Locally Restricted Colorings of Graphs

Sanming Zhou*
Department of Mathematics and Statistics
The University of Melbourne
Parkville, VIC 3010, Australia
email: *smzhou@ms.unimelb.edu.au*

Abstract

Let G be a simple graph and f a function from the vertices of G to the set of positive integers. An (f, n)-coloring of G is an assignment of n colors to the vertices of G such that each vertex x is adjacent to less than $f(x)$ vertices with the same color as x. The minimum n such that an (f, n)-coloring of G exists is defined to be the f-chromatic number of G. In this paper, we address a study of this kind of locally restricted coloring.

1 Introduction

The purpose of this paper is to address a study of the following generalized coloring for graphs. Let $G = (V(G), E(G))$ be a simple graph and let $f : V(G) \to \mathbb{N}$ be a function from the vertices of G to the set $\mathbb{N}$ of positive integers. A subset X of $V(G)$ is said to be an *f-independent set* [14] if each $x \in X$ is adjacent to less than $f(x)$ vertices in X. A partition of $V(G)$ into n (color) classes each is an f-independent set of G is said to be an *(f, n)-coloring* of G (or an *f-coloring* of G if the number n of colors used is of less importance in the context). We define the *f-chromatic number* of G, denoted by $\chi_f(G)$, to be the minimum integer n such that an (f, n)-coloring of G exists.

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This locally restricted coloring is one kind of conditional coloring (see e.g. [6]) for graphs and is closely related to the following existing coloring models. We notice first that, in the case where $f = k + 1$ is a constant function, for an integer $k \geq 0$, an (f, n)-coloring is a partition of $V(G)$ into n classes each induces a subgraph of maximum degree at most k, and in this case we denote $\chi_f(G)$ by $\chi_{k+1}(G)$. This coloring model, known as defective coloring [4], $(n, k)^{\Delta}$-coloring [5] and $(n, k)^*$-coloring [13] in the literature, received extensive study in recent years. For a set C of n colors and a function $g : V(G) \times C \to \mathbb{N} \cup \{0\}$, Woodall [13, Section 5] studied the coloring $c : V(G) \to C$ such that each $x \in V(G)$ is adjacent to at most $g(x, c(x))$ vertices with the same color $c(x)$ as itself. If, for each $x \in V(G)$, $g(x, i)+1 = f(x)$ is independent of the choice of $i \in C$, then such a coloring c is precisely an (f, n)-coloring of G defined above.

We start this paper with two examples in the next section. In Section 3, we will use some known results to derive two upper bounds for $\chi_f(G)$: The first one is a natural generalization of Welsh-Powell bound for the ordinary chromatic number $\chi(G)$, whilst the second one bears some similarity with Brooks theorem. In Section 4, we will concentrate on a study of the 2-chromatic number $\chi_2(G)$, which is of particular interest since each color class of a 2-coloring induces a subgraph consisting of independent vertices and independent edges.

Throughout the paper we always use G to denote a simple graph with $p = p(G)$ vertices and $q = q(G)$ edges. We use $\overline{G}$ to denote the complement graph of G and $G[X]$ to denote the subgraph of G induced by a subset $X \subseteq V(G)$. The degree in G of a vertex $x \in V(G)$ is denoted by $d_G(x)$ (or just $d(x)$ if no ambiguity exists), and the maximum degree of vertices of G is denoted by $\Delta(G)$. An f-coloring of G using $\chi_f(G)$ colors is said to be a *minimum f-coloring*. Clearly, if we define $f^*(x) = \min\{f(x), d(x) + 1\}$ for $x \in V(G)$, then $\chi_{f^*}(G) = \chi_f(G)$ and f^* is a *proper* function relative to G in the sense that $1 \leq f^*(x) \leq d(x)+1$ for all $x \in V(G)$. This indicates that we can restrict to proper functions f in the study of f-chromatic number. (However, this is not assumed in the following unless stated otherwise.) For a real number $a \in \mathbb{R}$, we denote by $\lfloor a \rfloor$ and $\lceil a \rceil$, respectively, the largest integer no more than a and the smallest integer no less than a. For other undefined terminologies for graphs, the reader is referred to [7].

2 Examples

For a sequence $\ell_1 \geq \cdots \geq \ell_p$ of positive integers, denote by $n(\ell_1, \ldots, \ell_p)$ the smallest integer n such that there exists a sequence $0 = i_0 < i_1 < \cdots < i_n = p$ with $i_t - i_{t-1} \leq \ell_{i_t}$ for $1 \leq t \leq n$. The following example determines the f-chromatic number of the complete graph K_p on p vertices.

Example 1 Suppose f is a proper function relative to K_p and let the integers $f(x), x \in V(K_p)$, be ordered in a non-decreasing sequence $\ell_1 \geq \cdots \geq \ell_p$. Then

$$\chi_f(K_p) = n(\ell_1, \ldots, \ell_p).$$

Proof Let $x_1 \prec \cdots \prec x_p$ be an order of the vertices of K_p with $f(x_i) = \ell_i$ for $1 \leq i \leq p$. Let $m(X) = \min_{x \in X} f(x)$ for $X \subseteq V(K_p)$. Let $\pi = \{V_1, \ldots, V_n\}$ be an (f, n)-coloring of K_p and set $i_t = |V_1| + \cdots + |V_t|$ for $1 \leq t \leq n$. Without loss of generality we may suppose that $m(V_1) \geq \cdots \geq m(V_n)$. Then, since each V_t is an f-independent set of K_p, we have $i_t - i_{t-1} = |V_t| \leq m(V_t)$ for $1 \leq t \leq n$, where we set $i_0 = 0$. Let $X_t = \{x_{i_{t-1}+1}, \ldots, x_{i_t}\}$ for $1 \leq t \leq n$ (note that $i_n = p$). Then one can see that $\ell_{i_t} = m(X_t) \geq m(V_t) \geq i_t - i_{t-1}$ for $1 \leq t \leq n$ and hence each X_t is an f-independent set of K_p. Therefore, $\{X_1, \ldots, X_n\}$ is an (f, n)-coloring of K_p using the same number of colors as π.

Conversely, for any sequence $0 = i_0 < i_1 < \cdots < i_n = p$ with $i_t - i_{t-1} \leq \ell_{i_t}$ for $1 \leq t \leq n$, the partition $\{X_1, \ldots, X_n\}$ defined by $X_t = \{x_{i_{t-1}+1}, \ldots, x_{i_t}\}$, for $1 \leq t \leq n$, is an (f, n)-coloring of K_p. Hence the result follows immediately from the definition of $n(\ell_1, \ldots, \ell_p)$. $\square$

Let $K_{\ell_1,\ldots,\ell_m}$ be the complete m-partite graph with ℓ_i vertices in the i-th part of the m-partition. The determination of $\chi_f(K_{\ell_1,\ldots,\ell_m})$ for a general proper function f seems to be more complicated. We have the following example for the 2-chromatic number of $K_{\ell_1,\ldots,\ell_m}$.

Example 2 Let $s = |\{i : \ell_i = 1, 1 \leq i \leq m\}|$. Then

$$\chi_2(K_{\ell_1,\ldots,\ell_m}) = m - \left\lfloor \frac{s}{2} \right\rfloor.$$

Proof Let $\{X_1, \ldots, X_m\}$ be the m-partition of $G = K_{\ell_1,\ldots,\ell_m}$. Let $\pi = \{V_1, \ldots, V_n\}$ be a minimum 2-coloring of G. Denote $J_i = \{j : V_i \cap X_j \neq \emptyset, 1 \leq j \leq m\}$ for $1 \leq i \leq n$. Then $1 \leq |J_i| \leq 2$ since otherwise $G[V_i]$ would contain triangles. Set $I_1 = \{1 \leq i \leq n : |J_i| = 1\}$ and $I_2 = \{1 \leq i \leq n : |J_i| = 2\}$, and call V_i a *first type color class* (*second type color class*, respectively) if $i \in I_1$ ($i \in I_2$, respectively). Then any second type color class V_i contains exactly one vertex from each X_j with $j \in J_i$ and hence $|V_i| = 2$. We choose π such that it contains the minimum number $|I_2|$ of second type color classes. Then there exists no j such that $j \in J_{i_1} \cap J_{i_2}$ for some $i_1 \in I_1$ and $i_2 \in I_2$. Suppose otherwise, then we can replace V_{i_1} by the whole X_j and delete all the possible vertices of X_j from each V_i with $i \in I_2$. In this way we get another minimum 2-coloring of G with fewer second type color classes, a contradiction. Thus, for each $1 \leq j \leq m$, either X_j is a first type color class of π, or each vertex of X_j is contained in a second type color class. We claim that each X_j with $|X_j| \geq 2$ falls into the former category. Suppose to the contrary that $X_j = \{x_1, \ldots, x_{\ell_j}\}$ with $\ell_j = |X_j| \geq 2$ and that each x_t belongs to a second type color class $\{x_t, y_t\}$ of π, $1 \leq t \leq \ell_j$. Then by removing from π all these color classes and adding the new color classes $X_j, \{y_1, y_2\}, \{y_3\}, \ldots, \{y_{\ell_j}\}$, we get another minimum 2-coloring of G with fewer second type color classes than π. This is a contradiction and hence we have proved that each non-singleton part X_j is a first type color class of π. Therefore, $\chi_2(G) = (m - s) + \lceil s/2 \rceil = m - \lfloor s/2 \rfloor$. □

3 Two upper bounds

Our first upper bound for $\chi_f(G)$ is a counterpart of the following Welsh-Powell upper bound [12] for $\chi(G)$:

$$\chi(G) \leq \max_{1 \leq i \leq p} \min\{i, d_i + 1\}, \tag{1}$$

where $d_1, \ldots, d_p$ is the degree sequence of G. It was shown in [16] that a similar upper bound holds for conditional chromatic numbers of finite sets. Let $S = \{x_1, \ldots, x_p\}$ be a finite set. A property P associated with the subsets of S is said to be *hereditary* if whenever $X \subseteq S$ has property P

then each subset of X has property P as well. The *P-chromatic number* $\chi_P(S)$ of S (see e.g. [15]) is defined to be the minimum integer n such that S can be partitioned into n subsets each with property P. The *P-degree* of x in S, denoted by $d_P(x, S)$, was defined in [16] to be the largest number of members in a family of minimal (under set-theoretic inclusion) subsets of S not possessing P such that any two distinct members in the family intersect precisely at $\{x\}$. It was proved in [16, Theorem 1] that

$$\chi_P(S) \leq \max_{1\leq i\leq p} \min\{i, d_P(x_i, S) + 1\}. \tag{2}$$

We observed that the property P of being an f-independent set of G is a hereditary property associated with the subsets of $V(G)$, that is, X is an f-independent set of G implies that each subset of X is also an f-independent set of G. In this case we call $d_P(x, V(G))$ the *f-degree* of $x \in V(G)$ in G and we denote it by $d_f(x, G)$. In other words, $d_f(x, G)$ is the maximum number of minimal non-f-independent sets whose pairwise intersections are $\{x\}$. From (2) above we get immediately the following upper bound for $\chi_f(G)$.

Theorem 1 *Let $V(G) = \{x_1, \ldots, x_p\}$, and let $f : V(G) \to \mathbb{N}$. Then*

$$\chi_f(G) \leq \max_{1\leq i\leq p} \min\{i, d_f(x_i, G) + 1\}. \tag{3}$$

In the particular case where $f = 1$, this upper bound gives rise to (1) since $\chi_1(G) = \chi(G)$ and the 1-degree $d_1(x, G)$ agrees with $d(x)$. As in the case of the general upper bound (2) (see [16]), the right-hand side of (3) is minimized when the vertices of G are ordered in such a way that $d_f(x_1, G) \geq \cdots \geq d_f(x_p, G)$.

The second upper bound we will give for $\chi_f(G)$ is closely related to the following elegant theorem which was stated without proof in [2, Lemma 2′] in an equivalent form. The proofs were given in [1, 8, 13] and a variant of the following form can be found in [13, Theorem 5.2].

Theorem 2 (see [1, 2, 8, 13]) *Let C be a set of colors and let $g : V(G) \times C \to \mathbb{R}$ satisfy $\sum_{i\in C} g(x, i) > d(x)$ for each $x \in V(G)$. Then there exists a coloring $c : V(G) \to C$ such that $d_{G[c^{-1}(i)]}(x) < g(x, i)$ for each vertex x of G colored with $i \in C$.*

We strengthen this result by proving the following theorem, which constructs clearly the coloring c guaranteed and implies an upper bound for $\chi_f(G)$. The following short proof is different from that given in [1, 8, 13]. Also it seems that it is not the unpublished proof of Borodin and Kostochka [2] since both [1] and [13] imply that in [2] induction on $|C|$ is exploited and in the case where $|C| = 2$ the required coloring $c : V(G) \to \{0, 1\}$ is achieved by maximizing the quantity $\frac{1}{2}\sum_{x \in V(G)}(g(x, c(x)) - g(x, 1 - c(x))) - t_c$, where t_c is the number of edges joining two vertices of the same color.

Theorem 2′ *Let C be a set of colors and let $g : V(G) \times C \to \mathbb{R}$ satisfy $\sum_{i \in C} g(x, i) > d(x)$ for each $x \in V(G)$. Let $\pi = \{V_i : i \in C\}$ be a partition of $V(G)$ such that $g_\pi = \sum_{i \in C}\sum_{x \in V_i}(g(x, i) - \frac{1}{2}d_{G[V_i]}(x))$ is as large as possible. Then $d_{G[V_i]}(x) < g(x, i)$ for each $i \in C$ and $x \in V_i$.*

Proof For each vertex x of G and each V_i (x is not necessarily in V_i), we denote by $d_i(x)$ the number of vertices in V_i adjacent to x. (In particular, if $x \in V_i$, then $d_i(x) = d_{G[V_i]}(x)$.) Suppose to the contrary that there exists a pair (x, j) with $x \in V_j$ such that $d_j(x) = d_{G[V_j]}(x) \geq g(x, j)$. Since $\sum_{i \in C} g(x, i) > d(x) = \sum_{i \in C} d_i(x)$ by our assumption, there exists $\ell \in C \setminus \{j\}$ such that $d_\ell(x) < g(x, \ell)$. Let $\sigma = \{W_i : i \in C\}$ be the partition of $V(G)$ defined by $W_j = V_j \setminus \{x\}, W_\ell = V_\ell \cup \{x\}$ and $W_i = V_i$ for $i \neq j, \ell$. Then for each vertex $y \in W_j$, $g(y, j) - \frac{1}{2}d_{G[W_j]}(y)$ equals to $g(y, j) - \frac{1}{2}(d_{G[V_j]}(y) - 1)$ if y is adjacent to x and $g(y, j) - \frac{1}{2}d_{G[V_j]}(y)$ otherwise. Similarly, for each $z \in W_\ell \setminus \{x\}$, $g(z, \ell) - \frac{1}{2}d_{G[W_\ell]}(z)$ equals to $g(z, \ell) - \frac{1}{2}(d_{G[V_\ell]}(z) + 1)$ if z is adjacent to x and $g(z, \ell) - \frac{1}{2}d_{G[V_\ell]}(z)$ otherwise. Therefore, we have

$$\begin{aligned} g_\sigma &= g_\pi + \{\tfrac{1}{2}d_j(x) - (g(x, j) - \tfrac{1}{2}d_j(x))\} \\ &\quad + \{(g(x, \ell) - \tfrac{1}{2}d_\ell(x)) - \tfrac{1}{2}d_\ell(x)\} \\ &= g_\pi + (d_j(x) - g(x, j)) + (g(x, \ell) - d_\ell(x)) \\ &> g_\pi. \end{aligned}$$

This contradicts our choice of π and hence the result is proved. □

Let us call

$$ds_f(x, G) = \left\lceil \frac{d(x) + 1}{f(x)} \right\rceil$$

the f-*density* of x in G. Then Theorem $2'$ implies the following upper bound for $\chi_f(G)$ in terms of the *maximum* f-*density* of G defined by

$$DS_f(G) = \max_{x \in V(G)} ds_f(x, G).$$

Theorem 3 *For any function* $f : V(G) \to \mathbb{N}$, *we have*

$$\chi_f(G) \leq DS_f(G). \tag{4}$$

In particular, we have

$$\chi_k(G) \leq \left\lceil \frac{\Delta(G)+1}{k} \right\rceil. \tag{5}$$

Proof Let n be a positive integer satisfying $n \geq (d(x)+1)/f(x)$ for each $x \in V(G)$. Let C be a set of n colors and set $g(x,i) = f(x)$ for each $i \in C$. Then $\sum_{i \in C} g(x,i) > d(x)$ for each $x \in V(G)$ and hence by Theorem $2'$ the partition $\pi = \{V_i : i \in C\}$ with $g_\pi = \sum_{x \in V(G)} f(x) - \sum_{i \in C} q(G[V_i])$ as large as possible is an (f,n)-coloring of G. Since the minimum such integer n is $DS_f(G)$, it follows that $\chi_f(G) \leq DS_f(G)$. $\Box$

This proof shows that the partition $\pi = \{V_1, \ldots, V_n\}$ of $V(G)$ with $\sum_{i=1}^n q(G[V_i])$ as small as possible can serve uniformly as an f-coloring of G for any f with $DS_f(G) \leq n$. The upper bounds (4) and (5) resemble the classical theorem of Brooks (see e.g. [7]), which says that $\chi(G) \leq \Delta(G)+1$ for any connected graph G with equality if and only if G is either a complete graph or an odd cycle. However, characterization of the extremal graphs for (4) or (5) seems to be much harder, even in the case where $k = 2$ (see Example 3 in the next section). As noticed in [5, Theorem 5(b)], (5) can be derived from [9, Theorem 1].

4 Results on 2-chromatic number

By definition, the 2-chromatic number $\chi_2(G)$ is the minimum number of classes into which $V(G)$ can be partitioned such that each class induces a subgraph whose connected components are either K_1 or K_2. Similarly, the 3-chromatic number $\chi_3(G)$ is the minimum number of classes into which

$V(G)$ can be partitioned such that each class induces a subgraph whose connected components are either paths or cycles. Therefore, $\chi_2(G)$ and $\chi_3(G)$ provide, respectively, upper and lower bounds for the *vertex linear arboricity* $\mathrm{vla}(G)$ of G, which was defined in [10] to be the minimum number of classes into which $V(G)$ can be partitioned such that each class induces a forest whose connected components are paths. Since $\lceil(\Delta(G)+1)/2\rceil = \lfloor\Delta(G)/2\rfloor + 1$, from (5) we get

$$\chi_3(G) \leq \mathrm{vla}(G) \leq \chi_2(G) \leq \lfloor\Delta(G)/2\rfloor + 1, \tag{6}$$

and hence any upper bound for $\chi_2(G)$ is also an upper bound for $\mathrm{vla}(G)$. In particular, by proving a result ([10, Theorem (1)]) which is equivalent to $\chi_2(G) \leq \lfloor\Delta(G)/2\rfloor + 1$, the author of [10] obtained the upper bound $\mathrm{vla}(G) \leq \lfloor\Delta(G)/2\rfloor + 1$ for $\mathrm{vla}(G)$ ([10, Theorem (2)]). Clearly, cycles C_p and complete graphs K_p are extremal graphs for $\chi_2(G) \leq \lfloor\Delta(G)/2\rfloor + 1$, and it was shown in [10, Theorem (3)] that these are the only extremal graphs for $\mathrm{vla}(G) \leq \lfloor\Delta(G)/2\rfloor + 1$ if G is connected and $\Delta(G) \geq 2$ is even. The following example indicates that there exist other families of infinitely many extremal graphs for $\chi_2(G) \leq \lfloor\Delta(G)/2\rfloor + 1$, and that behaviour of the extremal graphs for this upper bound seems to be unmanageable.

Example 3 Let $m \geq 1$ be an integer and let H be the graph obtained from K_{2m+1} by removing a matching $x_1x_2, \ldots, x_{2\ell-1}x_{2\ell}$ of $\ell \leq m$ edges. Let $T_1, \ldots, T_{2\ell}$ be vertex-disjoint trees (possibly K_1) each with maximum degree at most $2m$ and each has no common vertex with H. Identifying a degree-one vertex of T_i (or the unique vertex of T_i if $T_i = K_1$) with x_i for each i, we obtain a graph G with maximum degree $2m$ and one can check that $\chi_2(G) = \lfloor\Delta(G)/2\rfloor + 1 = m + 1$.

In the remaining part of this section, we will give a few lower and upper bounds for $\chi_2(G)$. First, we prove the following two lower bounds of $\chi_2(G)$ involving the independence number $\beta(\overline{G})$ of $\overline{G}$ and the edge independence number $\beta'(G)$ of G.

Theorem 4 *The following lower bounds for the* 2*-chromatic number hold:*

$$\chi_2(G) \geq \max\left\{\left\lceil \frac{\beta(\overline{G})}{2} \right\rceil, \left\lceil \frac{p - 2\beta'(G)}{\beta(G)} \right\rceil\right\} \tag{7}$$

$$\chi_2(G) \geq \left\lceil \frac{p^2}{p^2 - 2(q - \beta'(G))} \right\rceil. \tag{8}$$

Moreover, the equality in (8) occurs if and only if G is the graph obtained from a complete n-partite graph $K_{2\ell,\dots,2\ell}$ by adding a perfect matching (in such a case $n = \chi_2(G)$).

Proof Let $\{V_1, \dots, V_n\}$ be a minimal 2-coloring of G. Since the connected components of each $G[V_i]$ are either K_1 or K_2, we have $p = \sum_{i=1}^{n} |V_i| \leq n\beta(G) + 2\beta'(G)$, which implies $n \geq (p - 2\beta'(G))/\beta(G)$. Let X be a maximum independent set of $\overline{G}$. Then $G[X]$ is a complete subgraph of G with $\beta(\overline{G})$ vertices. So $\chi_2(G) \geq \chi_2(G[X]) = \lceil \beta(\overline{G})/2 \rceil$ and (7) is established.

Let $x_1 \prec \dots \prec x_p$ be an order of the vertices of G such that the vertices in V_i precede those in V_j whenever $i < j$. Let $A(G)$ be the adjacency matrix of G with rows and columns indexed by $x_1, \dots, x_p$ in this order. Then we can take $A(G)$ as a partitioned matrix so that the i-th principal submatrix A_i of $A(G)$ is the adjacency matrix of $G[V_i]$. Note that the number of 0-entries in $A(G)$ (A_i, respectively) is $p^2 - 2q$ ($|V_i|^2 - 2q(G[V_i])$, respectively). Applying Cauchy-Schwartz inequality, we have

$$\begin{aligned} p^2 - 2q &\geq \textstyle\sum_{i=1}^{n} |V_i|^2 - 2\sum_{i=1}^{n} q(G[V_i]) \\ &\geq \frac{(\sum_{i=1}^{n} |V_i|)^2}{n} - 2\beta'(G) \\ &= \frac{p^2}{n} - 2\beta'(G), \end{aligned}$$

which implies (8). If the equality in (8) occurs, then from the proof above we have

(i) $|V_1| = \cdots = |V_n| = p/n$ and any two vertices in distinct color classes are adjacent; and

(ii) $\beta'(G) = \sum_{i=1}^{n} q(G[V_i])$.

If n is even, then $p/2 = \beta'(G) = \sum_{i=1}^{n} q(G[V_i]) \leq n\lfloor p/2n \rfloor \leq p/2$, implying that $p/n = 2\ell$ is even and each $G[V_i]$ is an ℓ-matching. So G is the complete n-partite graph $K_{2\ell,\dots,2\ell}$ together with a perfect matching. If n is odd, let, say, $q(G[V_1]) = \max_{1\le i\le n} q(G[V_i])$. Then $\frac{p(n-1)}{2n} + q(G[V_1]) \le \beta'(G) = \sum_{i=1}^{n} q(G[V_i])$. Thus, $\frac{p(n-1)}{2n} \le \sum_{i=2}^{n} q(G[V_i]) \le (n-1)\lfloor p/2n\rfloor \le \frac{p(n-1)}{2n}$, implying that $p/n = 2\ell$ is even and each $G[V_i]$ consists of $p/2n$ independent edges. Therefore, G is again $K_{2\ell,\dots,2\ell}$ plus a perfect matching. Conversely, if G is a complete n-partite graph $K_{2\ell,\dots,2\ell}$ together with a perfect matching, then (8) gives $\chi_2(G) \ge n$ and the n-partition of $K_{2\ell,\dots,2\ell}$ is a 2-coloring of G. Thus, $\chi_2(G) = n$ and the equality in (8) occurs. □

Note that G and $\overline{G}$ cannot be extremal graphs for (8) simultaneously. Thus from (8) and the known result $\beta'(G) + \beta'(\overline{G}) \le 2\lfloor p/2\rfloor$ (see [3]) we get the following corollary.

Corollary 5

$$\frac{1}{\chi_2(G)} + \frac{1}{\chi_2(\overline{G})} < \begin{cases} \frac{p+3}{p}, & \text{if } p \text{ is even} \\ \frac{p+3}{p} - \frac{2}{p^2}, & \text{if } p \text{ is odd.} \end{cases}$$

When the number of edges of G is relatively small, we have the following upper bound for $\chi_2(G)$.

Theorem 6 *Suppose $q < \frac{1}{2}\binom{m+1}{2}$ for an integer m with $1 < m \le p$. Then*

$$\chi_2(G) \le \left\lceil \frac{m}{2} \right\rceil. \tag{9}$$

Proof We make induction on p. If $p = m$, then $\chi_2(G) \le \chi_2(K_m) = \lceil m/2\rceil$ since G is a spanning subgraph of K_m. Suppose (9) is true for any graph with $p-1 \ge m$ vertices and less than $\frac{1}{2}\binom{m+1}{2}$ edges. Let G be a graph with p vertices and $q < \frac{1}{2}\binom{m+1}{2}$ edges. Then there exists $x \in V(G)$ such that $d_G(x) \le \lceil m/2\rceil - 1$ since otherwise we would have $q \ge \frac{p}{2}\cdot\lceil m/2\rceil \ge m(m+1)/4 = \frac{1}{2}\binom{m+1}{2}$, a contradiction. Let $H = G - x$. Then $q(H) \le q(G) < \frac{1}{2}\binom{m+1}{2}$ and hence by the induction hypothesis we have $\chi_2(H) \le \lceil m/2\rceil$. Let $\{V_1, \dots, V_n\}$ be a minimum 2-coloring of H (where $n = \chi_2(H)$). If $n < \lceil m/2\rceil$, then obviously $\chi_2(G) \le \lceil m/2\rceil$ and we are done. If $n = \lceil m/2\rceil$, then since $d_G(x) \le \lceil m/2\rceil - 1$ there exists some V_i whose vertices are not adjacent to x. Thus $\{V_1, \dots, V_i \cup \{x\}, \dots, V_n\}$ is a $(2, \lceil m/2\rceil)$-coloring of G and the proof is complete. □

Corollary 7 *If* $q < \frac{p(p+1)}{4}$*, then*

$$\chi_2(G) \leq \begin{cases} \left\lceil \frac{\lceil \frac{1}{2}(\sqrt{16q+1}-1)\rceil + 1}{2} \right\rceil, & \text{if } q = \ell(4\ell+1) \text{ or } \ell(4\ell-1) \\ & \text{for some integer } \ell \\ \left\lceil \frac{\lceil \frac{1}{2}(\sqrt{16q+1}-1)\rceil}{2} \right\rceil, & \text{otherwise.} \end{cases} \tag{10}$$

Proof Since $q < p(p+1)/4$, there exists m such that $1 < m \leq p$ and $q < \frac{1}{2}\binom{m+1}{2}$. The minimum value of $\lceil m/2 \rceil$ for such integers m is the right-hand side of (10) and hence (10) follows from (9) immediately. □

The equalities in (9) and (10) are attained when, for example, $G = C_4$ and $m = 4$.

5 Problems

If $\{V_1, \ldots, V_n\}$ is an (f, n)-coloring of G and $\{W_1, \ldots, W_m\}$ is a (g, m)-coloring of $\overline{G}$, then clearly $\{V_i \cap W_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is an $(f + g - 1, nm)$-coloring of K_p, where $f + g - 1$ is the function defined by $(f + g - 1)(x) = f(x) + g(x) - 1$ for each vertex x. Therefore, we have

$$\chi_{f+g-1}(K_p) \leq \chi_f(G)\chi_g(\overline{G})$$

and hence

$$2\sqrt{\chi_{f+g-1}(K_p)} \leq \chi_f(G) + \chi_g(\overline{G}).$$

These can be viewed as generalizations of the "easy" parts of the following well-known Nordhaus-Gaddum inequalities [11]:

$$p \leq \chi(G)\chi(\overline{G}) \leq \left\lfloor \left(\frac{p+1}{2}\right)^2 \right\rfloor \tag{11}$$

$$\lceil 2\sqrt{p} \rceil \leq \chi(G) + \chi(\overline{G}) \leq p + 1. \tag{12}$$

Unfortunately, we have been unable to obtain the counterpart of the right-hand side of (12), although one can get a loose upper bound for $\chi_f(G) + \chi_g(\overline{G})$ from (4).

Problem 1 For given proper functions f, g relative to $G, \overline{G}$ respectively, find sharp upper bounds for $\chi_f(G) + \chi_g(\overline{G})$ in terms of f, g and some basic parameters of G and $\overline{G}$. In particular, find such upper bounds in the case where f, g are constant functions.

Denote by $\mathcal{F}(G)$ the lattice of proper functions relative to G with the join "$\vee$" and meet "$\wedge$" defined by

$$(f \vee g)(x) = \max\{f(x), g(x)\}$$

$$(f \wedge g)(x) = \min\{f(x), g(x)\}$$

for any $f, g \in \mathcal{F}(G)$ and $x \in V(G)$. It seems to the author that the following inequality is supported by a number of examples:

$$\chi_{f \vee g}(G) + \chi_{f \wedge g}(G) \leq \chi_f(G) + \chi_g(G). \tag{13}$$

Problem 2 Is (13) true for any simple graph G and any $f, g \in \mathcal{F}(G)$? If it is not true in general, under what circumstances can we guarantee that (13) is true ?

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