A distance labelling problem for hypercubes

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Abstract

Let $i_1 \geq i_2 \geq i_3 \geq 1$ be integers. An $L(i_1, i_2, i_3)$-labelling of a graph $G = (V, E)$ is a mapping $\phi : V \to \{0, 1, 2, \ldots\}$ such that $|\phi(u) - \phi(v)| \geq i_t$ for any $u, v \in V$ with $d(u, v) = t$, $t = 1, 2, 3$, where $d(u, v)$ is the distance in G between u and v. The integer $\phi(v)$ is called the label assigned to v under ϕ, and the difference between the largest and the smallest labels is called the span of ϕ. The problem of finding the minimum span, $\lambda_{i_1,i_2,i_3}(G)$, over all $L(i_1, i_2, i_3)$-labellings of G arose from channel assignment in cellular communication systems, and the related problem of finding the minimum number of labels used in an $L(i_1, i_2, i_3)$-labelling was originated from recent studies of the scalability of optical networks. In this paper we study the $L(i_1, i_2, i_3)$-labelling problem for hypercubes Q_d ($d \geq 3$) and obtain upper and lower bounds on $\lambda_{i_1,i_2,i_3}(Q_d)$ for any (i_1, i_2, i_3).

Key words: channel assignment, labelling, λ-number, distance-colouring, hypercube, binary code

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1 Introduction

Let $d \geq 1$ be an integer. The d-dimensional cube Q_d is the graph with vertices the binary code words of length d such that two vertices are adjacent if and only if they differ in exactly one position. Motivated by radio channel assignment and investigation of scalability of optical networks, a few labelling problems [5, 6, 10, 14, 15, 17, 18, 20, 21] on hypercubes with distance constraints have attracted considerable attention in recent years.

Let $G = (V, E)$ be a graph and $i_1 \geq i_2 \geq \cdots \geq i_k$ (≥ 1) a sequence of integers. An $L(i_1, i_2, \ldots, i_k)$-*labelling* of G is a mapping $\phi : V \to \{0, 1, 2, \ldots\}$ such that

$$|\phi(u) - \phi(v)| \geq i_t, \ t = 1, 2, \ldots, k \tag{1}$$

for any $u, v \in V$ with $d(u, v) = t$, where $d(u, v)$ is the distance in G between u and v. The integer $\phi(u)$ is called the *label* of u under ϕ, and $\mathrm{sp}(G; \phi) := \max_{v \in V(G)} \phi(v) - \min_{v \in V(G)} \phi(v)$ is called the *span* of ϕ. Without loss of generality we will always assume $\min_{v \in V(G)} \phi(v) = 0$, so that $\mathrm{sp}(G; \phi) = \max_{v \in V(G)} \phi(v)$. The minimum span over all $L(i_1, i_2, \ldots, i_k)$-labellings of G, namely,

$$\lambda_{i_1,i_2,\ldots,i_k}(G) := \min_{\phi} \mathrm{sp}(G; \phi),$$

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is called the $\lambda_{i_1,i_2,\ldots,i_k}$-*number* of G. A related invariant, $\chi_{i_1,i_2,\ldots,i_k}(G)$, is the minimum number of labels required in an $L(i_1, i_2, \ldots, i_k)$-labelling of G. In the context of channel assignment [6, 7, 8], G can be interpreted as an interference graph of a communication network, $\phi(u)$ is the channel assigned to transmitter u, (1) is the separation requirement for transmitters with distance at most k, and $\lambda_{i_1,i_2,\ldots,i_k}(G)$ is the minimum span of a channel assignment under such constraints. In a different scenario, we may think of packing vertices of G in a sufficiently large number of bins (say, at least $i_1|V|$ bins), which are labelled $0, 1, 2, \ldots$ sequentially, in such a way that, for $t = 1, 2, \ldots, k$, any two bins (possibly identical) with distance $< i_t$ do not contain distinct vertices with distance t or less in G. In this model, bin j contains vertices in $\phi^{-1}(j)$ for $j = 0, 1, 2, \ldots$, empty bins correspond to unused labels, and $\lambda_{i_1,i_2,\ldots,i_k}(G)$ is the minimum of the largest label of a used bin, with minimum over all possible ways of packing. An unused label between 0 and the largest label used is called a *hole*, and the meaning of a *no-hole* $L(i_1, i_2, \ldots, i_k)$-labelling is self-evident. So far most research on the $L(i_1, i_2, \ldots, i_k)$-labelling problem has focused on the case where $k = 2$; see [1] for a recent survey with over one hundred references on λ_{i_1,i_2} and related topics.

A related problem is to colour the vertices of a graph G such that any two vertices of distance at most k receive different colours. Such a colouring is called a $\bar{k}$-*colouring* in [17] and the minimum number of colours needed in a $\bar{k}$-colouring of G is denoted by $\chi_{\bar{k}}(G)$. Clearly,

$$\chi_{\bar{k}}(G) = \chi_{1,1,\ldots,1}(G) = \chi(G^k)$$

where G^k is the kth power of G and χ denotes the chromatic number. Thus the $\bar{k}$-colouring problem is the same as the colouring problem for power graphs. This problem has a long history [11] and is active in recent years (see e.g. [10, 14, 17, 20]) with motivation from studying the scalability of optical networks. Anthony W. To (personal communication) observed that, for any $i_1 \geq i_2 \geq \cdots \geq i_k \geq 1$, we have

$$\chi_{\bar{k}}(G) = \chi_{i_1,i_2,\ldots,i_k}(G). \tag{2}$$

In fact, since any $L(i_1, i_2, \ldots, i_k)$-labelling is an $L(1, 1, \ldots, 1)$-labelling, we have $\chi_{1,1,\ldots,1}(G) \leq \chi_{i_1,i_2,\ldots,i_k}(G)$. On the other hand, we can magnify any $L(1, 1, \ldots, 1)$-labelling ϕ to get an $L(i_1, i_2, \ldots, i_k)$-labelling, namely, $\psi(v) = i_1\phi(v)$ for $v \in V$. It is obvious that ψ uses the same number of labels as ϕ. Hence $\chi_{i_1,i_2,\ldots,i_k}(G) \leq \chi_{1,1,\ldots,1}(G)$ and (2) follows. Another observation is the following relation:

$$\lambda_{1,1,\ldots,1}(G) = \chi_{\bar{k}}(G) - 1. \tag{3}$$

In fact, any $L(1, 1, \ldots, 1)$-labelling of G with minimum span must be no-hole. Hence $\lambda_{1,1,\ldots,1}(G) \leq \chi_{\bar{k}}(G) - 1$. This together with $\lambda_{1,1,\ldots,1}(G) \geq \chi_{\bar{k}}(G) - 1$ implies (3).

1.1 Literature review

In [17, line 12, pp.185] Wan proved

$$d + 1 \leq \chi_{\bar{2}}(Q_d) \leq 2^{\lceil \log_2(d+1) \rceil} \tag{4}$$

and conjectured that the upper bound is the exact value of $\chi_{\bar{2}}(Q_d)$. According to [15], this conjecture was disproved by $13 \leq \chi_{\bar{2}}(Q_8) \leq 14$, which was obtained independently by Hougardy

[19] in 1991 and Royle [9, Section 9.7] in 1993. In [10], Kim, Du and Pardalos proved that

$$2d \le \chi_{\bar{3}}(Q_d) \le 2^{\lceil \log_2 d \rceil + 1}. \tag{5}$$

In the same paper they also gave an upper bound on $\chi_{\bar{k}}(Q_d)$ for $k > 3$, which was improved by Ngo, Du and Graham in [14]. In [15] it was proved that $\lim_{d\to\infty} \chi_{\bar{2}}(Q_d)/d = 1$ and $\lim_{d\to\infty} \chi_{\bar{3}}(Q_d)/d = 2$. All these results on Q_d were obtained via coding theory.

In [21, Theorem 2.5] it was proved that the upper bound in (4) is valid for the family $\mathcal{G}$ of connected graphs whose automorphism group contains a vertex-transitive Abelian subgroup. This came along with an upper bound [21, Theorem 2.5] on $\lambda_{i_1,i_2}(G)$ for any $G \in \mathcal{G}$, which can be restated as follows:

$$\lambda_{i_1,i_2}(G) \le 2^p \max\{i_2, \lceil i_1/2 \rceil\} + 2^q(i_1 - \max\{i_2, \lceil i_1/2 \rceil\}) - i_1, \tag{6}$$

where $p := \lceil \log_2(d+1) \rceil$ and $q := \max\{d+1+p-2^p, 0\}$ with d the degree of G. Since all hypercubes are members of $\mathcal{G}$, (6) applies to Q_d and in the special case where $(i_1, i_2) = (2, 1)$ it implies the upper bound on $\lambda_{2,1}(Q_d)$ obtained earlier in [18]. It is well known that Q_d is the Cartesian product $K_2 \Box K_2 \Box \cdots \Box K_2$ (d factors), where K_2 is the complete graph with two vertices. In general, the Cartesian product $K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_d}$ of complete graphs is called a Hamming graph, and results on its λ_{i_1,i_2}-number can be found in [2, 3, 4, 5, 21]. (See also [22] for a recent survey on distance-labelling problems for Hamming graphs and hypercubes.) In particular, Theorem 2.9 and Lemma 5.1 in [21] imply the following interesting "sandwich theorem" (which was not stated explicitly in [21]): Suppose $2i_2 \ge i_1 \ge 2$. Then for any positive integers n_1, n_2, d such that $n_1 > d \ge 2$, n_2 divides n_1 and each prime factor of n_1 is no less than d, any positive integers $n_3, \ldots, n_d$ which are less than or equal to n_2, and any subgraph G of $K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_d}$ which contains a copy of $K_{n_1} \Box K_{n_2}$ as a subgraph, we have $\lambda_{i_1,i_2}(G) = (n_1 n_2 - 1) i_2$. A similar sandwich result was recently obtained in [2] for $\lambda_{2,1}$, $\lambda_{1,1}$ and six other invariants for Hamming graphs under the condition that n_1 is sufficiently large with respect to $n_2, \ldots, n_d$.

1.2 Main results

In this paper we study the $L(i_1, i_2, i_3)$-problem for hypercubes. As above, denote

$$p = p(d) := \lceil \log_2(d+1) \rceil \tag{7}$$

$$q = q(d) := \max\{d + 1 + \lceil \log_2(d+1) \rceil - 2^{\lceil \log_2(d+1) \rceil}, 0\}. \tag{8}$$

Then $q \le p$ and

$$2^{p-1} \le d \le 2^p - 1.$$

Note that d is a power of 2 if and only if $d = 2^{p-1}$, that is, $d \ne 2^{p-1}$ if and only if d is not a power of 2.

The main result of this paper is the following theorem. An $L(i_1, i_2, i_3)$-labelling is said to be *balanced* if each label used is assigned to the same number of vertices.

Theorem 1 *Let $d \geq 3$ be an integer, and let p, q be as defined in (7), (8) respectively. Then, for any integers $i_1 \geq i_2 \geq i_3 \geq 1$,*

$$i_2(d-1) + i_1 \leq \lambda_{i_1,i_2,i_3}(Q_d) \leq \begin{cases} 2^p(i_3+n) + 2^q(i_1-n) - i_1, & d \neq 2^{p-1} \\ (2^p-2)n + i_1, & d = 2^{p-1}. \end{cases} \tag{9}$$

where $n := \max\{i_2, \lceil i_1/2 \rceil\}$, and we can give explicitly balanced $L(i_1,i_2,i_3)$-labellings of Q_d which use $2^{\lceil \log_2 d \rceil + 1}$ labels and have span the upper bound above. In addition, if $i_1 \leq 2$, then

$$\lambda_{i_1,i_2,i_3}(Q_d) \geq 2(d-1) + i_1. \tag{10}$$

The lower bound in (9) is simple. Nevertheless, it might be the best that we can hope for arbitrary $i_1 \geq i_2 \geq i_3 \geq 1$. Generally speaking, we believe that the upper bound in (9) is closer to the actual value of $\lambda_{i_1,i_2,i_3}(Q_d)$ than the lower bound.

In view of (3), a consequence of Theorem 1 is the upper bound (5) on $\chi_{\bar{3}}(Q_d)$. Moreover, the proof of Theorem 1 will provide a method of generating $\bar{3}$-colourings of Q_d with $2^{\lceil \log_2 d \rceil + 1}$ colours in a systematic way. In general, this method can produce many such "near-optimal" $\bar{3}$-colourings by varying a set of vectors of $V(p,2)$ satisfying certain conditions (see section 5 for details), where $V(d,2)$ is the d-dimensional linear space over the Galois field GF(2). A specific $\bar{3}$-colouring of Q_d with $2^{\lceil \log_2 d \rceil + 1}$ colours was given in [14, Section 3] by using Hamming code.

In the case where $d \neq 2^{p-1}$, the leading term of the upper bound in (9) is $2^p(i_3+n)$, which is strictly less than $2(i_3+n)d$. In the case where $d = 2^{p-1}$, the upper bound in (9) is $2n(d-1)+i_1$, which is independent of i_3. For $(i_1,i_2,i_3) = (1,1,1)$, the lower bound in (10) is $2d-1$, and the upper bound in (9) is $2^{p+1}-1 = 2^{\lceil \log_2 d \rceil + 1} - 1$ if $d \neq 2^{p-1}$, and $2^p - 1 = 2d-1 = 2^{\lceil \log_2 d \rceil + 1} - 1$ if $d = 2^{p-1}$. Thus, in view of (3), when $(i_1,i_2,i_3) = (1,1,1)$, (9)-(10) gives (5) exactly. Moreover, in this case $\lambda_{1,1,1}(Q_d) = 2d-1$ for $d = 2^{p-1}$, and hence the upper bound (9) and the lower bound (10) are attained. The next small instance is $(i_1,i_2,i_3) = (2,1,1)$, for which we have the following consequence of Theorem 1. Again, the upper bound (9) and the lower bound (10) are attained when $(i_1,i_2,i_3) = (2,1,1)$ and $d = 2^{p-1}$.

Corollary 1 *Let $d \geq 3$, and let p, q be as in (7), (8) respectively. If $d \neq 2^{p-1}$, then*

$$2d \leq \lambda_{2,1,1}(Q_d) \leq 2^{p+1} + 2^q - 2; \tag{11}$$

if $d = 2^{p-1}$, then

$$\lambda_{2,1,1}(Q_d) = 2d \tag{12}$$

and moreover Q_d admits a balanced $L(2,1,1)$-labelling with span $2d$ and exactly one hole.

Theorem 1 will be proved in the next two sections and Corollary 1 will be proved in section 4. In section 5 we will summarize the procedure of generating the $L(i_1,i_2,i_3)$-labellings promised in Theorem 1, and conclude the paper with a few remarks.

2 Lower bounds

Different techniques will be exploited in proving the lower and upper bounds in (9). For the lower bounds, which are the easier part of Theorem 1, a pure combinatorial argument will be used. For a vertex u of Q_d, let $Q_d(u)$ denote the neighbourhood of u in Q_d.

Proof of Theorem 1 (lower bounds) Let ϕ be an $L(i_1, i_2, i_3)$-labelling of Q_d, and u a 0-labelled vertex of Q_d. Then $\phi(v) \geq i_1$ for $v \in Q_d(u)$, and $|\phi(v) - \phi(v')| \geq i_2$ for distinct $v, v' \in Q_d(u)$. Hence $\mathrm{sp}(Q_d; \phi) \geq \phi(v^*) := \max_{v \in Q_d(u)} \phi(v) \geq i_2(d-1)+i_1$. Thus, $\lambda_{i_1,i_2,i_3}(Q_d) \geq i_2(d-1) + i_1$ by the arbitrariness of ϕ.

Suppose $i_1 \leq 2$ in the remaining proof. Clearly, there are $\phi(v^*) - d$ labels in $\{1, 2, \ldots, \phi(v^*)\}$ which are not used by any vertex in $Q_d(u)$. Call them "unused labels". For $w \in Q_d(v^*) \setminus \{u\}$, since the distance in Q_d between w and any vertex in $Q_d(u) \cup \{u\}$ is at most 3, any "used label" is forbidden for w; in other words, w should receive an unused label or a label larger than $\phi(v^*)$.

Case 1: $\phi(v^*) \geq 2d - 1$. In this case, there are enough unused labels for the $d - 1$ vertices in $Q_d(v^*) \setminus \{u\}$. In the case where at least one vertex in $Q_d(v^*) \setminus \{u\}$ receives a label which is larger than $\phi(v^*)$, this label must be at least as large as $\phi(v^*) + i_1$, and hence $\mathrm{sp}(Q_d; \phi) \geq (2d-1) + i_1 > 2(d-1) + i_1$. Thus, we may assume that all vertices in $Q_d(v^*) \setminus \{u\}$ receive unused labels. If $\phi(v^*) \geq 2d$, then $\mathrm{sp}(Q_d; \phi) \geq \phi(v^*) \geq 2d \geq 2(d-1) + i_1$ since $i_1 \leq 2$. Assume then that $\phi(v^*) = 2d - 1$. Then the $\phi(v^*) - d$ $(= d - 1)$ unused labels are all used up by the $d-1$ vertices in $Q_d(v^*) \setminus \{u\}$. However, for $w \in Q_d(v^*) \setminus \{u\}$ and $x \in Q_d(w) \setminus Q_d(u)$, the distance between x and any vertex in $(Q_d(v^*) \setminus \{u\}) \cup \{v^*\}$ is at most 3. Thus, x must receive a label which is larger than $\phi(v^*)$, and hence $\mathrm{sp}(Q_d; \phi) \geq \phi(v^*) + 1 = 2d \geq 2(d-1) + i_1$.

Case 2: $\phi(v^*) < 2d - 1$. In this case at least $(d-1) - (\phi(v^*) - d) = (2d-1) - \phi(v^*) \geq 1$ vertices in $Q_d(v^*) \setminus \{u\}$ receive labels larger than $\phi(v^*)$. In fact, such labels must be at least as large as $\phi(v^*)+i_1$, and also they are pairwise distinct (with mutual separation at least i_2). Thus, the largest label assigned to a vertex in $Q_d(v^*) \setminus \{u\}$ is at least $(\phi(v^*)+i_1)+\{(2d-2)-\phi(v^*)\} = 2(d-1) + i_1$, and hence $\mathrm{sp}(Q_d; \phi) \geq 2(d-1) + i_1$.

In each case above we have proved that $\mathrm{sp}(Q_d; \phi) \geq 2(d-1) + i_1$. Since ϕ is an arbitrary $L(i_1, i_2, i_3)$-labelling of Q_d, it follows that $\lambda_{i_1,i_2,i_3}(Q_d) \geq 2(d-1)+i_1$ when $i_1 \leq 2$, and the proof of the lower bounds is complete. □

3 Upper bounds

To establish the upper bounds in (9) we will use a group-theoretic approach, which bears some similarity with the one for $L(i_1, i_2)$-labellings introduced by the author in [21]. The terminology on groups used in the proof is standard; see e.g. [16]. Let Γ be a finite group. A subset Ω of Γ is called a *Cayley set* if $1_\Gamma \notin \Omega$ and $\alpha \in \Omega$ implies $\alpha^{-1} \in \Omega$, where 1_Γ is the identity element of Γ. Given such a pair (Γ, Ω), the *Cayley graph* of Γ with respect to Ω, denoted by $\mathrm{Cay}(\Gamma, \Omega)$, is the graph with vertices the elements of Γ in which $\alpha, \beta \in \Gamma$ are adjacent if and only if $\alpha\beta^{-1} \in \Omega$. Thus, for any $\alpha, \beta \in \Gamma$, there is a path in $\mathrm{Cay}(\Gamma, \Omega)$ joining α and β if and only if $\alpha\beta^{-1} \in \langle\Omega\rangle$, where $\langle\Omega\rangle$ is the subgroup of Γ generated by Ω. In particular, $\mathrm{Cay}(\Gamma, \Omega)$ is connected if and only if Ω is a generating set of Γ. In the case where Γ is an Abelian group of order at least three, it is well known that any connected Cayley graph on Γ is Hamiltonian (see e.g. [13, Corollary 3.2]). In particular, any connected Cayley graph on a finite Abelian group contains a *Hamiltonian path*, that is, a path visiting every vertex exactly once.

Recall that $V(d, 2)$ is the d-dimensional linear space over $\mathrm{GF}(2)$. In the following vectors of $V(d, 2)$ are taken as row vectors, and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d$ denotes the standard basis of $V(d, 2)$, where

$\mathbf{e}_j = (0, \ldots, 1, \ldots, 0)$ with the jth coordinate 1 and all other coordinates 0, for $j = 1, 2, \ldots, d$. It is well known that the additive group of $V(d, 2)$ is isomorphic to the elementary Abelian 2-group $\mathbb{Z}_2^d$, and that Q_d is isomorphic to the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^d, S)$, where

$$S := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\}.$$

Henceforth we will identify Q_d with $\mathrm{Cay}(\mathbb{Z}_2^d, S)$. Thus, for $\mathbf{u}, \mathbf{v} \in V(d, 2)$, the *distance* in Q_d between $\mathbf{u}$ and $\mathbf{v}$, $d(\mathbf{u}, \mathbf{v})$, is the *Hamming distance* between $\mathbf{u}$ and $\mathbf{v}$, that is, the number of coordinates in which $\mathbf{u}$ and $\mathbf{v}$ differ. A vector $\mathbf{u} = (u_1, u_2, \ldots, u_d) \in V(d, 2)$ is called *even* or *odd* according as $\sum_{i=1}^{d} u_i = 0$ or 1, that is, the number of 1's in the coordinates of $\mathbf{u}$ is even or odd.

The strategy that we are going to use to establish the upper bounds in Theorem 1 can be explained as follows. We first choose a subspace N of $V(d, 2)$ which is defined as the null space of a $d \times p$ matrix A over GF(2) (see (13) below). In the language of coding theory, N can be viewed as a binary linear $(d, d-p)$-code [12, Chapter 8] with the transpose of A as the parity-check matrix. In the following it is convenient to take N as an additive subgroup of $\mathbb{Z}_2^d$. Thus we have a natural partition of $\mathbb{Z}_2^d$ into cosets $N + \mathbf{u}$ where $\mathbf{u} \in \mathbb{Z}_2^d$, and we consider the quotient graph G of Q_d with respect to this partition. (Given a graph H and a partition $\mathcal{P}$ of its vertex set, the *quotient graph* of H with respect to $\mathcal{P}$ is defined to have vertex set $\mathcal{P}$ such that $B, C \in \mathcal{P}$ are adjacent if and only if there exists at least one edge of H between B and C.) We will choose A judiciously such that any two vectors in the same coset are distance ≥ 3 apart in Q_d, and distance ≥ 4 apart if in addition they have the same parity (even or odd). Thus we may label the vectors in the same coset by two labels, one for even vectors and the other for odd vectors. (For the special case where $d = 2^{p-1}$, one label is enough for each coset if we choose a different matrix A judiciously.) The complement of G is a Cayley graph on the Abelian group $\mathbb{Z}_2^d/N$, and hence each of its components contains a Hamiltonian path. We will label the cosets on such a path successively and make the span as small as we can.

3.1 Preparations

Now we start the technical detail. As before we assume that $d \geq 3$ and p, q are as defined in (7), (8) respectively. Since $d \leq 2^p - 1$ and $V(p, 2)$ has $2^p - 1$ non-zero vectors, there exists a $d \times p$ matrix A over GF(2) such that $\mathrm{rank}(A) = p$ and the rows $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d$ of A are pairwise distinct and non-zero vectors of $V(p, 2)$. Later we will specify the choice of these vectors together with a set of other vectors in $V(p, 2)$. Let

$$N := \{\mathbf{x} \in V(d, 2) : \mathbf{x}A = \mathbf{0}_p\} \tag{13}$$

be the null space of A, where $\mathbf{0}_p$ is the zero-vector of $V(p, 2)$. Since $\mathrm{rank}(A) = p$, N is a $(d-p)$-dimensional subspace of $V(d, 2)$. The additive group of N, denoted by the same notation, is a subgroup of $\mathbb{Z}_2^d$ with $|\mathbb{Z}_2^d : N| = 2^p$, and thus $\mathbb{Z}_2^d$ is partitioned into 2^p cosets $N + \mathbf{u}$ of N in $\mathbb{Z}_2^d$, where $\mathbf{u} \in V(d, 2)$. Since $\mathbf{e}_j A = \mathbf{a}_j$ (the jth row of A), and since the rows of A are non-zero and pairwise distinct, it follows that $\mathbf{e}_j \notin N$ for $j = 1, 2, \ldots, d$ and $\mathbf{e}_j + \mathbf{e}_{j'} \notin N$ when $j \neq j'$.

The following lemmas are valid for any $d \times p$ matrix A over GF(2) such that $\mathrm{rank}(A) = p$ and the rows of A are non-zero and pairwise distinct. These lemmas will be used in the proof of the upper bounds in (9) for specifically chosen A.

Lemma 1 *For any $N+\mathbf{u} \in \mathbb{Z}_2^d/N$ and any two distinct elements $\mathbf{x}+\mathbf{u}, \mathbf{x}'+\mathbf{u}$ of $N+\mathbf{u}$ (where $\mathbf{u} \in V(d,2)$ and $\mathbf{x}, \mathbf{x}' \in N$ with $\mathbf{x} \neq \mathbf{x}'$), we have $d(\mathbf{x}+\mathbf{u}, \mathbf{x}'+\mathbf{u}) \geq 3$.*

Proof Clearly, $d(\mathbf{x}+\mathbf{u}, \mathbf{x}'+\mathbf{u}) = d(\mathbf{x}, \mathbf{x}') = \mathbf{x}-\mathbf{x}'$. Thus, for $\mathbf{x}+\mathbf{u} \in N+\mathbf{u}$ and $\mathbf{x}'+\mathbf{u} \in N+\mathbf{u}$ with $\mathbf{x} \neq \mathbf{x}'$, we have $d(\mathbf{x}+\mathbf{u}, \mathbf{x}'+\mathbf{u}) \geq 3$ since $d(\mathbf{x}, \mathbf{x}') = 1$ implies $\mathbf{x}-\mathbf{x}' = \mathbf{e}_j \in N$, and $d(\mathbf{x}, \mathbf{x}') = 2$ implies $\mathbf{x}-\mathbf{x}' = \mathbf{e}_j + \mathbf{e}_{j'} \in N$. □

Let

$$N^{(0)} := \{\mathbf{x} \in N : \ \mathbf{x} \text{ is even}\}, \ N^{(1)} := \{\mathbf{x} \in N : \ \mathbf{x} \text{ is odd}\}$$

and

$$N^{(0)} + \mathbf{u} := \{\mathbf{x}+\mathbf{u} : \mathbf{x} \in N^{(0)}\}, \ N^{(1)} + \mathbf{u} := \{\mathbf{x}+\mathbf{u} : \mathbf{x} \in N^{(1)}\}. \quad (14)$$

Lemma 2 *For any $N+\mathbf{u} \in \mathbb{Z}_2^d/N$, $\{N^{(0)}+\mathbf{u}, N^{(1)}+\mathbf{u}\}$ is a partition of $N+\mathbf{u}$. Moreover, for any two distinct elements $\mathbf{x}+\mathbf{u}, \mathbf{x}'+\mathbf{u}$ in the same part of $\{N^{(0)}+\mathbf{u}, N^{(1)}+\mathbf{u}\}$ (where $\mathbf{x}, \mathbf{x}' \in N$ with $\mathbf{x} \neq \mathbf{x}'$), we have $d(\mathbf{x}+\mathbf{u}, \mathbf{x}'+\mathbf{u}) \geq 4$.*

Proof Since $\{N^{(0)}, N^{(1)}\}$ is a partition of N, it follows immediately that $\{N^{(0)}+\mathbf{u}, N^{(1)}+\mathbf{u}\}$ is a partition of $N+\mathbf{u}$. By Lemma 1 we have $d(\mathbf{x}+\mathbf{u}, \mathbf{x}'+\mathbf{u}) \geq 3$. If $d(\mathbf{x}+\mathbf{u}, \mathbf{x}'+\mathbf{u}) = 3$, then $d(\mathbf{x}, \mathbf{x}') = d(\mathbf{x}+\mathbf{u}, \mathbf{x}'+\mathbf{u}) = 3$, and hence $\mathbf{x}$, $\mathbf{x}'$ differ in precisely three coordinates. Thus, $\mathbf{x}$ and $\mathbf{x}'$ must have different parity, which is a contradiction. Hence Lemma 2 is established. □

Denote

$$S/N := \{N + \mathbf{e}_j : j = 1, 2, \ldots, d\}.$$

Since $\mathbf{e}_j \notin N$ for each j, the identity element N of $\mathbb{Z}_2^d/N$ is not in S/N. Also, it is clear that S/N is closed under taking inverse. Hence both S/N and $S^*/N := (\mathbb{Z}_2^d/N) \setminus ((S/N) \cup \{N\})$ are Cayley sets of $\mathbb{Z}_2^d/N$. Let

$$G := \mathrm{Cay}(\mathbb{Z}_2^d/N, S/N), \ G^* := \mathrm{Cay}(\mathbb{Z}_2^d/N, S^*/N)$$

be the corresponding Cayley graphs. Since $(S/N) \cup (S^*/N)$ is a partition of $(\mathbb{Z}_2^d/N) \setminus \{N\}$, we have the following lemma.

Lemma 3 *G and G^* are complementary graphs of each other, that is, $N+\mathbf{u}, N+\mathbf{v} \in \mathbb{Z}_2^d/N$ are adjacent in G if and only if they are not adjacent in G^*.*

The next lemma tells us the relationship between the adjacency relations of G and Q_d.

Lemma 4 *Let $N+\mathbf{u}, N+\mathbf{v} \in \mathbb{Z}_2^d/N$ be distinct cosets, where $\mathbf{u}-\mathbf{v} \notin N$. Then $N+\mathbf{u}$ and $N+\mathbf{v}$ are adjacent in G if and only if there exist $\mathbf{u}' \in N+\mathbf{u}$ and $\mathbf{v}' \in N+\mathbf{v}$ such that $\mathbf{u}'$ and $\mathbf{v}'$ are adjacent in Q_d. In other words, G is the quotient graph of Q_d with respect to the partition $\mathbb{Z}_2^d/N$ of $\mathbb{Z}_2^d$.*

Proof If $N+\mathbf{u}$ and $N+\mathbf{v}$ are adjacent in G, then $(N+\mathbf{u})-(N+\mathbf{v}) = N+(\mathbf{u}-\mathbf{v}) \in S/N$ and hence $\mathbf{x}+(\mathbf{u}-\mathbf{v}) = \mathbf{e}_j$ for some $\mathbf{x} \in N$ and j. Thus, $\mathbf{x}+\mathbf{u} \in N+\mathbf{u}$ and $\mathbf{v} \in N+\mathbf{v}$ are adjacent in Q_d. Conversely, if $\mathbf{x}+\mathbf{u} \in N+\mathbf{u}$, $\mathbf{y}+\mathbf{v} \in N+\mathbf{v}$ are adjacent in Q_d for some $\mathbf{x}, \mathbf{y} \in N$, then $(\mathbf{x}+\mathbf{u})-(\mathbf{y}+\mathbf{v}) = \mathbf{e}_j$ for some j and hence $(N+\mathbf{u})-(N+\mathbf{v}) \in S/N$, that is, $N+\mathbf{u}$ and $N+\mathbf{v}$ are adjacent in G. □

Lemma 5 *Each coset of* $\langle S^*/N\rangle$ *in* $\mathbb{Z}_2^d/N$ *is a connected component of* G^**, and vice versa.*

Proof From the definition of a Cayley graph, we have: $N+\mathbf{u}$ and $N+\mathbf{v}$ are in the same connected component of $G^* \Leftrightarrow (N+\mathbf{u})-(N+\mathbf{v}) \in \langle S^*/N\rangle \Leftrightarrow N+\mathbf{u}$ and $N+\mathbf{v}$ are in the same coset of $\langle S^*/N\rangle$ in $\mathbb{Z}_2^d/N$. □

Since G^* is a Cayley graph, its components must be isomorphic to each other, and they are all isomorphic to $\mathrm{Cay}(\langle S^*/N\rangle, S^*/N)$. Thus, G^* has r components each with order s, where r, s are defined by

$$r := |(\mathbb{Z}_2^d/N) : \langle S^*/N\rangle|, \;\; s := |\langle S^*/N\rangle|. \tag{15}$$

Clearly, we have

$$rs = |\mathbb{Z}_2^d : N| = 2^p. \tag{16}$$

Let $G_1^*, G_2^*, \ldots, G_r^*$ denote the connected components of G^*. Since $\langle S^*/N\rangle$ is Abelian, from [13, Corollary 3.2], $\mathrm{Cay}(\langle S^*/N\rangle, S^*/N)$ contains a Hamiltonian path, and so does every G_i^*. (See the first paragraph of this section.) For $i = 1, 2, \ldots, r$, let

$$N+\mathbf{u}_{i,1}, N+\mathbf{u}_{i,2}, \ldots, N+\mathbf{u}_{i,s} \tag{17}$$

be a Hamiltonian path of G_i^*. Note that by Lemmas 3-4 there exists no edge of Q_d between $N+\mathbf{u}_{i,j}$ and $N+\mathbf{u}_{i,j+1}$ for $j = 1, 2, \ldots, s-1$. Note also that $\{N+\mathbf{u}_{i,j} : i = 1, 2, \ldots, r, j = 1, 2, \ldots, s\}$ is a partition of $\mathbb{Z}_2^d$.

3.2 Upper bounds

Equipped with the results above, we are now ready to prove the upper bounds in (9).

Proof of Theorem 1 (upper bound) We use the notations above and distinguish the following two cases: (a) $d \neq 2^{p-1}$; (b) $d = 2^{p-1}$.

General Case: $d \neq 2^{p-1}$. Since $d + (p-q) = \min\{2^p - 1, d+p\} \leq 2^p - 1$, we can choose pairwise distinct non-zero vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{p-q}$ of $V(p, 2)$ such that

(i) $\mathrm{rank}(A) = p$; and

(ii) $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{p-q}$ are independent,

where A is the $d \times p$ matrix with rows $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d$. To be specific let us choose such vectors in the following way: if $q \geq 1$, then for $t = 1, 2, \ldots, p-q$ let $\mathbf{b}_t$ be the vector with the jth coordinate 0 if $j < t$ and 1 if $j \geq t$; if $q = 0$, then define $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{p-1}$ in the same way and set $\mathbf{b}_p = (1, 0, \ldots, 0, 1)$. (The case $p = q$ occurs precisely when $d + 1 = 2^p$, and in this case we leave $\mathbf{b}_0$ undefined.) Choose $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ to be the standard basis of $V(p, 2)$, and then choose distinct non-zero vectors $\mathbf{a}_{p+1}, \ldots, \mathbf{a}_d$ from $V(p, 2) \setminus \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{p-q}\}$. Then $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p, \mathbf{a}_{p+1}, \ldots, \mathbf{a}_d, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{p-q}$ meet all the requirements above.

With the specific choice above, Lemmas 1-5 are all valid for A and its null space N, and we will use them in the following. Note that $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ are odd. Since $d > 2^{p-1}$ but $V(p, 2)$ contains 2^{p-1} odd vectors only, there exists at least one even vector among $\mathbf{a}_{p+1}, \ldots, \mathbf{a}_d$. Without loss of generality we may suppose that $\mathbf{a}_d$ is even, that is, $\sum_{j=1}^{p} a_{dj} = 0$, where we denote $\mathbf{a}_i = (a_{i1}, a_{i2}, \ldots, a_{ip})$ for each i. Let $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in V(d, 2)$. If $\mathbf{x} \in N$, then $\sum_{i=1}^{d} x_i = \sum_{i=p+1}^{d} (1 + \sum_{j=1}^{p} a_{ij}) x_i$ by the specific choice of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d$. Setting $(x_{p+1}, \ldots, x_{d-1}, x_d) = (0, \ldots, 0, 1)$, we get $\sum_{i=1}^{d} x_i = 1$ and hence $\mathbf{x}$ is odd. Thus, $N^{(1)} \neq \emptyset$. Let $M = \{\mathbf{x} \in V(d, 2) : \sum_{i=1}^{d} x_i = 0\}$, so that $N^{(0)} = N \cap M$. Since $\emptyset \neq N^{(1)} = N \setminus M$ and M is a $(d-1)$-dimensional subspace of $V(d, 2)$, we must have $M + N = V(d, 2)$. Thus, since the dimension of N is $d - p$, it follows from the dimension formula that $N^{(0)}$ is a $(d - p - 1)$-dimensional subspace of $V(d, 2)$, and therefore $|N^{(0)}| = |N^{(1)}| = |N|/2$.

Now that $\text{rank}(A) = p$, there exists $\mathbf{x}_j \in V(d, 2)$ such that $\mathbf{x}_j A = \mathbf{b}_j$, for $j = 1, 2, \ldots, p - q$. Since $\mathbf{b}_j \neq \mathbf{a}_i = \mathbf{e}_i A$ for $i = 1, 2, \ldots, d$ and $j = 1, 2, \ldots, p - q$, we have $\mathbf{x}_j - \mathbf{e}_i \notin N$ and hence $\mathbf{x}_j \in \mathbb{Z}_2^d \setminus (N + S)$, where $N + S := \{\mathbf{x} + \mathbf{e}_i : \mathbf{x} \in N, \mathbf{e}_i \in S\}$. One can check that $\langle S^*/N \rangle = \langle \mathbb{Z}_2^d \setminus (N + S) \rangle / N$. Hence $N + \mathbf{x}_j \in \langle S^*/N \rangle$ for $j = 1, 2, \ldots, p - q$. Since by (ii) $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{p-q}$ are independent in $V(p, 2)$, $N + \mathbf{x}_1, N + \mathbf{x}_2, \ldots, N + \mathbf{x}_{p-q}$ are independent in the quotient space $V(d, 2)/N$. Therefore, $s = |\langle S^*/N \rangle| \geq |\langle N + \mathbf{x}_1, N + \mathbf{x}_2, \ldots, N + \mathbf{x}_{p-q} \rangle| \geq 2^{p-q}$, and thus by (16) we have

$$r \leq 2^q. \tag{18}$$

Recall that $n := \max\{i_2, \lceil i_1/2 \rceil\}$. Define ϕ to be the labelling of Q_d such that, for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$, all the elements in $N^{(0)} + \mathbf{u}_{i,j}$ are labelled by

$$(i - 1)\{(s - 1)n + i_1\} + (j - 1)n + \{(i - 1)s + (j - 1)\} i_3 \tag{19}$$

and all the elements in $N^{(1)} + \mathbf{u}_{i,j}$ are labelled by

$$(i - 1)\{(s - 1)n + i_1\} + (j - 1)n + \{(i - 1)s + j\} i_3. \tag{20}$$

Clearly, for any fixed i, the labels assigned to the elements in $N + \mathbf{u}_{i,j}$ increase with j. The smallest and the largest labels assigned to an element of $\bigcup_{j=1}^{s} (N + \mathbf{u}_{i,j})$ are $(i-1)(s-1)n + (i-1)i_1 + (i-1)i_3 s$ and $i(s-1)n + (i-1)i_1 + i i_3 s$, respectively. Thus, the labels increase with i, and if $i \neq i'$, then for any $\mathbf{u} \in \bigcup_{j=1}^{s} (N + \mathbf{u}_{i,j})$ and $\mathbf{u}' \in \bigcup_{j=1}^{s} (N + \mathbf{u}_{i',j})$ we have $|\phi(\mathbf{u}) - \phi(\mathbf{u}')| \geq i_1$. Thus, since $i_1 \geq i_2 \geq i_3$, (1) is satisfied by such pairs $(\mathbf{u}, \mathbf{u}')$ for $t = 1, 2, 3$, regardless of the distance in Q_d between $\mathbf{u}$ and $\mathbf{u}'$.

Now let us consider $N + \mathbf{u}_{i,j}$, $N + \mathbf{u}_{i,j'}$ in the same connected component of G^*. If $\mathbf{u} \in N + \mathbf{u}_{i,j}$ and $\mathbf{u}' \in N + \mathbf{u}_{i,j'}$ are adjacent in Q_d, then by Lemma 4, $N + \mathbf{u}_{i,j}$, $N + \mathbf{u}_{i,j'}$ are adjacent in G, and hence by Lemma 3 they are not adjacent in G^*. Thus, due to the Hamiltonian path (17) of G_i^*, we have $|j - j'| \geq 2$ and hence $|\phi(\mathbf{u}) - \phi(\mathbf{u}')| \geq |j - j'| n \geq 2n \geq i_1$. If $d(\mathbf{u}, \mathbf{u}') = 2$, then by Lemma 1 we have $j \neq j'$ and hence $|\phi(\mathbf{u}) - \phi(\mathbf{u}')| \geq |j - j'| n \geq n \geq i_2$. Finally, suppose $d(\mathbf{u}, \mathbf{u}') = 3$. If $j \neq j'$, then $|\phi(\mathbf{u}) - \phi(\mathbf{u}')| \geq i_2$ $(\geq i_3)$ as above. If $j = j'$, then by Lemma 2, one of $\mathbf{u}$, $\mathbf{u}'$ is in $N^{(0)} + \mathbf{u}_{i,j}$ and the other one is in $N^{(1)} + \mathbf{u}_{i,j}$, and hence $|\phi(\mathbf{u}) - \phi(\mathbf{u}')| = i_3$.

In summary, we have proved that ϕ is an $L(i_1, i_2, i_3)$-labelling of Q_d. Noting that $rs = 2^p$ and $r \leq 2^q$ by (16) and (18), we have

$$\text{sp}(Q_d; \phi) = rs(i_3 + n) + r(i_1 - n) - i_1 \leq 2^p(i_3 + n) + 2^q(i_1 - n) - i_1 \tag{21}$$

and hence the first upper bound in (9) follows. The number of labels used by ϕ is $2rs = 2^{\lceil \log_2 d \rceil + 1}$ since $2^{p-1} < d \leq 2^p - 1$. Recall that $N^{(0)}$ and $N^{(1)}$ each contains half of the elements of N.

Hence $N^{(0)} + \mathbf{u}_{i,j}$ and $N^{(1)} + \mathbf{u}_{i,j}$ each contains half of the elements of $N + \mathbf{u}_{i,j}$. Therefore, ϕ is balanced.

Special Case: $d = 2^{p-1}$. Note that we could apply the labelling ϕ above to the case where $d = 2^{p-1}$. However, it does not produce the desired bounds on $\lambda_{i_1,i_2,i_3}(Q_d)$ and $\chi_{\bar{3}}(Q_d)$ in this case. In fact, the case $d = 2^{p-1}$ is quite special and deserves a more careful treatment. In this case, $q = \max\{p + 1 - 2^{p-1}, 0\} = 0$ (note that $p \geq 3$ as $d \geq 3$), and we are going to choose $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$ in a different way. Let us first choose $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ to be a set of even vectors of $V(p, 2)$ with rank $p - 1$. (For instance, we may choose $\mathbf{b}_t$ to be the vector of $V(p, 2)$ with the tth and $(t + 1)$th coordinates 1 and all other coordinates 0, for $t = 1, 2, \ldots, p$, with t modulo p.) Then choose $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ to be the standard basis of $V(p, 2)$. Note that $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ is no longer independent, but it contains $p - 1$ independent vectors due to the requirement on its rank. (As a matter of fact, any set of p even vectors of $V(p, 2)$ must be dependent because the corresponding determinant is equal to 0.) Note also that $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ are all odd. Since in total there are 2^{p-1} odd vectors in $V(p, 2)$ and $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$ are all even, there are exactly $2^{p-1} - p$ $(= d - p)$ odd vectors in $V(p, 2) \setminus \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$, and hence we can choose $\mathbf{a}_{p+1}, \ldots, \mathbf{a}_d$ to be these odd vectors. Then $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$ are non-zero and pairwise distinct such that $\mathrm{rank}(A) = p$, where A is the $d \times p$ matrix with rows $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d$. For this A and its null space N, Lemmas 1-5 are all valid. Using the same notation as before, since $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ contains $p - 1$ instead of p independent vectors, we have $s = |\langle S^*/N \rangle| \geq |\langle N + \mathbf{x}_1, N + \mathbf{x}_2, \ldots, N + \mathbf{x}_p \rangle| \geq 2^{p-1}$. Hence, instead of (18), we have $r \leq 2$ by (16). Thus, again by (16), we have either $(r, s) = (1, 2^p)$ or $(r, s) = (2, 2^{p-1})$. Since $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ is the standard basis of $V(p, 2)$, for each $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in N$ we have $\sum_{i=1}^{d} x_i = \sum_{i=p+1}^{d} (1 + \sum_{j=1}^{p} a_{ij}) x_i$, where $\mathbf{a}_i = (a_{i1}, a_{i2}, \ldots, a_{ip})$ for each i. Since all $\mathbf{a}_i$'s are odd, we have $\sum_{j=1}^{p} a_{ij} = 1$ for $i = p + 1, \ldots, d$, and hence $\sum_{i=1}^{d} x_i = 0$. Thus, N consists of even vectors only, that is, $N^{(0)} = N$ and $N^{(1)} = \emptyset$. Hence, by Lemmas 1-2, for each coset $N + \mathbf{u} \in \mathbb{Z}_2^d/N$, the distance in Q_d between any two elements in $N + \mathbf{u}$ is at least 4, and so we need only one label for $N + \mathbf{u}$. In the case where $(r, s) = (1, 2^p)$, from Lemmas 4-5 the labelling under which all elements in $N + \mathbf{u}_{1,j}$ are labelled by

$$(j - 1)n, \ j = 1, 2, \ldots, 2^p \tag{22}$$

is an $L(i_1, i_2, i_3)$-labelling of Q_d with span $(2^p - 1)n \leq (2^p - 2)n + i_1$. In the case where $(r, s) = (2, 2^{p-1})$, we may label all elements in $N + \mathbf{u}_{1,j}$ by

$$(j - 1)n, \ j = 1, 2, \ldots, 2^{p-1} \tag{23}$$

and all elements in $N + \mathbf{u}_{2,j}$ by

$$(2^{p-1} - 1)n + i_1 + (j - 1)n, \ j = 1, 2, \ldots, 2^{p-1}. \tag{24}$$

Using Lemmas 4-5, one can verify that this is an $L(i_1, i_2, i_3)$-labelling of Q_d with span $(2^p - 2)n + i_1$. Thus, in each case the second bound in (9) has been established, and moreover the $L(i_1, i_2, i_3)$-labelling above uses $rs = 2^p = 2^{\lceil \log_2(d+1) \rceil} = 2^{\lceil \log_2 d \rceil + 1}$ labels. (Note that $\lceil \log_2(d + 1) \rceil = \lceil \log_2 d \rceil + 1$ when $d = 2^{p-1}$.) Moreover, in each case the labelling above is balanced with each label used by $2^{2^{p-1}-p}$ elements. $\square$

Up to now we have completed the proof of Theorem 1.

4 Proof of Corollary 1

Proof of Corollary 1 Let $(i_1, i_2, i_3) = (2, 1, 1)$. Then $n = 1$ and hence (11) follows from (9)-(10) when $d \neq 2^{p-1}$. In the case where $d = 2^{p-1}$, the upper bound in (9) and the lower bound in (10) are both equal to $2d$, and hence (12) follows. In this latter case, since $\chi_{\bar{3}}(Q_d) = 2d$ by (5), any optimal $L(2, 1, 1)$-labelling of Q_d contains exactly one hole. Hence we must have $(r, s) = (2, 2^{p-1})$ when $d = 2^{p-1}$ (where r, s are as defined in (15)), and the labelling defined by (23)-(24) is a balanced optimal $L(2, 1, 1)$-labelling of Q_d with exactly one hole. □

5 Remarks

The proof of Theorem 1 implies the following procedure for generating $L(i_1, i_2, i_3)$-labellings of Q_d which have span the upper bound in (9) and use $2^{\lceil \log_2 d \rceil + 1}$ labels. Let p, q be as in (7), (8) respectively.

1. In the case where $d \neq 2^{p-1}$, choose pairwise distinct non-zero vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d$, $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{p-q}$ of $V(p, 2)$ satisfying (i) and (ii); in the case where $d = 2^{p-1}$, choose $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d$ to be all odd vectors of $V(p, 2)$, and let $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ be any set of even vectors of $V(p, 2)$ with rank $p - 1$.

2. Compute the null space N of the $d \times q$ matrix with rows $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d$; and compute $N^{(0)} + \mathbf{u}$ and $N^{(1)} + \mathbf{u}$ for each $N + \mathbf{u} \in \mathbb{Z}_2^d/N$ using (14).

3. Let $G^* := \mathrm{Cay}(\mathbb{Z}_2^d/N, S^*/N)$, where $S^*/N := (\mathbb{Z}_2^d/N) \setminus ((S/N) \cup \{N\})$. Find a Hamiltonian path (17) in each connected component of G^*.

4. In the case where $d \neq 2^{p-1}$, label the elements of $\mathbb{Z}_2^d$ using (19)-(20); in the case where $d = 2^{p-1}$, label the elements of $\mathbb{Z}_2^d$ using (22) if $r = 1$, or (23)-(24) if $r = 2$.

For $(i_1, i_2, i_3) = (1, 1, 1)$ and $d = 2^{p-1}$, since $\lambda_{1,1,1}(Q_d) = 2d - 1$ by (9) and (10), and $\chi_{\bar{3}}(Q_d) = 2d$ by (3) and (5), any optimal $L(1, 1, 1)$-labelling must be no-hole. Since $i_1 = n = 1$ in this case, the labelling given by (22) (when $r = 1$) or (23)-(24) (when $r = 2$) is a no-hole balanced $L(1, 1, 1)$-labelling with span $2d - 1$ (hence optimal).

The lower bound $\lambda_{i_1,i_2,i_3}(Q_d) \geq i_2(d-1) + i_1$ in (9) is quite crude, and there is room to obtain better lower bounds for specific values of i_1, i_2 and i_3.

Comparing (9) and (6), the upper bound for $\lambda_{i_1,i_2,i_3}(Q_d)$ is larger than the upper bound for $\lambda_{i_1,i_2}(Q_d)$ by $2^p i_3$ when $d \neq 2^{p-1}$, and by $\min\{i_1 - i_2, \lfloor i_1/2 \rfloor\}$ when $d = 2^{p-1}$.

Finally, in view of (21) the first upper bound in (9) can be improved as $2^p(i_3+n)+(2^p/s)(i_1 - n) - i_1$, where $s = |\langle (\mathbb{Z}_2^d/N) \setminus ((S/N) \cup \{N\}) \rangle|$ with N as defined in (13) for the chosen matrix A.

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