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CROSS RATIO GRAPHS

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ABSTRACT

A family of arc-transitive graphs is studied. The vertices of these graphs are ordered pairs of distinct points from a finite projective line, and adjacency is defined in terms of the cross ratio. A uniform description of the graphs is given, their automorphism groups are determined, the problem of isomorphism between graphs in the family is solved, some combinatorial properties are explored, and the graphs are characterised as a certain class of arc-transitive graphs. Some of these graphs have arisen as examples in studies of arc-transitive graphs with complete quotients and arc-transitive graphs which 'almost cover' a 2-arc transitive graph.

1. *Introduction*

The purpose of this paper is to study a family of finite arc-transitive graphs which admit an arc-transitive action of a projective linear group, and for which there is a natural graph epimorphism onto a complete graph. The name 'cross ratio graphs' is used because, for many of the graphs in the family, adjacency is defined in terms of the cross ratio of certain quadruples of elements of a finite projective line. Various subfamilies of this family of graphs have arisen as examples in studies of arc-transitive graphs with complete quotients [**6**, **7**], and arc-transitive graphs which 'almost cover' a 2-arc transitive graph [**17**]. In this paper we give a uniform description of the graphs in the family, we determine their automorphism groups and specify which pairs of graphs in the family are isomorphic, and we explore some of their combinatorial properties. In addition we give the following characterisation of cross ratio graphs as a certain class of arc-transitive graphs (see Theorem 5.1 for a more precise statement of this result).

THEOREM 1.1. *Suppose that $q = p^r$, where p is a prime, $r \geqslant 1$, and $q \geqslant 3$, and suppose that Γ is a G-arc-transitive graph with vertices the ordered pairs of distinct points from the projective line* $\mathrm{PG}(1, q)$, *where G is a* 3*-transitive subgroup of* $\mathrm{P\Gamma L}(2, q)$. *Then $\Gamma \cong (q+1) \cdot K_q$, or $\Gamma \cong \binom{q+1}{2} \cdot K_2$, or Γ is a cross ratio graph.*

We have been unable to locate a description of the 3-transitive subgroups of $\mathrm{P\Gamma L}(2, q)$ which contains sufficient detail for our purposes, and so we first give such a description in Section 2. In Section 3 we give the definitions of the cross ratio graphs, and in Section 4 we study some of their structure, in particular identifying two complete quotients of each graph. In Section 5 we prove Theorem 5.1 from which Theorem 1.1 follows immediately. Then in Section 6 we determine the full automorphism groups of these graphs, proving that some of them admit a decomposition as a non-trivial lexicograph product. The proofs of several of the results in this section depend on the finite simple group classification to identify overgroups of $\mathrm{PSL}(2, q)$ in certain symmetric groups. We are grateful to Alice

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Niemeyer for her assistance in the use of GRAPE [**15**] for computing some of the automorphism groups for small values of the parameters. For convenience, a summary of the results concerning the automorphism groups is given as Theorem 7.1 in the final section, and in this section we determine precisely which pairs of cross ratio graphs are isomorphic.

2. *The* 3-*transitive subgroups of* $\mathrm{P\Gamma L}(2,q)$

Let $q = p^r$ be a prime power. The projective line $\mathrm{PG}(1,q)$ over the field $\mathrm{GF}(q)$ of order q can be identified with the set $\mathrm{GF}(q) \cup \{\infty\}$, where ∞ satisfies the usual arithmetic rules such as $1/\infty = 0$, $\infty + y = \infty$, etc. The two-dimensional projective group $\mathrm{PGL}(2,q)$ then consists of all fractional linear transformations

$$t_{a,b,c,d}: z \longmapsto \frac{az+b}{cz+d} \quad (\text{with } a,\, b,\, c,\, d \in \mathrm{GF}(q), \text{ and } ad-bc \neq 0)$$

of $\mathrm{PG}(1,q)$ (see, for example, [**5**, p. 242]). Note that $t_{a,b,c,d} = t_{a',b',c',d'}$ if and only if the 4-tuple (a,b,c,d) is a non-zero multiple of (a',b',c',d'). The group $\mathrm{PGL}(2,q)$ is sharply 3-transitive in this action on $\mathrm{PG}(1,q)$, that is, it is 3-transitive and only the identity element $t_{1,0,0,1}$ fixes three elements of $\mathrm{PG}(1,q)$. The Frobenius automorphism $\psi: x \longmapsto x^p$ of $\mathrm{GF}(q)$ induces an automorphism of $\mathrm{PGL}(2,q)$ by $\psi: t_{a,b,c,d} \longmapsto t_{a^p,b^p,c^p,d^p}$, and the group generated by $\mathrm{PGL}(2,q)$ and ψ is the semidirect product $\mathrm{PGL}(2,q)\cdot\langle\psi\rangle$ and is denoted by $\mathrm{P\Gamma L}(2,q)$. It is the automorphism group of $\mathrm{PGL}(2,q)$, and it too acts on $\mathrm{PG}(1,q)$ (with $\psi: z \longmapsto z^p$, where $\infty^p = \infty$). The purpose of this section is to determine precisely which subgroups of $\mathrm{P\Gamma L}(2,q)$ act 3-transitively on $\mathrm{PG}(1,q)$. (A *primitive element* of $\mathrm{GF}(q)$ is a generator of the multiplicative group of non-zero elements of $\mathrm{GF}(q)$.)

THEOREM 2.1. *Let $q = p^r$, where p is prime and $r \geqslant 1$. Then a subgroup G of $\mathrm{P\Gamma L}(2,q)$ is 3-transitive on $\mathrm{PG}(1,q)$ if and only if one of the following holds:*

(a) *$G = \mathrm{PGL}(2,q)\cdot\langle\psi^s\rangle$ for some divisor s of r;*

(b) *p is odd, r is even, and $G = \mathrm{M}(s,q) := \langle \mathrm{PSL}(2,q), \psi^s t_{a,0,0,1}\rangle$, for some divisor s of $r/2$ (where a is a primitive element of $\mathrm{GF}(q)$).*

Proof. We use without further reference information contained in [**9**, II.8]. If q is prime then $\mathrm{P\Gamma L}(2,q) = \mathrm{PGL}(2,q)$ is sharply 3-transitive, so (a) holds. Thus we may assume that $r \geqslant 2$, and in particular that $q \geqslant 4$, so the socle $Y := \mathrm{PSL}(2,q)$ of $X := \mathrm{P\Gamma L}(2,q)$ is a non-abelian simple group contained in G. If q is even then $Y = \mathrm{PGL}(2,q)$ and again (a) holds. Thus we may assume that q is odd and that $G \cap \mathrm{PGL}(2,q) = Y$. Now Y consists of all $t_{a,b,c,d}$ with $ad-bc$ a square in $\mathrm{GF}(q)$, and Y is 2-transitive on $\mathrm{PG}(1,q)$. Therefore $X = YX_{\infty 0}$, and similarly $G = YG_{\infty 0}$. Thus $\bar{X} := X/Y \cong X_{\infty 0}/Y_{\infty 0}$, and $\bar{G} := G/Y \cong G_{\infty 0}/Y_{\infty 0}$. We shall identify $\bar{X}$ with $X_{\infty 0}/Y_{\infty 0}$. Now $\mathrm{PGL}(2,q)_{\infty 0} = \langle t_{a,0,0,1}\rangle$, where a is a primitive element of $\mathrm{GF}(q)$, and $X_{\infty 0}$ is the semidirect product $\langle t_{a,0,0,1}\rangle\cdot\langle\psi\rangle$ (and $(t_{a,0,0,1})^\psi = t_{a',0,0,1}$ where $a' = a^p$). Also $Y_{\infty 0} = \langle (t_{a,0,0,1})^2\rangle$, and so $\bar{X} \cong \langle\bar{a}\rangle \times \langle\psi\rangle$, where $\bar{a} = t_{a,0,0,1}Y_{\infty 0}$ and $\bar{a}$ has order 2.

The subgroups of X containing Y are in one-to-one correspondence with the subgroups of $\bar{X}$, and we determine the 3-transitive subgroups of X by studying $\bar{X}$. Since G does not contain $\mathrm{PGL}(2,q)$, the element $\bar{a} \notin \bar{G}$, and hence $\bar{G}$ projects faithfully onto $\langle\psi\rangle$. Thus $\bar{G}$ is cyclic, and so $\bar{G} = \langle\psi^s\bar{a}^\varepsilon\rangle$, for some divisor s of r and $\varepsilon = 0$ or 1. Also, since $(\psi^s\bar{a}^\varepsilon)^{r/s} = \bar{a}^{\varepsilon r/s}$ and $\bar{G}$ does not contain $\bar{a}$, it follows that $\varepsilon r/s$ is even. An element of $G_{\infty 0}$ corresponding to the generator $\psi^s \bar{a}^\varepsilon$ of $\bar{G}$ has the form $\psi^s(t_{a,0,0,1})^{\varepsilon+2k}$ for some integer k. Since $G_{\infty 0}$ is transitive on $\mathrm{PG}(1,q)\backslash\{\infty, 0\}$, and since the two orbits of $Y_{\infty 0}$ in $\mathrm{PG}(1,q)\backslash\{\infty, 0\}$ are the set of squares, and the set of non-squares, the generator $\psi^s(t_{a,0,0,1})^{\varepsilon+2k}$ of $G_{\infty 0}$ maps 1 to a non-square. The image of 1 under this element is $a^{\varepsilon+2k}$, and hence G is 3-transitive if and only if $\varepsilon = 1$. Since $\varepsilon r/s$ is even, we must also have r/s even. □

COROLLARY 2.2. *Let* $G \leqslant \mathrm{P\Gamma L}(2,q)$ *be* 3-*transitive on* $\mathrm{PG}(1,q)$ *as in Theorem* 2.1. *Then* $G_{\infty 01}$ *is* $\langle \psi^s \rangle$ *if* G *is as in part* (a)*, and is* $\langle \psi^{2s} \rangle$ *if* G *is as in part* (b).

Proof. Since $\mathrm{PGL}(2,q)$ is sharply 3-transitive, it is clear that the stabiliser of $\infty, 0$ and 1 in $G = \mathrm{PGL}(2,q)\cdot\langle\psi^s\rangle$ is $G_{\infty 01} = \langle \psi^s \rangle$ (where s divides r). On the other hand, suppose that $G = \mathrm{M}(s,q) = \langle \mathrm{PSL}(2,q), \psi^s t_{a,0,0,1}\rangle$ (where r is even and s divides $r/2$). Then $|G_{\infty 01}| = r/2s$. Since $G_{\infty 01} \leqslant \mathrm{P\Gamma L}(2,q)_{\infty 01} = \langle \psi \rangle$, it follows that $G_{\infty 01} = \langle \psi^{2s} \rangle$. □

3. *Definition of the cross ratio graphs*

In this section we give the definitions of the cross ratio graphs. For each of these graphs the vertex set may be identified with the set

$$V(q) := \{xy \mid x, y \in \mathrm{PG}(1,q),\ x \neq y\} \tag{1}$$

of ordered pairs of distinct points from $\mathrm{PG}(1,q)$, for some prime power $q \geqslant 3$. Note that we write xy for the ordered pair (x,y), where x,y are distinct elements of $\mathrm{PG}(1,q)$. Each of the cross ratio graphs with vertex set $V(q)$ admits a 3-transitive subgroup of $\mathrm{P\Gamma L}(2,q)$ acting naturally on $V(q)$ as a subgroup of automorphisms. Just as the 3-transitive subgroups of $\mathrm{P\Gamma L}(2,q)$ occurring in Theorem 2.1 are of two types, so also the cross ratio graphs are of two types, according to whether the largest subgroup of $\mathrm{P\Gamma L}(2,q)$ acting as automorphisms occurs in case (a) or (b) of Theorem 2.1, and the graphs are said to be of *untwisted* or *twisted* type respectively.

The untwisted cross ratio graphs can be defined directly in terms of the cross ratio. For distinct elements $a,b,c,d \in \mathrm{PG}(1,q)$, the *cross ratio* is defined as

$$c(a,b;c,d) := \frac{(a-c)(b-d)}{(a-d)(b-c)} \tag{2}$$

(see, for example, [**8**, p. 23]) with the usual convention that $x-\infty = -\infty$, $\infty - x = \infty, (\infty\cdot x)/(\infty\cdot y) = x/y$, etc. The cross ratio can take all values in $\mathrm{PG}(1,q)$ except 0, 1 and ∞, and is invariant under the action of $\mathrm{PGL}(2,q)$, while under the action of ψ we have

$$c(a^\psi, b^\psi; c^\psi, d^\psi) = c(a,b;c,d)^\psi.$$

Moreover, $\mathrm{PGL}(2,q)$ is transitive on the set of 4-tuples with a fixed cross ratio. Let F denote the prime subfield $\mathrm{GF}(p)$ of the field $\mathrm{GF}(q)$, where, as in Section 2, $q = p^r$ with p a prime and $r \geqslant 1$. The following notation will be used throughout the paper.

DEFINITION 3.1. For each element $d \in \mathrm{GF}(q)$, $d \neq 0, 1$, the subfield $F[d]$ of $\mathrm{GF}(q)$ generated by d is $\mathrm{GF}(p^{s(d)})$, for some divisor $s(d)$ of r. Given d, for each divisor s of $s(d)$, the $\langle\psi^s\rangle$-orbit containing d is denoted $B(d,s) := \{d^{\psi^{si}} \mid 0 \leqslant i < s(d)/s\}$, and has size $s(d)/s$.

We define the untwisted cross ratio graphs as follows.

DEFINITION 3.2. Let $q = p^r$ for a prime p and $r \geqslant 1$, and let $d, s(d), s$, and $B(d,s)$ be as in Definition 3.1. Then the *untwisted cross ratio graph* $\Gamma = \mathrm{CR}(q; d, s)$ is defined as the graph with vertex set $V(q)$, such that $\{uv, wx\}$ is an edge if and only if u, v, w, x are distinct elements of $\mathrm{PG}(1, q)$, and $c(u, v; w, x) \in B(d, s)$.

For a vertex γ of a graph Γ, the set of vertices joined to γ is denoted by $\Gamma(\gamma)$.

REMARK 3.3. (a) Since $c(u, v; w, x) = c(w, x; u, v)$, for all distinct 4-tuples u, v, w, x, it follows that the adjacency relation is symmetric, so the graph $\mathrm{CR}(q; d, s)$ is well-defined as an undirected graph. From the properties of the cross ratio mentioned above it is clear that $\mathrm{CR}(q; d, s)$ admits the 3-transitive subgroup $G := \mathrm{PGL}(2, q) \cdot \langle \psi^s \rangle$ of $\mathrm{P\Gamma L}(2, q)$ as a subgroup of automorphisms, and that G acts transitively on arcs (ordered pairs of adjacent vertices).

(b) From the definition of $\Gamma = \mathrm{CR}(q; d, s)$, a vertex $1x \in \Gamma(\infty 0)$ if and only if $c(\infty, 0; 1, x) \in B(d, s)$, that is, if and only if $x \in B(d, s)$. Since G is transitive on the arcs of Γ, the set $\Gamma(\infty 0)$ is the orbit of $G_{\infty 0} = \langle t_{a,0,0,1}, \psi^s \rangle$ containing $1d$ (where a is a primitive element of $\mathrm{GF}(q)$), and hence consists of all the vertices xy, for $x \in \mathrm{GF}(q) \setminus \{0\}$, such that $y \in B(d, s)\, x$.

(c) The subfamily of untwisted cross ratio graphs defined in [**6**, Example 2.4] were given with a slightly different notation. The graph $\mathrm{CR}(q, d)$ in [**6**] is called here $\mathrm{CR}(q; d, s(d))$. The assertion made towards the end of [**6**, Example 2.4] concerning isomorphisms between the graphs $\mathrm{CR}(2^p, d)$ for various d was not fully established in that paper, since only isomorphisms induced by elements of $\mathrm{P\Gamma L}(2, 2^p)$ were considered there. The assertion is in fact true and follows from our determination of the full automorphism groups of these graphs in Section 6.

Next we turn to the cross ratio graphs corresponding to the groups defined in part (b) of Theorem 2.1. They are defined as orbital graphs for the transitive actions of these groups on $V(q)$. For a transitive group G on a set Ω, there is a natural action of G induced on $\Omega \times \Omega$ given by $(\beta, \gamma)^g := (\beta^g, \gamma^g)$ for $\beta, \gamma \in \Omega, g \in G$. The G-orbits in this induced action are called orbitals for G in Ω. For each orbital Δ, not equal to the trivial orbital $\{(\beta, \beta) \mid \beta \in \Omega\}$, the associated *orbital graph* is defined as the directed graph with vertex set Ω and edge set Δ. If Δ is *self-paired*, that is, if $(\beta, \gamma) \in \Delta$ implies that $(\gamma, \beta) \in \Delta$, then this orbital graph may be regarded as an undirected graph. It follows from the definition that the group G acts as an arc-transitive group of automorphisms of each of its orbital graphs. Conversely, any G-arc-transitive graph Γ with vertex set Ω is isomorphic to an orbital graph for G on Ω. We denote the G-orbital containing the pair (β, γ) by $(\beta, \gamma)^G$.

DEFINITION 3.4. Let $q = p^r$, where p is an odd prime and r is an even integer, and let $d, s(d), s$ and $B(d, s)$ be as in Definition 3.1, and such that both $s(d)$ and s are even, and $d - 1$ is a square. Then the *twisted cross ratio graph* $\mathrm{TCR}(q; d, s)$ is defined as the orbital graph for the group $G := \mathrm{M}(s/2, q)$ (defined in Theorem 2.1(b)) acting on $V(q)$ corresponding to the orbital $(\infty 0, 1d)^G$.

REMARK 3.5. (a) Since $d - 1$ is a square, the element $t_{1,-d,1,-1} \in \mathrm{PSL}(2, q) \subseteq \mathrm{M}(s/2, q)$; it interchanges ∞ and 1, and interchanges 0 and d, and hence interchanges $(\infty 0, 1d)$ and $(1d, \infty 0)$. Hence the orbital $(\infty 0, 1d)^G$ is self-paired and so $\mathrm{TCR}(q; d, s)$ is well-defined as an undirected graph.

(b) By its definition as an orbital graph for $G \geqslant \mathrm{M}(s/2, q)$, $\Gamma := \mathrm{TCR}(q; d, s)$ admits G as an arc-transitive subgroup of automorphisms. The set of vertices $\Gamma(\infty 0)$ adjacent to $\infty 0$ is therefore the orbit of $G_{\infty 0} = \langle (t_{a,0,0,1})^2, \psi^{s/2} t_{a,0,0,1} \rangle$ containing $1d$. A straightforward computation shows that $\Gamma(\infty 0)$ consists of all the vertices xy, for $x \in \mathrm{GF}(q) \backslash \{0\}$, such that

$$y \in \begin{cases} B(d,s)\, x & \text{if } x \text{ is a square} \\ B(d,s)^{\psi^{s/2}} x & \text{if } x \text{ is not a square.} \end{cases}$$

Note that, since $B(d, s)$ is the $\langle \psi^s \rangle$-orbit containing d, the sets $B(d, s)$ and $B(d, s)^{\psi^{s/2}}$ are disjoint (and their union is $B(d, s/2)$).

4. *Quotients of cross ratio graphs*

We derive a little extra information about the automorphism groups of the cross ratio graphs Γ, and their adjacency relations. For a partition $\mathscr{P}$ of the vertex set of Γ, the *quotient graph* $\Gamma_{\mathscr{P}}$ of Γ relative to $\mathscr{P}$ is the graph with vertex set $\mathscr{P}$ such that $\{P_1, P_2\}$ is an edge of $\Gamma_{\mathscr{P}}$ if and only if there exist $x_1 \in P_1$ and $x_2 \in P_2$ such that $\{x_1, x_2\}$ is an edge of Γ. We are interested in partitions $\mathscr{P}$ which are invariant under the group G used to define Γ. To help determine such partitions, we record the following information about the action of $\mathrm{P\Gamma L}(2, q)$ on $V(q)$.

PROPOSITION 4.1. *Let $q = p^r$ where p is a prime, $r \geqslant 1$ and $q \geqslant 3$, and consider the action of* $\mathrm{P\Gamma L}(2, q)$ *on* $V(q)$.

(a) *The centraliser of* $\mathrm{P\Gamma L}(2, q)$ *in the symmetric group* $\mathrm{Sym}(V(q))$ *is* $C = \langle \alpha \rangle$, *where* $\alpha: xy \longmapsto yx$ (*for* $xy \in V(q)$).

(b) $\mathrm{P\Gamma L}(2, q)$ *preserves the following three non-trivial block systems in* $V(q)$:

$$\mathscr{B} := \{B(x) \mid x \in \mathrm{PG}(1, q)\}, \text{ where } B(x) = \{xy \mid y \in \mathrm{PG}(1, q), y \neq x\}$$
$$\mathscr{B}' := \{B'(x) \mid x \in \mathrm{PG}(1, q)\}, \text{ where } B'(x) = \{yx \mid y \in \mathrm{PG}(1, q), y \neq x\}$$
$$\mathscr{A} := \{\{xy, yx\} \mid x, y \in \mathrm{PGL}(2, q), x \neq y\},$$

and if $q \neq 3, 5$ then these are the only non-trivial block systems in $V(q)$ preserved by any 3-*transitive subgroup of* $\mathrm{P\Gamma L}(2, q)$. *Moreover, C preserves $\mathscr{A}$ and interchanges $\mathscr{B}$ and $\mathscr{B}'$.*

Proof. In the action of $\mathrm{P\Gamma L}(2, q)$ on $V(q)$, the stabiliser of the point $\infty 0$ fixes exactly two points, namely $\infty 0$ and 0∞. Thus $|C| = 2$, and C is generated by α (see, for example, [**5**, Theorem 4.2A]). It is straightforward to check that $\mathrm{P\Gamma L}(2, q)$ preserves the block systems $\mathscr{B}$, $\mathscr{B}'$ and $\mathscr{A}$, and that C preserves $\mathscr{A}$ and interchanges $\mathscr{B}$ and $\mathscr{B}'$. To prove the remaining assertion we suppose that $q \neq 3, 5$. Let G be a 3-transitive subgroup of $\mathrm{P\Gamma L}(2, q)$, so in particular G contains $Y = \mathrm{PSL}(2, q)$. Let D be a non-trivial block of imprimitivity for G in $V(q)$ such that D contains $\infty 0$. Then the set stabiliser G_D is a proper subgroup of G which properly contains $H := G_{\infty 0}$. Since D is also a block of imprimitivity for Y, we have $H \cap Y < Y_D < Y$, and $H \cap Y \cong Z_{(q-1)/\gcd(2, q-1)}$. If $q > 11$, or if $q = 4$ or 8, then the only overgroups of $H \cap Y$ in Y are the setwise stabilisers of the blocks containing $\infty 0$ from the three block systems $\mathscr{B}$, $\mathscr{B}'$ and $\mathscr{A}$ (see [**4**, Section 260]), and hence for these values of q, there are no other non-trivial block systems preserved by Y, and hence no others preserved by G. The remaining values of q to consider are $q = 7, 9, 11$. In these cases, if G contains

$\mathrm{PGL}(2,q)$, then the only overgroups of H in G are the three corresponding to the block systems $\mathcal{B}, \mathcal{B}'$ and $\mathcal{A}$. If G does not contain $\mathrm{PGL}(2,q)$, then $q=9$ and $G = \mathrm{M}_{10}$, $H = Q_8$. Again the only overgroups of H in G are the three corresponding to the block systems $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{A}$. □

We use this result to analyse the structure of the cross ratio graphs, and in particular to identify two quotient graphs which are complete graphs of order $q+1$.

THEOREM 4.2. *Let $q = p^r$ for a prime p and $r \geqslant 1$ with $q \geqslant 3$, let d, $s(d)$, s and $B(d,s)$ be as in Definition* 3.1*, and let $\mathcal{B}$, $\mathcal{B}'$, $\mathcal{A}$ and $C = \langle \alpha \rangle$ be as in Proposition* 4.1. *Let $\Gamma = \mathrm{CR}(q; d, s)$ or $\Gamma = \mathrm{TCR}(q; d, s)$, where in the latter case r, $s(d)$, s are even and $d-1$ is a square. Then:*

(a) *$\Gamma(\infty 0) \cap B(1) = \{1x \mid x \in B(d,s)\}$ is of size $s(d)/s$, and Γ has valency $(q-1)\,s(d)/s$.*

(b) *For $x, y \in \mathrm{PG}(1,q)$ with $x \neq y$, the only vertex of $B(x)$ which is not joined by an edge to a vertex of $B(y)$ is xy. Similarly, the only vertex of $B'(x)$ which is not joined by an edge to a vertex of $B'(y)$ is yx. Also, the quotient graphs satisfy $\Gamma_{\mathcal{B}} \cong \Gamma_{\mathcal{B}'} \cong K_{q+1}$.*

(c) *If $\Gamma = \mathrm{CR}(q; d, s)$ then $\mathrm{Aut}(\Gamma) \cap (C \times \mathrm{P\Gamma L}(2,q)) = C \times (\mathrm{PGL}(2,q) \cdot \langle \psi^s \rangle)$, while if $\Gamma = \mathrm{TCR}(q; d, s)$ then*

$$\mathrm{Aut}(\Gamma) \cap (C \times \mathrm{P\Gamma L}(2,q)) = \begin{cases} C \times \mathrm{M}(s/2, q) & \text{if } d \text{ is a square} \\ \langle M(s/2,q), \alpha\psi^{s/2} \rangle & \text{if } d \text{ is not a square.} \end{cases}$$

In all cases, if $q \neq 3, 5$, then the only non-trivial partition of $V(q)$ invariant under this group is $\mathcal{A}$.

(d) *If Γ' is $\mathrm{CR}(q; d', s')$ or $\mathrm{TCR}(q; d', s')$ (for appropriate d', s'), then there exists an element $\varphi \in C \times \mathrm{P\Gamma L}(2,q)$ which defines an isomorphism from Γ to Γ' if and only if d and d' are in the same $\langle \psi \rangle$-orbit, $s = s'$, and Γ, Γ' are either both untwisted graphs or both twisted graphs. Moreover $\Gamma = \Gamma'$ if and only if, in addition, d and d' are in the same $\langle \psi^s \rangle$-orbit.*

Proof. Part (a) follows from Remarks 3.3(b) and 3.5(b) since $\Gamma(\infty 0)$ consists of $|B(d,s)| = s(d)/s$ vertices from $B(w)$, for each $w \in \mathrm{GF}(q)\backslash\{0\}$. Since $\mathrm{PSL}(2,q)$ is transitive on $V(q)$ it is sufficient to prove the first assertion of part (b) in the case $x = \infty$, $y = 0$. By part (a), the vertex $\infty 0$ is adjacent to no vertex of $B(0)$, but is adjacent to $1d$. Let $w \in \mathrm{GF}(q)$ with $w \neq 0$. Then each of $t_{-w,w,0,1}$ and $\psi^{s/2} t_{-w,w,0,1}$ maps $\infty, 0, 1$ to $\infty, w, 0$ respectively, and hence maps the edge $\{\infty 0, 1d\}$ to a pair $\{\infty w, 0x\}$, for some x. Since $t_{-w,w,0,1} \in \mathrm{PGL}(2,q)$, and since $\mathrm{M}(s/2,q)$ contains $t_{-w,w,0,1}$ if $-w$ is a square, and contains $\psi^{s/2} t_{-w,w,0,1}$ otherwise, it follows that ∞w is adjacent to at least one vertex of $B(0)$. This proves the first part of (b). Since both $\mathrm{PGL}(2,q)$ and $\mathrm{M}(s/2,q)$ are 2-transitive on $\mathcal{B}$ it follows that $\Gamma_{\mathcal{B}} \cong K_{q+1}$. The assertions concerning $\mathcal{B}'$ follow similarly.

For part (c) we assume first that $\Gamma = \mathrm{CR}(q; d, s)$. Now $\{uv, wx\}$ is an edge of Γ if and only if $c(u,v; w,x) \in B(d,s)$, and since

$$c(v,u; x,w) = \frac{(v-x)(u-w)}{(v-w)(u-x)} = c(u,v; w,x),$$

it follows that $\{vu, xw\}$ is also an edge. Thus α preserves edges, and hence, by Remark 3.3(a), $\mathrm{Aut}(\Gamma)$ contains $H := C \times (\mathrm{PGL}(2,q) \cdot \langle \psi^s \rangle)$. Suppose that $\varphi \in \mathrm{Aut}(\Gamma) \cap$

$(C\times \mathrm{P\Gamma L}(2,q))$, but $\varphi \notin H$. Since $\psi, \psi^2, \ldots, \psi^s$ is a transversal for H in $C\times \mathrm{P\Gamma L}(2,q)$, we may assume that $\varphi=\psi^i$ for some i with $1\leqslant i<s$. Then φ fixes $\Gamma(\infty 0)\cap B(1)$ setwise, that is, ψ^i fixes $B(d,s)$ setwise. It follows that $B(d,s)$ is invariant under the action of $\langle \psi^s, \psi^i\rangle$, and this group is equal to $\langle\psi^j\rangle$ where $j=\gcd(s,i)$. Since $i<s$ we have $j<s$, and since j divides s, j also divides $s(d)$. By the definition of $B(d,s)$ (as an orbit under the action of $\langle\psi^s\rangle$) it follows that $B(d,s)$ is an orbit under the action of $\langle\psi^j\rangle$. However this latter orbit comprises the field elements $d, d^{\psi^j}, \ldots, d^{\psi^{j(k-1)}}$, where k is the least positive integer such that $d^{\psi^{jk}}=d$, that is, $d^{p^{jk}-1}=1$. Since d is a generator of $\mathrm{GF}(p^{s(d)})$, this means that $s(d)$ divides jk, that is, $s(d)/j$ divides k. Thus $|B(d,s)|=k\geqslant s(d)/j>s(d)/s$, contradicting part (a). Hence $\mathrm{Aut}(\Gamma)\cap(C\times \mathrm{P\Gamma L}(2,q))=H$. By Proposition 4.1, if $q\neq 3,5$ then the only non-trivial block system in $V(q)$ preserved by $C\times \mathrm{PGL}(2,q)$ is $\mathscr{A}$.

Now suppose that $\Gamma=\mathrm{TCR}(q;d,s)$. Since r is even, it follows that -1 is a square, and hence $t_{a,0,0,1}\notin \mathrm{PSL}(2,q)$ (where a is a primitive element of $\mathrm{GF}(q)$). Now $t_{a,0,0,1}$ maps the edge $\{\infty 0, 1d\}$ to the pair $\{\infty 0, ad'\}$, where $d'=ad$, and by Remark 3.5(b), this pair is not an edge. It follows that $\mathrm{Aut}(\Gamma)$ does not contain $\mathrm{PGL}(2,q)$, and therefore, by Theorem 2.1, $\mathrm{Aut}(\Gamma)\cap \mathrm{P\Gamma L}(2,q)=\mathrm{M}(s',q)$, for some s' dividing r. Next suppose that $\psi^i\in\mathrm{Aut}(\Gamma)$, for some divisor i of r. Then ψ^i maps the edge $\{\infty 0, 1d\}$ to the edge $\{\infty 0, 1d^{\psi^i}\}$. By Remark 3.5(b), $d^{\psi^i}\in B(d,s)$, and it follows that i is a multiple of s. Therefore $\mathrm{Aut}(\Gamma)\cap \mathrm{P\Gamma L}(2,q)=\mathrm{M}(s/2,q)$. Since $\mathrm{M}(s/2,q)$ is transitive on the arcs of Γ, the generator α of C is an automorphism of Γ if and only if $e:=\{0\infty, d1\}$ (the image of $\{\infty 0, 1d\}$ under α) is an edge, and e is an edge if and only if the image e' of e under $t_{0,1,1,0}\in \mathrm{PSL}(2,q)$ is an edge. (Note that $t_{0,1,1,0}\in\mathrm{PSL}(2,q)$ since -1 is a square.) The image e' is $\{\infty 0, d^{-1}1\}$, and by Remark 3.5(b) it is an edge if and only if

$$1\in\begin{cases} B(d,s)\,d^{-1} & \text{if } d \text{ is a square}\\ B(d,s)^{\psi^{s/2}}\,d^{-1} & \text{if } d \text{ is not a square.}\end{cases}$$

Since $d\in B(d,s)$ it follows that α is an automorphism if and only if d is a square. Thus if d is a square we have proved that $\mathrm{Aut}(\Gamma)\cap(C\times\mathrm{P\Gamma L}(2,q))=C\times \mathrm{M}(s/2,q)$. Now suppose that d is not a square. Then $\alpha\notin\mathrm{Aut}(\Gamma)$. In this case the image of $\{\infty 0, 1d\}$ under $\alpha\psi^{s/2}$ is $\{0\infty, d^{\psi^{s/2}}1\}$, and this is an edge if and only if its image $e:=\{\infty 0, d^{-\psi^{s/2}}1\}$ under $t_{0,1,-1,0}$ is an edge. By Remark 3.5(b), since d is not a square, e is an edge if and only if $1\in B(d,s)^{\psi^{s/2}}d^{-\psi^{s/2}}$, and this condition holds since $d\in B(d,s)$. Thus $\alpha\psi^{s/2}$ is in $\mathrm{Aut}(\Gamma)$, and it follows that $\mathrm{Aut}(\Gamma)\cap(C\times\mathrm{P\Gamma L}(2,q))=\langle \mathrm{M}(s/2,q), \alpha\psi^{\sigma/2}\rangle$. Thus, for all d, $\mathrm{Aut}(\Gamma)\cap(C\times\mathrm{P\Gamma L}(2,q))$ interchanges $\mathscr{B}$ and $\mathscr{B}'$, and so, by Proposition 4.1, since here $q\neq 3,5$, the only non-trivial partition of $V(q)$ which is invariant under this group is $\mathscr{A}$. Thus part (c) is proved.

Finally suppose that $\varphi\in C\times\mathrm{P\Gamma L}(2,q)$ induces an isomorphism from Γ to Γ', where $\Gamma'=\mathrm{CR}(q;d',s')$ or $\mathrm{TCR}(q;d',s')$. Since the quotient group of $C\times\mathrm{P\Gamma L}(2,q)$ modulo $\mathrm{PSL}(2,q)$ is abelian, $H:=\mathrm{Aut}(\Gamma)\cap(C\times\mathrm{P\Gamma L}(2,q))$ is normal in $C\times\mathrm{P\Gamma L}(2,q)$ and hence is normalised by φ. It follows that $\mathrm{Aut}(\Gamma')\cap(C\times\mathrm{P\Gamma L}(2,q))=\mathrm{Aut}(\Gamma)\cap(C\times\mathrm{P\Gamma L}(2,q))$, and in particular, by part (c), either Γ and Γ' are both untwisted graphs, or they are both twisted graphs. Also $s'=s$. Since Γ and Γ' must have the same valency, we have also $s(d')=s(d)$. Since H is arc-transitive on Γ, we may assume that φ maps the arc $(\infty 0, 1d)$ of Γ to the arc $(\infty 0, 1d')$ of Γ'. In particular φ fixes the vertex $\infty 0$ and maps $1d$ to a vertex in $B(1)$. The subset of $C\times\mathrm{P\Gamma L}(2,q)$ of elements with this property is the subgroup $\langle\psi, \alpha t_{0,d,1,0}\rangle$, and so φ lies in this subgroup (which contains $\langle\psi\rangle$ as a subgroup of index 2). Since $\alpha t_{0,d,1,0}$ fixes the vertex $1d$, it follows that φ maps $1d$ to $1d^{\psi^i}$ for some i. Hence d and d' are in the same $\langle\psi\rangle$-orbit. The graphs Γ and Γ' will be equal if and only if $d^{\psi^i}\in B(d,s)$, that is, i is a multiple of s. Conversely, if $d'=d^{\psi^i}$ and $s'=s$, then ψ^i maps $\mathrm{CR}(q;d,s)$ to $\mathrm{CR}(q;d',s')$, and $\mathrm{TCR}(q;d,s)$ to $\mathrm{TCR}(q;d',s')$. Thus (d) is proved. $\square$

We end this section by noting a combinatorial property of the cross ratio graphs which highlights the relationship between the partitions $\mathscr{B}$ and $\mathscr{B}'$ of $V(q)$. For a graph Γ and a subset U of the vertex set V of Γ, $\Gamma(U)$ denotes the set of vertices in $V\setminus U$ which are joined by an edge to some vertex of U.

LEMMA 4.3. *For* $\Gamma=\mathrm{CR}(q;d,s)$ *or* $\mathrm{TCR}(q;d,s)$, *we have, for each* $x\in\mathrm{PG}(1,q)$, $V(q)\setminus\Gamma(B(x))=B(x)\cup B'(x)=V(q)\setminus\Gamma(B'(x))$.

Proof. By Theorem 4.2, the set $\Gamma(B(x))$ contains $B(y)\setminus\{yx\}$, but not yx, for any $y\neq x$, and hence $\Gamma(B(x))=V(q)\setminus(B(x)\cup B'(x))$. The second equality is proved in a similar manner. $\square$

5. *Characterising the cross ratio graphs*

First we characterise the cross ratio graphs as a certain family of arc-transitive graphs. It was in this situation that they first arose in [**6**]. Theorem 1.1 follows immediately from Theorem 5.1.

THEOREM 5.1. *Suppose that* $q=p^r$, *where* p *is a prime,* $r\geqslant 1$, *and* $q\geqslant 3$, *and suppose that* Γ *is a* G*-arc-transitive graph with vertex set* $V(q)$ (*as defined in* (1)), *where* G *is a* 3*-transitive subgroup of* $\mathrm{P\Gamma L}(2,q)$ *with the induced natural action on* $V(q)$. *Then one of the following holds:*

(a) $\Gamma\cong(q+1)\cdot K_q$, *with connected components being either the sets* $B(x)$ *or the sets* $B'(x)$, *for* $x\in\mathrm{PG}(1,q)$ (*where these sets are as defined in Proposition* 4.1)*;*

(b) $\Gamma\cong\binom{q+1}{2}\cdot K_2$ *with connected components the pairs* $\{xy,yx\}$, $x\neq y$*;*

(c) Γ *is isomorphic to* $\mathrm{CR}(q;d,s)$ *or* $\mathrm{TCR}(q;d,s)$ *for some* d,s.

In part (c) of this theorem we show that, for $\Gamma=\mathrm{CR}(q;d,s)$ or $\Gamma=\mathrm{TCR}(q;d,s)$, the group G is equal to $\mathrm{PGL}(2,q)\cdot\langle\psi^{s'}\rangle$ or $\mathrm{M}(s'/2,q)$ respectively, for some divisor s' of r such that $\gcd(s(d),s')=s$.

Proof of Theorem 5.1. As discussed in Section 3, Γ is an orbital graph for some non-trivial self-paired G-orbital in $V(q)$. This orbital is the G-orbit on ordered pairs containing $(\infty 0,xy)$ for some $x,y\in\mathrm{PG}(1,q)$, with $x\neq y$. If $x=\infty$ then (a) holds with components the blocks $B(u)$ for $u\in\mathrm{PG}(1,q)$, if $y=0$ then (a) holds with components the blocks $B'(u)$ for $u\in\mathrm{PG}(1,q)$, and if $xy=0\infty$ then (b) holds. Thus suppose that $x\neq\infty$, $y\neq 0$, and $xy\neq 0\infty$. Suppose that $x=0$, so that $y\neq\infty$. Any element of G which maps $\infty 0$ to $0y$ must map $0y$ to yz, for some z, and hence there is no element of G which interchanges $\infty 0$ and $0y$, contradicting the arc-transitivity of G. Hence $x\neq 0$ and similarly $y\neq\infty$, so $\infty,0,x,y$ are pairwise distinct. Since G is 3-transitive on $\mathrm{PG}(1,q)$, we may assume that $x=1$.

By Theorem 2.1, for some divisor s of r, we have $G=\mathrm{PGL}(2,q)\cdot\langle\psi^s\rangle$, or $G=\mathrm{M}(s/2,q)$, where in the latter case p is odd and both r and s are even. By Corollary

2.2, $G_{\infty 01} = \langle \psi^s \rangle$, and since G is 3-transitive on PG$(1, q)$, and transitive on arcs of Γ, it follows that $\langle \psi^s \rangle$ is transitive on the vertices of $\Gamma(\infty 0) \cap \mathrm{B}(1)$. Thus this set consists of all pairs $1y'$, for $y' \in A(y) := \{ y^{\psi^{si}} \mid \text{for some } i\}$. We can determine $\Gamma(\infty 0)$ since it is the orbit of $G_{\infty 0}$ containing $1y$. If $G = \mathrm{PGL}(2, q) \cdot \langle \psi^s \rangle$ then $\Gamma(\infty 0)$ consists of the pairs uv where $v \in A(y)u$. If $G = \mathrm{M}(s/2, q)$, then $\Gamma(\infty 0)$ consists of the pairs uv where $v \in A(y)u$ if u is a square, and where $v \in A(y)^{\psi^{s/2}} u$ if u is not a square.

The set $A(y)$ is contained in the subfield $F[y] = \mathrm{GF}(p^{s(y)})$ generated by y, and so each element of $A(y)$ is left invariant by $\psi^{s(y)}$. Moreover $\psi^{s(y)}$ maps squares to squares. It follows that $\Gamma(\infty 0)$ is left invariant by $\langle G_{\infty 0}, \psi^{s(y)} \rangle$ and hence that $\langle G, \psi^{s(y)} \rangle$ leaves the G-orbital of arcs of Γ invariant, that is, $\langle G, \psi^{s(y)} \rangle$ is contained in Aut(Γ). Thus we may assume that $\psi^{s(y)} \in G$, and hence that s divides $s(y)$. This means that $A(y)$ is the set $B(y, s)$, see Definition 3.1. If $G = \mathrm{PGL}(2, q) \cdot \langle \psi^s \rangle$ then we have shown that the set of vertices adjacent to $\infty 0$ is the same for Γ and CR$(q; y, s)$, and they admit the same arc-transitive group G. Hence in this case $\Gamma = \mathrm{CR}(q; y, s)$.

Suppose therefore that $G = \mathrm{M}(s/2, q)$. Since G is arc-transitive on Γ, some element $g = \psi^i t_{a,b,c,d}$ of G interchanges $\infty 0$ and $1y$. Since g interchanges ∞ and 1, and maps 0 to y, we have $g = \psi^i t_{1,-y,1,-1}$. Then, since g maps y to 0, we have $y^{\psi^i} = y$, and hence $s(y)$ divides i. Since s divides $s(y)$, this means that s divides i, and hence $\psi^i \in G$. Therefore $t_{1,-y,1,-1} \in G \cap \mathrm{PGL}(2, q) = \mathrm{PSL}(2, q)$, and so $y - 1$ is a square. Therefore the graph TCR$(q; y, s)$ is defined, and we have shown that the set of vertices adjacent to $\infty 0$ is the same for Γ and TCR$(q; y, s)$, and they admit the same arc-transitive group M$(s/2, q)$. Hence in this case $\Gamma = \mathrm{TCR}(q; y, s)$. $\square$

6. *Combinatorial structure and automorphisms*

In many of the graphs CR$(q; d, s)$ and TCR$(q; d, s)$ the centraliser C of PΓL$(2, q)$ is admitted as a group of automorphisms. When this happens we have at least two edges between the blocks $A_1 := \{\infty 0, 0\infty\}$ and $A_2 := \{1d, d1\}$ of $\mathscr{A}$ (defined in Proposition 4.1), namely the edges $\{\infty 0, 1d\}$ and $\{0\infty, d1\}$. Occasionally all four possible edges joining a vertex of A_1 to a vertex of A_2 are present. In this case the graph has the structure of a lexicographic product of two smaller graphs, and we can determine its automorphism group precisely. Our first task is to identify when this situation arises. If Σ and Δ are graphs with vertex sets V_Σ and V_Δ, then the *lexicographic product* $\Sigma[\Delta]$ of Σ and Δ is the graph with vertex set $V_\Sigma \times V_\Delta$ such that (σ, δ) and (σ', δ') are joined by an edge if and only if either $\{\sigma, \sigma'\}$ is an edge of Σ, or $\sigma = \sigma'$ and $\{\delta, \delta'\}$ is an edge of Δ.

Proposition 6.1. *Let $q = p^r$, where p is a prime p, $r \geqslant 1$, and $q \geqslant 3$. Suppose that Γ is a cross ratio graph* CR$(q; d, s)$ *or* TCR$(q; d, s)$, *for some d, s. Then $\Gamma \cong \Gamma_{\mathscr{A}}[\overline{K_2}]$ if and only if one of the following holds:*

(a) $\Gamma = \mathrm{CR}(q; d, s)$, *and*
 (i) *either p is odd, $d = -1$;*
 (ii) *or $s(d)$ is divisible by $2s$;*

(b) $\Gamma = \mathrm{TCR}(q; d, s)$, *$d$ (as well as $d - 1$) is a square, s is even, and $2s$ divides $s(d)$.*

Proof. Suppose that Γ satisfies one of the conditions above. By Theorem 4.2, Aut(Γ) contains $C \times G$, for some 3-transitive subgroup G of PΓL$(2, q)$. Since G is transitive on the arcs of $\Gamma_{\mathscr{A}}$, and since C interchanges the vertices in each block of $\mathscr{A}$, in order to prove that $\Gamma \cong \Gamma_{\mathscr{A}}[\overline{K_2}]$ it is sufficient to prove that the vertex $\infty 0$ is adjacent to $d1$. By Remark 3.3 or 3.5, this is the case provided that $1 \in B(d, s)\, d$. This is clearly true if $d = -1$, while in the other cases, since $s(d)$ is even, it follows from the fact that $d^{p^{s(d)/2}+1} = 1$, and s divides $s(d)/2$.

Conversely, suppose that $\Gamma \cong \Gamma_{\mathscr{A}}[\overline{K_2}]$. Then in particular $C \leqslant \mathrm{Aut}(\Gamma)$, so by Theorem 4.2, d is a square in the case where $\Gamma = \mathrm{TCR}(q; d, s)$. Since $\infty 0$ is adjacent to $d1$, it follows from Remark 3.3 or 3.5 that $1 \in B(d, s)\, d$, that is, $d^{p^{is}+1} = 1$ for some integer i such that $0 \leqslant i < s(d)/s$. If this holds for $i = 0$ then $d = -1$, and in this case $\Gamma = \mathrm{CR}(q; d, s)$ (since for the twisted graphs the parameter $s(d)$ is even). Thus we may suppose that $i \neq 0$. Then $d^{p^{2is}-1} = 1$ but $d^{p^{is}-1} \neq 1$, and hence $s(d)$ divides $2is$ but $s(d)$ does not divide is. Thus $2is = ks(d)$ for some odd integer k. Since s divides $s(d)$, and in addition s is even in the case of $\mathrm{TCR}(q; d, s)$, the result follows. $\square$

For $q = 3$ there is only one cross ratio graph, namely $\mathrm{CR}(3; 2, 1)$ and it is not difficult to show that this graph is the disjoint union of three cycles of length 4, namely $(\infty 0, 12, 0\infty, 21)$, $(\infty 1, 20, 1\infty, 02)$ and $(\infty 2, 01, 2\infty, 10)$. This is the smallest example of a cross ratio graph being a lexicographic product; the quotient graph $\Gamma_{\mathscr{A}}$ is $3 \cdot C_2$, and $\mathrm{CR}(3; 2, 1) = (3 \cdot C_2)[\overline{K_2}] = 3 \cdot (C_2[\overline{K_2}]) = 3 \cdot C_4$. Also for $q = 5$, the quotient graph $\Gamma_{\mathscr{A}}$ for $\Gamma = \mathrm{CR}(5; 4, 1)$ is $5 \cdot C_3$ and $\mathrm{CR}(5; 4, 1) = (5 \cdot C_3)[\overline{K_2}] = 5 \cdot (C_3[\overline{K_2}])$. It turns out that all other cross ratio graphs are connected.

PROPOSITION 6.2. *Let $q = p^r$, where p is a prime $p, r \geqslant 1$, and $q \geqslant 3$. Suppose that Γ is a cross ratio graph $\mathrm{CR}(q; d, s)$ or $\mathrm{TCR}(q; d, s)$, for some d, s. Then one of the following holds:*

(a) $\Gamma = \mathrm{CR}(3; 2, 1) = 3 \cdot C_4$;
(b) $\Gamma = \mathrm{CR}(5; 4, 1) = 5 \cdot (C_3[\overline{K_2}])$;
(c) Γ *is connected.*

Proof. By the remarks above, we may suppose that $q \geqslant 4$. Let $G = \mathrm{Aut}(\Gamma) \cap (C \times \mathrm{P\Gamma L}(2, q))$. If Γ is not connected then the connected components form a G-invariant partition of $V(q)$. By Theorem 4.2, if $q \neq 5$, then the only non-trivial G-invariant partition of $V(q)$ is $\mathscr{A}$, but the blocks of $\mathscr{A}$ (of size 2) are not connected components of Γ. Hence if $q \neq 5$ then Γ is connected. Thus suppose that $q = 5$, and Γ is not as in case (b). Then $\Gamma = \mathrm{CR}(5; d, 1)$ with $d = 2$ or 3. By Theorem 4.2, $G = C \times \mathrm{PGL}(2, 5)$, and Γ has valency 4. Also by Proposition 6.1, Γ is not $\Gamma_{\mathscr{A}}[\overline{K_2}]$, so $\infty 0$ is adjacent to vertices in four distinct blocks of $\mathscr{A}$. Thus $\Gamma_{\mathscr{A}}$ has valency at least 4, and its connected components therefore have size at least 5. Since $\mathrm{PSL}(2, 5)$ is transitive on $\mathscr{A}$ (of degree 15) and has no proper subgroup of index 3, it follows that the quotient graph $\Gamma_{\mathscr{A}}$ is connected. Thus Γ has one or two components, and since $\mathrm{PSL}(2, 5)$ is vertex-transitive on Γ and has no subgroup of index 2, it follows that Γ is connected. $\square$

We now determine the full automorphism groups of the cross ratio graphs which are lexicographic products. For a simple group S, let $m(S)$ denote the minimal index of a proper subgroup of S. In the next theorem, and other results in this section we use results from [**10**, **12**] which depend on the finite simple group classification.

THEOREM 6.3. *Let $q = p^r$, where p is a prime, $r \geqslant 1$, and $q \geqslant 3$. Suppose that Γ is a cross ratio graph $\mathrm{CR}(q; d, s)$ or $\mathrm{TCR}(q; d, s)$ such that $\Gamma \cong \Gamma_{\mathscr{A}}[\overline{K_2}]$, for some d, s as in Proposition 6.1. Then one of the following holds:*

(a) $\Gamma = \mathrm{CR}(3;2,1) = 3\cdot C_4$ *and* $\mathrm{Aut}(\Gamma) = D_8 \,\mathrm{wr}\, S_3$;

(b) $\Gamma = \mathrm{CR}(5;4,1) = 5\cdot(C_3[\overline{K_2}])$ *and* $\mathrm{Aut}(\Gamma) = (S_2 \,\mathrm{wr}\, S_3) \,\mathrm{wr}\, S_5$;

(c) $\mathrm{Aut}(\Gamma) = S_2 \,\mathrm{wr}\, \mathrm{Aut}(\Gamma_{\mathscr{A}}) \cong S_2^{(q+1)q/2}\cdot\mathrm{Aut}(\Gamma_{\mathscr{A}})$ *and* $\mathrm{Aut}(\Gamma_{\mathscr{A}}) = \mathrm{Aut}(\Gamma) \cap \mathrm{P\Gamma L}(2,q)$.

Proof. The automorphism groups of the graphs in parts (a) and (b) are easily seen to be as stated. Thus by Proposition 6.2, we may assume that Γ is connected, and by Proposition 6.1 we may assume that $q \neq 3, 5$. Now $\mathrm{Aut}(\Gamma)$ contains $S_2 \,\mathrm{wr}\, \mathrm{Aut}(\Gamma_{\mathscr{A}})$ and equality holds provided $\mathrm{Aut}(\Gamma)$ preserves the partition $\mathscr{A}$. By Theorem 4.2 the only non-trivial partition of $V(q)$ which may be preserved by $\mathrm{Aut}(\Gamma)$ is $\mathscr{A}$. Consequently, if $\mathrm{Aut}(\Gamma)$ does not preserve $\mathscr{A}$, then it is primitive on $V(q)$. If $\mathrm{Aut}(\Gamma)$ were primitive on $V(q)$, then since $S_2 \,\mathrm{wr}\, \mathrm{Aut}(\Gamma_{\mathscr{A}})$ contains a transposition, $\mathrm{Aut}(\Gamma)$ would be the full symmetric group on $V(q)$, which is not the case. Hence $\mathrm{Aut}(\Gamma) = S_2 \,\mathrm{wr}\, \mathrm{Aut}(\Gamma_{\mathscr{A}})$. Now the group $\mathrm{Aut}(\Gamma_{\mathscr{A}})$ is primitive on $\mathscr{A}$ and contains a primitive subgroup G which is a 3-transitive subgroup of $\mathrm{P\Gamma L}(2,q)$ acting on unordered pairs from $\mathrm{PG}(1,q)$.

Suppose that $\mathrm{Aut}(\Gamma_{\mathscr{A}}) \neq G$, and let $G < A < \mathrm{Aut}(\Gamma_{\mathscr{A}})$ with G a maximal subgroup of A. In particular $q \neq 4$. Then it follows from [**10**] that one of the following holds:

(i) $A = A_{q+1}$ or S_{q+1} on pairs;

(ii) $q = 2^r$, and for some prime divisor r_0 of r, $\mathrm{soc}(A) = \mathrm{Sp}(2r_0, 2^{r/r_0})$ acting on $[\mathrm{Sp}(2r_0, 2^{r/r_0}) : \mathbf{O}^+(2r_0, 2^{r/r_0})]$;

(iii) $q = 7$, and $A = \mathrm{U}(3,3)\cdot 2$.

In case (i), the action of A is on unordered pairs from a set of size $q+1$; the neighbours of $\{\infty 0\}$ in $\Gamma_{\mathscr{A}}$ consist of pairs disjoint from $\{\infty 0\}$ and $(A_{q+1})_{\infty 0}$ is transitive on such pairs, so the valency of $\Gamma_{\mathscr{A}}$ is $\binom{q-1}{2}$, which contradicts the fact that this valency is $(q-1)s(d)/2s$ (see Theorem 4.2).

In case (ii), the stabiliser A_0 in A of $\{\infty 0\}$ is an orthogonal group of type $\mathbf{O}^+(2r_0, 2^{r/r_0})$. Suppose first that r_0 is odd. Then the derived group A_0' is the simple group $\Omega^+(2r_0, 2^{r/r_0})$, and by [**16**, 18.2], $\Omega^+(2r_0, 2^{r/r_0})$ must be involved in the action of A_0 on $\Gamma_{\mathscr{A}}(\{\infty 0\})$ (since A is primitive on $\mathscr{A}$). However $\Gamma_{\mathscr{A}}(\{\infty 0\})$ has size $(q-1)s(d)/2s$, which is less than $m(A_0')$ when $r_0 \geqslant 5$ (see [**3**]). If $r_0 = 3$, then $\Omega^+(6, 2^{r/3}) \cong \mathrm{PSL}(4, 2^{r/3})$, and the valency of $\Gamma_{\mathscr{A}}$ is $(2^r-1)k$ with $k = s(d)/2s$ an integer less than r. There is no subgroup of $\mathrm{PSL}(4, 2^{r/3})$ of index $(2^r-1)k$ (see for example, [**14**, Theorem 3.1], noting that the indices of the possible subgroups are all divisible by $(2^{r/3})^2$). This leaves the case $r_0 = 2$; here $\Omega^+(4, 2^{r/2}) \cong \mathrm{PSL}(2, 2^{r/2}) \times \mathrm{PSL}(2, 2^{r/2})$, and $A_0 \cap \mathrm{PSL}(2,q) = D_{2(q-1)}$ acts on $\Gamma_{\mathscr{A}}(\{\infty 0\})$ with orbits of length $q-1$ or $2(q-1)$. Moreover $(A_0 \cap \mathrm{PSL}(2,q))'$ intersects the two simple direct factors of $\Omega^+(4, 2^{r/2})$ in cyclic groups of orders $2^{r/2} \pm 1$. Hence both simple direct factors are involved in the action of A_0 on $\Gamma_{\mathscr{A}}(\{\infty 0\})$. However $\Omega^+(4, 2^{r/2})$ has no transitive representation of degree $q-1$ or $2(q-1)$ in which a stabiliser meets each direct factor in a proper subgroup.

In case (iii), $r = 1$ and so by Proposition 6.1, $d = -1$ and $\Gamma_{\mathscr{A}}$ has valency $(q-1)/2 = 3$. The stabiliser A_0 in A of $\{\infty 0\}$ is a parabolic subgroup $3^{1+2}:8:2$ and has no subgroup of index 3. Thus we have shown that in all cases $\mathrm{Aut}(\Gamma) = G$. □

We may now assume that, for $\Gamma = \mathrm{CR}(q;d,s)$ or $\mathrm{TCR}(q;d,s)$, each vertex is adjacent to at most one vertex of any block of $\mathscr{A}$. We say that a graph Γ is an *r-fold cover* of its quotient graph $\Gamma_{\mathscr{A}}$ if the blocks of $\mathscr{A}$ all have size r and, for each pair

A, A' of adjacent blocks of $\Gamma_{\mathscr{A}}$, the subgraph of Γ induced on $A \cup A'$ is a perfect matching $r \cdot K_2$. A 2-fold cover is usually called a *double cover*. The case $q = 5$ is unusual in that the subgroup of $\mathrm{Aut}(\Gamma)$ studied in Theorem 4.2 may have extra block systems in $V(5)$. Therefore we deal with this case next.

THEOREM 6.4. *The graphs* $\Gamma = \mathrm{CR}(5; d, 1)$*, with* $d = 2$ *or* 3*, are double covers of* $\Gamma_{\mathscr{A}}$*, the line graph of the Petersen graph, and* $\mathrm{Aut}(\Gamma) = C \times \mathrm{PGL}(2, 5)$.

We thank Brendan McKay and Gordon Royle for helping us to identify these two graphs. The first one $\mathrm{CR}(5; 2, 1)$ is the line graph of the tensor product $P * K_2$ where P is the Petersen graph. The graph $P * K_2$ has 20 vertices and valency 3; its adjacency matrix is

$$\begin{pmatrix} 0 & A_P \\ A_P & 0 \end{pmatrix}$$

where A_P is the adjacency matrix of the Petersen graph. The other graph $\mathrm{CR}(5; 3, 1)$ has girth 5 and so is not a line graph. Both graphs appear in the list of vertex-transitive graphs of order less than 31 to be found at http://www.cs.uwa.edu.au/gordon/remote/trans/index.html, which extends the work described in [**13**]. They are two of the eight connected vertex-transitive graphs of order 30 and valency 4 which are not Cayley graphs.

Proof of Theorem 6.4. Let $\Gamma = \mathrm{CR}(5; d, 1)$ with $d = 2$ or 3, so $\mathrm{Aut}(\Gamma)$ contains $G = C \times \mathrm{PGL}(2, 5)$. Here Γ has valency 4, and $G_{\infty 0} \cong D_8$ has orbits of lengths $1, 1, 4, 4, 4, 8, 8$ in $V(5)$. Also the set of points at distance 2 from $\infty 0$ in Γ is one of the orbits of length 8. A small computation shows that Γ is a double cover of its quotient graph $\Gamma_{\mathscr{A}}$, and $\Gamma_{\mathscr{A}}$ is distance transitive with intersection array $\{4, 2, 1; 1, 1, 4\}$. Thus $\Gamma_{\mathscr{A}}$ is the line graph of the Petersen graph (see [**1**, p. 222]), with automorphism group $\mathrm{PGL}(2, 5)$. This means that the subgroup of $\mathrm{Aut}(\Gamma)$ which preserves $\mathscr{A}$ is precisely $G = C \times \mathrm{PGL}(2, 5)$.

Suppose that $\mathrm{Aut}(\Gamma)$ does not preserve $\mathscr{A}$. If $\mathrm{Aut}(\Gamma)$ is primitive on $V(5)$ of degree 30, then (see [**5**]), its socle is $\mathrm{PSL}(2, 29)$ or A_{30}, both of which are 2-transitive, contradicting the fact that Γ is not a complete graph. Thus $\mathrm{Aut}(\Gamma)$ is imprimitive. Let B be a non-trivial block of imprimitivity for $\mathrm{Aut}(\Gamma)$ in $V(5)$ containing $\infty 0$. If $0\infty \notin B$ then, as B is a union of $G_{\infty 0}$-orbits, and $|B|$ divides 30, it follows that $|B| = 5$. However this means that B meets five blocks of $\mathscr{A}$ which form a block for G in $\mathscr{A}$. However G has no blocks of imprimitivity in $\mathscr{A}$ of size 5. Hence $0\infty \in B$. Since B is a block for G, it is a union of a subset $B_{\mathscr{A}}$ of blocks of $\mathscr{A}$, and $B_{\mathscr{A}}$ is a non-trivial block for G in $\Gamma_{\mathscr{A}}$. Thus $B_{\mathscr{A}}$ is an antipodal block for $\Gamma_{\mathscr{A}}$ of size 3, and $|B| = 6$. Moreover B is the unique non-trivial block for $\mathrm{Aut}(\Gamma)$ containing $\infty 0$, and so its stabiliser induces a primitive subgroup of S_6 on it. Such a subgroup has order divisible by 5, and so $|\mathrm{Aut}(\Gamma)|$ is divisible by 25. Thus a Sylow 5-subgroup of $\mathrm{Aut}(\Gamma)$ has orbits of length 5, 25, and hence there is a 5-subgroup S with five fixed points, and five orbits of length 5. We may assume that S fixes $\infty 0$. Then S fixes setwise the set of four points adjacent to $\infty 0$, and eight points at distance 2 from $\infty 0$. This contradiction proves that $\mathrm{Aut}(\Gamma)$ is as stated. □

Now we may assume in addition that $q \neq 3, 5$, and hence by Theorem 4.2, that the unique non-trivial block system for $\mathrm{Aut}(\Gamma) \cap (C \times \mathrm{P\Gamma L}(2, q))$ in $V(q)$ is $\mathscr{A}$. We prove first that this subgroup is the largest subgroup of $\mathrm{Aut}(\Gamma)$ which preserves $\mathscr{A}$.

PROPOSITION 6.5. *Let $q = p^r$, where p is a prime, $r \geqslant 1$, and $q \neq 3, 5$. Suppose that Γ is a cross ratio graph* $\mathrm{CR}(q; d, s)$ *or* $\mathrm{TCR}(q; d, s)$, *for some d, s, such that Γ is not isomorphic to $\Gamma_{\mathscr{A}}[\overline{K_2}]$. Then the stabiliser in* $\mathrm{Aut}(\Gamma)$ *of the block system $\mathscr{A}$ is the subgroup* $\mathrm{Aut}(\Gamma) \cap (C \times \mathrm{P\Gamma L}(2, q))$.

Proof. Suppose that the assertion is false for Γ, and let A be a subgroup of $\mathrm{Aut}(\Gamma)$ which preserves $\mathscr{A}$, and which contains G as a maximal subgroup. Let K denote the kernel of the action induced by A on $\mathscr{A}$. Since $\infty 0$ is joined to at most one vertex of each block of $\mathscr{A}$, the stabiliser $K_{\infty 0}$ fixes pointwise each vertex adjacent to $\infty 0$, and since Γ is connected, by Proposition 6.2, it follows that $K_{\infty 0} = 1$. Thus $|K| \leqslant 2$. In fact $K = C \cap \mathrm{Aut}(\Gamma)$, so $K = C$ unless $\Gamma = \mathrm{TCR}(q; d, s)$ and d is not a square, in which case $K = 1$. Then since $A \neq G$, we must have $q \neq 4$. Let $\bar{A} = A/K$, the subgroup of $\mathrm{Sym}(\mathscr{A})$ induced by A, and similarly let $\bar{G} = GK/K$. By Theorem 4.2, $\bar{G} = \mathrm{PGL}(2, q) \cdot \langle \psi^s \rangle$ if $\Gamma = \mathrm{CR}(q; d, s)$, and if $\Gamma = \mathrm{TCR}(q; d, s)$ then $\bar{G} = \mathrm{M}(q, s/2)$ if d is a square, and $\bar{G} = \mathrm{PGL}(2, q) \cdot \langle \psi^{s/2} \rangle$ if d is not a square (see Theorem 4.2). In all cases $\bar{G}$ is primitive on $\mathscr{A}$. Thus we have an inclusion $\bar{G} < \bar{A}$ of primitive groups with $\bar{G}$ maximal in $\bar{A}$. By [**10**], $\bar{A}$ is one of the groups listed in (i)–(iii) in the proof of Theorem 6.3. In this case we note that there are one or two edges of Γ between adjacent blocks of $\Gamma_{\mathscr{A}}$, and hence $\Gamma_{\mathscr{A}}$ has valency $v_{\mathscr{A}} = 2(q-1)\,s(d)/s$ or $(q-1)\,s(d)/s$ respectively, and in either case $v_{\mathscr{A}}$ is a multiple of $q-1$.

In case (i), $\bar{A} = A_{q+1}$ or S_{q+1} acting an unordered pairs from a set of size $q+1$, and arguing as in the proof of Theorem 6.3, $\Gamma_{\mathscr{A}}$ has valency $\binom{q-1}{2}$. Since $v_{\mathscr{A}} = (q-1)\,k$, where k divides $2r$, it follows that $q = 8$, and both $\Gamma_{\mathscr{A}}$ and Γ have valency $7s(d)/s = 21$, so $s(d) = 3$, $s = 1$. By [**2**, p. 37], the central product $2 \cdot A_9$ has no transitive permutation representation of degree 72, and hence A contains a subgroup A_9 acting on $V(q)$ (as on the set of ordered pairs of distinct points from $\mathrm{PG}(1, 8)$). However the stabiliser in A_9 of $\infty 0$ is transitive on the 42 pairs xy with x, y distinct from $\infty, 0$. Hence A_9 does not preserve adjacency in Γ.

In case (ii), $q = 2^r \geqslant 8$ and $\mathrm{soc}(\bar{A}) = \mathrm{Sp}(2r_0, 2^{r/r_0})$, for some prime divisor r_0 of r, and if r_0 is odd then the derived group $\bar{A}'_{\{\infty 0\}}$ of $\bar{A}_{\{\infty 0\}}$ is the simple group $\Omega^+(2r_0, 2^{r/r_0})$ and acts non-trivially on $\Gamma_{\mathscr{A}}(\{\infty 0\})$ of degree $v_{\mathscr{A}} = (q-1)\,k$, where $k = 2s(d)/s$ or $s(d)/s$, an integer at most $2r$. If $r = r_0$ then the action of $\bar{A}$ on $\mathscr{A}$ is 2-transitive, which is not allowed since $\Gamma_{\mathscr{A}}$ is not a complete graph. Hence $r > r_0$. If $r_0 \geqslant 5$ then, by [**3**], $v_{\mathscr{A}}$ is less than $m(\bar{A}'_{\{\infty 0\}})$, so $r_0 \leqslant 3$. If $r_0 = 3$, then $\Omega^+(6, 2^{r/3}) \cong \mathrm{PSL}(4, 2^{r/3})$, but there is no subgroup of $\mathrm{PSL}(4, 2^{r/3})$ of index $(2^r - 1)\,k$ with $k \leqslant 2r$ (see [**14**, Theorem 3.1]). This leaves the case $r_0 = 2$; here $\Omega^+(4, 2^{r/2}) \cong \mathrm{PSL}(2, 2^{r/2}) \times \mathrm{PSL}(2, 2^{r/2})$, and the argument in the proof of Theorem 6.3 works in this case also, proving that there are no possibilities. Finally in case (iii), $q = 7$ so $r = s(d) = s = 1$, $\mathrm{Aut}(\Gamma)$ contains C and so $v_{\mathscr{A}} = 6$. By [**2**], the subgroup $\bar{A}_{\{\infty 0\}} = 3^{1+2}{:}8{:}2$, which has no subgroup of index 6. $\square$

Finally we determine $\mathrm{Aut}(\Gamma)$. The case $q = 7$ required special attention. We are grateful to Alice Niemeyer for helping us investigate the automorphism groups of the graphs $\mathrm{CR}(7; d, 1)$ using GRAPE [**15**].

THEOREM 6.6. *Let $q = p^r$, where p is a prime, $r \geqslant 1$, and $q \neq 3, 5$. Suppose that Γ is a cross ratio graph* $\mathrm{CR}(q; d, s)$ *or* $\mathrm{TCR}(q; d, s)$, *for some d, s, such that Γ is not isomorphic to $\Gamma_{\mathscr{A}}[\overline{K_2}]$. Then* $\mathrm{Aut}(\Gamma) \cap (C \times \mathrm{P\Gamma L}(2, q))$ (*as determined in Theorem* 4.2).

Proof. Suppose now that $\mathrm{Aut}(\Gamma) \neq G$ where $G = \mathrm{Aut}(\Gamma) \cap (C \times \mathrm{P\Gamma L}(2,q))$, and let $A \leqslant \mathrm{Aut}(\Gamma)$ be a subgroup which contains G as a maximal subgroup. Since $\mathscr{A}$ is the unique non-trivial block system preserved by G, it follows from Proposition 6.5 that A is primitive on $V(q)$. Let S denote the socle of G', so $S = \mathrm{PSL}(2,q)$, and let N denote the socle of A. Note that since $q \neq 3,5$, either $(q, m(S))$ ($m(S)$ defined above) is one of $(7,7),(9,6),(11,11)$, or $m(S) = q+1$.

Since $|V(q)| = q(q+1)$ is not a prime power, it follows that N is not elementary abelian, and hence $N = T^k$ for some positive integer k and non-abelian simple group T. Suppose first that $k \geqslant 2$. Then, by the O'Nan–Scott theorem (see [**11**]), it follows that $q(q+1)$ is either n^k where $n = |T:R|$ for some proper subgroup R of T, or $|T|^l$ with $k/2 \leqslant l \leqslant k$. In particular, $q \neq 7,9,11$, and so $m(S) = q+1$. If $S \nleqslant N$, then S permutes the simple direct factors of N non-trivially. This implies that $k \geqslant m(S) = q+1$, but then $q(q+1)$ cannot have the required form. Thus $S \leqslant N$, and S is a subgroup of the direct product $S_1 \times \ldots \times S_k$ of the projections of S onto the k simple direct factors of N. Since S is simple this means that S is isomorphic to a subgroup of T. In particular $q(q+1) < |S| \leqslant |T|$, and so $q(q+1) = n^k$ with $n = |T:R| \geqslant m(T)$. In this case the action of A is said to be of product type and the stabiliser $N_{\infty 0}$ is $R_1 \times \ldots \times R_k$ with each $R_i \cong R$. If $S_i \leqslant R_i$ for each i, then $N_{\infty 0} \geqslant S_1 \times \ldots \times S_k \geqslant S$, which is not the case. Hence for some i, the subgroup $R_i \cap S_i$ is a proper subgroup of S_i and we have $S_i \cong S$ and $m(S) \leqslant |S_i:(R_i \cap S_i)| \leqslant |T:R| = n$. This however implies that $n \geqslant q+1$ so $q(q+1) \neq n^k$.

Thus $N = T$ is a non-abelian simple group, and since A/N is therefore soluble we have $S \leqslant N$. We have a maximal factorisation $A = A_{\infty 0} G$ with $|A:A_{\infty 0}| = q(q+1)$. By [**12**], it follows that $q = 7$, and $\mathrm{soc}(A) = \mathrm{PSL}(3,4)$, $G = C \times \mathrm{PGL}(2,7)$. However, for each of the graphs $\Gamma = \mathrm{CR}(7;d,1)$ with $d \neq -1$, computation using GRAPE [**15**] showed that $\mathrm{Aut}(\Gamma) = C \times \mathrm{PGL}(2,7)$. □

7. *Summary, and isomorphisms of cross ratio graphs*

The previous section contains a determination of the full automorphism groups of all the cross ratio graphs. Here we give a summary of those results, and then determine all occurrences of isomorphism between cross ratio graphs. The results of the next theorem follow immediately from the results proved in Section 6.

THEOREM 7.1 (Summary of automorphism group results). *Let $q = p^r$ for a prime p and $r \geqslant 1$. Let $d \in \mathrm{GF}(q)$, $d \neq 0,1$, and let s be a divisor of $s(d)$. Let Γ be a cross ratio graph (twisted or untwisted) defined on $V(q)$.*

(a) *If $\Gamma = \mathrm{CR}(q;d,s)$ then one of the following holds:*

(i) $\Gamma = \mathrm{CR}(3;2,1) = 3 \cdot C_4$ *and* $\mathrm{Aut}(\Gamma) = D_8 \,\mathrm{wr}\, S_3$.

(ii) $\Gamma = \mathrm{CR}(5;4,1) = 5 \cdot (C_3[\overline{K_2}])$ *and* $\mathrm{Aut}(\Gamma) = (S_2 \,\mathrm{wr}\, S_3) \,\mathrm{wr}\, S_5$.

(iii) *q is odd, $q \geqslant 7$, $d = -1$ (so $s = 1$), $\Gamma = \mathrm{CR}(q;-1,1) \cong \Gamma_{\mathscr{A}}[\overline{K_2}]$ is connected, and we have* $\mathrm{Aut}(\Gamma) = S_2 \,\mathrm{wr}\, \mathrm{P\Gamma L}(2,q)$.

(iv) *$s(d)/s$ is even, $\Gamma = \mathrm{CR}(q;d,s) \cong \Gamma_{\mathscr{A}}[\overline{K_2}]$ is connected, and we have* $\mathrm{Aut}(\Gamma) = S_2 \,\mathrm{wr} (\mathrm{PGL}(2,q) \cdot \langle \psi^s \rangle)$.

(v) *$s(d)/s$ is odd, $d \neq -1$, $\Gamma = \mathrm{CR}(q;d,s)$ is connected and is a cover of $\Gamma_{\mathscr{A}}$, and we have* $\mathrm{Aut}(\Gamma) = C \times (\mathrm{PGL}(2,q) \cdot \langle \psi^s \rangle)$.

(b) *If $\Gamma = \mathrm{TCR}(q;d,s)$, so in addition q is odd, s is even, and $d-1$ is a square, then one of the following holds:*

(i) *d is a square, $s(d)/s$ is even, $\Gamma = \mathrm{TCR}(q;d,s) \cong \Gamma_{\mathscr{A}}[\overline{K_2}]$ is connected, and* $\mathrm{Aut}(\Gamma) = S_2 \,\mathrm{wr}\, \mathrm{M}(s/2, q)$.

(ii) *d is a square, $s(d)/s$ is odd, $\Gamma = \mathrm{TCR}(q;d,s)$ is connected and is a cover of $\Gamma_{\mathscr{A}}$, and we have* $\mathrm{Aut}(\Gamma) = C \times \mathrm{M}(s/2, q)$.

(iii) *d is not a square, $\Gamma = \mathrm{TCR}(q;d,s)$ is connected (and there is at most one edge of Γ between any pair of blocks of $\mathscr{A}$), and* $\mathrm{Aut}(\Gamma) = \langle \mathrm{M}(s/2, q), \alpha\psi^{s/2} \rangle$.

Finally we examine isomorphism between cross ratio graphs.

THEOREM 7.2 (Isomorphism of cross ratio graphs). *Suppose that $\Gamma = \mathrm{CR}(q;d,s)$ or $\mathrm{TCR}(q;d,s)$, and that $\Gamma' = \mathrm{CR}(q';d',s')$ or $\mathrm{TCR}(q';d',s')$, for appropriate parameters q, d, s, q', d', s'. Suppose further that Γ and Γ' are isomorphic. Then $q = q'$, $s(d) = s(d')$, $s = s'$, and Γ, Γ' are either both untwisted graphs or both twisted graphs. Moreover, either d and d' are in the same $\langle\psi\rangle$-orbit, or Γ and Γ' are lexicographic products in case* (a) (iv) *or* (b) (i) *of Theorem* 7.1, *and d^{-1} is in the same $\langle\psi\rangle$-orbit as d'. Conversely, if either of these sets of conditions holds, then $\Gamma \cong \Gamma'$.*

Proof. Suppose that Γ and Γ' are isomorphic and let $\varphi : V(q) \longrightarrow V(q')$ be an isomorphism. Since $|V(q)| = |V(q')|$, we have $q = q'$, and so $\varphi \in \mathrm{Sym}(V(q))$, and $\mathrm{Aut}(\Gamma)^{\varphi} = \mathrm{Aut}(\Gamma')$. Also, since Γ and Γ' have the same valency, $s(d)/s = s(d')/s'$. We consider the possibilities for Γ according to the case of Theorem 7.1 it belongs to, and use the fact that $\mathrm{Aut}(\Gamma)^{\varphi} = \mathrm{Aut}(\Gamma')$. If Γ is in case (a) (i) or (ii) then Γ is disconnected, and so Γ' is disconnected and hence is equal to Γ, and $d = d' = -1$. If Γ is in case (a) (iii), then considering the automorphism groups of Γ and Γ' specified in Theorem 7.1, we see that Γ' also lies in case (a) (iii), so $d = d' = -1$ and $\Gamma = \Gamma'$. (Since -1 is fixed by ψ, the converse assertion holds in these cases also.) If Γ is in case (a) (v), then consideration of the automorphism groups again implies that Γ' also lies in case (a) (v) and $s' = s$ so $s(d') = s(d)$. Here φ normalises $\mathrm{Aut}(\Gamma)$, so $\varphi \in C \times \mathrm{P\Gamma L}(2, q)$, and by Theorem 4.2, d, d' are in the same $\langle\psi\rangle$-orbit, and this condition is sufficient for isomorphism.

The remaining case for untwisted cross ratio graphs Γ is the case where Γ lies in (a) (iv), and this case requires more analysis. Consideration of the automorphism groups again implies that Γ' also lies in case (a) (iv) and $s' = s$ so $s(d') = s(d)$. Here φ normalises $S_2 \,\mathrm{wr}(\mathrm{PGL}(2, q) \cdot \langle\psi^s\rangle)$, so $\varphi \in S_2 \,\mathrm{wr}\, \mathrm{P\Gamma L}(2, q)$. We may assume that φ maps the arc $(\infty 0, 1d)$ of Γ to the arc $(\infty 0, 1d')$ of Γ'. The subset of $S_2 \,\mathrm{wr}\, \mathrm{P\Gamma L}(2, q)$ which maps the ordered pair $(\infty 0, 1d)$ to $(\infty 0, 1e)$, for some e, consists of elements φ of the form $\varphi_1\varphi_2$ where φ_1 lies in the base group of the wreath product, and $\varphi_2 \in \mathrm{P\Gamma L}(2, q)$ maps $\infty 0$ to either $\infty 0$ or 0∞, and φ_2 maps $1d$ to either $1e$ or $e1$. The possible elements φ_2 form a union of four cosets of $\langle\psi\rangle$, namely $\langle\psi\rangle$, $t_{1,0,0,d}\langle\psi\rangle$, $t_{0,1,1,0}\langle\psi\rangle$, and $t_{0,d,1,0}\langle\psi\rangle$. The possibilities for e are the $\langle\psi\rangle$-orbits of either d or d^{-1}. Conversely if d' is in the $\langle\psi\rangle$-orbit of d or d^{-1}, then there exists an element $\varphi_2 \in \mathrm{P\Gamma L}(2, q)$ which fixes $\infty 0$ and maps $1d$ to $1d'$ or $d'1$, and hence there is an element $\varphi \in S_2 \,\mathrm{wr}\, \mathrm{P\Gamma L}(2, q)$ which induces an isomorphism from Γ to Γ'.

Now we may assume that both Γ and Γ' are of twisted type, and since $\mathrm{Aut}(\Gamma)^{\varphi} = \mathrm{Aut}(\Gamma')$, both graphs lie in the same case of Theorem 7.1, and we have $s' = s$, $s(d') = s(d)$, and φ normalises $\mathrm{Aut}(\Gamma)$. If the graphs lie in case (b) (ii) or (iii), then all the assertions follow from Theorem 4.2. This leaves case (b) (i), where we have $\varphi \in S_2 \,\mathrm{wr}\, \mathrm{P\Gamma L}(2, q)$. As in the previous paragraph we may assume that φ maps the arc $(\infty 0, 1d)$

of Γ to the arc $(\infty 0, 1d')$ of Γ', and so $\varphi = \varphi_1 \varphi_2$ with φ_2 in one of the specified cosets of $\langle \psi \rangle$. The argument above holds here also, proving that d' is in the same $\langle \psi \rangle$-orbit as d or d^{-1}, and that this condition is sufficient for isomorphism of Γ and Γ'. $\square$

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