ are pairwise disjoint, we have

$$f(E(G_1 + \ldots + G_n)) = [k_1, 2\sum_{t=1}^{n-1} l_t + k_n + q_n - 1] \setminus \bigcup_{i=1}^{n-1} [2\sum_{t=1}^{i-1} l_t + k_i + q_i, 2\sum_{t=1}^{i} l_t + k_{i+1} - 1]$$
$$= [k_1, 2\sum_{t=1}^{n-1} l_t + k_n + q_n - 1] \therefore \bigcup_{i=1}^{n-1} \{2\sum_{t=1}^{i-1} l_t + l_i + (k_i + q_i - l_i + j) :$$
$$j = 0, 1, \ldots, k_{i+1} - k_i - q_i + 2l_i - 1\}.$$

Since

$$0 \leq k_i + q_i - l_i$$
$$\leq k_i + q_i - l_i + j$$

$$\le k_i + q_i - l_i + (k_{i+1} - k_i - q_i + 2l_i - 1)$$
$$= k_{i+1} + l_i - 1$$
$$\le k_{i+1} + (q_i + q_{i+1} - k_{i+1} - 1) - 1$$
$$= q_i + q_{i+1} - 2$$

and $f_i(V(G_i)) = [0, q_i' - 1]$, which is deduced from $q_i = p_i - 1$ and f_i is injective, we have

$$E_{ij} = \{uv : u \in V(G_i), v \in V(G_{i+1}), f_i(u) + f_{i+1}(v) = k_i + q_i - l_i + j\} \neq \varnothing$$

for each i and j. Let e_{ij} be any edge in E_{ij} and $E = \{e_{ij} : 1 \le i \le n-1,\ 0 \le j \le k_{i+1} - k_i - q_i + 2l_i - 1\}$. Then $H = (G_1 + \ldots + G_n) \cup E = H(\sum_{i=1}^{n} p_i, q_n + k_n - k_1 + 2\sum_{i=1}^{n-1} l_i)$, $f_{\max} \le \sum_{i=1}^{n-1} l_i + q_n' - 1 \le \sum_{i=1}^{n-1} l_i + q_n \le |E(H)| - 1$, and $f^+(E(H)) = [k_1, q_n + k_n + 2\sum_{i=1}^{n-1} l_i - 1]$. We conclude that f is a k_1-sequential labeling of H.

THEOREM 20. *Let $G_i = G_i(p_i, p_i)$ be strongly k_i-harmonious, $k_1 \le \ldots \le k_n$. If $p_i \le l_i \le \min\{k_i + p_i,\ p_i + p_{i+1} - k_{i+1} - 1\}$, $1 \le i \le n$, then we can add to $G_1 + \ldots + G_n$ $k_{i+1} - k_i - p_i + 2l_i$ edges which connect over G_i and G_{i+1}, $1 \le i \le n-1$, such that the resultant graph is strongly k_1-harmonious.*

THEOREM 21. *Let $G_i = G_i(p_i, p_i - 1)$ be strongly k_i-elegant, $1 \le i \le n$, $k_1 \le \ldots \le k_n$. If $q_i \le l_i \le \min\{k_i + q_i, q_i + q_{i+1} - k_{i+1} + 1\}$, $1 \le i \le n-1$, then we can add to $G_1 + \ldots + G_n$ $k_{i+1} - k_i - q_i + 2l_i$ edges connecting over G_i and G_{i+1}, $1 \le i \le n-1$, such that the resultant graph is strongly k_1-elegant.*

The two theorems above can be proved by using similar method as used in the proof of Theorem 19. Applying this method to strongly k-indexable graphs we have

THEOREM 22. *Let $G_i = G_i(p_i, q_i)$ be strongly k_i-indexable, $1 \le i \le n$, $k_1 \le \ldots \le k_n$. Suppose $p_i \le \min\{k_i + q_i, p_i + p_{i+1} - k_{i+1} - 1\}$ for each i, $1 \le i \le n-1$. Then we can add to $G_1 + \ldots + G_n$ $k_{i+1} - k_i + 2p_i - q_i$ edges connecting over G_i and G_{i+1}, $1 \le i \le n-1$, such that the resultant graph is strongly k_1-indexable.*

COROLLARY 5. *Let $G = G(p, q)$ be a strongly k-indexable graph and $G_1, \ldots, G_n$ be n copies of G which are pairwise vertex disjoint. If $p - q \le k \le p - 1$, then in $G_1 + \ldots + G_n$ we can add $2p - q$ (≥ 3) edges connecting over G_i and G_{i+1}, $1 \le i \le n-1$, such that the resultant graph is also strongly k-indexable.*

5. REMARKS

(1) The methods described in the previous sections are essentially constructive. In Setion 2, we

have identified nonadjacent pairs of vertices that could be made adjacent or edges that could be deleted to get new labeled graphs. From the procedure of the proofs for the theorems in Section 4 we know how we can add new edges. Because of these constructive characteristics, our methods can be easily realized on computers.

(2) The methods can be used repeatedly.

(3) The sets E_{ij} in the proofs of Corollaries 3 and 5 and Theorem 18 can be augmented, thus enlarging the range of the chosen edges. Let $t_i = \max\{0, 2i-n\}$ for $i = 1, \ldots, n-1$. If f_1 is a k-sequential labeling of T in Corollary 3, let

$$E'_{ij} = \{uv : (u, v) \in \bigcup_{t=t_i+1}^{i} V(T_t) \times V(T_{2i+1-t}),\ f_1(u) + f_1(v) = k + p - l + j - 1\}$$

and $E \subseteq \bigcup_{i=1}^{n-1} \bigcup_{j=0}^{2l-p} E'_{ij}$ be such that $|E \cap E'_{ij}| = 1$ for every E'_{ij}. Then $(T_1 + \ldots + T_n) \cup E$ is k-sequential. In Theorem 17 one can take $E'_{ij} = \{uv : (u, v) \in \bigcup_{t=t_i+1}^{i} V(G_t) \times V(G_{2i+1-t}), f(u) + f(v) = k + q - l + j\}$ in place of E_{ij}. In Corollary 5 one can take

$$E'_{ij} = \{uv : (u, v) \in \bigcup_{t=t_i+1}^{i} V(G_t) \times V(G_{2i+1-t}), f_1(u) + f_1(v) = k + q - p + j\}$$

instead of E_{ij}, where f_1 is a strong k-indexer of G.

(4) We can estimate how many different (upto isomorphism) labeled graphs one can obtain by using the methods given in Section 4. For example, in Theorem 16 a lower bound is

$$\prod_{i=1}^{n} \prod_{j=0}^{k_{i+1}-k_i-p_i+2l_i} r_{ij},$$

where

$$r_{ij} = |E_{ij}| = \begin{cases} k_i + p_i - l_i + j, & \text{if } k_i + p_i - l_i - 1 + j \le \min\{p_i, p_{i+1}\} \\ \min\{p_i, p_{i+1}\}, & \text{if } \min\{p_i, p_{i+1}\} - 1 \le k_i + p_i - l_i - 1 + j \le \max\{p_i, p_{i+1}\} - 1 \\ p_{i+1} - k_i + l_i - j, & \text{if } k_i + p_i - l_i - 1 + j \ge \max\{p_i, p_{i+1} > \min\{p_i, p_{i+1}\}. \end{cases}$$

Considering the remark (3) above this bound can be improved in the case of Corollary 3. In Theorem 17 the lower bound is $\prod_{i=1}^{n-1} \prod_{j=0}^{2l-q-1} |E'_{ij}| = \prod_{i=1}^{n-1} \prod_{j=0}^{2l-q-1} (i - t_i)|E_{ij}| \ge \prod_{i=1}^{n-1} \prod_{j=0}^{2l-q-1} (2(i - t_i)$ $= [2^{n-1} \prod_{i=1}^{n-1} (i - t_i)]^{2l-q}$. In the case of Corollary 5 the lower bound is $\prod_{i=1}^{n-1} \prod_{j=0}^{2p-q-1} (i - t_i) r_{ij}$, where

$$r_{ij} = |E_{ij}| = \begin{cases} k+q-p+j+1, & \text{if } k+q-p+j \le p-1 \\ 3p-q-k-j+1, & \text{if } k+q-p+j \ge p. \end{cases}$$

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