

Routing permutations and involutions on optical ring networks: complexity results and solution to an open problem

Jinjiang Yuan^{a,1}, Jian Ying Zhang^b, Sanming Zhou^{c,*,2}

^a Department of Mathematics, Zhengzhou University, Zhengzhou, Henan 450052, PR China

^b Faculty of Information and Communication Technologies, Swinburne University of Technology, Hawthorn, Victoria 3122, Australia

^c Department of Mathematics and Statistics, The University of Melbourne, Parkville, VIC 3010, Australia

Received 28 November 2005; accepted 16 July 2006

Available online 11 October 2006

Abstract

Given a network G and a demand D of communication requests on G , a routing for (G, D) is a set of directed paths of G , each from the source to the destination of one request of D . The ROUTING AND WAVELENGTH ASSIGNMENT PROBLEM seeks a routing \mathcal{R} for (G, D) and an assignment of wavelengths to the directed paths in \mathcal{R} such that the number of wavelengths used is minimized, subject to that any two directed paths with at least one common arc receive distinct wavelengths. In the case where G is a ring, this problem is known as the RING ROUTING AND WAVELENGTH ASSIGNMENT PROBLEM (RRWA). If in addition D is symmetric (that is, $(s, t) \in D$ implies $(t, s) \in D$) and the directed paths for requests (s, t) and (t, s) are required to be reverse of each other, then the problem is called the SYMMETRIC RING ROUTING AND WAVELENGTH ASSIGNMENT PROBLEM (SRRWA). A demand is called a permutation demand if, for each vertex v of G , the number of requests with source v and the number of requests with destination v are the same and are equal to 0 or 1. A symmetric permutation demand is called an involution demand. In this paper we prove that both RRWA and SRRWA are \mathbb{NP} -complete even when restricted to involution demands. As a consequence RRWA is \mathbb{NP} -complete when restricted to permutation demands. For general demands we prove that RRWA and SRRWA can be solved in polynomial time when the number of wavelengths is fixed. Finally, we answer in the negative an open problem posed by Gargano and Vaccaro and construct infinitely many counterexamples using involution demands.

© 2006 Elsevier B.V. All rights reserved.

Keywords: Routing; Ring; Wavelength assignment; Permutation; Involution

1. Introduction

The ROUTING AND WAVELENGTH ASSIGNMENT PROBLEM has been receiving extensive attention due to its importance in optical networking and telecommunication. In this paper we study this problem for optical ring networks and answer an open question.

* Corresponding author.

E-mail address: smzhou@ms.unimelb.edu.au (S. Zhou).

¹ Supported by the National Natural Science Foundation of China under grant number 10371112.

² Supported by a Discovery Project Grant (DP0558677) from the Australian Research Council and a Melbourne Early Career Researcher Grant from The University of Melbourne.

An optical network can be modelled by an undirected graph $G = (V(G), E(G))$, in which vertices represent processors, routers or memory modules and edges represent optical fibre links. It is usually assumed that each link allows data streams to transmit in both directions. (Physically this is realized by providing two optical fibres.) Thus, we will take G as a *symmetric* directed graph in which there is an *arc* from u to v if and only if there is an arc from v to u , for $u, v \in V(G)$. An ordered pair of distinct vertices of G is called a *communication request* on G , and a set of such requests is called a *demand*. In practice a request (s, t) corresponds to a data stream to be transmitted from s to t , which are called the *source* and *destination* of (s, t) respectively, and a demand corresponds to communications to be realized in one round of routing. Taking each request (s, t) as the arc from s to t , we may identify a demand D on G with a directed graph having the same vertex set $V(G)$ as G . In this directed graph, an isolated vertex is neither a source nor a destination of any request in D , and vice versa. Depending on properties of D , we may have various types of demands. For example, if D is a complete directed graph on $V(G)$ (that is, $D = \{(s, t) : s, t \in V(G), s \neq t\}$), then it is an *all-to-all* demand; if D has no isolated vertex and all arcs of D emanate from the same vertex, then it is a *one-to-all* demand. These two types of demands have been well studied in the literature in the context of routing and wavelength assignment; see e.g. [6,14]. A demand on G is said to be a *permutation demand* [6,10,14,20,22] if each connected component of the corresponding directed graph is a directed cycle (possibly with length 2) or an isolated vertex. In other words, a demand is a permutation demand if and only if, for each $v \in V(G)$, the number of requests with source v and the number of requests with destination v are the same and are equal to 0 or 1. A permutation demand D can be identified with the permutation which permutes u to v for each $(u, v) \in D$ and fixes each of the isolated vertices of D . Thus, permutation demands are in one-to-one correspondence with permutations of vertices of G .

Given a demand D on an optical network G , realization of D consists of the following two steps. First, for each $(s, t) \in D$, we need to design a directed path $R(s, t)$ of G from s to t ; a set

$$\mathcal{R} = \{R(s, t) : (s, t) \in D\}$$

of such directed paths is called a *routing* for D . For each arc a of G , the *load* of a under \mathcal{R} , denoted by $\bar{\pi}(\mathcal{R}, a)$, is the number of directed paths in \mathcal{R} which traverse a in its direction. Let

$$\bar{\pi}(\mathcal{R}) = \max\{\bar{\pi}(\mathcal{R}, a) : a \text{ an arc of } G\}$$

be the maximum load of an arc of G under \mathcal{R} . Define

$$\bar{\pi}(D) = \min\{\bar{\pi}(\mathcal{R}) : \mathcal{R} \text{ a routing of } D\}.$$

In the literature $\bar{\pi}(D)$ is called the *arc-load* [6,9,14,24] of G with respect to D , or the *arc-congestion* [19,23] of (G, D) . In the case where D is all-to-all, $\bar{\pi}(D)$ is known as the *arc-forwarding index* [19] of G .

Using the Wavelength Division Multiplexing (WDM) technology, the high bandwidth of optic fibres is partitioned into channels, each of which supports a different wavelength. In this way multiple data streams can be transmitted concurrently along the same fibre provided that they are assigned different wavelengths. In accordance with this, in the second step of realizing a demand D we need to assign a wavelength to each directed path in \mathcal{R} such that two such paths having an arc in common receive distinct wavelengths. Let $\bar{w}(\mathcal{R})$ be the minimum number of wavelengths needed in such an assignment for \mathcal{R} . Since wavelengths are limited and costly resources, making effective use of them is an important issue in optical networking. Thus, we define [14]

$$\bar{w}(D) = \min\{\bar{w}(\mathcal{R}) : \mathcal{R} \text{ a routing for } D\}$$

and call it the *wavelength number* of D . Given (G, D) , the ROUTING AND WAVELENGTH ASSIGNMENT PROBLEM (RWA) seeks a routing \mathcal{R} for D and an assignment of wavelengths to the directed paths in \mathcal{R} such that $\bar{w}(\mathcal{R}) = \bar{w}(D)$, that is, the number of wavelengths used by \mathcal{R} is minimized.

The arc-congestion $\bar{\pi}(D)$ provides a natural lower bound on the wavelength number $\bar{w}(D)$. In fact, since $\bar{w}(\mathcal{R}) \geq \bar{\pi}(\mathcal{R}, a)$ for each arc a of G , we have $\bar{w}(\mathcal{R}) \geq \bar{\pi}(\mathcal{R})$. Hence, for any demand D on G ,

$$\bar{w}(D) \geq \bar{\pi}(D).$$

Taking wavelengths as colours, we may interpret wavelength assignments as proper (vertex) colourings of the *conflict graph* $G(\mathcal{R})$, which is defined [14] to have vertices the directed paths in \mathcal{R} such that two such paths are adjacent if

and only if they have at least one common arc. With this interpretation $\vec{w}(\mathcal{R})$ is [14] exactly the chromatic number of $G(\mathcal{R})$. Henceforth proper colourings of $G(\mathcal{R})$ will also be called *proper colourings* of \mathcal{R} .

The model described above is the (off-line) *directed model* for the routing and wavelength assignment problem. Note that all the definitions above have natural undirected analogues and thus we have the (off-line) *undirected model* for the problem. In this latter model, a network G is taken as an undirected graph, requests are routed along undirected paths, and $\pi(D)$ and $w(D)$ can be defined in a similar manner as above. Thus, for a (undirected) routing \mathcal{P} for D , $\pi(\mathcal{P})$ is the maximum number of paths in \mathcal{P} containing an edge of G , and two paths in \mathcal{P} having a common edge require different wavelengths, etc. As pointed out in [5] the directed model is more realistic considering the current technology [15].

There is a comprehensive literature on RWA and the related loading problem for both directed and undirected models; see e.g. [1,2,5,6,14,19,21,23–25]. Researchers have so far produced a large number of results for various types of networks and demands. Due to the importance of the ring structure in optical networking (e.g. SONET rings are among the standard configurations for optical networks), the case where $G = C_n$ (cycle with n vertices) is a ring has received special attention [2,5–7,9,23,24,26]. In this case RWA is called the RING ROUTING AND WAVELENGTH ASSIGNMENT PROBLEM (RRWA). Formally its recognition version can be stated as follows.

Ring Routing and Wavelength Assignment Problem (RRWA) Given a ring C_n , a demand D on C_n and a positive integer K , is there a routing \mathcal{R} for D such that $\vec{w}(\mathcal{R}) \leq K$?

In this paper we will focus on RRWA and the related arc-congestion $\vec{\pi}$. Before moving on to the main results of the paper, let us mention briefly some relevant results. For general demands on a ring it was proved in [12,24] respectively that the recognition problems for π and $\vec{\pi}$ can be solved in polynomial time. In the case where each request is associated with a non-negative integer weight, the “weighted” edge/arc-congestion can be defined by counting the total weight that an edge/arc carries. It was shown in [5,9] respectively that the weighted arc- and edge-congestion problems are \mathbb{NP} -complete for rings. For the same problems, polynomial time approximation schemes (PTAS) were found in [5,16] respectively, and polynomial time exact algorithms were given in [18,26] respectively under the assumption that the weights can be split and routed in two directions along the ring. Also, a fast approximation algorithm for the weighted edge-congestion was given in [21]. The reader is referred to [19,23] for upper bounds on π , $\vec{\pi}$, w , \vec{w} when the network is a ring or a hamiltonian decomposable graph.

For general demands on a ring, by using reduction from the well-known ARC COLOURING PROBLEM [13] it was proved independently in [11] and [24] that RRWA is \mathbb{NP} -complete and thus is computationally intractable. In [11] it was also proved that the undirected version of RRWA is \mathbb{NP} -complete, and in [24] a 2-approximation algorithm was given for the directed version. In view of these results it is natural to investigate whether RRWA is tractable when restricted to some special demands. As building blocks for arbitrary demands [6,14], permutation demands are arguably the most important special demands. In [19] it was mentioned without proof that RRWA is \mathbb{NP} -complete for permutations.

2. Main results

In this paper we will prove that RRWA is \mathbb{NP} -complete even when restricted to *involution demands*, which are defined to be permutation demands such that each component of the corresponding directed graphs is a directed cycle of length 2 or an isolated vertex. As a consequence we obtain immediately the \mathbb{NP} -completeness of RRWA for permutation demands [19]. Practically in an involution demand the two vertices in the same 2-cycle need to exchange information. Taken as a permutation, an involution demand is a product of disjoint transpositions, and hence is an involution of the group of permutations on the vertex set of the network. Recently involution demands on trees were studied in [8], where among other results it was proved that RWA is \mathbb{NP} -complete even when G is a tree and D is a very restrictive demand such as permutation, involution, etc. Independently, we came to the same notion of involution demands in our study of RRWA and an open problem posed in [14] (see Question 2.6 below). The first main result of this paper is the following theorem.

Theorem 2.1. *The RING ROUTING AND WAVELENGTH ASSIGNMENT PROBLEM is \mathbb{NP} -complete even when restricted to involution demands.*

To us this is unexpected because involution demands are so restrictive. Since involutions are permutations, [Theorem 2.1](#) implies the following corollary, which is stated in [\[19\]](#) without proof.

Corollary 2.2. *The RING ROUTING AND WAVELENGTH ASSIGNMENT PROBLEM is \mathbb{NP} -complete for permutation demands.*

As in the case of general demands, to prove [Theorem 2.1](#) we will make use of the \mathbb{NP} -completeness [\[13\]](#) of the ARC COLOURING PROBLEM (ACP). However, it seems that involution demands are much more complicated to deal with and that a direct reduction from ACP is unlikely to work. Instead we need to understand first the complexity of ACP for some very special instances, and this will be detailed in the next section.

A demand D is called *symmetric* if $(s, t) \in D$ implies $(t, s) \in D$. In view of this definition, involution demands can be characterized as permutation demands which are symmetric. For a symmetric demand D , a routing $\mathcal{R} = \{R(s, t) : (s, t) \in D\}$ for D is said to be *symmetric* if $R(t, s)$ is the reverse of $R(s, t)$. (The *reverse* of a directed path $v_1 v_2 \dots v_k$ is the directed path $v_k v_{k-1} \dots v_1$.) Given a symmetric demand D on a network G , the SYMMETRIC ROUTING AND WAVELENGTH ASSIGNMENT PROBLEM (SRWA) [\[8\]](#) asks for a symmetric routing \mathcal{R} and an assignment of wavelengths to the directed paths in \mathcal{R} such that the number of wavelengths used is minimized, subject to that any two directed paths in \mathcal{R} with at least one arc in common receive distinct wavelengths. In the case where $G = C_n$ is a ring the recognition version of this problem is as follows.

Symmetric Ring Routing and Wavelength Assignment Problem (SRRWA) Given a ring C_n , a symmetric demand D on C_n and a positive integer K , is there a symmetric routing \mathcal{R} for D such that $\vec{w}(\mathcal{R}) \leq K$?

Using essentially nothing more than what we need in the proof of [Theorem 2.1](#) we will prove the following result.

Theorem 2.3. *The SYMMETRIC RING ROUTING AND WAVELENGTH ASSIGNMENT PROBLEM is \mathbb{NP} -complete even when restricted to involution demands.*

Thus, both RRWA and SRRWA are computationally intractable even for involution demands. On the positive side, by exploiting a classic result [\[13\]](#) about the ARC COLOURING PROBLEM we will prove that, theoretically, both of them can be solved in polynomial time for general demands when the number of available wavelengths is fixed. These results, stated in the next two theorems, are in contrast with the ordinary graph colouring problem, which is intractable even when the number of colours is fixed. From a practical point of view it is reasonable to assume that the number of wavelengths available is fixed and limited, because technological difficulties prevent partitioning the bandwidth of optic fibres into a large number of channels.

Theorem 2.4. *For any fixed number K of wavelengths, the RING ROUTING AND WAVELENGTH ASSIGNMENT PROBLEM can be solved in $O(|D|^{4K} K! K \log K)$ time for any demand D .*

Theorem 2.5. *For any fixed number K of wavelengths, the SYMMETRIC RING ROUTING AND WAVELENGTH ASSIGNMENT PROBLEM can be solved in $O((|D|/2)^{2K} K! K \log K)$ time for any symmetric demand D .*

As mentioned earlier, involution demands arise also from our study of the following open problem [\[14, Problem 1\]](#).

Question 2.6. Is it true that, for any demand D on a graph G , there exists a routing \mathcal{R} for D such that

$$\vec{\pi}(\mathcal{R}) = \vec{\pi}(D), \quad \vec{w}(\mathcal{R}) = \vec{w}(D)$$

hold simultaneously?

The third main result of this paper settles this problem in the negative. As stated in [Theorem 2.7](#) below, we will prove that even when restricted to the case where G is a ring and D is an involution the answer to this question is negative. In contrast the gap between $\vec{\pi}(\mathcal{R})$ and $\vec{\pi}(D)$ can be arbitrarily large for those \mathcal{R} with $\vec{w}(\mathcal{R}) = \vec{w}(D)$.

Theorem 2.7. *For any positive integer K , there exists a ring C_n and an involution demand D on it such that $\vec{\pi}(\mathcal{R}) - \vec{\pi}(D) \geq K$ for any routing \mathcal{R} for D with $\vec{w}(\mathcal{R}) = \vec{w}(D)$. Moreover, we can explicitly construct infinitely many examples of (C_n, D) with these properties.*

The undirected version (π versus w) of Question 2.6 was posed in [3,4] and answered in the negative by Margara [17], who constructed one specific counterexample. Here for the directed case we find infinitely many counterexamples and moreover we show that $\vec{\pi}(\mathcal{R}) - \vec{\pi}(D)$ can be arbitrarily large.

The rest of this paper is organized as follows. In the next section we will prove that the well-known ARC COLOURING PROBLEM [13] is NP-complete even when restricted to some very special instances. Based on this we will prove Theorems 2.1 and 2.3 in Section 4. In Section 5 we will prove Theorems 2.4 and 2.5, and in Section 6 we will prove Theorem 2.7. The paper concludes with remarks and open problems.

3. Preliminaries

Throughout the paper we will use C_n to denote the ring (cycle) with n vertices. We assume that the vertices of C_n are v_0, v_1, \dots, v_{n-1} , and they are labelled clockwise, so that the edges of C_n are $e_i = v_i v_{i+1}$ and the arcs of C_n are $(v_i, v_{i+1}), (v_{i+1}, v_i), 0 \leq i \leq n - 1$. Here and in the remainder of the paper subscripts are taken modulo n . Whenever we say a path of C_n without the prefix “directed”, we mean an undirected path of C_n with length at least one. (In the literature [13] such paths are also called arcs of C_n . Since we have reserved the term “arcs” for ordered pairs of adjacent vertices, we will use the word “paths” instead in order to avoid possible confusion.) For a directed or undirected path P , we use $E(P)$ to denote the set of arcs of P (when P is directed) or the set of edges of P (when P is undirected). In either case the length of P is given by $\ell(P) = |E(P)|$.

Let \mathcal{P} be a family of paths of C_n , in which repeated paths are allowed. We may take \mathcal{P} as an undirected routing, and hence $\pi(\mathcal{P}, e_i)$ and $\pi(\mathcal{P})$ are well-defined. More explicitly, $\pi(\mathcal{P}, e_i)$ is the load of the edge e_i under \mathcal{P} (the number of paths of \mathcal{P} containing e_i) and $\pi(\mathcal{P})$ is the maximum load of an edge of C_n . The family \mathcal{P} is said to be uniform if all edges of C_n have the same load under \mathcal{P} , that is, $\pi(\mathcal{P}, e_i) = \pi(\mathcal{P})$ for $i = 0, 1, \dots, n - 1$. Similar to the directed case, the conflict graph $G(\mathcal{P})$ is the graph with vertices the paths in \mathcal{P} such that two paths are adjacent if and only if they have at least one common edge. Also, proper colourings of $G(\mathcal{P})$ are called proper colourings of \mathcal{P} , and they correspond to wavelength assignments to the paths in \mathcal{P} without conflict. Clearly, for a proper K -colouring of \mathcal{P} to exist we must have $K \geq \pi(\mathcal{P})$. The following problem and its NP-completeness are well-known in the literature [13].

Arc Colouring Problem (ACP) Given a family \mathcal{P} of paths of C_n and a positive integer $K \geq \pi(\mathcal{P})$, is there a proper K -colouring of \mathcal{P} ?

Theorem 3.1. [13] *The ARC COLOURING PROBLEM is NP-complete.*

In order to prove Theorems 2.1 and 2.3, we will strengthen Theorem 3.1 by showing that ACP is NP-complete even when restricted to certain very special instances; see Theorem 3.3 below. To prove this we will need the following result about the complexity of ACP for uniform path families.

Theorem 3.2. *The ARC COLOURING PROBLEM is NP-complete even when restricted to those instances (C_n, \mathcal{P}, K) such that \mathcal{P} is uniform, $n \geq 4K$, and $K = \pi(\mathcal{P})$.*

Proof. Suppose that we are given an instance of ACP, that is, a family \mathcal{P} of paths of C_n together with a positive integer $K \geq \pi(\mathcal{P})$. If $n < 4K$, then we replace the edge $e_{n-1} = v_{n-1}v_0$ of C_n by a path P of length $4K$, and replace e_{n-1} by P in each path of \mathcal{P} containing e_{n-1} . In this way we obtain a new instance $(C_{n'}, \mathcal{P}_1, K)$ such that $n' \geq 4K$, $\pi(\mathcal{P}_1, e_i) = \pi(\mathcal{P}, e_i)$ for $i = 0, 1, \dots, n - 2$, and the load under \mathcal{P}_1 of each edge of P is equal to $\pi(\mathcal{P}, e_{n-1})$. If $n \geq 4K$ then we merely set $C_{n'} = C_n$ and $\mathcal{P}_1 = \mathcal{P}$. In both cases, for each edge e of $C_{n'}$ we add $K - \pi(\mathcal{P}_1, e)$ copies of e (as paths of $C_{n'}$) to \mathcal{P}_1 , and denote the resultant path family by \mathcal{P}' . Then $(C_{n'}, \mathcal{P}', K)$ is such that \mathcal{P}' is uniform, $n' \geq 4K$ and $K = \pi(\mathcal{P}')$, and the transformation from (C_n, \mathcal{P}, K) to $(C_{n'}, \mathcal{P}', K)$ can be achieved in polynomial time.

If \mathcal{P} is K -colourable, then \mathcal{P}_1 is K -colourable, and moreover a proper K -colouring c of \mathcal{P} gives rise to a proper K -colouring c_1 of \mathcal{P}_1 . For each edge e of $C_{n'}$, the $\pi(\mathcal{P}_1, e)$ paths of \mathcal{P}_1 containing e use exactly $\pi(\mathcal{P}_1, e)$ colours

under c_1 . Thus, the $K - \pi(\mathcal{P}_1, e)$ copies of e can be coloured by the remaining $K - \pi(\mathcal{P}_1, e)$ colours. In this way we obtain a proper K -colouring of \mathcal{P}' , and hence \mathcal{P}' is K -colourable. Conversely, if \mathcal{P}' admits a proper K -colouring c' , then the restriction of c' to \mathcal{P}_1 is a proper K -colouring of \mathcal{P}_1 . This K -colouring of \mathcal{P}_1 then gives rise to a proper K -colouring of \mathcal{P} . Since by [Theorem 3.1](#) ACP is \mathbb{NP} -complete, [Theorem 3.2](#) follows immediately. \square

A path P of C_n is called a *short path* if the length of P is strictly less than $n/2$. A family \mathcal{P} of paths is said to be *short* if every member of \mathcal{P} is a short path of C_n . In the case where \mathcal{P} is both uniform and short we say that \mathcal{P} is a *uniform short path family*.

Uniform Short Arc Colouring Problem (USACP) Given a uniform short path family \mathcal{P} of C_n and a positive integer $K \geq \pi(\mathcal{P})$, is there a proper K -colouring of \mathcal{P} ?

Theorem 3.3. *The UNIFORM SHORT ARC COLOURING PROBLEM is \mathbb{NP} -complete even when restricted to those instances (C_n, \mathcal{P}, K) such that $n \geq 4K^2 + 1$, $K = \pi(\mathcal{P})$, each path in \mathcal{P} is of length at most $2K$, no two paths in \mathcal{P} are identical, and each vertex of C_n is an end-vertex of at least one path in \mathcal{P} .*

Proof. It is not difficult to see that USACP is in the class \mathbb{NP} . Let (C_n, \mathcal{P}, K) be an instance of ACP such that \mathcal{P} is uniform, $n \geq 4K$, and $K = \pi(\mathcal{P})$. By [Theorem 3.2](#) it suffices to construct in polynomial time an instance $(C_{n'}, \mathcal{P}', K)$ of USACP which satisfies the conditions in [Theorem 3.3](#) such that \mathcal{P} is K -colourable if and only if \mathcal{P}' is K -colourable.

Since \mathcal{P} is uniform, for each $i = 0, 1, \dots, n - 1$, there are exactly $K = \pi(\mathcal{P})$ paths in \mathcal{P} which contain e_i , that is, $|\{P \in \mathcal{P} : e_i \in E(P)\}| = K$. Let

$$\varphi_i : \{P \in \mathcal{P} : e_i \in E(P)\} \rightarrow \{1, 2, \dots, K\} \tag{1}$$

be an arbitrary but fixed bijection. In the sequel we may identify φ_i with a linear order on the paths of \mathcal{P} containing e_i : if $\varphi_i(P) = j$ then P is the j th path containing e_i .

Let $n_1 = (K + 1)n$. Let C_{n_1} be the cycle obtained from C_n by replacing each edge $e_i = v_i v_{i+1}$ ($0 \leq i \leq n - 1$) of C_n by a path

$$x(i, 0)x(i, 1)x(i, 2) \dots x(i, K)x(i, K + 1)$$

of length $K + 1$, where $x(i, 0) = v_i$ and $x(i, K + 1) = x(i + 1, 0) = v_{i+1}$ for each i . For each path P in \mathcal{P} , say P is the path of length $\ell = \ell(P)$ from v_i to $v_{i+\ell}$ clockwise around C_n , we define $\ell + 1$ paths P_0, P_1, \dots, P_ℓ of C_{n_1} such that

- (i) P_0 is the path from v_i to $x(i, \varphi_i(P))$ clockwise around C_{n_1} ;
- (ii) P_k is the path from $x(i + k - 1, \varphi_{i+k-1}(P))$ to $x(i + k, \varphi_{i+k}(P))$ clockwise around C_{n_1} , $1 \leq k \leq \ell - 1$; and
- (iii) P_ℓ is the path from $x(i + \ell - 1, \varphi_{i+\ell-1}(P))$ to $v_{i+\ell}$ clockwise around C_{n_1} .

In other words, we subdivide the path P of C_{n_1} from v_i to $v_{i+\ell}$ clockwise by the vertices $x(i + k, \varphi_{i+k}(P))$, $0 \leq k \leq \ell - 1$, and thus obtain $\ell + 1$ new paths P_0, P_1, \dots, P_ℓ . These new paths have length at most $2K$, and hence are short paths of C_{n_1} since $2K < n_1/2$ by our assumption $n \geq 4K$. Define \mathcal{P}_1 to be the family of such new paths with P running over all paths in \mathcal{P} , that is,

$$\mathcal{P}_1 = \bigcup_{P \in \mathcal{P}} \{P_i : 0 \leq i \leq \ell(P)\}.$$

Since \mathcal{P} is uniform and $K = \pi(\mathcal{P})$, \mathcal{P}_1 is a uniform short path family of C_{n_1} with $K = \pi(\mathcal{P}_1)$.

From the construction of \mathcal{P}_1 , each vertex of C_{n_1} other than v_i , $0 \leq i \leq n - 1$, is an end-vertex of at least one path in \mathcal{P}_1 . Moreover, v_i is not an end-vertex of any path in \mathcal{P}_1 if and only if v_i is not an end-vertex of any path in \mathcal{P} . Let

$$V = \{v_i : 0 \leq i \leq n - 1, v_i \text{ is not an end-vertex of any path in } \mathcal{P}_1\}.$$

Then $|V| \leq n - 2$. Define $C_{n'}$ to be the cycle obtained from C_{n_1} by skipping the vertices in V , that is,

$$C_{n'} = C_{n_1} - V + \{x(i - 1, K)x(i, 1) : v_i \in V\}$$

where $n' = n_1 - |V| = (K + 1)n - |V|$. Note that $n' \geq (K + 1)n - (n - 2) \geq 4K^2 + 2$ as $n \geq 4K$ by our assumption. Define \mathcal{P}' to be the family of paths of $C_{n'}$ obtained from the paths in \mathcal{P}_1 by skipping the vertices in V . (Thus, the path $x(i - 1, K)x(i, 0)x(i, 1)$ is replaced by the edge $x(i - 1, K)x(i, 1)$ in each of the paths of \mathcal{P}_1 passing through $v_i = x(i, 0) \in V$.) Since \mathcal{P}_1 is uniform and $K = \pi(\mathcal{P}_1)$, \mathcal{P}' is uniform with $K = \pi(\mathcal{P}')$. Also, each path in \mathcal{P}' is of length at most $2K$, no two paths in \mathcal{P}' are identical, and each vertex of $C_{n'}$ is an end-vertex of at least one path in \mathcal{P}' . Thus, $(C_{n'}, \mathcal{P}', K)$ is an instance of USACP which satisfies all the conditions in [Theorem 3.3](#). Clearly, the construction of this instance from (C_n, \mathcal{P}, K) is accomplished in polynomial time. From the construction one can check that two paths in \mathcal{P}' have a common edge if and only if their corresponding paths in \mathcal{P}_1 have a common edge. Therefore, we have

Claim 1. \mathcal{P}' admits a proper K -colouring if and only if \mathcal{P}_1 admits a proper K -colouring.

From this it remains to show that \mathcal{P} admits a proper K -colouring if and only if \mathcal{P}_1 admits a proper K -colouring. The “only if” part of this statement is trivial: for any proper K -colouring c of \mathcal{P} , the colouring of \mathcal{P}_1 such that all P_i , $0 \leq i \leq \ell(P)$, receive the same colour as P under c is a proper K -colouring of \mathcal{P}_1 . In the following we will show the “if” part of this statement.

Suppose that \mathcal{P}_1 admits a proper K -colouring c_1 . For $0 \leq i \leq n - 1$ and $1 \leq q \leq K$, since φ_i (defined in (1)) is bijective, the vertex $x(i, q)$ is the common end-vertex of exactly two paths in \mathcal{P}_1 ; and these two paths are P_k, P_{k+1} for some $P \in \mathcal{P}$ and integer k with $0 \leq k \leq \ell(P) - 1$, $v_i \in V(P_k)$ and $v_{i+1} \in V(P_{k+1})$. We claim that

Claim 2. The two paths in \mathcal{P}_1 with $x(i, q)$ as their common end-vertex must receive the same colour under c_1 .

To prove this let us consider the K paths in \mathcal{P} containing the edge $e_i = v_i v_{i+1}$. Let P^1, P^2, \dots, P^K be the linear order on $\{P \in \mathcal{P} : e_i \in E(P)\}$ induced by φ_i . That is, $\varphi_i(P^q) = q$ for $1 \leq q \leq K$. Assume that the two paths in \mathcal{P}_1 with $x(i, q)$ as their common end-vertex are $P_{k(q)}^q$ and $P_{k(q)+1}^q$ for some $k(q)$ with $0 \leq k(q) \leq \ell(P^q) - 1$. Note that the path from v_i to $x(i, q)$ clockwise around $C_{n'}$ is contained in the path $P_{k(q)}^q$, and that the path from $x(i, q)$ to v_{i+1} clockwise around $C_{n'}$ is contained in the path $P_{k(q)+1}^q$. Thus, the subgraph H of the conflict graph $G(\mathcal{P}_1)$ of \mathcal{P}_1 induced by

$$\{P_{k(q)}^q, P_{k(q)+1}^q : 1 \leq q \leq K\}$$

has the following property: for $0 \leq p \leq K$, every path in

$$\mathcal{P}^{(p)} := \{P_{k(q)+1}^q : 1 \leq q \leq p\} \cup \{P_{k(q)}^q : p + 1 \leq q \leq K\}$$

contains the edge $x(i, p)x(i, p + 1)$. Hence $\mathcal{P}^{(p)}$ is a K -clique of H for each p with $0 \leq p \leq K$. This implies that, for $1 \leq p \leq K$, under the colouring c_1 of \mathcal{P}_1 the set of colours used by $\{P_{k(q)+1}^q : 1 \leq q \leq p\}$ is identical to the set of colours used by $\{P_{k(q)}^q : 1 \leq q \leq p\}$. Consequently, for each q with $1 \leq q \leq K$, $P_{k(q)}^q$ and $P_{k(q)+1}^q$ receive the same colour under c_1 , and [Claim 2](#) is proved.

From [Claim 2](#) it follows that, for any $P \in \mathcal{P}$, under c_1 the paths $P_0, P_2, \dots, P_{\ell(P)}$ of \mathcal{P}_1 must receive the same colour, which can be taken as a colour for P . Thus, c_1 induces a proper K -colouring of \mathcal{P} , and the proof of [Theorem 3.3](#) is complete. \square

4. Proof of Theorems 2.1 and 2.3

Equipped with [Theorem 3.3](#) we are now ready to prove [Theorems 2.1 and 2.3](#). Recall that we label the vertices of the cycle C_n by v_0, v_1, \dots, v_{n-1} clockwise around the cycle. For a given request (s, t) on C_n , denote by $P^+(s, t)$ the directed (s, t) -path clockwise around C_n and $P^-(s, t)$ the directed (s, t) -path anticlockwise around C_n .

Proof of Theorem 2.1. Let (C_n, \mathcal{P}, K) be an instance of USACP satisfying the conditions of [Theorem 3.3](#). That is, $n \geq 4K^2 + 1$, $K = \pi(\mathcal{P})$, each path in \mathcal{P} is of length at most $2K$, no two paths in \mathcal{P} are identical, and each vertex of C_n is an end-vertex of at least one path in \mathcal{P} . From [Theorem 3.3](#) it suffices to construct in polynomial time an instance

$(C_{n'}, D, K)$ of RRWA such that \mathcal{P} admits a proper K -colouring if and only if there exists a routing \mathcal{R} for D which admits a proper K -colouring (that is, $\bar{w}(\mathcal{R}) \leq K$).

For $0 \leq i \leq n - 1$, denote

$$\mathcal{P}^{(i)} = \{P \in \mathcal{P}: v_i \text{ is an end-vertex of } P\}$$

and

$$p_i = |\mathcal{P}^{(i)}|.$$

Then $1 \leq p_i \leq 2K$ by our assumptions on (C_n, \mathcal{P}, K) . For each i , let

$$\varphi_i: \mathcal{P}^{(i)} \rightarrow \{1, 2, \dots, p_i\} \quad (2)$$

be an arbitrary but fixed bijection such that $\varphi_i(P_1) < \varphi_i(P_2)$ whenever $v_{i-1}v_i \in E(P_1)$ and $v_i v_{i+1} \in E(P_2)$ for $P_1, P_2 \in \mathcal{P}^{(i)}$.

Let $n' = \sum_{0 \leq i \leq n-1} p_i$. Then $n' \geq n \geq 4K^2 + 1$. Let $C_{n'}$ be the cycle obtained from C_n by replacing each edge $e_i = v_i v_{i+1}$ ($0 \leq i \leq n - 1$) of C_n with a path

$$P_i = x(i, 1)x(i, 2) \dots x(i, p_i)x(i, p_i + 1)$$

of length p_i , where $x(i, 1) = v_i$ and $x(i, p_i + 1) = x(i + 1, 1) = v_{i+1}$ for each i . Thus, we have

$$C_{n'} = x(0, 1) \dots x(0, p_0)x(1, 1) \dots x(1, p_1) \dots x(n - 1, 1) \dots x(n - 1, p_{n-1})x(0, 1).$$

For each path $P \in \mathcal{P}$, say, P is the path from v_i to v_j clockwise around C_n , define $f(P)$ to be the path of $C_{n'}$ from $x(i, \varphi_i(P))$ to $x(j, \varphi_j(P))$ clockwise around $C_{n'}$. Set

$$\mathcal{P}' = \{f(P): P \in \mathcal{P}\}$$

so that f is a bijection from \mathcal{P} to \mathcal{P}' . Let

$$B = \{x(i, p_i)x(i + 1, 1) \in E(C_{n'}): 0 \leq i \leq n - 1\}.$$

Then \mathcal{P}' has the following properties:

- (i) each vertex of $C_{n'}$ is an end-vertex of exactly one path in \mathcal{P}' ;
- (ii) each path in \mathcal{P}' is of length at most $2K^2$ and contains at most $2K$ edges in B ;
- (iii) $\pi(\mathcal{P}', e) = K$ for each edge e in B ;
- (iv) two paths $P_1, P_2 \in \mathcal{P}$ have a common edge in C_n if and only if the two corresponding paths $f(P_1), f(P_2) \in \mathcal{P}'$ have a common edge in $C_{n'}$.

From (iv) it follows that

Claim 3. \mathcal{P} has a proper K -colouring if and only if \mathcal{P}' has a proper K -colouring.

If s and t are the end-vertices of a path in \mathcal{P}' , then by (i) they are the end-vertices of a unique path, denoted by $P(s, t)$, in \mathcal{P}' . Hence

$$D = \{(s, t): s \text{ and } t \text{ are the two end-vertices of a path in } \mathcal{P}'\}$$

defines an involution demand on $C_{n'}$, and $(C_{n'}, D, K)$ is an instance of RRWA. Clearly, the construction of $(C_{n'}, D, K)$ from (C_n, \mathcal{P}, K) can be achieved in polynomial time. Thus, to prove [Theorem 2.1](#), it remains to show that

(\star) \mathcal{P} has a proper K -colouring if and only if there is a routing \mathcal{R} for D such that \mathcal{R} has a proper K -colouring.

For $(s, t) \in D$, since $n' \geq 4K^2 + 1$ and because of (ii), one of the two directed (s, t) -paths of $C_{n'}$ has length at most $2K^2$ (shorter way), and the other one has length at least $2K^2 + 1$ (longer way). Let

$$B^* = \{(x(i, p_i), x(i + 1, 1)), (x(i + 1, 1), x(i, p_i)): 0 \leq i \leq n - 1\}.$$

Then $B^* = \{(v, u), (u, v) : uv \in B\}$ and $|B^*| = 2|B| = 2n$. From (ii) we have

Claim 4. Let \mathcal{R} be a routing for D on $C_{n'}$, and let $(s, t) \in D$. If (s, t) is routed along the shorter way, then $|E(R(s, t)) \cap B^*| = |E(P(s, t)) \cap B| \leq 2K$; otherwise $|E(R(s, t)) \cap B^*| = n - |E(P(s, t)) \cap B| \geq 2K + 1$.

Let us prove the “ \Rightarrow ” part of (\star) first. If \mathcal{P} has a K -colouring, then \mathcal{P}' has a K -colouring c . Define $\mathcal{R} = \{R(s, t) : (s, t) \in D\}$ by

$$R(s, t) = \begin{cases} P^+(s, t), & |E(P^+(s, t)) \cap B^*| \leq 2K \\ P^-(s, t), & \text{otherwise.} \end{cases}$$

Then under the routing \mathcal{R} each $(s, t) \in D$ is routed along the shorter way, that is, $R(s, t)$ is an orientation of $P(s, t)$. Therefore, a proper K -colouring of the routing \mathcal{R} can be obtained by assigning $R(s, t)$ the colour of $P(s, t) \in \mathcal{P}'$ under the colouring c .

To prove the “ \Leftarrow ” part of (\star) we assume that there is a routing \mathcal{R} for the demand D such that \mathcal{R} has a proper K -colouring c' . First, we show that $|E(R(s, t)) \cap B^*| \leq 2K$ for all $(s, t) \in D$. Suppose to the contrary that $|E(R(s_0, t_0)) \cap B^*| \geq 2K + 1$ for some $(s_0, t_0) \in D$. Then

$$|E(R(s_0, t_0)) \cap B^*| \geq |E(P(s_0, t_0)) \cap B| + 1.$$

Since $|E(R(s, t)) \cap B^*| \geq |E(P(s, t)) \cap B|$ for all $(s, t) \in D$, we have

$$\sum_{(s,t) \in D} |E(R(s, t)) \cap B^*| \geq 2 \sum_{P \in \mathcal{P}} |E(P) \cap B| + 1.$$

Since $\sum_{P \in \mathcal{P}} |E(P) \cap B| = nK$ by (iii), we have

$$\sum_{(s,t) \in D} |E(R(s, t)) \cap B^*| \geq 2nK + 1. \tag{3}$$

From this it follows that $\bar{\pi}(\mathcal{R}) \geq K + 1$, and thus \mathcal{R} does not admit any proper K -colouring, a contradiction. Therefore, we have proved $|E(R(s, t)) \cap B^*| = |E(P(s, t)) \cap B| \leq 2K$ for all $(s, t) \in D$. Consequently, (s, t) is routed along the shorter way; in other words, $R(s, t)$ is an orientation of $P(s, t)$. (This also implies that \mathcal{R} is a symmetric routing for D .) Now one can see that under \mathcal{R} only one of (s, t) and (t, s) is routed clockwise around the ring. The colouring for the clockwise directed paths in \mathcal{R} corresponds exactly to a proper K -colouring of \mathcal{P}' . Thus, by Claim 3, \mathcal{P} admits a proper K -colouring. From Theorem 3.3 it then follows that RRWA is \mathbb{NP} -complete for involution demands, and the proof of Theorem 2.1 is complete. \square

Proof of Theorem 2.3. Note that $(C_{n'}, D, K)$ constructed in the proof above is also an instance of SRRWA. In the proof of the “ \Rightarrow ” part of (\star) , $R(s, t)$ and $R(t, s)$ are reverse of each other. Hence $\mathcal{R} = \{R(s, t) : (s, t) \in D\}$ is a symmetric routing for D , and a proper K -colouring of \mathcal{R} can be obtained by assigning $R(s, t)$ and $R(t, s)$ the colour of $P(s, t) \in \mathcal{P}'$ under the colouring c . In proving the “ \Leftarrow ” part of (\star) , we have shown that the routing \mathcal{R} must be symmetric; see the paragraph following (3). Thus, the proof of Theorem 2.1 also serves as a proof of Theorem 2.3. \square

From the proof of Theorem 2.1 we also obtain the following by-product.

Theorem 4.1. The ARC COLOURING PROBLEM is \mathbb{NP} -complete even when restricted to those instances (C_n, \mathcal{P}, K) such that $K = \pi(\mathcal{P})$, each path of \mathcal{P} is short, and each vertex of C_n is an end-vertex of exactly one path in \mathcal{P} .

5. Proof of Theorems 2.4 and 2.5

In this section a demand is allowed to have multiple requests. The following known result is essential for our proof of Theorems 2.4 and 2.5.

Lemma 5.1. [13] For any fixed number K of colours, the ARC COLOURING PROBLEM can be solved in $O(|\mathcal{P}|K!K \log K)$ time.

We will also need the following simple observation.

Lemma 5.2. Suppose that D is a symmetric demand on C_n and \mathcal{R} is a symmetric routing for D . For $(s, t), (t, s) \in D$, let $P(s, t)$ be the path on C_n corresponding to $R(s, t)$ and $R(t, s)$. Let

$$\mathcal{P} = \{P(s, t): (s, t), (t, s) \in D\}.$$

Then, for any positive integer K , \mathcal{R} has a proper K -colouring if and only if \mathcal{P} has a proper K -colouring. Furthermore, a proper K -colouring of \mathcal{P} can be transformed in linear time to a proper K -colouring of \mathcal{R} .

Proof of Theorem 2.5. Let D be a symmetric demand on C_n . Without loss of generality we may assume that n is odd since otherwise we can replace one edge of C_n by a path of length 2. Under this assumption any directed path of C_n is of length less than $n/2$ or greater than $n/2$, and consequently any request is routed along the longer path or the shorter path around the ring C_n . For any routing \mathcal{R} for D , define

$$M(\mathcal{R}) = \left\{ \{s, t\}: (s, t), (t, s) \in D \text{ and } \ell(R(s, t)) > \frac{n}{2} \right\}.$$

Then there are exactly $2|M(\mathcal{R})|$ requests in D which are routed along the longer path around the ring. To guarantee that K colours are sufficient for \mathcal{R} , the load of the routing for these $2|M(\mathcal{R})|$ requests can not exceed K . Thus, we have

$$2|M(\mathcal{R})| \left(\frac{n+1}{2} \right) \leq 2nK$$

and so

$$|M(\mathcal{R})| \leq 2K - 1.$$

Consider all possible symmetric routings \mathcal{R} with $|M(\mathcal{R})| \leq 2K - 1$. There are at most $(|D|/2)^{2K-1}$ different ways of routing symmetrically the longer paths. (Note that $|D|$ is even since D is symmetric.) From this and Lemmas 5.1–5.2, the result of Theorem 2.5 follows. \square

Proof of Theorem 2.4. Similarly, for RRWA, to guarantee that K colours are sufficient, the load of the routing for the requests routed along longer paths can not exceed K . Analogous to the proof above, one can show that the number of requests routed along longer paths can not exceed $4K - 1$. Thus, we have at most $|D|^{4K-1}$ different ways of routing the longer paths. For each of them the clockwise directed paths and the anticlockwise directed paths can be coloured separately. The result of Theorem 2.4 then follows. \square

6. Proof of Theorem 2.7

Proof of Theorem 2.7. Let K be any given positive integer. Let $k = 2K$ and $n = 12k$. Define $D = \bigcup_{i=1}^6 D_i$, where

$$\begin{aligned} D_1 &= \{(v_{i-1}, v_{6k-i}), (v_{6k-i}, v_{i-1}): 1 \leq i \leq k\}, \\ D_2 &= \{(v_{4k+i-1}, v_{9k+i-1}), (v_{9k+i-1}, v_{4k+i-1}): 1 \leq i \leq k\}, \\ D_3 &= \{(v_{8k+i-1}, v_{k+i-1}), (v_{k+i-1}, v_{8k+i-1}): 1 \leq i \leq k\}, \\ D_4 &= \{(v_{2k+i-1}, v_{4k-i}), (v_{4k-i}, v_{2k+i-1}): 1 \leq i \leq k\}, \\ D_5 &= \{(v_{6k+i-1}, v_{8k-i}), (v_{8k-i}, v_{6k+i-1}): 1 \leq i \leq k\}, \\ D_6 &= \{(v_{10k+i-1}, v_{12k-i}), (v_{12k-i}, v_{10k+i-1}): 1 \leq i \leq k\}. \end{aligned}$$

Clearly, $|D| = 12k$, $|D_i| = 2k$ for $i = 1, 2, \dots, 6$, and D is an involution demand on C_n . Since no request of D involves two antipodal vertices of C_n , each request of D must be routed along the shorter path (with length $< n/2$) or longer path (with length $> n/2$) around C_n . Let

$$B = \{(v_{3k-1}, v_{3k}), (v_{3k}, v_{3k-1}), (v_{7k-1}, v_{7k}), (v_{7k}, v_{7k-1}), (v_{11k-1}, v_{11k}), (v_{11k}, v_{11k-1})\}.$$

Claim 5. $\vec{\pi}(D) = 2k$.

In fact, if all the requests in D are routed along shorter paths around the ring, then the maximum load on an arc is $2k$. Thus, we have $\vec{\pi}(D) \leq 2k$. On the other hand, for a routing \mathcal{R} for D with $\vec{\pi}(\mathcal{R}) = \vec{\pi}(D)$, each of the $12k$ directed paths in \mathcal{R} contains at least one arc in B . Thus, under \mathcal{R} the maximum load of an arc in B is at least $12k/6 = 2k$. Therefore, $\vec{\pi}(D) \geq 2k$ and Claim 5 follows.

Claim 6. $\vec{w}(D) \leq \frac{5k}{2}$.

In fact, we can partition D_1 into two subsets $D_1^{(1)}$ and $D_1^{(2)}$, where

$$D_1^{(1)} = \left\{ (v_{i-1}, v_{6k-i}), (v_{6k-i}, v_{i-1}) : 1 \leq i \leq \frac{k}{2} \right\},$$

$$D_1^{(2)} = \left\{ (v_{i-1}, v_{6k-i}), (v_{6k-i}, v_{i-1}) : \frac{k}{2} + 1 \leq i \leq k \right\}.$$

Let \mathcal{R} be the routing for D such that the requests in $D_1^{(1)}$ are routed along the longer path, and the remaining requests in D are routed along the shorter path. Then any directed path in $\mathcal{R}|_{D_1^{(1)}}$ has no common arc with any directed path in $\mathcal{R}|_{D_1^{(2)}}$. (For $F \subseteq D$ we denote $\mathcal{R}|_F = \{R(s, t) \in \mathcal{R} : (s, t) \in F\}$.) It follows that the directed paths in $\mathcal{R}|_{D_1}$ can be coloured using $k/2$ colours. In the conflict graph $G(\mathcal{R})$ of \mathcal{R} , the directed paths in $\mathcal{R}|_{D_2 \cup D_4 \cup D_6}$ induce six cliques each with size k , and so can be coloured by k other colours. Similarly, in $G(\mathcal{R})$ the directed paths in $\mathcal{R}|_{D_3 \cup D_5}$ induce four cliques each with size k , and so can be coloured by k more colours. Altogether we use $5k/2$ colours to colour \mathcal{R} properly, and Claim 6 is proved.

Claim 7. For any routing \mathcal{R} for D with $\vec{w}(\mathcal{R}) = \vec{w}(D)$, we must have $\vec{\pi}(\mathcal{R}) \geq \frac{5k}{2}$.

To prove this let $\mathcal{R} = \{R(s, t) : (s, t) \in D\}$ be a routing for D with $\vec{w}(\mathcal{R}) = \vec{w}(D)$. For $(s, t) \in D$, define

$$\varphi(s, t) = \begin{cases} 0, & \text{if } R(s, t) \text{ is along the shorter path around the ring,} \\ 1, & \text{if } R(s, t) \text{ is along the longer path around the ring,} \end{cases}$$

$$\mu(s, t) = \begin{cases} 0, & \text{if } R(s, t) \text{ is clockwise around the ring,} \\ 1, & \text{if } R(s, t) \text{ is anticlockwise around the ring.} \end{cases}$$

Then $R(s, t)$ contains exactly $\varphi(s, t) + 1$ arcs in B . For $i = 1, \dots, 6$ and $p, q = 0, 1$, set

$$D_i(p, q) = \{(s, t) \in D_i : \varphi(s, t) = p, \mu(s, t) = q\}.$$

Consider the following subsets of D :

$$D_1(0, 0) \cup D_2(0, 0) \cup D_2(1, 0) \cup D_3(0, 0) \cup D_3(1, 0),$$

$$D_1(1, 0) \cup D_2(0, 0) \cup D_2(1, 0) \cup D_3(0, 0) \cup D_3(1, 0),$$

$$D_1(0, 1) \cup D_2(0, 1) \cup D_2(1, 1) \cup D_3(0, 1) \cup D_3(1, 1),$$

$$D_1(1, 1) \cup D_2(0, 1) \cup D_2(1, 1) \cup D_3(0, 1) \cup D_3(1, 1).$$

For each of them the corresponding set of directed paths in \mathcal{R} induces a clique of $G(\mathcal{R})$. Hence by Claim 6 the cardinality of each of these subsets of D is at most $5k/2$. This, together with the fact $|D_2| = |D_3| = 2k$, implies that

$$|D_1(0, 0) \cup D_1(0, 1)| \leq k, \quad |D_1(1, 0) \cup D_1(1, 1)| \leq k.$$

But $2k = |D_1| = |D_1(0, 0) \cup D_1(0, 1)| + |D_1(1, 0) \cup D_1(1, 1)|$, so we have

$$|D_1(0, 0) \cup D_1(0, 1)| = |D_1(1, 0) \cup D_1(1, 1)| = k.$$

Now consider the following 4-element subset of B :

$$B_0 = \{(v_{7k-1}, v_{7k}), (v_{7k}, v_{7k-1}), (v_{11k-1}, v_{11k}), (v_{11k}, v_{11k-1})\}.$$

Each of the k directed paths in $\mathcal{R}|_{D_1(1,0) \cup D_1(1,1)}$ contains at least two arcs in B_0 , and each of the $8k$ directed paths in $\mathcal{R}|_{D_2 \cup D_3 \cup D_5 \cup D_6}$ contains at least one arc in B_0 . Hence the maximum load of an arc in B_0 under \mathcal{R} is at least $(2k + 8k)/4 = 5k/2$, and Claim 7 is proved.

Claims 6 and 7 together imply that, for any routing \mathcal{R} with $\vec{w}(\mathcal{R}) = \vec{w}(D)$, we have $5k/2 \geq \vec{w}(D) = \vec{w}(\mathcal{R}) \geq \vec{\pi}(\mathcal{R}) \geq 5k/2$. Thus, for such \mathcal{R} we have $\vec{w}(D) = \vec{w}(\mathcal{R}) = \vec{\pi}(\mathcal{R}) = 5k/2$, and hence $\vec{\pi}(\mathcal{R}) - \vec{\pi}(D) = k/2 = K$ by Claim 5. This completes the proof of Theorem 2.7. \square

From the proof above one can see that the result of Theorem 2.7 is valid even when \mathcal{R} is required to be a symmetric routing. That is, there exists a ring and an involution demand D on it such that $\vec{\pi}(\mathcal{R}) - \vec{\pi}(D) \geq K$ for any symmetric routing \mathcal{R} for D with $\vec{w}(\mathcal{R}) = \vec{w}(D)$.

7. Concluding remarks

In this paper we have shown that, even when restricted to involution demands, both RRWA and SRRWA are \mathbb{NP} -complete and hence admit no polynomial time exact algorithms unless $\mathbb{P} = \mathbb{NP}$. For practical purposes it is significant to design polynomial time approximation algorithms for RRWA and SRRWA. Note that for general demands we have a 2-approximation algorithm [24] for RRWA. For permutation and/or involution demands we wonder if one can find an approximation algorithm for RRWA with better performance ratio. In particular, for involutions we would like to ask the following question. (An optimization problem is said to have a polynomial time approximation scheme (PTAS) if, for any rational $\varepsilon > 0$, there is a polynomial time approximation algorithm with performance ratio at most $1 + \varepsilon$.)

Question 7.1. Does RRWA (SRRWA) admit a PTAS when restricted to involution demands? How about permutation demands?

Finally, as a follow-up to Question 2.6 and Theorem 2.7 we pose the following question.

Question 7.2. Under what conditions on (G, D) is there a routing \mathcal{R} for D such that $\vec{\pi}(\mathcal{R}) = \vec{\pi}(D)$ and $\vec{w}(\mathcal{R}) = \vec{w}(D)$ hold simultaneously?

References

- [1] A. Aggarwal, A. Bar-Noy, D. Coppersmith, R. Ramaswami, B. Schieber, M. Sudan, Efficient routing in optical networks, *Journal of the ACM* 46 (1996) 973–1001.
- [2] O. Baudon, G. Fertin, I. Havel, Routing permutations and 2-1 routing requests in the hypercube, *Discrete Applied Mathematics* 113 (2001) 43–58.
- [3] B. Beauquier, J.-C. Bermond, L. Gargano, P. Hell, S. Perennes, U. Vaccaro, Graph problems arising from wavelength-routing in all-optical networks, in: *Proc. of the 2nd Workshop on Optics in Computer Science (WOCS'97)*, Geneva, Switzerland, April 1997.
- [4] B. Beauquier, S. Perennes, D. Toth, All-to-all routing and colouring in weighted trees of rings, in: *Proc. of the 11th ACM Symposium on Parallel Algorithms and Architectures (SPAA'99)*, Saint-Malo, France, 1999, pp. 185–190.
- [5] L. Becchetti, M.D. Ianni, A. Marchetti-Spaccamela, Approximation algorithms for routing and call scheduling in all-optical chains and rings, *Theoretical Computer Science* 287 (2002) 429–448.
- [6] J.-C. Bermond, L. Gargano, S. Perennes, A. Rescigno, U. Vaccaro, Efficient collective communications in optical networks, *Theoretical Computer Science* 233 (2000) 165–189.
- [7] T. Carpenter, S. Cosares, I. Sanjeev, Demand routing and slotting on ring networks, Technical Report, Bellcore Preprint, 1997.
- [8] S. Corteel, M. Valencia-Pabon, D. Gardy, D. Barth, A. Denise, The permutation-path coloring problem on trees, *Theoretical Computer Science* 297 (2003) 119–143.
- [9] S. Cosares, I. Sanjeev, An optimization problem related balancing loads on SONET rings, *Telecommunications Systems* 3 (1994) 165–181.
- [10] T. Dobravec, B. Robic, J. Zerovnik, Permutation routing in double-loop networks: design and empirical evaluation, *Journal of Systems Architecture* 48 (2003) 387–402.
- [11] T. Erlebach, K. Jansen, The complexity of path coloring and call scheduling, *Theoretical Computer Science* 255 (2001) 33–50.
- [12] A. Frank, Edge-disjoint paths in planar graphs, *Journal of Combinatorial Theory (B)* 38 (1985) 164–178.
- [13] M.R. Garey, D.S. Johnson, G.L. Miller, C.H. Papadimitriou, The complexity of colouring circular arcs and chords, *SIAM Journal on Algebraic Discrete Methods* 1 (1980) 216–227.
- [14] L. Gargano, U. Vaccaro, Routing in all-optical networks: algorithmic and graph-theoretic problems, in: *Numbers, Information and Complexity*, Kluwer Academic Publishers, The Netherlands, 2000, pp. 555–578.
- [15] P.E. Green, *Fibre-Optic Communication Networks*, Prentice-Hall, Englewood Cliffs, NJ, 1992.

- [16] S. Khanna, A polynomial time approximation scheme for the SONET ring loading problem, *Bell Labs Technocal Journal* (Spring 1997) 36–41.
- [17] L. Margara, Wavelength assignment problem in all-optical rings revisited, in: *Proc. of the 2nd International Workshop on Approximation and Randomization Algorithms in Communication Networks (ARACNE 2001)*, in: *Proceedings in Informatics*, vol. 12, Carleton Scientific, 2001, pp. 55–64.
- [18] Y.S. Myung, An efficient algorithm for the ring loading problem with integer demand splitting, *SIAM Journal on Discrete Mathematics* 14 (2001) 291–298.
- [19] M.S. Paterson, H. Schröder, O. Sýkora, I. Vrto, Permutation communication in all-optical rings, *Parallel Processing Letters* 12 (2002) 23–29.
- [20] M. Ramras, Routing Permutations on a graph, *Networks* 23 (1993) 391–398.
- [21] A. Schrijver, P. Seymour, P. Winkler, The ring loading problem, *SIAM Review* 41 (1999) 777–791.
- [22] X. Shen, Q. Hu, H. Hai, X. Wang, Optimal routing of permutations of rings, in: *Lecture Notes in Computer Science*, vol. 834, Springer, Berlin, 1994, pp. 360–368.
- [23] L. Stacho, J. Širáň, S. Zhou, Routing balanced communications on Hamiltonian decomposable networks, *Parallel Processing Letters* 14 (2004) 377–385.
- [24] G. Wilfong, P. Winkler, Ring routing and wavelength translation, in: *Proceedings of the 9th ACM-SIAM Symposium on Discrete Algorithms*, San Francisco, CA, 1998, pp. 333–341.
- [25] P. Winkler, L. Zhang, Wavelength assignment and generalized interval graph coloring, in: *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms* (Baltimore, MD, 2003), New York, 2003, pp. 830–831.
- [26] J. Yuan, S. Zhou, Polynomial time solvability of the weighted ring arc-loading problem with integer splitting, *Journal of Interconnection Networks* 5 (2004) 193–200.