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Finite symmetric graphs with two-arc transitive quotients

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Abstract

This paper forms part of a study of 2-arc transitivity for finite imprimitive symmetric graphs, namely for graphs Γ admitting an automorphism group G that is transitive on ordered pairs of adjacent vertices, and leaves invariant a nontrivial vertex partition $\mathscr{B}$. Such a group G is also transitive on the ordered pairs of adjacent vertices of the quotient graph $\Gamma_{\mathscr{B}}$ corresponding to $\mathscr{B}$. If in addition G is transitive on the 2-arcs of Γ (that is, on vertex triples (α, β, γ) such that α, β and β, γ are adjacent and $\alpha \neq \gamma$), then G is not in general transitive on the 2-arcs of $\Gamma_{\mathscr{B}}$, although it does have this property in the special case where $\mathscr{B}$ is the orbit set of a vertex-intransitive normal subgroup of G. On the other hand, G is sometimes transitive on the 2-arcs of $\Gamma_{\mathscr{B}}$ even if it is not transitive on the 2-arcs of Γ. We study conditions under which G is transitive on the 2-arcs of $\Gamma_{\mathscr{B}}$. Our conditions relate to the structure of the bipartite graph induced on $B \cup C$ for adjacent blocks B, C of $\mathscr{B}$, and a graph structure induced on B. We obtain necessary and sufficient conditions for G to be transitive on the 2-arcs of one or both of $\Gamma, \Gamma_{\mathscr{B}}$ for certain families of imprimitive symmetric graphs.

Keywords: Symmetric graph; Two-arc transitive graph; Quotient graph; Three-arc graph

1. Introduction

The family of finite 2-arc transitive graphs has been studied intensively ever since the publication of the seminal results of Tutte [19,20]. Although most quotients of 2-arc

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transitive graphs are not themselves 2-arc transitive, it was shown by the second author [16] that all normal quotients of 2-arc transitive graphs are 2-arc transitive and are covered by the original graph. This result led to the study, and in some cases, classification of various families of 2-arc transitive graphs defined by the kind of 2-arc transitive automorphism groups they admit, see for example [1,5–7,10,12,13,15].

The finite 2-arc transitive graphs form part of the family of finite symmetric graphs, and our paper arose from studying this larger class of graphs from a geometric viewpoint initiated by Gardiner and the second author in [8]. We observed that some situations arising from this geometric approach led to a 2-arc transitive group action on a quotient of a symmetric graph even when the original graph was not 2-arc transitive. To us this was unexpected, as most quotients of symmetric graphs (even, as we mentioned in the previous paragraph, most quotients of 2-arc transitive graphs) are not themselves 2-arc transitive. Thus the questions that inspired and guided our investigations, and for which we obtain partial answers in this paper are:

Question 1.1. *When does a quotient of a symmetric graph admit a natural* 2*-arc transitive group action*? *If there is such a quotient, what information does this give us about the original graph*?

We give a brief introduction below to the geometric framework from [8] for studying finite symmetric graphs. This enables us to explain the context of our work. Our main results are Theorems 1.3 and 1.4, stated in Section 1.2 below.

1.1. Symmetric graphs: the context of our investigations

Let Γ be a finite graph and s a positive integer. An *s-arc* of Γ is a sequence $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of $s+1$ vertices of Γ such that α_i, α_{i+1} are adjacent for $i = 0, \ldots, s-1$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $i = 1, \ldots, s-1$. If Γ admits a group G of automorphisms such that G is transitive on the vertex set $V(\Gamma)$ of Γ and, under the induced action, transitive on the set $\mathtt{Arc}_s(\Gamma)$ of s-arcs of Γ, then Γ is said to be (G, s)*-arc transitive*. A 1-arc is usually called an *arc*, and a $(G, 1)$-arc transitive graph is called a *G-symmetric graph*. We will use $\mathtt{Arc}(\Gamma)$ in place of $\mathtt{Arc}_1(\Gamma)$. Let H be a group acting transitively on a finite set Ω. A partition $\mathcal{B}$ of Ω is said to be *H-invariant* if $B^h \in \mathcal{B}$ for all $B \in \mathcal{B}$ and $h \in H$, where $B^h := \{\alpha^h : \alpha \in B\}$. If the *trivial partitions* $\{\{\alpha\} : \alpha \in \Omega\}$ and $\{\Omega\}$ are the only H-invariant partitions of Ω, then H is said to be *primitive* on Ω; otherwise H is *imprimitive* on Ω. If every nontrivial normal subgroup of H is transitive on Ω, then H is said to be *quasi-primitive* on Ω.

For most G-symmetric graphs Γ, the group G acts imprimitively on $V(\Gamma)$, and Γ is called an *imprimitive G-symmetric graph*. In this case, $V(\Gamma)$ admits a nontrivial G-invariant partition $\mathcal{B}$. For any partition $\mathcal{B}$ of $V(\Gamma)$, we define the *quotient graph* $\Gamma_{\mathcal{B}}$ of Γ with respect to $\mathcal{B}$ as the graph with vertex set $\mathcal{B}$ in which $B, C \in \mathcal{B}$ are adjacent if and only if there exists an edge of Γ joining a vertex of B to a vertex of C. If Γ is G-symmetric, $\mathcal{B}$ is G-invariant, and $\Gamma_{\mathcal{B}}$ has at least one edge, then each block of $\mathcal{B}$ is an independent set of Γ and $\Gamma_{\mathcal{B}}$ is G-symmetric (although G may not act faithfully on $V(\Gamma_{\mathcal{B}}) = \mathcal{B}$, see e.g. [2, Proposition 22.1] or [16]). In this case, for blocks B, C adjacent in $\Gamma_{\mathcal{B}}$, the subgraph of Γ induced by $B \cup C$ with isolated vertices deleted is a bipartite graph, denoted by $\Gamma[B, C]$.

Since $\mathcal{B}$ is G-invariant, up to isomorphism, $\Gamma[B, C]$ is independent of the choice of adjacent blocks B and C. We will use k to denote the size of each part of the bipartition of $\Gamma[B, C]$, and $v := |B|$ the size of blocks of $\mathcal{B}$, so that $1 \leqslant k \leqslant v$. Also for $\alpha \in V(\Gamma)$ we denote by $\Gamma(\alpha)$ the *neighbourhood* of α in Γ, that is, the set of vertices adjacent to α, and set $\Gamma(B) := \bigcup_{\alpha \in B} \Gamma(\alpha)$. Similarly $\Gamma_{\mathcal{B}}(B)$ denotes the neighbourhood of B in $\Gamma_{\mathcal{B}}$.

As remarked above, for a $(G, 2)$-arc transitive graph Γ, $\Gamma_{\mathcal{B}}$ is not usually $(G, 2)$-arc transitive, and several examples of this situation are given in [5,6] with $G = \mathrm{Sz}(q)$ and $\mathrm{Ree}(q)$. However, it was shown in [16] that $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if $\mathcal{B}$ is *G-normal* (that is, if $\mathcal{B}$ is the orbit set of a vertex-intransitive normal subgroup of G), and also if Γ is *small* (that is, $\Gamma(C) \cap \Gamma(D) \cap B \neq \emptyset$ whenever $C, D \in \Gamma_{\mathcal{B}}(B)$). In both of these cases, $k = v$ and Γ is a *v-fold cover* of $\Gamma_{\mathcal{B}}$, that is, $\Gamma[B, C] = v \cdot K_2$ forms a matching of v edges.

The aim of this paper is to study other types of G-invariant partitions $\mathcal{B}$ for which the quotient graph $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive, whether or not the original graph Γ has this property. The investigation was motivated by a result in [11] where, for $k = v - 1 \geqslant 2$, necessary and sufficient conditions were obtained for $\Gamma_{\mathcal{B}}$ to be $(G, 2)$-arc transitive, and a useful construction (see the start of Section 1.2) was given for imprimitive G-symmetric graphs Γ with this property, starting with a $(G, 2)$-arc transitive graph. Our first main result, Theorem 1.3, gives necessary and sufficient conditions for $\Gamma_{\mathcal{B}}$ to be $(G, 2)$-arc transitive in the case where $k = v - 2 \geqslant 1$.

The stabiliser in G of a vertex $\alpha \in V(\Gamma)$ is denoted by G_α. For a group theoretic property **P**, a G-symmetric graph Γ is said to be *G-locally* **P** if, for some vertex α (and hence for each vertex α), the permutation group induced by G_α on $\Gamma(\alpha)$ has property **P**. We will also call the property of $(G, 2)$-arc transitivity a local property since a G-symmetric graph Γ is $(G, 2)$-arc transitive if and only if it is G-locally 2-transitive. Since a 2-transitive group is primitive and a primitive group is quasi-primitive, it follows that $(G, 2)$-arc transitivity implies G-local primitivity, which in turn implies G-local quasi-primitivity. Thus none of these local properties is in general inherited by a quotient graph $\Gamma_{\mathcal{B}}$. In analogy with the case of $(G, 2)$-arc transitivity discussed above, we pose the following question for local properties of symmetric graphs.

Question 1.2. *For a G-symmetric graph Γ with a nontrivial G-invariant partition $\mathcal{B}$ such that Γ is G-locally* **P**, *for some property* **P**, *under what conditions is $\Gamma_{\mathcal{B}}$ also G-locally* **P**?

Each of the properties $(G, 2)$-arc transitivity, G-local primitivity and G-local quasi-primitivity is inherited by $\Gamma_{\mathcal{B}}$ from Γ if $\mathcal{B}$ is G-normal or if $\mathcal{B}$ is small with at least three blocks, (see [16] or [17, Theorem 4.1]). This fact leads to a useful induction scheme, which in a sense reduces the study of nonbipartite $(G, 2)$-arc transitive graphs, G-locally primitive graphs and G-locally quasi-primitive graphs to that of the corresponding graphs with G quasi-primitive on $V(\Gamma)$ (see [17, Section 4] for details). In the case where $k = v$ and Γ is G-locally primitive, Γ is a v-fold cover of $\Gamma_{\mathcal{B}}$. Moreover if $k = v - 1 \geqslant 2$ and Γ is G-locally primitive, then $\Gamma_{\mathcal{B}}$ is even $(G, 2)$-arc transitive. Also in this case Γ is an *almost cover* of $\Gamma_{\mathcal{B}}$ in the sense that $\Gamma[B, C] \cong (v - 1) \cdot K_2$. (See [8, Theorem 5.4] and its amended form [25, Corollary 4.2].)

Here, in Theorem 1.4, we answer Question 1.2 in the case $k = v - 2 \geqslant 1$. For such a graph $(\Gamma, \mathcal{B})$, there are exactly two vertices of B that are not adjacent to any vertex of C. We denote by $\langle B, C\rangle$ the set of these two vertices of B. There is a naturally defined multigraph Γ^B with vertex set B and with an edge joining the two vertices of $\langle B, C\rangle$ for each $C \in \Gamma_{\mathcal{B}}(B)$ (see Section 2 for details). Thus we "decompose" the graph Γ into the "product" of three "factor graphs", namely $\Gamma_{\mathcal{B}}$, $\Gamma[B, C]$ and Γ^B, which mirror the structure of Γ. This decomposition is in accordance with the geometric approach to imprimitive graphs introduced in [8] and further developed in [9,11,21,22,23,24,25]. In our situation the third component Γ^B plays the role of the 1-design $\mathcal{D}(B)$ introduced in [8]: for a G-symmetric graph Γ admitting a nontrivial G-invariant partition $\mathcal{B}$, $\mathcal{D}(B)$ is defined to have point set $B \in \mathcal{B}$ and blocks $\Gamma(C) \cap B$ (with possible repetitions) for all $C \in \Gamma_{\mathcal{B}}(B)$, with incidence given by inclusion. If $k = v - 2 \geqslant 1$, then $\mathcal{D}(B)$ and the multigraph Γ^B determine each other. We will use $\mathtt{Simple}(\Gamma^B)$ to denote the underlying simple graph of Γ^B.

For notation and terminology for graphs and groups not defined in the paper, the reader is referred to [2,4], respectively. For a group G acting on Ω and for $X \subseteq \Omega$, we denote by G_X and $G_{(X)}$ the setwise and pointwise stabilisers of X in G, respectively. For $\alpha \in \Omega$, the subset $\alpha^G := \{\alpha^g : g \in G\}$ of Ω is called the *G-orbit* on Ω containing α. The group G induces an action on the cartesian product $\Omega \times \Omega$, and a G-orbit on $\Omega \times \Omega$ is called a *G-orbital* on Ω. The action of G on Ω is said to be *faithful* if $G_{(\Omega)} = 1$, and *regular* if it is transitive and the stabiliser $G_\alpha = 1$ for $\alpha \in \Omega$. Suppose the group G acts on two sets Ω_1 and Ω_2. If there exists a bijection $\psi : \Omega_1 \to \Omega_2$ such that $\psi(\alpha^g) = (\psi(\alpha))^g$ for all $\alpha \in \Omega_1$ and $g \in G$, then the actions of G on Ω_1 and Ω_2 are said to be *permutationally equivalent*. By a graph we mean a simple graph, whereas a multigraph means multiple edges may exist. The valency of a regular graph or multigraph Γ is denoted by $\mathtt{val}(\Gamma)$, and the union of n vertex-disjoint copies of Γ is denoted by $n \cdot \Gamma$. For two graphs Γ and Σ, the *lexicographic product* of Γ by Σ, denoted by $\Gamma[\Sigma]$, is the graph with vertex set $V(\Gamma) \times V(\Sigma)$ such that (α, β), (γ, δ) are adjacent if and only if either α, γ are adjacent in Γ, or $\alpha = \gamma$ and β, δ are adjacent in Σ.

1.2. Main results

The construction introduced in [11] is as follows. For a regular graph Σ, a subset Δ of $\mathtt{Arc}_3(\Sigma)$ is called *self-paired* if $(\tau, \sigma, \sigma', \tau') \in \Delta$ implies $(\tau', \sigma', \sigma, \tau) \in \Delta$. For such a Δ, the 3-*arc graph* $\Xi(\Sigma, \Delta)$ of Σ with respect to Δ is the graph with vertex set $\mathtt{Arc}(\Sigma)$ in which (σ, τ), (σ', τ') are adjacent if and only if $(\tau, \sigma, \sigma', \tau') \in \Delta$. The reader is referred to [11,22,23,25] for results concerning this construction.

Our first main result tells us when $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if $k = v - 2 \geqslant 1$. It also gives us information about the structure of Γ. Recall that, for $C \in \Gamma_{\mathcal{B}}(B)$, $\langle B, C\rangle$ denotes the pair of vertices of B that are not adjacent to any vertex of C. Set

$$\mathcal{P} = \{\langle B, C\rangle \mid (B, C) \in \mathtt{Arc}(\Gamma_{\mathcal{B}})\}. \tag{1}$$

In general $\mathcal{P}$ is just a set of pairs of vertices of Γ, with repetitions allowed. However in certain cases, when we ignore multiplicities, $\mathcal{P}$ is a partition of $V(\Gamma)$.

Theorem 1.3. *Suppose Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - 2 \geqslant 1$. Then $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if Γ^B is a* *simple graph and one of the following occurs*:

(a) $v = 3$, *and* $\Gamma^B \cong K_3$ *is* G_B-*symmetric*;
(b) $v \geqslant 4$, v *is even, and* $\Gamma^B \cong (v/2) \cdot K_2$.

Moreover, *in case* (a) *we have* $\Gamma \cong (|V(\Gamma)|/2) \cdot K_2$, $\Gamma_{\mathcal{B}}$ *is trivalent*, *and any finite trivalent* $(G, 2)$-*arc transitive graph can occur as* $\Gamma_{\mathcal{B}}$. *In case* (b), *the set* $\mathcal{P}$ *defined in* (1) *is a partition of* $V(\Gamma)$ *with block size* 2 *that is a refinement of* $\mathcal{B}$; *furthermore*, *if* $v = 4$ *then* $\Gamma_{\mathcal{B}} \cong s \cdot C_t$ *for some integers* $s \geqslant 1$ *and* $t \geqslant 3$, *and either* $\Gamma \cong 2st \cdot K_2$ *or* $\Gamma \cong st \cdot C_4$; *if* $v \geqslant 6$ *then* $\Gamma_{\mathcal{P}}$ *is isomorphic to a* 3-*arc graph of* $\Gamma_{\mathcal{B}}$ *with respect to a self-paired G-orbit on* $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$.

The reader is referred to Remark 3.4 in Section 3 for the reconstruction of Γ from $\Gamma_{\mathcal{B}}$ via $\Gamma_{\mathcal{P}}$ in the main case $v \geqslant 6$. The next theorem shows that, if $k = v - 2$ and v is sufficiently large, then $(G, 2)$-arc transitivity is inherited by $\Gamma_{\mathcal{B}}$ from Γ, and moreover we can derive a lot of structural information about Γ, $\Gamma_{\mathcal{B}}$ and $\Gamma_{\mathcal{P}}$. A graph Σ is said to be $(G, 3)$-*arc regular*, where $G \leqslant \mathrm{Aut}(\Sigma)$, if G is regular on the set of 3-arcs of Σ. Following [2, Proposition 18.1], a trivalent graph is $(G, 3)$-arc regular if and only if it is $(G, 3)$- but not $(G, 4)$-arc transitive. For an integer $n \geqslant 4$ and a connected graph Σ of girth at least 4, if there exists a set $\mathcal{E}$ of n-cycles of Σ such that each 2-arc of Σ is contained in a unique n-cycle of $\mathcal{E}$, then Σ is called [14] a *near n-gonal graph* with respect to $\mathcal{E}$.

Theorem 1.4. *Suppose* Γ *is a G-symmetric graph admitting a nontrivial G-invariant partition* $\mathcal{B}$ *such that* $k = v - 2 \geqslant 3$, *where* $G \leqslant \mathrm{Aut}(\Gamma)$. *Suppose further that* Γ *is* $(G, 2)$-*arc transitive. Then* $v = 2\hat{v}$ *is even*, $\Gamma^B \cong \hat{v} \cdot K_2$, *and* $\Gamma_{\mathcal{B}}$ *is* $(G, 2)$-*arc transitive of valency* $\hat{v}$. *Moreover, one of the following holds, where* $\mathcal{P}$ *is the partition defined in* (1).

(a) $\Gamma_{\mathcal{P}}$ *is* $(G, 2)$-*arc transitive and is an almost cover of* $\Gamma_{\mathcal{B}}$. *Also* Γ *is a* 2-*fold cover of* $\Gamma_{\mathcal{P}}$ *and has valency* $\hat{v} - 1$. *If* $\Gamma_{\mathcal{B}}$ *is connected, then either* $\Gamma_{\mathcal{B}}$ *is a complete graph and* $\Gamma_{\mathcal{P}}$ *is known explicitly, or* $\Gamma_{\mathcal{B}}$ *is a near n-gonal graph with respect to a G-orbit on n-cycles of* $\Gamma_{\mathcal{B}}$, *for some even integer* $n \geqslant 4$.
(b) $\hat{v} = 3$, $\Gamma = s \cdot C_t$ *for some* s, t *with* $t \geqslant 3$, $\Gamma_{\mathcal{B}}$ *is a* $(G, 3)$-*arc transitive trivalent graph, and* $\Gamma_{\mathcal{P}} = \Xi(\Gamma_{\mathcal{B}}, \Delta)$ *which is* 4-*valent and is not* $(G, 2)$-*arc transitive, where* Δ *is the set of all* 3-*arcs of* $\Gamma_{\mathcal{B}}$. *If* $\Gamma_{\mathcal{B}}$ *is connected then it is* $(G, 3)$-*arc regular, and moreover any connected trivalent* $(G, 3)$-*arc regular graph can occur as* $\Gamma_{\mathcal{B}}$.

Remark 1.5. (a) For the cases of smaller k, namely $k = v - 2 = 1$ or 2, we can determine at least one of the graphs Γ, $\Gamma_{\mathcal{B}}$, Γ^B and $\Gamma[B, C]$, and we know exactly when $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. See Theorem 4.5 for a full account of the results.

(b) The assumption that $\Gamma_{\mathcal{B}}$ is connected does not sacrifice generality: it is satisfied in particular when Γ is connected. The possibilities for $\Gamma_{\mathcal{P}}$ in part (a) with $\Gamma_{\mathcal{B}}$ a complete graph were classified in [9], or see [23, Theorem 3.19]. For each possibility in part (a) for $\Gamma_{\mathcal{P}}$ with given 2-arc transitive automorphism group H, there is at least one graph Γ satisfying the hypotheses of the theorem. For example just form the standard (unconnected) 2-fold cover by taking two vertex-disjoint copies of $\Gamma_{\mathcal{P}}$ and $G = H \times Z_2$.

(c) A delicate construction will be given in Construction 4.2 to prove the statement of the last sentence of Theorem 1.4(b).

The proofs of Theorems 1.3 and 1.4 will be given in Sections 3 and 4, respectively. At one point of the proof of Theorem 1.3 we use the main result of [11], and for the proof of Theorem 1.4 (a) we need some results in [9] and the main result of [25]. The key information needed in our proofs is the structure of Γ^B, which will be explained in Section 2. We will show that the underlying simple graph $\mathtt{Simple}(\Gamma^B)$ of Γ^B is G_B-vertex-transitive and G_B-edge-transitive (and G_B-symmetric in some cases), where $\mathtt{Simple}(\Gamma^B)$ is defined to be the graph obtained from Γ^B by identifying multi-edges. Moreover, $\mathtt{Simple}(\Gamma^B)$ is either connected or isomorphic to the graph $(v/2) \cdot K_2$ (see Theorem 2.1).

In a subsequent paper [18] we will focus on the case where Γ^B is simple and is a cycle. In this case we show that there are close relationships between $\Gamma_{\mathcal{B}}$ and maps on closed orientable surfaces. See [18] for details.

2. The structure of the graph Γ^B

From now on we assume that Γ is a G-symmetric graph admitting a G-invariant partition $\mathcal{B}$ such that $k = v - 2 \geqslant 1$. Recall that, for $(B, C) \in \mathtt{Arc}(\Gamma_{\mathcal{B}})$, $\langle B, C\rangle$ denotes the pair of vertices of B that are not adjacent to any vertex in C, $\mathcal{P}$ is the set of these pairs, and Γ^B is the multigraph with vertex set B and with an edge joining the two vertices of $\langle B, C\rangle$, for all $C \in \Gamma_{\mathcal{B}}(B)$. Thus, counting multiple edges, the number of edges of Γ^B is equal to the valency $b := \mathtt{val}(\Gamma_{\mathcal{B}})$ of $\Gamma_{\mathcal{B}}$. Since Γ is G-symmetric, up to isomorphism Γ^B is independent of the choice of B. Also, the *multiplicity* of each edge of Γ^B is a constant, m say. In this section, we will study the structure of Γ^B and its influence on that of Γ. The following theorem is fundamental to our subsequent discussion. It asserts that the underlying simple graph $\mathtt{Simple}(\Gamma^B)$ of Γ^B is a G_B-vertex-transitive and G_B-edge-transitive graph. Moreover, $\mathtt{Simple}(\Gamma^B)$ is either connected or its edges form a perfect matching. For $\alpha \in V(\Gamma)$ we denote $\Gamma_{\mathcal{B}}(\alpha) := \{C \in \mathcal{B} : \alpha \in \Gamma(C)\}$, and set $r := |\Gamma_{\mathcal{B}}(\alpha)|$. We use $\mathtt{gcd}(\cdot, \cdot)$ to denote the greatest common divisor.

Theorem 2.1. *Suppose Γ is a G-symmetric graph, where $G \leqslant \mathrm{Aut}(\Gamma)$, admitting a non-trivial G-invariant partition $\mathcal{B}$ such that $k = v - 2 \geqslant 1$. Then G_B acts on Γ^B as a group of automorphisms, and $\mathtt{Simple}(\Gamma^B)$ is a G_B-vertex-transitive and G_B-edge-transitive graph. Also, $\mathtt{val}(\Gamma^B) = b - r = 2b/v$, $b = v \cdot \mathtt{val}(\Gamma^B)/2$, $r = (v-2) \cdot \mathtt{val}(\Gamma^B)/2$, and m divides $\mathtt{gcd}(b - r, 2b)$. Moreover, one of the following occurs.*

(a) *Γ^B is connected; in this case G is faithful on $\mathcal{B}$ and on $\mathtt{Arc}(\Gamma_{\mathcal{B}})$ in its induced actions.*
(b) *v is even, $\mathtt{Simple}(\Gamma^B) \cong (v/2) \cdot K_2$, and hence is G_B-symmetric. In this case, the following hold:*
   (i) *$\mathcal{P}$ (defined in (1)) is the set of connected components of Γ^B for $B \in \mathcal{B}$, and so is a G-invariant partition of $V(\Gamma)$ which has block size 2 and refines $\mathcal{B}$, and is such that $G_{(\mathcal{B})} = G_{(\mathcal{P})}$;*
   (ii) *the mapping φ interchanging the two vertices in each block of $\mathcal{P}$ is an involution that centralises G and leaves $\mathcal{B}$ invariant, thus $\langle G, \varphi\rangle \leqslant \mathbb{Z}_2 \,\mathrm{wr}\, \mathrm{Aut}(\Gamma_{\mathcal{P}})$, and one*

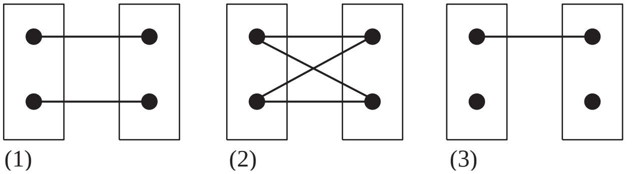

Fig. 1. The three possibilities in Theorem 2.1(b)(ii): (1) $\Gamma[P, Q] = 2 \cdot K_2$; (2) $\Gamma[P, Q] = K_{2,2}$; (3) $\Gamma[P, Q] = K_2$.

of the following occurs:

(1) Γ *is a* 2-*fold cover of* $\Gamma_{\mathcal{P}}$, $\varphi \in \mathrm{Aut}(\Gamma)$;
(2) $\Gamma \cong \Gamma_{\mathcal{P}}[\overline{K}_2]$, $\varphi \in \mathrm{Aut}(\Gamma)$;
(3) $\Gamma[P, Q] \cong K_2$ *for adjacent blocks* P, Q of $\mathcal{P}$, $\varphi \notin \mathrm{Aut}(\Gamma)$, *and G is faithful on* $\mathcal{B}$ *and* $\mathcal{P}$.

The three cases in (b)(ii) above are illustrated in Fig. 1. Note that in the second case Γ is determined completely by $\Gamma_{\mathcal{P}}$. Theorem 2.1 has the following consequence:

Corollary 2.2. *Let* Γ, G, $\mathcal{B}$ *be as above. Suppose G is not faithful on* $\mathcal{B}$. *Then* $\mathtt{Simple}(\Gamma^B) \cong (v/2) \cdot K_2$, *and the set of* $G_{(\mathcal{B})}$*-orbits on* $V(\Gamma)$ *is the partition* $\mathcal{P}$. *Furthermore, either* (1) *or* (2) *in Theorem* 2.1(b)(ii) *holds; and if* (1) *holds and* $\Gamma_{\mathcal{P}}$ *is connected then* $G_{(\mathcal{B})} = \langle \varphi \rangle \cong \mathbb{Z}_2$.

Proof of Theorem 2.1. It is easy to check that the induced action of G_B on B preserves the adjacency of Γ^B, and so Γ^B admits G_B as a group of automorphisms and G_B is transitive on B. For two arcs (α, β), (α, γ) of $\mathtt{Simple}(\Gamma^B)$ with $\beta \neq \gamma$, there exist $C, D \in \Gamma_{\mathcal{B}}(B)$ such that $\langle B, C \rangle = \{\alpha, \beta\}$ and $\langle B, D \rangle = \{\alpha, \gamma\}$. It follows from the definition of $\langle B, C \rangle$ and $\langle B, D \rangle$ that γ is adjacent to a vertex δ in C, and β is adjacent to a vertex ε in D. By the G-symmetry of Γ, there exists $g \in G$ such that $(\gamma, \delta)^g = (\beta, \varepsilon)$. This implies that $g \in G_B$ and $C^g = D$. Hence $\langle B, C \rangle^g = \langle B, D \rangle$, and therefore $\{\alpha, \beta\}^g = \{\alpha, \gamma\}$. Since G_B is transitive on B, it follows that $\mathtt{Simple}(\Gamma^B)$ is G_B-vertex- and G_B-edge-transitive. Note that each vertex in B is incident with all but $\mathtt{val}(\Gamma^B)$ blocks of $\Gamma_{\mathcal{B}}(B)$. So we have $r + \mathtt{val}(\Gamma^B) = b$. Counting the number of edges of Γ incident with a vertex of B we have $vr = b(v-2)$. It follows that $\mathtt{val}(\Gamma^B) = b - r = 2b/v$, $b = v \cdot \mathtt{val}(\Gamma^B)/2$ and $r = (v-2) \cdot \mathtt{val}(\Gamma^B)/2$. Since m is a divisor of $\mathtt{val}(\Gamma^B)$, it is a divisor of $b - r$ and $2b$.

The G_B-vertex- and G_B-edge-transitivity of $\mathtt{Simple}(\Gamma^B)$ implies that the connected components of Γ^B, say $B^{(1)}, \ldots, B^{(\omega)}$, form a G_B-invariant partition of B. From this it is straightforward to show that the set $\mathcal{Q}$ of such components, for B running over $\mathcal{B}$, is a G-invariant partition of $V(\Gamma)$ and is a refinement of $\mathcal{B}$. We claim that either $\mathcal{Q} = \mathcal{B}$ (that is, Γ^B is connected) and (a) holds, or $\mathcal{Q}$ has block size 2.

Suppose first that each block $B^{(i)}$ of $\mathcal{Q}$ contains at least three vertices. Let $C \in \mathcal{B}$ be such that $\langle B, C \rangle \subset B^{(1)}$. We shall prove that Γ^B is connected. Suppose to the contrary that Γ^B has at least two connected components and let $\alpha \in B^{(1)} \setminus \langle B, C \rangle$, and $\gamma \in B^{(2)}$. Recall that each vertex in $B \setminus \langle B, C \rangle$ is adjacent to some vertex in $C \setminus \langle C, B \rangle$. In particular, there exist $\beta, \delta \in C \setminus \langle C, B\rangle$ that are adjacent to α, γ, respectively. By the G-symmetry of Γ, there exists $g \in G$ such that $(\alpha, \beta)^g = (\gamma, \delta)$. Since $\mathcal{B}$ is G-invariant, this implies that g fixes B and C setwise. Thus g fixes setwise the component $B^{(1)}$ of Γ^B containing $\langle B, C\rangle$, and hence $\gamma = \alpha^g \in (B^{(1)})^g = B^{(1)}$ which is a contradiction. Thus Γ^B is connected. Thus, since $v = |B| \geqslant 3$, distinct vertices of B are incident with distinct sets of edges of Γ^B. This in turn implies $\Gamma_{\mathcal{B}}(\alpha) \neq \Gamma_{\mathcal{B}}(\beta)$ for distinct $\alpha, \beta \in B$. Therefore, if an element g of G fixes setwise all the blocks of $\mathcal{B}$, then for each $B \in \mathcal{B}$, g fixes each vertex in B and hence g fixes each vertex of Γ. Therefore $g = 1$ and G is faithful on $\mathcal{B}$. This in turn implies that G is faithful on $\mathtt{Arc}(\Gamma_{\mathcal{B}})$ in its induced action, and hence (a) holds.

In the remaining case $\mathcal{Q}$ has block size 2, that is, each component of Γ^B contains only two vertices which are joined by m multiple edges. In this case it is clear that v is even, $\mathtt{val}(\Gamma^B) = m$, $\mathcal{Q} = \mathcal{P} = \{\langle B, C\rangle : (B, C) \in \mathtt{Arc}(\Gamma_{\mathcal{B}})\}$ (ignoring the multiplicity of each $\langle B, C\rangle$), and $\mathtt{Simple}(\Gamma^B)$ is isomorphic to $(v/2) \cdot K_2$ and hence is G_B-symmetric. Since $\mathcal{P}$ is a refinement of $\mathcal{B}$ and since $\mathcal{B}$ is G-invariant, we have $G_{(\mathcal{P})} \subseteq G_{(\mathcal{B})}$. On the other hand, if $g \in G_{(\mathcal{B})}$, then g fixes setwise each block of $\mathcal{B}$ and hence fixes setwise each block $\langle B, C\rangle$ of $\mathcal{P}$. So $g \in G_{(\mathcal{P})}$. Therefore $G_{(\mathcal{B})} = G_{(\mathcal{P})}$. Clearly the mapping φ interchanging the two vertices in each block $\langle B, C\rangle$ of $\mathcal{P}$ is an involution and leaves $\mathcal{B}$ invariant. For any $\alpha \in V(\Gamma)$ and $g \in G$, let $\beta = \alpha^g$, $\alpha' = \alpha^\varphi$ and $\beta' = \beta^\varphi$, so that $\{\alpha, \alpha'\}$ and $\{\beta, \beta'\}$ are blocks of $\mathcal{P}$. Since $\mathcal{P}$ is G-invariant we have $\beta' = (\alpha')^g$. Hence $\alpha^{\varphi g} = (\alpha')^g = \beta' = \beta^\varphi = \alpha^{g\varphi}$. Since this holds for an arbitrary vertex α it follows that $\varphi g = g\varphi$ and hence φ centralises G. Now φ fixes each block of $\mathcal{P}$ setwise and so $\langle G, \varphi\rangle \leqslant Z_2 \operatorname{wr} \operatorname{Aut}(\Gamma_{\mathcal{P}})$. Since the bipartite graph $\Gamma[P, Q]$ is $G_{\{P,Q\}}$-symmetric, for adjacent blocks P, Q of $\mathcal{P}$, it follows that one of the following occurs: (1) $\Gamma[P, Q] \cong 2 \cdot K_2$, that is, Γ is a 2-fold cover of $\Gamma_{\mathcal{P}}$; (2) $\Gamma[P, Q] \cong K_{2,2}$, that is, $\Gamma \cong \Gamma_{\mathcal{P}}[\overline{K}_2]$; (3) $\Gamma[P, Q] \cong K_2$. In the first two cases, it is easy to see that φ preserves the adjacency of Γ, so $\varphi \in \operatorname{Aut}(\Gamma)$. In the last case (3), φ maps adjacent vertices to nonadjacent vertices, which implies that $\varphi \notin \operatorname{Aut}(\Gamma)$ and G is faithful on $\mathcal{P}$. Since $G_{(\mathcal{B})} = G_{(\mathcal{P})}$, in this case G is faithful on $\mathcal{B}$ as well. $\square$

Proof of Corollary 2.2. Since G is not faithful on $\mathcal{B}$, case (b) of Theorem 2.1 holds, and hence $\mathtt{Simple}(\Gamma^B)$ is $(v/2) \cdot K_2$ and $G_{(\mathcal{B})} = G_{(\mathcal{P})}$. Since the blocks of $\mathcal{P}$ have size 2 and $G_{(\mathcal{B})} = G_{(\mathcal{P})} \neq 1$, it follows that the set of $G_{(\mathcal{B})}$-orbits in $V(\Gamma)$ is $\mathcal{P}$. Also, since $G_{(\mathcal{B})} \neq 1$, one of (1) or (2) of Theorem 2.1 (b)(ii) holds. Suppose that (1) holds and $\Gamma_{\mathcal{P}}$ is connected. Let $g \in G_{(\mathcal{B})} \setminus \{1\}$. Then g interchanges the two vertices in some block of $\mathcal{P}$, say $P = \{\alpha, \beta\}$. Let $Q = \{\gamma, \delta\} \in \mathcal{P}$ be adjacent to P, say α is adjacent to γ and β is adjacent to δ in Γ. Since g interchanges α and β and fixes Q setwise, the G-symmetry of Γ implies that g interchanges γ and δ as well. Similarly, g interchanges the two vertices in any block of $\mathcal{P}$ adjacent to Q. Continuing this process, we conclude that, since $\Gamma_{\mathcal{P}}$ is connected, g interchanges the two vertices in each block of $\mathcal{P}$, that is, $g = \varphi$. Thus $G_{(\mathcal{B})} = \langle\varphi\rangle$. $\square$

Remark 2.3. (a) From $b = v \cdot \mathtt{val}(\Gamma^B)/2$ it follows that $\mathtt{val}(\Gamma^B)$ is even if v is odd, and that $v/\mathtt{gcd}(v, 2)$ is a divisor of b.

(b) In the three possibilities (1)–(3) of Theorem 2.1(b) (ii) the parameters with respect to $\mathcal{P}$ are (1) $v_{\mathcal{P}} = k_{\mathcal{P}} = 2$, $r_{\mathcal{P}} = b_{\mathcal{P}} = \mathtt{val}(\Gamma)$; (2) $v_{\mathcal{P}} = k_{\mathcal{P}} = 2$, $r_{\mathcal{P}} = b_{\mathcal{P}} = \mathtt{val}(\Gamma)/2$; and (3) $v_{\mathcal{P}} = 2$, $k_{\mathcal{P}} = 1$, $r_{\mathcal{P}} = \mathtt{val}(\Gamma)$, $b_{\mathcal{P}} = \mathtt{val}(\Gamma)/2$, respectively.

(c) A generic construction of a family of examples satisfying Theorem 2.1 (a) is given in Example 2.4 below. These examples will be used in the next two sections. In this family the graph Γ has connected components of size 2, but the quotient $\Gamma_{\mathcal{B}}$ may be an arbitrary symmetric trivalent graph. The smallest member of this family of graphs is shown in Fig. 2.

(d) As seen in Theorem 2.1, G_B is vertex- and edge-transitive on the underlying graph $\mathtt{Simple}(\Gamma^B)$. However, G_B may not be symmetric on $\mathtt{Simple}(\Gamma^B)$. In fact, it is even possible to have different subgroups $G, H \leqslant \mathrm{Aut}(\Gamma)$ with G, H both symmetric on Γ and leaving $\mathcal{B}$ invariant such that $\mathtt{Simple}(\Gamma^B)$ is G_B-symmetric but not H_B-symmetric. See Example 2.4 (b) below for a simple example of this situation.

Example 2.4. (a) Let Σ be any trivalent G-symmetric graph. Define Γ to be the graph with vertex set $\mathtt{Arc}(\Sigma)$ and edge set $\{\{(\sigma,\tau),(\tau,\sigma)\} : (\sigma,\tau) \in \mathtt{Arc}(\Sigma)\}$. Then $\Gamma = n \cdot K_2$, where $n = |\mathtt{Arc}(\Sigma)|/2 = 3|V(\Sigma)|/2$, and Γ is a G-symmetric graph admitting $\mathcal{B} = \{B(\sigma) : \sigma \in V(\Sigma)\}$ as a G-invariant partition, where $B(\sigma) = \{(\sigma,\tau) : \tau \in \Sigma(\sigma)\}$. For this partition we have $k = v - 2 = 1$, $\Gamma^B \cong K_3$ for $B = B(\sigma)$, and $\Gamma_{\mathcal{B}} \cong \Sigma$ via the bijection $B(\sigma) \mapsto \sigma$. Let $\Sigma(\sigma) = \{\tau, \eta, \varepsilon\}$. From the definition of Γ we have: Σ is $(G,2)$-arc transitive $\Leftrightarrow G_{\sigma\tau}$ is transitive on $\Sigma(\sigma)\setminus\{\tau\} = \{\eta, \varepsilon\} \Leftrightarrow G_{\sigma\tau}$ is transitive on $\{(\sigma,\eta),(\sigma,\varepsilon)\} \Leftrightarrow G_{\sigma\tau} = (G_B)_{(\sigma,\tau)}$ is transitive on the neighbourhood of (σ,τ) in Γ^B (where $B = B(\sigma)$) $\Leftrightarrow \Gamma^B$ is G_B-symmetric.

The construction above can produce all G-symmetric graphs Γ admitting a G-invariant partition $\mathcal{B}$ such that $k = v - 2 = 1$ and $\Gamma^B \cong K_3$: for if Γ is such a graph then we construct as follows an isomorphism from Γ to the graph Γ' obtained by applying this construction to the quotient $\Gamma_{\mathcal{B}}$. For this pair $(\Gamma, \mathcal{B})$, we have $b = 3$ and $m = r = 1$ by Theorem 2.1, and hence $\Gamma_{\mathcal{B}}$ is trivalent. For each $(B, C) \in \mathtt{Arc}(\Gamma_{\mathcal{B}})$, there is a unique vertex, say α, in B which is adjacent to a vertex in C, and $\alpha \mapsto (B, C)$ defines a bijection from $V(\Gamma)$ to $\mathtt{Arc}(\Gamma_{\mathcal{B}})$. It can be verified that this bijection is an isomorphism from Γ to Γ'. Moreover, $\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive $\Leftrightarrow G_B$ is doubly transitive on $\Gamma_{\mathcal{B}}(B) \Leftrightarrow G_{BC}$ is transitive on $\Gamma_{\mathcal{B}}(B)\setminus\{C\} \Leftrightarrow (G_B)_\alpha\ (= G_\alpha = G_{BC})$ is transitive on $\Gamma^B(\alpha) \Leftrightarrow \Gamma^B$ is G_B-symmetric.

(b) Let $\Sigma = K_4$ and $G = S_4$ (symmetric group) or A_4 (alternating group), with G acting on $V(\Sigma)$ in its natural action. Then Σ is trivalent and G-symmetric, and the construction above gives rise to $\Gamma = 6 \cdot K_2$, see Fig. 2. Let $B = B(\sigma)$. If $G = S_4$ then $G_B^B \cong S_3$ and Γ^B is G_B-symmetric; whilst if $G = A_4$ then $G_B^B \cong \mathbb{Z}_3$ and Γ^B is not G_B-symmetric. Note that in the former case $\Gamma_{\mathcal{B}} \cong K_4$ is $(G,2)$-arc transitive but Γ contains no 2-arcs.

In view of the above, one might ask for conditions under which $\mathtt{Simple}(\Gamma^B)$ is G_B-symmetric. The following lemma provides a sufficient condition for this to be true. For $\alpha \in B$, we denote by $(\Gamma^B(\alpha))^c$ the "complementary neighbourhood graph" defined to have vertex set $\Gamma^B(\alpha)$ (the neighbourhood of α in Γ^B) and edges of the form $\{\beta, \gamma\}$ with $\beta, \gamma \in \Gamma^B(\alpha)$ not adjacent in Γ^B.

Lemma 2.5. *Suppose Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - 2 \geqslant 1$. If $(\Gamma^B(\alpha))^c$ is connected, then $\mathtt{Simple}(\Gamma^B)$ is G_B-symmetric. In particular, if Γ^B has no triangles then $\mathtt{Simple}(\Gamma^B)$ is G_B-symmetric.*

Proof. Let $(\alpha, \beta), (\alpha, \gamma)$ be distinct arcs of $\mathtt{Simple}(\Gamma^B)$. As shown in the first paragraph of the proof of Theorem 2.1, there exists $g \in G_B$ such that $\gamma^g = \beta$ and $\{\alpha, \beta\}^g = \{\alpha, \gamma\}$. If $\alpha^g = \gamma$, then $\beta^g = \alpha$ and hence $(\gamma, \beta) = (\alpha, \gamma)^g$, implying that β, γ are adjacent in Γ^B.

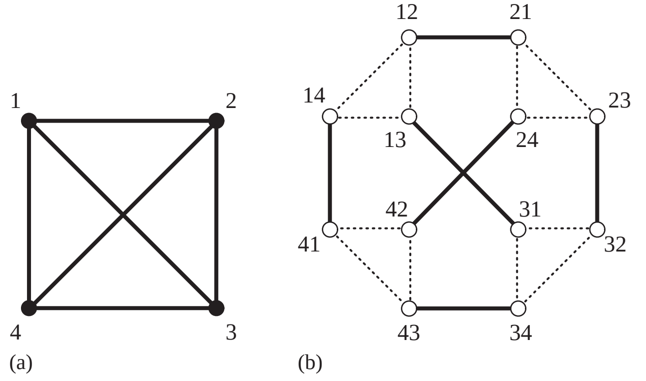

Fig. 2. (a) $\Sigma = K_4$; (b) $\Gamma = 6 \cdot K_2$, with highlighted edges, is obtained from K_4 by using the construction in Example 2.4.

In other words, if β, γ are not adjacent in Γ^B, then we have $g \in G_\alpha$ and $\beta^g = \gamma$. In the case where β, γ are adjacent in Γ^B, the connectedness of $(\Gamma^B(\alpha))^c$ assures that there exists a path $\beta = \beta_0, \beta_1, \ldots \beta_n = \gamma$ of $(\Gamma^B(\alpha))^c$ joining β and γ. So β_{i-1}, β_i are not adjacent in Γ^B for $1 \leqslant i \leqslant n$, and thus the argument above ensures that there exists $g_i \in G_\alpha$ such that ${\beta_{i-1}}^{g_i} = \beta_i$. Setting $g = g_1 \cdots g_n$, then $g \in G_\alpha$ and $\beta^g = \gamma$, and the proof is complete. □

3. Proof of Theorem 1.3

Throughout this section, we shall assume that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - 2 \geqslant 1$. Before proceeding to the proof of Theorem 1.3 we derive some general information about such a graph Γ. Let $B \in \mathcal{B}$. Recall that m denotes the multiplicity of each edge of Γ^B. For each (unordered) pair α, β of adjacent vertices of Γ^B, we define an m-element subset $\langle\alpha, \beta\rangle$ of $\Gamma_{\mathcal{B}}(B)$ by

$$\langle\alpha, \beta\rangle := \{C \in \Gamma_{\mathcal{B}}(B) : \langle B, C\rangle = \{\alpha, \beta\}\}, \text{ and}$$
$$\text{set } \mathcal{L}(B) := \{\langle\alpha, \beta\rangle : \alpha, \beta \text{ adjacent in } \Gamma^B\}.$$

It follows from the definition that $\alpha \in \langle B, C\rangle \Leftrightarrow \langle B, C\rangle = \{\alpha, \beta\}$ for some β in $B \Leftrightarrow C \in \langle\alpha, \beta\rangle$ for some β in B. Thus, each block C of $\Gamma_{\mathcal{B}}(B)$ belongs to one and only one member of $\mathcal{L}(B)$. The proof of the following lemma is straightforward and hence is omitted.

Lemma 3.1. *The set $\mathcal{L}(B)$ is a G_B-invariant partition of $\Gamma_{\mathcal{B}}(B)$, and the induced action of G_B on $\mathcal{L}(B)$ is permutationally equivalent to the action of G_B on the edge set of* $\texttt{Simple}(\Gamma^B)$. *In particular,*

(a) *if $m = 1$ (that is, Γ^B is simple), then the actions of G_B on $\Gamma_{\mathcal{B}}(B)$ and on the edges of Γ^B are permutationally equivalent; and*

(b) *if* $m \geqslant 2$, *then* $\Gamma_{\mathcal{B}}$ *is G-locally imprimitive and in particular* $\Gamma_{\mathcal{B}}$ *is not* $(G, 2)$*-arc transitive*.

In case (b) of Theorem 2.1, we have $\mathtt{Simple}(\Gamma^B) \cong (v/2) \cdot K_2$ and $V(\Gamma)$ admits a second G-invariant partition $\mathcal{P}$ that is a refinement of $\mathcal{B}$ and has block size 2. Let $\widehat{B} := \{P \in \mathcal{P} : P \subset B\}$ and $\widehat{\mathcal{B}} := \{\widehat{B} : B \in \mathcal{B}\}$. Recall that, for a G-symmetric graph $(\Gamma, \mathcal{B})$, $\mathcal{D}(B)$ is the 1-design with point set B and blocks (with possible repetitions) $\Gamma(C) \cap B$ for all $C \in \Gamma_{\mathcal{B}}(B)$. The following lemma can be easily verified.

Lemma 3.2. *Suppose that* Γ^B *is disconnected, so case* (b) *of Theorem* 2.1 *holds. Then the following hold*.

(a) $\widehat{\mathcal{B}}$ *is a G-invariant partition of* $\mathcal{P}$, *and the parameters* $\hat{v}, \hat{k}, \hat{b}, \hat{r}$ *with respect to* $(\Gamma_{\mathcal{P}}, \widehat{\mathcal{B}})$ *satisfy* $\hat{v} = v/2$, $\hat{k} = \hat{v} - 1$, $\hat{b} = b$ *and* $\hat{r} = r$.
(b) $(\Gamma_{\mathcal{P}})_{\widehat{\mathcal{B}}} \cong \Gamma_{\mathcal{B}}$.
(c) $\mathcal{D}(\widehat{B})$ *has no repeated blocks if and only if* $\mathcal{D}(B)$ *has no repeated blocks, which in turn is true if and only if* Γ^B *is simple* (*that is,* $\Gamma^B \cong (v/2) \cdot K_2$).

The definition of a 3-arc graph was given at the beginning of Section 1.2. It was proved in [11, Theorem1] that, if $k = v - 1 \geqslant 2$, then $\mathcal{D}(B)$ contains no repeated blocks if and only if $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive, and in this case Γ is isomorphic to a 3-arc graph of $\Gamma_{\mathcal{B}}$ with respect to a self-paired G-orbit on 3-arcs of $\Gamma_{\mathcal{B}}$. Applying this to $(\Gamma_{\mathcal{P}}, \widehat{\mathcal{B}})$ and using Lemma 3.2, we obtain the following result which proves Theorem 1.3 for large v. Note that we need $v \geqslant 6$ in the proof to ensure that $\hat{v} \geqslant 3$.

Theorem 3.3. *Suppose* Γ *is a G-symmetric graph admitting a nontrivial G-invariant partition* $\mathcal{B}$ *such that* $k = v - 2 \geqslant 4$ *and* Γ^B *is disconnected. Then* $\Gamma_{\mathcal{B}}$ *is* $(G, 2)$*-arc transitive if and only if* Γ^B *is a simple graph* (*that is,* $\Gamma^B \cong (v/2) \cdot K_2$), *and in this case* $\Gamma_{\mathcal{P}}$ (*with* $\mathcal{P}$ *as given in Theorem* 2.1 (b)) *is isomorphic to a* 3*-arc graph* $\Xi(\Gamma_{\mathcal{B}}, \Delta)$ *of* $\Gamma_{\mathcal{B}}$ *with respect to some self-paired G-orbit* Δ *on* $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$.

Proof. We use Lemma 3.2 without mentioning each time. Since $v \geqslant 6$, $\widehat{\mathcal{B}}$ is a G-invariant partition of $\mathcal{P}$ (the vertex set of $\Gamma_{\mathcal{P}}$) with $\hat{k} = \hat{v} - 1 \geqslant 2$. Applying [11, Theorem 1] we have: $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive $\Leftrightarrow$ $(\Gamma_{\mathcal{P}})_{\widehat{\mathcal{B}}}$ is $(G, 2)$-arc transitive $\Leftrightarrow$ $\mathcal{D}(\widehat{B})$ contains no repeated blocks $\Leftrightarrow$ $\mathcal{D}(B)$ contains no repeated blocks $\Leftrightarrow$ Γ^B is a simple graph. Moreover, in this case we know by [11, Theorem 1] that $\Gamma_{\mathcal{P}} \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ for some self-paired G-orbit Δ on $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$. $\square$

We are now ready to prove Theorem 1.3. The proof uses repeatedly the fact that a G-vertex-transitive graph Σ is $(G, 2)$-arc transitive if and only if G_σ is doubly transitive on $\Sigma(\sigma)$ for some $\sigma \in V(\Sigma)$.

Proof of Theorem 1.3. Suppose first that $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. Then G_B is 2-transitive on $\Gamma_{\mathcal{B}}(B)$ (see the comments above), and Lemma 3.1(b) implies that Γ^B is a simple graph. Suppose that $v = 3$. Then $\Gamma^B \cong K_3$ and thus $b = \mathtt{val}(\Gamma_{\mathcal{B}}) = 3$ and $r = 1$ by

Theorem 2.1. It follows that $\Gamma \cong (|V(\Gamma)|/2) \cdot K_2$, $\Gamma_{\mathcal{B}}$ is trivalent, and Γ^B is G_B-symmetric from the discussion in Example 2.4(a). (The G_B-symmetry of Γ^B also follows from Lemma 3.1(a), the 2-transitivity of G_B on $\Gamma_{\mathcal{B}}(B)$ and the simplicity of Γ^B.) Moreover, for any finite trivalent $(G, 2)$-arc transitive graph Σ, the graph Γ constructed in Example 2.4 together with the partition $\mathcal{B}$ therein satisfies the conditions of Theorem 1.3 and is such that $v = 3$ and $\Gamma_{\mathcal{B}} \cong \Sigma$.

Now suppose that $v \geqslant 4$. Then Γ^B must contain two independent edges (that is, sharing no common vertex). However, G_B is 2-transitive on the edges of Γ^B, so every pair of edges of Γ^B is independent. In other words, v is even and $\Gamma^B \cong (v/2) \cdot K_2$, and in particular Γ^B is G_B-symmetric. By Theorem 2.1 (b), $V(\Gamma)$ admits a G-invariant partition $\mathcal{P}$ with the desired properties. Moreover, if $v = 4$ then $b = v/2 = 2$ and so $\Gamma_{\mathcal{B}} \cong s \cdot C_t$ for some integers $s \geqslant 1$ and $t \geqslant 3$. Thus, since $\Gamma[B, C]$ is either $2 \cdot K_2$ or C_4, we have $\Gamma \cong 2st \cdot K_2$ or $\Gamma \cong st \cdot C_4$, respectively. If $v \geqslant 6$ then by Theorem 3.3, $\Gamma_{\mathcal{P}}$ is isomorphic to a 3-arc graph of $\Gamma_{\mathcal{B}}$ with respect to some self-paired G-orbit on $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$.

To complete the proof we must prove that if Γ^B is simple and either (a) or (b) of Theorem 1.3 holds, then $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. Suppose then that Γ^B is simple and suppose first that (a) holds. Then $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive by the discussion in Example 2.4(a). (Another proof: Since $\Gamma^B \cong K_3$ is G_B-symmetric, G_B is 2-transitive on the edges of Γ^B, and hence G_B is 2-transitive on $\Gamma_{\mathcal{B}}(B)$ by Lemma 3.1(a).) Finally suppose that (b) holds. If $v = 4$ then $\Gamma^B \cong 2 \cdot K_2$, $b = v/2 = 2$ and so $\Gamma_{\mathcal{B}}$ is a union of disjoint cycles. Then since $\Gamma_{\mathcal{B}}$ is G-symmetric, it is $(G, 2)$-arc transitive. In the general case $v \geqslant 6$, v is even, and $\Gamma^B \cong (v/2) \cdot K_2$. Here the $(G, 2)$-arc transitivity of $\Gamma_{\mathcal{B}}$ follows from Theorem 3.3. This completes the proof. □

Remark 3.4. (a) In the general case of Theorem 1.3 where $v \geqslant 6$, an interesting situation arises. The group G_B is not 2-transitive on B, for otherwise Γ^B would be a complete graph, but the vertices of B can be paired in such a way that G_B is 2-transitive on the set of these pairs. See Example 4.4 for such a triple $(\Gamma, G, \mathcal{B})$ with $v = 6$.

(b) Also in this case Γ can be constructed from $\Gamma_{\mathcal{B}}$ via the following two steps:

(i) Construct $\Gamma_{\mathcal{P}}$ from $\Gamma_{\mathcal{B}}$ using the 3-arc graph construction.
(ii) Then construct Γ from $\Gamma_{\mathcal{P}}$.

The first step is under our control in a sense. For the second step, recall that we have only three possibilities listed in (b)(ii) of Theorem 2.1. In Theorem 2.1 (b)(ii)(1), Γ is a 2-fold cover of $\Gamma_{\mathcal{P}}$, and so we can make use of a standard covering graph construction (see [2, Chapter 19]). In Theorem 2.1 (b)(ii)(2), $\Gamma \cong \Gamma_{\mathcal{P}}[\overline{K}_2]$ so Γ is known. The remaining case, Theorem 2.1 (b)(ii)(3), is very hard to manage in general. An attempt to construct imprimitive symmetric graphs with at most one edge between any two blocks is given in [24, Section 4].

(c) Under the assumption that $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive, Theorem 1.3 shows that possibility (a) of Theorem 2.1 occurs if and only if $v = 3$, and in this case Γ is determined uniquely by $\Gamma_{\mathcal{B}}$. Here $\Gamma^B \cong K_3$, and the permutation group induced by G_B on B is S_3 since Γ^B is G_B-symmetric. The smallest example of such graphs is the G-symmetric graph $\Gamma = 6 \cdot K_2$ with $(G, 2)$-arc transitive quotient $\Gamma_{\mathcal{B}} \cong K_4$, where $G = S_4$, which was constructed in Example 2.4(b).

4. Proof of Theorem 1.4

The main result of this section, Theorem 4.5, determines precisely when $\Gamma_{\mathcal{B}}$ inherits $(G, 2)$-arc transitivity from Γ in the case where $k = v - 2 \geqslant 1$. This is a rather technical and detailed result and Theorem 1.4 follows immediately from it. In the proof we use the following lemma which is a direct consequence of [16, Lemma 2.2(a)] since our condition $k \geqslant (v+1)/2$ implies that $\mathcal{B}$ is small, that is, $\Gamma(C) \cap \Gamma(D) \cap B \neq \emptyset$ for all $C, D \in \Gamma_{\mathcal{B}}(B)$.

Lemma 4.1. *Suppose that Γ is a $(G, 2)$-arc transitive graph admitting a nontrivial G-invariant partition $\mathcal{B}$ of block size v such that the size k of each part of the bipartite graph $\Gamma[B, C]$ is greater than $v/2$. Then $\Gamma_{\mathcal{B}}$ is also $(G, 2)$-arc transitive.*

In the proof of Theorem 4.5 we will also use the following construction. Following [3] we define an *s-path* in a graph as an s-arc identified with its reverse s-arc. Thus, an s-path is an undirected walk of length s in which successive edges are distinct.

Construction 4.2. Let Σ be a connected trivalent $(G, 3)$-arc regular graph, where $G \leqslant \mathrm{Aut}(\Sigma)$. Let $\sigma \in V(\Sigma)$, and let $\Sigma(\sigma) = \{\tau, \eta, \varepsilon\}$. Let $\Sigma(\eta) = \{\sigma, \lambda, \mu\}$ and $\Sigma(\varepsilon) = \{\sigma, \xi, \zeta\}$. Since Σ has girth at least 4 ([2, Proposition 17.2]), we have $\{\lambda, \mu\} \cap \{\tau, \varepsilon\} = \emptyset$ and $\{\xi, \zeta\} \cap \{\tau, \eta\} = \emptyset$. (But it may happen that $\{\lambda, \mu\} \cap \{\xi, \zeta\} \neq \emptyset$.) By the $(G, 3)$-arc transitivity of Σ, there exists $g \in G$ such that $(\mu, \eta, \sigma, \varepsilon)^g = (\tau, \sigma, \eta, \lambda)$. Set $\theta = \xi^g$. Define Ω to be the orbit under G of the 4-path $\mu\eta\sigma\varepsilon\xi$. Then (see Lemma 4.3), $\tau\sigma\eta\lambda\theta \in \Omega$, and $\Delta := (\mu\eta\sigma\varepsilon\xi, \tau\sigma\eta\lambda\theta)^G$ is a self-paired G-orbital on Ω. Define Γ to be the graph with vertex set Ω and arc set Δ. See Fig. 3 for an illustration of this construction.

Lemma 4.3. *With the notation of Construction 4.2,*

(a) *$\tau\sigma\eta\lambda\theta \in \Omega$, and $\Delta := (\mu\eta\sigma\varepsilon\xi, \tau\sigma\eta\lambda\theta)^G$ is a self-paired G-orbital on Ω;*
(b) *the graph Γ is G-symmetric admitting the G-invariant partition $\mathcal{B} := \{B(\sigma) : \sigma \in V(\Sigma)\}$, where $B(\sigma)$ is the set of 4-paths of Ω with middle vertex σ. Moreover, $|B(\sigma)| = 6$, $\Gamma_{\mathcal{B}} \cong \Sigma$ and $\Gamma, G, \mathcal{B}$ satisfy all the conditions of Theorem 1.4*(b).

Proof. In fact, we have $G_{\eta\sigma} \cong (\mathbb{Z}_2)^2$ by [2, 18f], and hence each nonidentity element of $G_{\eta\sigma}$ is an involution. Since $\varepsilon^g = \lambda$, we have $\tau^g \neq \lambda$, and similarly $\lambda^g \neq \tau$. But g swaps σ and η and hence swaps $\Sigma(\sigma)$ and $\Sigma(\eta)$, so it follows that $\tau^g = \mu$ and $\lambda^g = \varepsilon$. Thus, we have $(\tau, \sigma, \eta, \lambda)^g = (\mu, \eta, \sigma, \varepsilon)$. Set $\phi = \zeta^g$. Then $\theta^g = \xi$ and $\phi^g = \zeta$ for otherwise we would have $(\mu, \eta, \sigma, \varepsilon, \xi)^{g^2} = (\mu, \eta, \sigma, \varepsilon, \zeta)$, which implies that Σ is $(G, 4)$-arc transitive, a contradiction. So g swaps $(\mu, \eta, \sigma, \varepsilon, \xi)$ and $(\tau, \sigma, \eta, \lambda, \theta)$, and (a) follows. Thus the graph Γ with vertex set Ω and arc set Δ is well-defined and undirected. One can check that Γ is a G-symmetric graph of valency 2. Hence $\Gamma = s \cdot C_t$ for some s, t with $t > 2$ and Γ is $(G, 2)$-arc transitive. One can also check that $\mathcal{B} := \{B(\sigma) : \sigma \in V(\Sigma)\}$ is a G-invariant partition of Ω with block size $|B(\sigma)| = 6$ such that $\Gamma_{\mathcal{B}} \cong \Sigma$ via the bijection $B(\sigma) \mapsto \sigma$, for $\sigma \in V(\Sigma)$. For any two adjacent blocks $B(\sigma)$ and $B(\eta)$, there are exactly two vertices in $B(\sigma)$, namely $\xi\varepsilon\sigma\tau *$ and $\zeta\varepsilon\sigma\tau\star$, which are not adjacent to any vertex in $B(\eta)$, where $*, \star$ are the two vertices in $\Sigma(\tau)$ other than σ. Moreover, $\Gamma[B(\sigma), B(\eta)] \cong 4 \cdot K_2$, and two

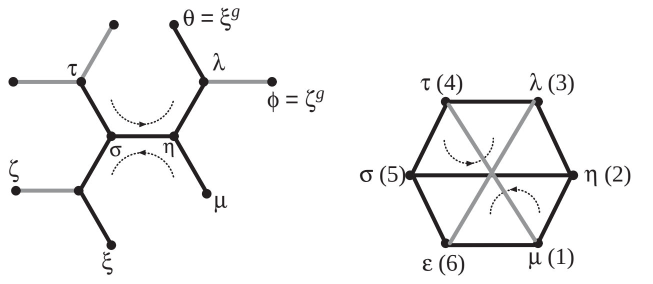

Fig. 3. Illustration of Construction 4.2. The right-hand side shows the case where $\Sigma = K_{3,3}$, which is trivalent $(G, 3)$-arc regular, where $G = \mathrm{Aut}(\Sigma) = S_3 \,\mathrm{wr}\, \mathbb{Z}_2$.

vertices of Γ are in the same block of the partition $\mathcal{P}$ (defined in (1)) if and only if they correspond to two 4-paths in Ω with the same 2nd, 3rd and 4th coordinates. Therefore, Γ, G and $\mathcal{B}$ satisfy all the conditions of Theorem 1.4(b). □

Note that the graph Γ is determined entirely by the 4-path $\mu\eta\sigma\varepsilon\xi$ of Σ, and the graph constructed in the same way by using the 4-path $\mu\eta\sigma\varepsilon\zeta$ is isomorphic to Γ via the bijection generated by $\mu\eta\sigma\varepsilon\xi \mapsto \mu\eta\sigma\varepsilon\zeta$. The construction above is a special case of a more general construction, called the flag graph construction, introduced in [24,23] by the third author. In line with this general construction [24], we may interpret the 4-path $\mu\eta\sigma\varepsilon\xi$ as the flag $(\sigma\tau, \{\sigma\tau, \eta\mu, \varepsilon\xi\})$ of the triple system with point set $\mathtt{Arc}(\Sigma)$ and block set $\{\sigma\tau, \eta\mu, \varepsilon\xi\}^G$. Also, Construction 4.2 bears some similarity with the 3-arc construction [11] which we used in the proof of Theorem 1.3. In fact, a 3-arc $(\tau, \sigma, \sigma', \tau')$ of a graph Σ can be identified with the ordered pair $((\sigma, \tau), (\sigma', \tau'))$ of arcs of Σ, and thus the self-paired G-orbit Δ on $\mathtt{Arc}_3(\Sigma)$ required to construct the 3-arc graph $\Xi(\Sigma, \Delta)$ can be identified with a self-paired G-orbital on $\mathtt{Arc}(\Sigma)$. Here in Construction 4.2 we use a self-paired G-orbital on the set of 4-paths of the graph Σ. Let us illustrate this construction by the following example.

Example 4.4. The smallest trivalent $(G, 3)$-arc regular graph is $\Sigma = K_{3,3}$ with $G = \mathrm{Aut}(\Sigma) = S_3 \,\mathrm{wr}\, \mathbb{Z}_2$. As shown in Fig. 3, we label the vertices of Σ by $1, 2, 3, 4, 5, 6$ such that $\{\{1, 3, 5\}, \{2, 4, 6\}\}$ is the bipartition of Σ. Since Σ is $(G, 3)$-arc regular, G has two orbits on the set of 4-paths of Σ. The first orbit is the set Ω_0 of 4-paths $\alpha\beta\gamma\delta\alpha$, where $\alpha, \beta, \gamma, \delta \in \{1, 2, \ldots, 6\}$ are pairwise distinct and any two consecutive terms in $\alpha\beta\gamma\delta\alpha$ have different parity, and the second orbit is the set Ω_1 of 4-paths $\alpha\beta\gamma\delta\tau$, where $\alpha, \beta, \gamma, \delta, \tau \in \{1, 2, \ldots, 6\}$ are pairwise distinct and any two consecutive terms in $\alpha\beta\gamma\delta\tau$ have different parity. (Note that $\alpha\beta\gamma\delta\alpha$ and $\alpha\delta\gamma\beta\alpha$ represent the same 4-path, and so do $\alpha\beta\gamma\delta\tau$ and $\tau\delta\gamma\beta\alpha$.)

The graphs obtained by applying Construction 4.2 to Ω_0 and Ω_1 are both isomorphic to $9 \cdot C_4$, and moreover the situation mentioned in Remark 3.4(a) occurs. In the following we will give details for the case of Ω_0. One can check that $\Delta = (12561, 45234)^G$ is the only self-paired G-orbital on Ω_0, and hence the graph Γ obtained from Construction 4.2 by using Ω_0 is unique. Also, one can check that, in the G-invariant partition $\mathcal{B} = \{B(i) : i = 1, 2, \ldots, 6\}$ of Ω_0, each block $B(i)$ consists of the six 4-paths $\alpha\beta\gamma\delta\alpha$ with $\gamma = i$. Since Γ has arc set Δ, the edges of Γ have the form $\{\alpha i j \beta\alpha, \gamma j i \delta\gamma\}$, where $i, j \in \{1, 2, \ldots, 6\}$ have different

Table 1
Results table for Theorem 4.5

Line	v	Γ	Γ^B	$\Gamma_{\mathcal{B}}$ is $(G,2)$-arc transitive ?
1	3	$(\lvert V(\Gamma)\rvert/2)\cdot K_2$	K_3	if and only if Γ^B is G_B-symmetric
2	3	$m=\mathtt{val}(\Gamma)\geqslant 2$	—	no; $G_B^{\Gamma_{\mathcal{B}}(B)}$ has a set of 3 blocks of imprimitivity of size m
3	4	$2st\cdot K_2$ $s\geqslant 1, t\geqslant 3$	$2\cdot K_2$	yes; $\Gamma_{\mathcal{B}}\cong s\cdot C_t$
4	4	$st\cdot C_4$ $s\geqslant 1, t\geqslant 3$	$2\cdot K_2$	yes; $\Gamma_{\mathcal{B}}\cong s\cdot C_t$
5	4	—	C_4 or K_4	no
6	4	$m=\mathtt{val}(\Gamma)\geqslant 2$	—	no; $G_B^{\Gamma_{\mathcal{B}}(B)}$ has a set of 2 blocks of imprimitivity of size m
7	>4	$v=2\hat{v}$ is even	$\hat{v}\cdot K_2$	yes; and $\mathtt{val}(\Gamma_{\mathcal{B}})=\hat{v}$

parity, say, j is even and i is odd, and $\{\{\alpha,\beta\},\{\gamma,\delta\},\{i,j\}\}$ is a partition of $\{1,2,\ldots,6\}$. Consequently, for even j and odd i, $\Gamma[B(i),B(j)]=4\cdot K_2$, $\langle B(j),B(i)\rangle=\{\alpha * j\beta\alpha\in\Omega_0 : \alpha\in\{2,4,6\}\setminus\{j\}, \{*,\beta\}=\{1,3,5\}\setminus\{i\}\}$, and $\langle B(i),B(j)\rangle=\{\gamma * i\delta\gamma\in\Omega_0 : \gamma\in\{1,3,5\}\setminus\{i\}, \{*,\delta\}=\{2,4,6\}\setminus\{j\}\}$. It follows that $\Gamma=9\cdot C_4$ and $\Gamma^{B(i)}=3\cdot K_2$ for each $i\in\{1,2,\ldots,6\}$. (The 4-cycle 12561, 45234, 12361, 43654, 12561 is a typical component of Γ.) Each edge of $\Gamma^{B(i)}$ joins the two elements of some $\langle B(i),B(j)\rangle$, where $j\in\{1,2,\ldots,6\}$ and j has parity opposite to i. Since $\Gamma^{B(i)}=3\cdot K_2$ and $\Gamma^{B(i)}$ admits $G_{B(i)}$ as a group of automorphisms, $G_{B(i)}$ is not 2-transitive on $B(i)$. On the other hand, since $\Gamma_{\mathcal{B}}\cong\Sigma$ is $(G,2)$-arc transitive, from Lemma 3.1(a) it follows that $G_{B(i)}$ is 2-transitive on the edges of $\Gamma^{B(i)}$. That is, $G_{B(i)}$ is 2-transitive on the three pairs $\langle B(i),B(j)\rangle$ of elements of $B(i)$.

We now state Theorem 4.5. Recall that m denotes the multiplicity of each edge of Γ^B.

Theorem 4.5. *Suppose that Γ is a $(G,2)$-arc transitive graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $k=v-2\geqslant 1$, where $G\leqslant \mathrm{Aut}(\Gamma)$. Then one of the lines of Table 1 holds, and $\Gamma[B,C]\cong(v-2)\cdot K_2$ for adjacent blocks B, C of $\mathcal{B}$ in all cases except line 4 where $\Gamma[B,C]\cong C_4$. Moreover, examples exist for each of the lines of Table 1, and further, if $v>4$, then one of the following holds, where $\mathcal{P}$ is the partition defined in (1).*

(a) *$\Gamma_{\mathcal{P}}$ is $(G,2)$-arc transitive and is an almost cover of $\Gamma_{\mathcal{B}}$. Also Γ is a 2-fold cover of $\Gamma_{\mathcal{P}}$ and has valency $\hat{v}-1$. If $\Gamma_{\mathcal{B}}$ is connected, then either $\Gamma_{\mathcal{B}}$ is a complete graph and $\Gamma_{\mathcal{P}}$ is known explicitly, or $\Gamma_{\mathcal{B}}$ is a near n-gonal graph with respect to a G-orbit on n-cycles of $\Gamma_{\mathcal{B}}$, for some even integer $n\geqslant 4$.*
(b) *$\hat{v}=3$, $\Gamma=s\cdot C_t$ for some s,t with $t\geqslant 3$, $\Gamma_{\mathcal{B}}$ is a $(G,3)$-arc transitive trivalent graph, and $\Gamma_{\mathcal{P}}=\Xi(\Gamma_{\mathcal{B}},\Delta)$ which is 4-valent and is not $(G,2)$-arc transitive, where Δ is the set of all 3-arcs of $\Gamma_{\mathcal{B}}$. If $\Gamma_{\mathcal{B}}$ is connected then it is $(G,3)$-arc regular, and moreover any connected trivalent $(G,3)$-arc regular graph can occur as $\Gamma_{\mathcal{B}}$.*

Proof. We first point out that examples exists for each line of Table 1. In fact, the graphs Γ in line 1 of Table 1 are precisely the graphs obtained from trivalent G-symmetric graphs Σ by using the construction in Example 2.4. The graph Γ in line 3 of Table 1 is isomorphic to a 2-fold cover of the graph Π with vertex set $\mathtt{Arc}(s \cdot C_t)$ and edge set $\{\{(\sigma, \tau), (\tau, \sigma)\} : (\sigma, \tau) \in \mathtt{Arc}(s \cdot C_t)\}$. Similarly, the graph in line 4 of the table is isomorphic to the lexicographic product of Π by $\overline{K}_2$ (empty graph on two vertices). The graphs constructed in Construction 4.2 and Lemma 4.3 are examples for line 7 of Table 1. In Examples 4.6–4.8 we will construct graphs in lines 2, 5 and 6 of Table 1, respectively. Thus, the existence of graphs in each line of Table 1 is established.

Now let us proceed to the core part of the proof. Suppose that Γ is a $(G, 2)$-arc transitive graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $k = v - 2 \geqslant 1$, where $G \leqslant \mathrm{Aut}(\Gamma)$. Let $B \in \mathcal{B}$ and $\alpha \in B$. Since Γ is $(G, 2)$-arc transitive, G_α is 2-transitive on $\Gamma(\alpha)$ and hence is primitive on $\Gamma(\alpha)$. It is easily checked that $\{\Gamma(\alpha) \cap C : C \in \Gamma_{\mathcal{B}}(\alpha)\}$ is a G_α-invariant partition of $\Gamma(\alpha)$. Therefore this partition is trivial, that is, either $|\Gamma(\alpha) \cap C| = 1$, or $\Gamma(\alpha) \cap C = \Gamma(\alpha)$.

Suppose that $\Gamma(\alpha) \subseteq C$ and $\mathtt{val}(\Gamma) \geqslant 2$. Then $r = |\Gamma_{\mathcal{B}}(\alpha)| = |\{C\}| = 1$ and $k = v - 2 \geqslant \mathtt{val}(\Gamma) \geqslant 2$. Since Γ is G-symmetric, C contains $\Gamma(\gamma)$ for each of the $k = v - 2$ vertices $\gamma \in B \setminus \langle B, C\rangle$. Let $\beta \in \langle B, C\rangle$. Then $\Gamma_{\mathcal{B}}(\beta) = \{D\}$ for some $D \in \Gamma_{\mathcal{B}}(B)$, and D contains $\Gamma(\delta)$ for $v - 2$ vertices δ of B. It follows that $v = 4$, $m = 1$, $\Gamma^B \cong 2 \cdot K_2$, and $\Gamma[B, C] \cong K_{2,2} = C_4$. Thus both Γ and $\Gamma_{\mathcal{B}}$ have valency 2. So $\Gamma_{\mathcal{B}} \cong s \cdot C_t$ for some $s \geqslant 1$, $t \geqslant 3$, and as $\Gamma[B, C] \cong C_4$, we have $\Gamma \cong st \cdot C_4$ and line 4 in Table 1 holds.

From now on we will assume that $|\Gamma(\alpha) \cap C| = 1$, and hence that $\Gamma[B, C] \cong (v-2) \cdot K_2$ for adjacent blocks B, C of $\mathcal{B}$. This implies that, for each part $\langle \beta, \gamma\rangle = \{C_1, \ldots, C_m\}$ of $\mathcal{L}(B)$ (defined at the start of Section 3), and each $\alpha \in B \setminus \{\beta, \gamma\}$, each of the C_i contains a unique vertex of $\Gamma(\alpha)$, and $\bigcup_{i=1}^m (\Gamma(\alpha) \cap C_i)$ is a block of imprimitivity for the action of G_α on $\Gamma(\alpha)$. Since G_α is primitive on $\Gamma(\alpha)$, this block is trivial so either $m = 1$ or $m = |\Gamma(\alpha)|$. Suppose that $m = |\Gamma(\alpha)| \geqslant 2$. Then $\Gamma_{\mathcal{B}}$ is G-locally imprimitive and in particular is not $(G, 2)$-arc transitive by Lemma 3.1. Also $\Gamma(\alpha) \subseteq \bigcup_{i=1}^m C_i$ for each of the $v - 2$ vertices $\alpha \in B \setminus \{\beta, \gamma\}$, so by Theorem 1.3, $v = 3$ or 4. If $v = 3$, then G_B preserves on $\Gamma_{\mathcal{B}}(B)$ a set of three blocks of imprimitivity of size $m = \mathtt{val}(\Gamma)$, and line 2 in Table 1 holds. If $v = 4$, then we must have $\mathtt{Simple}(\Gamma^B) \cong 2 \cdot K_2$, so G_B preserves on $\Gamma_{\mathcal{B}}(B)$ a set of two blocks of imprimitivity of size $m = \mathtt{val}(\Gamma) \geqslant 2$, and line 6 in Table 1 holds.

Thus we may assume that $m = 1$, that is, Γ^B is simple. Suppose that $v = 3$. Then $\Gamma^B \cong K_3$, $b = 3$, $r = 1$, and hence $\Gamma \cong (|V(\Gamma)|/2) \cdot K_2$. By Theorem 1.3, $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive if and only if Γ^B is G_B-symmetric, and thus line 1 in Table 1 holds. Suppose next that $v = 4$. Then $\Gamma^B \cong 2 \cdot K_2$, C_4 or K_4, and $b = 2$, 4 or 6 and $r = 1$, 2 or 3 respectively. In the first case there exist integers $s \geqslant 1$, $t \geqslant 3$ such that $\Gamma_{\mathcal{B}}, \Gamma$ are as described in line 3 of Table 1, and $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. In the last two cases $\Gamma_{\mathcal{B}}$ is not $(G, 2)$-arc transitive by Theorem 1.3, and thus line 5 in Table 1 holds.

We may therefore assume that $v > 4$, $\Gamma[B, C] \cong (v - 2) \cdot K_2$, and Γ^B is simple. Let $\mathcal{P}$ be as in (1). Then $k = v - 2 \geqslant (v + 1)/2$ and hence by Lemma 4.1, $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive. By Theorem 1.3, $v = 2\hat{v}$ is even, $\Gamma^B \cong \hat{v} \cdot K_2$, and hence the valency of $\Gamma_{\mathcal{B}}$ is $\hat{v}$. Thus, line 7 in Table 1 holds. Also from Theorem 1.3, $\Gamma_{\mathcal{P}}$ is isomorphic to a 3-arc graph $\Xi(\Gamma_{\mathcal{B}}, \Delta)$ of $\Gamma_{\mathcal{B}}$ with respect to a self-paired G-orbit Δ on $\mathtt{Arc}_3(\Gamma_{\mathcal{B}})$. Since the edges of $\Gamma[B, C]$ form a matching, only cases (1) or (3) of Theorem 2.1(b)(ii) can occur. Let $\widehat{\mathcal{B}}$ be the partition of $\mathcal{P}$ defined before Lemma 3.2. As we noted there, $\widehat{\mathcal{B}}$ is G-invariant and has blocks of size $\hat{v} = v/2 \geqslant 3$.

Suppose that case (1) of Theorem 2.1(b)(ii) holds so that Γ is a 2-fold cover of $\Gamma_{\mathcal{P}}$. By Lemma 4.1 applied to $\mathcal{P}$, $\Gamma_{\mathcal{P}}$ is $(G, 2)$-arc transitive. Also in this case $\Gamma_{\mathcal{P}}$ is an almost cover of $\Gamma_{\mathcal{B}}$. If $\Gamma_{\mathcal{B}}$ is a complete graph, then all possibilities for the 3-arc graphs of $\Gamma_{\mathcal{B}}$ were classified in [9] (see also [23, Theorem 3.19] for an explicit list), and thus $\Gamma_{\mathcal{P}}$ is known explicitly. If $\Gamma_{\mathcal{B}}$ is connected but not complete, then it has girth at least 4 as it is $(G, 2)$-arc transitive. By [25, Theorem1.1] there exists an even integer $n \geqslant 4$ such that $\Gamma_{\mathcal{B}}$ is a near n-gonal graph with respect to a G-orbit on n-cycles of $\Gamma_{\mathcal{B}}$. Thus (a) holds.

Finally suppose that case (3) of Theorem 2.1(b)(ii) holds, so that G is faithful on $\mathcal{B}$. Since $\Gamma^B \cong (v/2) \cdot K_2$, it follows that, for $(B, C) \in \mathtt{Arc}(\Gamma_{\mathcal{B}})$, $\Gamma_{\mathcal{P}}[\widehat{B}, \widehat{C}]$ has valency 2, and hence is $r \cdot C_u$ where $r \geqslant 1$ and u is even and $u \geqslant 4$. Then ru is the number of edges of Γ between B and C so $ru = v - 2$. Also Γ has valency $\hat{v} - 1$.

We claim that G_B is 3-transitive on $\Gamma_{\mathcal{B}}(B)$ of degree $\hat{v}$. By Lemma 3.1(a), the actions of G_B on $\Gamma_{\mathcal{B}}(B)$ and the edges of Γ^B are permutationally equivalent, so $\Gamma_{\mathcal{B}}$ has valency $\hat{v}$. If $C \in \Gamma_{\mathcal{B}}(B)$ and $\alpha \in \langle B, C\rangle$, then G_α has index 2 in $G_{B\,C}$. Then since G_α is 2-transitive on $\Gamma(\alpha)$ and since each of the blocks of $\Gamma_{\mathcal{B}}(B) \setminus \{C\}$ contains exactly one point of $\Gamma(\alpha)$, it follows that G_α is 2-transitive on $\Gamma_{\mathcal{B}}(B) \setminus \{C\}$ and hence G_B is 3-transitive on $\Gamma_{\mathcal{B}}(B)$. This proves the claim. It follows that G_B is 2-transitive and hence primitive on $B \setminus \langle B, C\rangle$. Since G_B induces a group of automorphisms of $\Gamma_{\mathcal{P}}[\widehat{B}, \widehat{C}]$, this implies that $\Gamma_{\mathcal{P}}[\widehat{B}, \widehat{C}] = C_u$ (that is, $r = 1$) and $u/2 \leqslant 3$. Thus $\hat{v} = 1 + u/2 = 3$ or 4. In the following we will show further that the case $\hat{v} = 4$ cannot happen.

Suppose for a contradiction that $\hat{v} = 4$. Let $G_1(B)$ denote the kernel of the action of G_B on $\Gamma_{\mathcal{B}}(B)$. We claim that $G_1(B)$ acts trivially on the connected component of $\Gamma_{\mathcal{B}}$ containing B. Suppose to the contrary that this is not so. Then $G_1(B)$ must act non-trivially on $\Gamma_{\mathcal{B}}(C)$ for $C \in \Gamma_{\mathcal{B}}(B)$. Now $G_1(B)$ is normal in G_{BC}, and G_{BC} induces S_3 on $\Gamma_{\mathcal{B}}(B) \setminus \{C\}$, and hence $G_1(B)$ is transitive on $\Gamma_{\mathcal{B}}(B) \setminus \{C\}$. Therefore $G_1(B)$ is transitive on the 3 blocks of $\mathcal{P}$ in $C \setminus \langle C, B\rangle$. This contradicts the fact that G_B induces a subgroup of D_{12} on $\Gamma_{\mathcal{P}}[\widehat{B}, \widehat{C}] \cong C_6$. Thus $G_1(B)$ is trivial on the connected component of $\Gamma_{\mathcal{B}}$ containing B. In particular $G_1(B)$ fixes setwise the blocks of $\mathcal{P}$ in $B \cup C$, and because of the nature of $\Gamma[B, C]$ we see that $G_1(B)$ must fix $B \cup C$ pointwise. Thus $G_1(B) \leqslant G_\alpha < G_B$, and so $|G_B : G_1(B)|$ is divisible by $8\,|G_\alpha : G_1(B)|$. However, $G_B/G_1(B) \cong S_4$ and hence $|G_\alpha : G_1(B)|$ divides 3. This is a contradiction since G_α induces S_3 on $\Gamma(\alpha)$ and its actions on $\Gamma(\alpha)$ and $\Gamma_{\mathcal{B}}(B) \setminus \{C\}$ are permutationally equivalent.

Now the only possibility is $\hat{v} = 3$. In this case $\Gamma_{\mathcal{P}}[\widehat{B}, \widehat{C}] \cong C_4 \cong K_{2,2}$, so $\Gamma_{\mathcal{B}}$ is $(G, 3)$-arc transitive and $\Gamma_{\mathcal{P}} \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ with Δ the set of all 3-arcs of $\Gamma_{\mathcal{B}}$ (see [11]). Since $\Gamma_{\mathcal{B}}$ is trivalent, it follows from the definition of a 3-arc graph that $\Gamma_{\mathcal{P}}$ is 4-valent. Moreover, since the edges of $\Gamma_{\mathcal{P}}[\widehat{B}, \widehat{C}]$ do not form a matching, $\Gamma_{\mathcal{P}}$ cannot be $(G, 2)$-arc transitive. Also in this case Γ has valency 2 so $\Gamma = s \cdot C_t$ for some s, t with $t > 2$. In the following we will prove that $\Gamma_{\mathcal{B}}$ is $(G, 3)$-arc regular if it is connected. We note first that each vertex α of Γ can be labelled by a 4-path of $\Gamma_{\mathcal{B}}$ in the following way. Let $\alpha \in \langle B, C\rangle$ and $\Gamma(\alpha) = \{\gamma, \delta\}$, and let $\gamma \in D$ and $\delta \in E$, where B, C, D, E are blocks of $\mathcal{B}$ such that $\Gamma_{\mathcal{B}}(B) = \{C, D, E\}$. Let $\gamma \in \langle D, F\rangle$ and $\delta \in \langle E, I\rangle$, for blocks F, I of $\mathcal{B}$. The 3-arc transitivity of $\Gamma_{\mathcal{B}}$ implies that its girth is at least 4 [2, Proposition 17.2]. Hence $D \neq I$ and $E \neq F$ (but $F = I$ may happen), and (F, D, B, E, I) is a 4-arc of $\Gamma_{\mathcal{B}}$. Now we label the vertex α by the 4-path $FDBEI$ of

$\Gamma_{\mathcal{B}}$. One can verify that the set Ω of such 4-paths, for α running over $V(\Gamma)$, is a G-orbit on the set of all 4-paths of $\Gamma_{\mathcal{B}}$. Moreover, the actions of G on $V(\Gamma)$ and Ω are permutationally equivalent via this labelling. Since α is the unique vertex whose labelling 4-path has prefix $FDBE$, it follows that any element g of G fixing the 3-arc (F, D, B, E) must fix each of F, D, B, E, I setwise and hence fix the 3-arc (D, B, E, I). Since the connectedness of $\Gamma_{\mathcal{B}}$ implies that each block of $\mathcal{B}$ appears in at least one member of Ω, repeating this procedure we know that g fixes setwise each block of $\mathcal{B}$. Since G is faithful on $\mathcal{B}$ by Corollary 2.2, as mentioned earlier, it follows that $g = 1$. Therefore, $\Gamma_{\mathcal{B}}$ is $(G, 3)$-arc regular and hence is not $(G, 4)$-arc transitive. Conversely, by Construction 4.2, any connected trivalent $(G, 3)$-arc regular graph Σ gives rise to a $(G, 2)$-arc transitive graph $\Gamma = s \cdot C_t$ which admits a G-invariant partition $\mathcal{B}$ such that $k = v - 2 = 4$ and $\Gamma_{\mathcal{B}} \cong \Sigma$. Thus (b) holds, and the proof is complete. $\square$

Example 4.6. (a) The graphs in line 2 of Table 1 have $k = v - 2 = 1$ (and $\mathtt{val}(\Gamma_{\mathcal{B}}) = 3m$ by Theorem 2.1), and hence can be constructed by using the flag graph construction given in [24, Section 4]. In particular, such graphs Γ with $\Gamma_{\mathcal{B}}$ a complete graph (that is, $\Gamma_{\mathcal{B}} \cong K_{3m+1}$) are $(G, 2)$-arc transitive graphs constructed in [24, Corollary 4.4(b)] (with $v = 3$ and $r = m \geqslant 2$). For instance, in [24, Corollary 4.4(b)] we may take the design $\mathcal{D}$ to be the Fano plane $\mathrm{PG}(2, 2)$ and set $G = \mathrm{PGL}(3, 2)$. Then, since in $\mathrm{PG}(2, 2)$ any two points lie in exactly one line, from [24, Example 4.5(b)] we get the graph Γ with vertices the (point, line)-flags of $\mathrm{PG}(2, 2)$ such that two such flags are adjacent if and only if they have the same line entry. Thus, we have $\Gamma \cong 7 \cdot K_3$, and Γ is a $(G, 2)$-arc transitive graph admitting $\mathcal{B} = \{B(\sigma) : \sigma \text{ a point of } \mathrm{PG}(2, 2)\}$ as a G-invariant partition, where $B(\sigma)$ is the set of (point, line)-flags of $\mathrm{PG}(2, 2)$ with point entry σ. For $\mathcal{B}$ we have $k = v - 2 = 1$ and $m = 2$, and Γ satisfies the conditions in line 2 of Table 1.

(b) We can also construct an infinite family of graphs satisfying line 2 of Table 1 by using a different approach. Let t be a prime with $t \equiv 1 \pmod 3$. Let Γ be the graph with vertex set $\mathbb{Z}_3 \times \mathbb{Z}_t$ and edge set $\{\{(i, j), (i, j \pm 1)\} : i \in \mathbb{Z}_3, j \in \mathbb{Z}_t\}$. Then $\Gamma \cong 3 \cdot C_t$. Since $t \equiv 1 \pmod 3$, we may choose $a \in \mathbb{Z}_t$ with order 3 in $\mathbb{Z}_t^*$, so that $a^3 \equiv 1 \pmod t$ and $1, -1, a, -a, a^2, -a^2$ are pairwise distinct $(\mathrm{mod}\, t)$. Let x, y, z be permutations defined by $(i, j)^x = (i, j + a^i)$, $(i, j)^y = (i, -j)$, $(i, j)^z = (i + 1, j)$, for $(i, j) \in \mathbb{Z}_3 \times \mathbb{Z}_t$. Then $x, y, z \in \mathrm{Aut}(\Gamma)$, and hence $G := \langle x, y, z \rangle \leqslant \mathrm{Aut}(\Gamma)$. Clearly, x, y, z have orders $t, 2, 3$, respectively, and $(yx)^2 = 1$. Hence $\langle x, y \rangle \cong D_{2t}$ on each of the three components of Γ. Also, $xz = zx$ and $yz = zy$. Thus, $|G| = 6t$ and G is transitive on $\mathbb{Z}_3 \times \mathbb{Z}_t$. The stabiliser of $(0, 0)$ in G is $\langle y \rangle \cong \mathbb{Z}_2$, and it acts doubly transitively on $\Gamma((0, 0)) = \{(0, 1), (0, -1)\}$. It follows that Γ is a $(G, 2)$-arc transitive graph. Since t is a prime and $a^3 \equiv 1 \pmod t$, $\mathcal{B} = \{B_j : j \in \mathbb{Z}_t\}$ is a partition of $\mathbb{Z}_3 \times \mathbb{Z}_t$ (the vertex set of Γ) with block size $v = |B_j| = 3$, where $B_j = \{(0, j), (1, aj), (2, a^2 j)\}$. Moreover, $\mathcal{B}$ is invariant under G since $B_j^x = B_{j+1}$, $B_j^y = B_{-j}$ and $B_j^z = B_{a^2 j}$ for each $j \in \mathbb{Z}_t$. Two blocks $B_j, B_{j'}$ of $\mathcal{B}$ are adjacent in $\Gamma_{\mathcal{B}}$ if and only if $j' - j \equiv \pm 1 \pmod t$, or $a(j' - j) \equiv \pm 1 \pmod t$, or $a^2(j' - j) \equiv \pm 1$ $(\mathrm{mod}\, t)$. Thus, for $\mathcal{B}$ we have $k = v - 2 = 1$, $m = 2$ and $\mathtt{val}(\Gamma_{\mathcal{B}}) = 6$. In particular, $\Gamma_{\mathcal{B}}(B_0) = \{B_1, B_{-1}, B_{a^2}, B_{-a^2}, B_a, B_{-a}\}$ with $(0, 0) \in B_0$ adjacent to $(0, 1) \in B_1$ and $(0, -1) \in B_{-1}$, $(1, 0) \in B_0$ to $(1, 1) \in B_{a^2}$ and $(1, -1) \in B_{-a^2}$, and $(2, 0) \in B_0$ to $(2, 1) \in B_a$ and $(2, -1) \in B_{-a}$. Furthermore, $\{\{B_1, B_{-1}\}, \{B_{a^2}, B_{-a^2}\}, \{B_a, B_{-a}\}\}$ is a G_{B_0}-invariant partition of $\Gamma_{\mathcal{B}}(B_0)$ with block size $m = 2$. Therefore, Γ satisfies the conditions in line 2 of Table 1. The smallest graph thus constructed is $\Gamma \cong 3 \cdot C_7$, which is obtained by taking $t = 7$ and $a = 2$. For this smallest graph we have $\Gamma_{\mathcal{B}} \cong K_7$, and the block B_j ($j \in \mathbb{Z}_7$) of $\mathcal{B}$ is given by the jth column of the following 3×7 array:

$$\begin{array}{ccccccc} 00 & 01 & 02 & 03 & 04 & 05 & 06 \\ 10 & 12 & 14 & 16 & 11 & 13 & 15 \\ 20 & 24 & 21 & 25 & 22 & 26 & 23 \end{array}$$

In conclusion, in the next two examples we will construct graphs in lines 5 and 6 of Table 1 by using a method similar to that used in Example 4.6(b).

Example 4.7. Let Γ be the graph with vertex set $\mathbb{Z}_4 \times \mathbb{Z}_8$ and edge set $\{\{(i, j), (i, j \pm 1)\} : i \in \mathbb{Z}_4, j \in \mathbb{Z}_8\}$. Then $\Gamma \cong 4 \cdot C_8$. Let x, y, z be permutations on $\mathbb{Z}_4 \times \mathbb{Z}_8$ defined by

$$(i, j)^x = \begin{cases} (i, j + 1 + 2i), & i = 0, 2, \\ (i + 2, j + 1 + 2i), & i = 1, 3 \end{cases}$$

$$(i, j)^y = \begin{cases} (i, -j), & i = 0, 2, \\ (i + 2, -j), & i = 1, 3 \end{cases}$$

and $(i, j)^z = (i + 1, j)$, for $i \in \mathbb{Z}_4$ and $j \in \mathbb{Z}_8$. (Thus, $(0, j)^x = (0, j + 1)$, $(1, j)^x = (3, j + 3)$, $(2, j)^x = (2, j + 5)$, $(3, j)^x = (1, j + 7)$ and $(i, j)^{x^2} = (i, j + 2)$ for all i, j.) Then x, y, z all preserve the adjacency of Γ and hence $G := \langle x, y, z\rangle \leqslant \mathrm{Aut}(\Gamma)$. Clearly, x, y, z have orders 8, 2, 4, respectively. One can check that $y^{-1}xy = x^{-1}$ and $y^{-1}zy = z^{-1}$, and so y and z^2 commute. One can also check that $z^{-1}xz = x^{-1}z^2$ and $z^2xz^2 = x^5$. It follows that $H := \langle x, z\rangle$ and $H_0 := \langle x, z^2\rangle$ are normal in G, and H_0 has order 16. Note that $|G : H| = 2$ and $|H : H_0| = 2$. Hence $|G| = 64$. Clearly, G is transitive on the vertex set $\mathbb{Z}_4 \times \mathbb{Z}_8$ of Γ. Thus, the stabiliser of $(0, 0)$ in G has order $64/(4 \times 8) = 2$ and hence is equal to $\langle y\rangle$. Since $\langle y\rangle$ is doubly transitive on $\Gamma((0, 0)) = \{(0, 1), (0, 7)\}$, it follows that Γ is $(G, 2)$-arc transitive. Let $B_0 = (0, 0)^{\langle z\rangle} = \{(0, 0), (1, 0), (2, 0), (3, 0)\}$, $B_2 = B_0^{x^2} = \{(0, 2), (1, 2), (2, 2), (3, 2)\}$, $B_4 = B_0^{x^4} = \{(0, 4), (1, 4), (2, 4), (3, 4)\}$, $B_6 = B_0^{x^6} = \{(0, 6), (1, 6), (2, 6), (3, 6)\}$, $B_1 = B_0^x = \{(0, 1), (1, 7), (2, 5), (3, 3)\}$, $B_3 = B_2^x = \{(0, 3), (1, 1), (2, 7), (3, 5)\}$, $B_5 = B_4^x = \{(0, 5), (1, 3), (2, 1), (3, 7)\}$ and $B_7 = B_0^x = \{(0, 7), (1, 5), (2, 3), (3, 1)\}$. Then y fixes B_0 and B_4 and swaps B_2 and B_6, B_1 and B_7, and B_3 and B_5. Also, z fixes B_{2i} for $i = 0, 1, 2, 3$, and cycles B_1 to B_3, B_3 to B_5, B_5 to B_7 and B_7 to B_1. Hence $\mathcal{B} = \{B_\ell : \ell = 0, 1, \ldots, 7\}$ is a G-invariant partition of $\mathbb{Z}_4 \times \mathbb{Z}_8$. By the definition of Γ, $(0, 0)$ is adjacent to $(0, 1) \in B_1$ and $(0, 7) \in B_7$, $(1, 0)$ to $(1, 1) \in B_3$ and $(1, 7) \in B_1$, $(2, 0)$ to $(2, 1) \in B_5$ and $(2, 7) \in B_3$, and $(3, 0)$ to $(3, 1) \in B_7$ and $(3, 7) \in B_5$. Hence $\Gamma_{\mathcal{B}}(B_0) = \{B_1, B_3, B_5, B_7\}$, $\Gamma^{B_0} \cong C_4$, $\Gamma[B_0, B_1] \cong 2 \cdot K_2$ and $k = v - 2 = 2$. Moreover, $G_{B_0} = \langle y, z\rangle \cong D_8$, and from the actions of y and z on $\mathcal{B}$ it follows that $\{\{B_1, B_5\}, \{B_3, B_7\}\}$ is a G_{B_0}-invariant partition of $\Gamma_{\mathcal{B}}(B_0)$. Therefore, $\Gamma_{\mathcal{B}}$ is not $(G, 2)$-arc transitive, and Γ satisfies all conditions in line 5 of Table 1. In addition, we have $\Gamma_{\mathcal{B}} \cong K_{4,4}$.

Example 4.8. Let t be a prime such that $t \equiv 1 \pmod 4$. Let a be an element of $\mathbb{Z}_t^*$ with order 4, so that $a^4 \equiv 1$ and $a^2 \equiv -1 \pmod t$. Define Γ to be the graph with vertex set $\mathbb{Z}_4 \times \mathbb{Z}_t$ and edge set $\{\{(i, j), (i, j \pm 1)\} : i \in \mathbb{Z}_4, j \in \mathbb{Z}_t\}$. Then $\Gamma \cong 4 \cdot C_t$. Let x, y, z be permutations defined by $(i, j)^x = (i, j + a^i)$, $(i, j)^y = (i, -j)$, $(i, j)^z = (i + 1, j)$, for $(i, j) \in \mathbb{Z}_4 \times \mathbb{Z}_t$. Then $x, y, z \in \mathrm{Aut}(\Gamma)$ and hence $G := \langle x, y, z\rangle \leqslant \mathrm{Aut}(\Gamma)$. From the definitions of x, y, z it follows that they have orders t, 2, 4, respectively, and that G is transitive on $\mathbb{Z}_4 \times \mathbb{Z}_t$. Also, the stabiliser of $(0, 0)$ in G is $\langle y\rangle$. Since $\langle y\rangle \cong \mathbb{Z}_2$ is doubly transitive on $\Gamma((0, 0)) = \{(0, 1), (0, -1)\}$, if follows that Γ is $(G, 2)$-arc transitive. Let $B_j = \{(0, j), (1, ja), (2, -j), (3, -ja)\}$ for $j \in \mathbb{Z}_t$. Then $B_j = B_0^{x^j}$ and $B_j^x = B_{j+1}$. For each j, we have $B_j^z = B_{-aj}$ and y swaps B_j and B_{-j}. Hence $\mathcal{B}$ is a G-invariant partition of $\mathbb{Z}_4 \times \mathbb{Z}_t$ and $G_{B_0} \cong \langle y, z\rangle$. From the definition of Γ, $(0, 0) \in B_0$ is adjacent to $(0, 1) \in B_1$ and $(0, -1) \in B_{-1}$, $(1, 0) \in B_0$ to $(1, 1) \in B_{-a}$ and $(1, -1) \in B_a$, $(2, 0) \in B_0$ to $(2, 1) \in B_{-1}$ and $(2, -1) \in B_1$, and $(3, 0) \in B_0$ to $(3, 1) \in B_a$ and $(3, -1) \in B_{-a}$. It follows that $\Gamma_{\mathcal{B}}(B_0) = \{B_1, B_{-1}, B_a, B_{-a}\}$, $\Gamma[B_0, B_1] \cong 2 \cdot K_2$, and Γ^{B_0} is isomorphic to $2 \cdot K_2$ with each edge repeated twice. Thus, for $\mathcal{B}$ we have $k = v - 2 = 2$ and $m = \mathtt{val}(\Gamma) = 2$. Note that y swaps B_1 and B_{-1}, and B_a and B_{-a}, and that $B_1^z = B_{-a}$, $B_{-1}^z = B_a$, $B_a^z = B_1$ and $B_{-a}^z = B_{-1}$. Thus, since $G_{B_0} \cong \langle y, z\rangle$, $\{\{B_1, B_{-1}\}, \{B_a, B_{-a}\}\}$ is a G_{B_0}-invariant partition of $\Gamma_{\mathcal{B}}(B_0)$ with block size $m = 2$. Consequently, $\Gamma_{\mathcal{B}}$ is not $(G, 2)$-arc transitive, and Γ satisfies all conditions in line 6 of Table 1. The smallest graph thus constructed is $\Gamma \cong 4 \cdot C_5$, which is obtained by taking $t = 5$ and $a = 2$. For this graph the block B_j ($j \in \mathbb{Z}_5$) of $\mathcal{B}$ is given by the jth column of the following 4×5 array:

00	01	02	03	04
10	12	14	11	13
20	24	23	22	21
30	33	31	34	32

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