Monatsh. Math. 139, 69–81 (2003)
DOI 10.1007/s00605-002-0538-4

Monatshefte für
Mathematik
Printed in Austria

Symmetric Graphs and Flag Graphs

By

Sanming Zhou*

The University of Melbourne, Parkville, Australia

Received April 24, 2001; in revised form October 9, 2002
Published online May 9, 2003

Abstract. With any G-symmetric graph Γ admitting a nontrivial G-invariant partition $\mathcal{B}$, we may associate a natural "cross-sectional" geometry, namely the 1-design $\mathcal{D}(B) = (B, \Gamma_{\mathcal{B}}(B), \mathrm{I})$ in which $\alpha \mathrm{I} C$ for $\alpha \in B$ and $C \in \Gamma_{\mathcal{B}}(B)$ if and only if α is adjacent to at least one vertex in C, where $B \in \mathcal{B}$ and $\Gamma_{\mathcal{B}}(B)$ is the neighbourhood of B in the quotient graph $\Gamma_{\mathcal{B}}$ of Γ with respect to $\mathcal{B}$. In a vast number of cases, the dual 1-design of $\mathcal{D}(B)$ contains no repeated blocks, that is, distinct vertices of B are incident in $\mathcal{D}(B)$ with distinct subsets of blocks of $\Gamma_{\mathcal{B}}(B)$. The purpose of this paper is to give a general construction of such graphs, and then prove that it produces all of them. In particular, we show that such graphs can be reconstructed from $\Gamma_{\mathcal{B}}$ and the induced action of G on $\mathcal{B}$. The construction reveals a close connection between such graphs and certain G-point-transitive and G-block-transitive 1-designs. By using this construction we give a characterization of G-symmetric graphs such that there is at most one edge between any two blocks of $\mathcal{B}$. This leads to, in a subsequent paper, a construction of G-symmetric graphs $(\Gamma, \mathcal{B})$ such that $|B| \geqslant 3$ and each $C \in \Gamma_{\mathcal{B}}(B)$ is incident in $\mathcal{D}(B)$ with $|B| - 1$ vertices of B.

2000 Mathematics Subject Classification: 05C25, 05B05, 05E99
Key words: Symmetric graph, flag graph, design

1. Introduction

Let Γ be a finite graph and G a finite group. If G acts on the vertex set $V(\Gamma)$ of Γ such that G preserves the adjacency of Γ, then Γ is said to *admit* G as a group of automorphisms. If such a group G is transitive on $V(\Gamma)$ and, in its induced action, is transitive on the set $\mathrm{Arc}(\Gamma)$ of arcs of Γ, then Γ is said to be a *G-symmetric graph*, where an *arc* is an ordered pair of adjacent vertices. The study of symmetric graphs has long been one of the main themes in the area of algebraic combinatorics. The reader may consult [2] for the history and basic results in this area.

In most cases, a G-symmetric graph Γ admits a *nontrivial G-invariant partition*, that is, a partition $\mathcal{B}$ of $V(\Gamma)$ such that $1 < |B| < |V(\Gamma)|$ and $B^g \in \mathcal{B}$ for $B \in \mathcal{B}$ and $g \in G$, where $B^g := \{\alpha^g : \alpha \in B\}$. In such a case Γ is called an *imprimitive G-symmetric graph*. From permutation group theory [3, Corollary 1.5A], this happens precisely when the stabilizer G_α in G of a vertex $\alpha \in V(\Gamma)$ is not a maximal subgroup of G. A standard approach to studying such a graph Γ is to analyse the *quotient graph* $\Gamma_{\mathcal{B}}$ of Γ with respect to $\mathcal{B}$, which is defined to be the graph with vertex set $\mathcal{B}$ in which $B, C \in \mathcal{B}$ are adjacent if and only if there exist $\alpha \in B$ and

*The work was supported by a discovery-project grant from the Australian Research Council.

$\beta \in C$ such that $\{\alpha, \beta\}$ is an edge of Γ. In the following we will always assume that $\Gamma_{\mathscr{B}}$ contains at least one edge, so each block of $\mathscr{B}$ is an independent set of Γ (see e.g. [2, Proposition 22.1] and [9]). This quotient graph $\Gamma_{\mathscr{B}}$ conveys a lot of information about the graph Γ and inherits some properties of Γ. For example, $\Gamma_{\mathscr{B}}$ is G-symmetric under the induced action (possibly unfaithful) of G on $\mathscr{B}$. Nevertheless, some important information is neglected by $\Gamma_{\mathscr{B}}$ since it does not tell us how adjacent blocks of $\mathscr{B}$ are joined by edges of Γ. Thus, instead of analysing $\Gamma_{\mathscr{B}}$ alone, it seems more helpful to analyse the triple $(\Gamma_{\mathscr{B}}, \Gamma[B,C], \mathscr{D}(B))$, and this approach was suggested in [4] and further developed in [5, 6], where $\Gamma[B,C]$ and $\mathscr{D}(B)$ are defined as follows. Let $\Gamma(\alpha)$ denote the *neighbourhood* of α in Γ, that is, the set of vertices of Γ adjacent to α. For $B \in \mathscr{B}$, define $\Gamma(B) := \bigcup_{\alpha \in B} \Gamma(\alpha)$, and denote by $\Gamma_{\mathscr{B}}(B)$ the neighbourhood of B in $\Gamma_{\mathscr{B}}$. For adjacent blocks B, C of $\mathscr{B}$, define $\Gamma[B,C]$ to be the induced bipartite subgraph of Γ with bipartition $\{\Gamma(C) \cap B, \Gamma(B) \cap C\}$. Define $\mathscr{D}(B) := (B, \Gamma_{\mathscr{B}}(B), \mathrm{I})$ to be the incidence structure in which $\alpha \mathrm{I} C$ for $\alpha \in B$ and $C \in \Gamma_{\mathscr{B}}(B)$ if and only if $\alpha \in \Gamma(C)$. Set

$$v := |B|, \quad k := |\Gamma(C) \cap B|, \quad r := |\Gamma_{\mathscr{B}}(\alpha)|$$

where $\Gamma_{\mathscr{B}}(\alpha) := \{C \in \Gamma_{\mathscr{B}}(B) : \alpha \mathrm{I} C\}$. Since Γ is G-symmetric, it is easily checked that $\mathscr{D}(B)$ is a 1-(v,k,r) design and, up to isomorphism, is independent of the choice of B. Also, up to isomorphism, $\Gamma[B,C]$ is independent of the choice of adjacent blocks B, C of $\mathscr{B}$. This approach is a geometric one in the sense that it involves the "cross-sectional" geometry $\mathscr{D}(B)$. Its usefulness lies on a thorough understanding of the three configurations above, as well as an attempt of reconstructing Γ from the triple $(\Gamma_{\mathscr{B}}, \Gamma[B,C], \mathscr{D}(B))$. The approach has been used effectively in the study of some classes of imprimitive symmetric graphs, see [4–6, 8, 10–12].

We find that it is significant to distinguish whether the dual 1-design $\mathscr{D}^*(B)$ of the "cross-sectional" geometry $\mathscr{D}(B)$ contains no repeated blocks. Here we may identify the "blocks" of $\mathscr{D}^*(B)$ with the subsets $\Gamma_{\mathscr{B}}(\alpha)$ of the "point set" $\Gamma_{\mathscr{B}}(B)$, for $\alpha \in B$, and we call two such "blocks" $\Gamma_{\mathscr{B}}(\beta), \Gamma_{\mathscr{B}}(\gamma)$ (for distinct $\beta, \gamma \in B$) *repeated* if $\Gamma_{\mathscr{B}}(\beta) = \Gamma_{\mathscr{B}}(\gamma)$. Both cases appear very often, and thus the class of imprimitive symmetric graphs can be divided into two large subclasses accordingly. Examples in the first subclass include G-symmetric graphs Γ with $k = v$, that is, Γ is a *multicover* of $\Gamma_{\mathscr{B}}$ (see [7]). In this case $\Gamma_{\mathscr{B}}(\beta) = \Gamma_{\mathscr{B}}(\gamma)$ holds for all $\beta, \gamma \in B$, and hence any two "blocks" of $\mathscr{D}^*(B)$ are repeated. This happens in particular when $\Gamma[B,C] \cong v \cdot K_2$ is a perfect matching between B and C; in this case Γ is a *cover* of $\Gamma_{\mathscr{B}}$, and such graphs Γ have been studied extensively in the literature (see e.g. [2]).

The second subclass consists of all imprimitive G-symmetric graphs Γ such that $\mathscr{D}^*(B)$ contains no repeated blocks. The purpose of this paper is to give a general construction of such graphs, and then prove that, up to isomorphism, it produces all of them. In particular, we will show that such graphs Γ can be reconstructed from $\Gamma_{\mathscr{B}}$ and the induced action of G on $\mathscr{B}$. The construction provides an approach to studying such graphs by using certain G-point-transitive and G-block-transitive 1-designs. In fact, the construction requires a 1-design $\mathscr{D}$ which admits G as a point- and block-transitive group of automorphisms. It also requires a "feasible" G-orbit Ω on the flags of $\mathscr{D}$, and a self-paired G-orbit Ψ on $\Omega \times \Omega$ which is "compatible" with Ω in the sense that $\sigma \neq \tau$ and σ, $\tau \in L \cap N$ for some (and hence all) $((\sigma, L), (\tau, N)) \in \Psi$ (see Definition 2.1). Given these the constructed graph $\Gamma(\mathscr{D}, \Omega, \Psi)$, called the G-flag graph of $\mathscr{D}$ with respect to (Ω, Ψ), is defined to have vertex set Ω and arc set Ψ. The vertices of such a graph admit a natural G-invariant partition, namely $\mathscr{B}(\Omega)$ defined in (1), Section 2. The main result in this paper is the following theorem, which shows that Γ is a G-symmetric graph with $\mathscr{D}^*(B)$ containing no repeated blocks if and only if it is isomorphic to such a G-flag graph.

Theorem 1.1. *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$ such that $\mathscr{D}^*(B)$ contains no repeated blocks. Let r be the block size of $\mathscr{D}^*(B)$, that is, $r = |\Gamma_{\mathscr{B}}(\alpha)|$. Then $\Gamma \cong \Gamma(\mathscr{D}, \Omega, \Psi)$ for a certain G-point-transitive and G-block-transitive 1-design $\mathscr{D}$ with block size $r+1$, a certain feasible G-orbit Ω on the flags of $\mathscr{D}$, and a certain self-paired G-orbit Ψ on $\Omega^{(2)}$ compatible with Ω.*

Conversely, for any G-point-transitive and G-block-transitive 1-design $\mathscr{D}$ with block size $r+1$, any feasible G-orbit Ω on the flags of $\mathscr{D}$, and any self-paired G-orbit Ψ on $\Omega^{(2)}$ compatible with Ω, the graph $\Gamma = \Gamma(\mathscr{D}, \Omega, \Psi)$, group G, partition $\mathscr{B} = \mathscr{B}(\Omega)$ and integer r satisfy all the conditions above.

The construction will be given in detail in Section 2, and Theorem 1.1 will be proved in Section 3. In Section 4, we will exploit this general construction to study symmetric graphs Γ such that there is at most one edge of Γ between any two blocks of $\mathscr{B}$. Such graphs arose naturally in the study of certain families of symmetric graphs [6, 8, 12]. This seemingly trivial case is notoriously difficult to manage, even in the case where in addition $\Gamma_{\mathscr{B}}$ is a complete graph (see [4, Section 4]). The behaviour of such graphs seems to be quite wild, and to the best knowledge of the author there is no useful description of such graphs up to now. As an application of our construction, we will provide such a description. In fact, in this case $\mathscr{D}^*(B)$ contains no repeated blocks, and hence Γ is isomorphic to a certain G-flag graph. We will further characterize such a graph as a G-flag graph $\Gamma(\mathscr{D}, \Omega, \Psi)$ with Ω satisfying some additional condition (see Theorem 4.3). This led to a construction [12] of G-symmetric graphs $(\Gamma, \mathscr{B})$ with $k = v - 1 \geqslant 2$ and, subsequently, classifications of certain subfamilies of such graphs, for G a projective or affine group over a finite field. Guided by our flag graph construction, in a recent work Iranmanesh, Praeger and the author gave an interesting construction [8, Construction 5.2] of some G-symmetric graphs $(\Gamma, \mathscr{B})$ with $\Gamma_{\mathscr{B}}$ a connected trivalent (G, 3)-arc regular graph. In that construction $\mathscr{D}$ is a triple system and the flags in Ω can be identified with certain 4-paths of $\Gamma_{\mathscr{B}}$. It is hoped that some other interesting families of symmetric graphs could be constructed or characterized by using the flag graph construction introduced in this paper and the knowledge on point- and block-transitive designs.

2. The Flag Graph Construction

For notation and terminology on designs, graphs and permutation groups, the reader is referred to [1–3], respectively. For a group G acting on two finite sets Ω_1, Ω_2 respectively, if there exists a bijection ρ from Ω_1 to Ω_2 such that $(\rho(\alpha))^g = \rho(\alpha^g)$

for any $\alpha \in \Omega_1$ and $g \in G$, then the actions of G on Ω_1 and Ω_2 are said to be *permutationally equivalent* with respect to ρ. An *incidence structure* is a triple $\mathscr{D} = (\mathscr{V}, \mathscr{E}, \mathrm{I})$, where $\mathscr{V}$, $\mathscr{E}$ are disjoint finite sets and I is a binary relation between $\mathscr{V}$ and $\mathscr{E}$, that is, $\mathrm{I} \subseteq \mathscr{V} \times \mathscr{E}$. The members of $\mathscr{V}$, $\mathscr{E}$ and I are called the *points*, *blocks* and *flags* of $\mathscr{D}$, respectively. If (α, X) is a flag, then we simply write $\alpha \mathrm{I} X$ and say that α, X are *incident* with each other. If two blocks are incident with the same set of points, then they are said to be *repeated blocks*. If all the blocks are incident with the same number (say k) of points, and all the points are incident with the same number (say r) of blocks, then $\mathscr{D}$ is said to be a 1-(v, k, r) *design*, where $v := |\mathscr{V}|$. A 1-(v, k, r) design $\mathscr{D}$ is said to be a t-(v, k, λ) *design*, for some integers $t \geqslant 2$ and $\lambda \geqslant 1$, if any t distinct points are incident with λ blocks simultaneously. The *dual* of a 1-(v, k, r) design $\mathscr{D} = (\mathscr{V}, \mathscr{E}, \mathrm{I})$ is the 1-(b, r, k) design $\mathscr{D}^* := (\mathscr{E}, \mathscr{V}, \mathrm{I}^*)$ with $X \mathrm{I}^* \alpha$ if and only if $\alpha \mathrm{I} X$, where b is the number of blocks of $\mathscr{D}$. If G is a group acting on $\mathscr{V}$ and $\mathscr{E}$, respectively, such that $\alpha \mathrm{I} X$ if and only if $\alpha^g \mathrm{I} X^g$, for $\alpha \in \mathscr{V}$, $X \in \mathscr{E}$ and $g \in G$, then we say that $\mathscr{D}$ *admits* G as a group of automorphisms. In this case G induces a natural action on the flags of $\mathscr{D}$. If G is transitive on the points (blocks, flags, respectively) of $\mathscr{D}$, then $\mathscr{D}$ is said to be *G-point-transitive* (*G-block-transitive*, *G-flag-transitive*, respectively). *In the following we assume without mentioning explicitly that the* 1*-designs used for our construction have no repeated blocks*. As usual in the literature, we may identify each block L of such a 1-design with the set of points incident with L.

We start our construction with the following general setting. Let $\mathscr{D}$ be a 1-design, and Ω a set of flags of $\mathscr{D}$. Let Ψ be a subset of the set $\Omega^{(2)}$ of ordered pairs of distinct flags in Ω. If Ψ is *self-paired*, that is, $((\sigma, L), (\tau, N)) \in \Psi$ implies $((\tau, N), (\sigma, L)) \in \Psi$, then we define the *flag graph* of $\mathscr{D}$ with respect to (Ω, Ψ), denoted by $\Gamma(\mathscr{D}, \Omega, \Psi)$, to be the graph with vertex set Ω in which two "vertices" (σ, L), $(\tau, N) \in \Omega$ are adjacent if and only if $((\sigma, L), (\tau, N)) \in \Psi$. The self-parity of Ψ guarantees that this graph is well-defined as an undirected graph. For a given point σ of $\mathscr{D}$, we denote by $\Omega(\sigma)$ the set of flags in Ω with point entry σ. Let $\mathscr{D}$ admit a group G of automorphisms. If Ω is a G-orbit on the flags of $\mathscr{D}$, then $\Omega(\sigma)$ is a G_σ-orbit on the flags of $\mathscr{D}$ with point entry σ, where G_σ is the stabilizer of σ in G. In this case, $\Gamma(\mathscr{D}, \Omega, \Psi)$ is G-vertex-transitive and its vertex set Ω admits

$$\mathscr{B}(\Omega) := \{\Omega(\sigma) : \sigma \text{ a point of } \mathscr{D}\} \tag{1}$$

as a natural G-invariant partition. If furthermore Ψ is a G-orbit on $\Omega^{(2)}$ (under the induced action), then $\Gamma(\mathscr{D}, \Gamma, \Psi)$ is G-symmetric. For a flag (σ, L) of $\mathscr{D}$, we use $G_{\sigma,L}$ to denote the subgroup of G fixing (σ, L), that is, the subgroup of G fixing σ and L setwise. Motivated by our study of G-symmetric graphs with $\mathscr{D}^*(B)$ containing no repeated blocks (see the next section), we require that Ω and Ψ satisfy the following conditions.

Definition 2.1. Let $\mathscr{D}$ be a G-point-transitive and G-block-transitive 1-design (with block size at least 2). Let σ be a point of $\mathscr{D}$. A G-orbit Ω on the flags of $\mathscr{D}$ is said to be *feasible* if

(a) $|\Omega(\sigma)| \geqslant 2$; and
(b) $G_{\sigma,L}$ is transitive on $L \setminus \{\sigma\}$, for some (and hence all) $(\sigma, L) \in \Omega$.

For such a feasible Ω, a G-orbit Ψ on $\Omega^{(2)}$ is said to be *compatible* with Ω if (c) $\sigma \neq \tau$ and $\sigma, \tau \in L \cap N$ for some (and hence all) $((\sigma, L), (\tau, N)) \in \Psi$.

If Ω is feasible and Ψ is self-paired and compatible with Ω, then for brevity we call $\Gamma(\mathscr{D}, \Omega, \Psi)$ the *G-flag graph* of $\mathscr{D}$ with respect to (Ω, Ψ).

Note that, since G is transitive on the points of $\mathscr{D}$, the validity of (a), (b) above does not depend on the choice of the point σ. In the following we will consider only G-flag graphs $\Gamma(\mathscr{D}, \Omega, \Psi)$. For such a pair (Ω, Ψ) and $((\sigma, L), (\tau, N)) \in \Psi$, it may happen that $L = N$ (see Example 4.5 in Section 4), and in this case each block of $\mathscr{D}$ induces a subgraph consisting of some connected components of $\Gamma(\mathscr{D}, \Omega, \Psi)$.

3. Proof of Theorem 1.1

To prove Theorem 1.1 we need some preliminary results. From the following discussion we can see that the flag graph construction given in the previous section comes in a natural way.

Let Γ be a G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$. Then the incidence structure $\mathscr{D}(B) = (B, \Gamma_{\mathscr{B}}(B), \mathrm{I})$ is a 1-(v, k, r) design, where $v = |B|$, $k = |\Gamma(C) \cap B|$ (for $C \in \Gamma_{\mathscr{B}}(B)$) and $r = |\Gamma_{\mathscr{B}}(\alpha)|$ are as defined in the introduction. The setwise stabilizer G_B of B in G induces natural actions on B and $\Gamma_{\mathscr{B}}(B)$. Moreover, $\mathscr{D}(B)$ admits G_B as a point-, block- and flag-transitive group of automorphisms [4]. Thus the dual 1-design $\mathscr{D}^*(B) := (\Gamma_{\mathscr{B}}(B), B, \mathrm{I}^*)$ of $\mathscr{D}(B)$ also admits G_B as a point-, block- and flag-transitive group of automorphisms. The "points" of $\mathscr{D}^*(B)$ are those blocks of $\mathscr{B}$ which are adjacent to B. We may identify the "blocks" of $\mathscr{D}^*(B)$ with the subsets $\Gamma_{\mathscr{B}}(\alpha)$ of $\Gamma_{\mathscr{B}}(B)$, for $\alpha \in B$, and thus identify the "block" set of $\mathscr{D}^*(B)$ with

$$\mathbf{E}(B) := \{\Gamma_{\mathscr{B}}(\alpha) : \alpha \in B\}.$$

We observe that $\mathscr{D}^*(B)$ can be "expanded" to the following 1-design which admits G as a point- and block-transitive group of automorphisms. For each $\alpha \in V(\Gamma)$, let $B(\alpha)$ denote the block of $\mathscr{B}$ containing α, and set

$$\mathscr{L}(\alpha) := \{B(\alpha)\} \cup \Gamma_{\mathscr{B}}(\alpha).$$

Note that, for distinct vertices α, β of Γ, it may happen that $\mathscr{L}(\alpha) = \mathscr{L}(\beta)$. We use $\mathbf{L}$ to denote the set of all $\mathscr{L}(\alpha)$, for $\alpha \in V(\Gamma)$, with repeated ones identified. One can see that $\mathscr{L}(\alpha) = \mathscr{L}(\beta)$ if and only if $\mathscr{L}(\alpha^g) = \mathscr{L}(\beta^g)$ for any $g \in G$. Therefore, the action of G on $\mathscr{B}$ induces an action of G on $\mathbf{L}$ defined by $(\mathscr{L}(\alpha))^g := \mathscr{L}(\alpha^g)$, for $\alpha \in V(\Gamma)$ and $g \in G$. We define

$$\mathscr{D}(\Gamma, \mathscr{B}) := (\mathscr{B}, \mathbf{L})$$

to be the incidence structure in which B is incident with $\mathscr{L}(\alpha)$ if and only if $B \in \mathscr{L}(\alpha)$.

Lemma 3.1. *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$. Then*
(*a*) *$\mathscr{D}(\Gamma, \mathscr{B})$ is a* 1*-design with block size* $r + 1$; *and*
(*b*) *$\mathscr{D}(\Gamma, \mathscr{B})$ admits G as a point- and block-transitive group of automorphisms.*

Proof. It is clear that G is transitive on $\mathscr{B}$ and on $\mathbf{L}$, and that G preserves the incidence relation of $\mathscr{D}(\Gamma, \mathscr{B})$. So G induces a group of automorphisms of $\mathscr{D}(\Gamma, \mathscr{B})$, and each $B \in \mathscr{B}$ is incident with the same number of elements of $\mathbf{L}$. Thus $\mathscr{D}(\Gamma, \mathscr{B})$ is a G-point-transitive and G-block-transitive 1-design. Clearly, $\mathscr{D}(\Gamma, \mathscr{B})$ has block size $r + 1$. □

In a lot of cases, the 1-design $\mathscr{D}^*(B)$ contains no repeated blocks, that is, $\Gamma_{\mathscr{B}}(\alpha) \neq \Gamma_{\mathscr{B}}(\beta)$ for distinct vertices, α, $\beta \in B$. The main result (Theorem 1.1) of this paper states that the flag graph construction can produce all G-symmetric graphs with this property. The truth of this result lies on the fact that in this case the flags ($B(\alpha)$, $\mathscr{L}(\alpha)$) of $\mathscr{D}(\Gamma, \mathscr{B})$, for $\alpha \in V(\Gamma)$, are pairwise distinct, or equivalently, for each $B \in \mathscr{B}$ the members of

$$\mathbf{L}(B) := \{\mathscr{L}(\alpha) : \alpha \in B\}$$

are pairwise distinct. Therefore, in this case $V(\Gamma)$ can be identified with the subset

$$\Omega(\Gamma, \mathscr{B}) := \{(B(\alpha), \mathscr{L}(\alpha)) : \alpha \in V(\Gamma)\}$$

of flags of $\mathscr{D}(\Gamma, \mathscr{B})$ via $\alpha \mapsto (B(\alpha), \mathscr{L}(\alpha))$. Denote by $G_{B,\Gamma_{\mathscr{B}}(\alpha)}$ and $G_{B,\mathscr{L}(\alpha)}$ the setwise stabilizers of $\Gamma_{\mathscr{B}}(\alpha)$ and $\mathscr{L}(\alpha)$ in G_B, respectively. Note that, under the action of G on $\mathbf{L}$, the subgroup G_B of G leaves $\mathbf{L}(B)$ invariant, and hence G_B induces an action on $\mathbf{L}(B)$.

Lemma 3.2. *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$. Then $\Omega(\Gamma, \mathscr{B})$ is a G-orbit on the flags of $\mathscr{D}(\Gamma, \mathscr{B})$. The flags $(B(\alpha), \mathscr{L}(\alpha))$ for $\alpha \in V(\Gamma)$ are pairwise distinct if and only if $\mathscr{D}^*(B)$ contains no repeated blocks for $B \in \mathscr{B}$, and in this case the following hold.*

(*a*) *The mapping $\rho : \alpha \mapsto (B(\alpha), \mathscr{L}(\alpha))$, for $\alpha \in V(\Gamma)$, defines a bijection from $V(\Gamma)$ to $\Omega(\Gamma, \mathscr{B})$; and the actions of G on $V(\Gamma)$ and on $\Omega(\Gamma, \mathscr{B})$ are permutationally equivalent with respect to ρ.*

(*b*) *The action of G_B on B is permutationally equivalent to the actions of G_B on $\mathbf{E}(B)$, $\mathbf{L}(B)$ with respect to the bijections defined by $\alpha \mapsto \Gamma_{\mathscr{B}}(\alpha)$, $\alpha \mapsto \mathscr{L}(\alpha)$, for $\alpha \in B$, respectively. Hence we have $G_{B,\Gamma_{\mathscr{B}}(\alpha)} = G_{B,\mathscr{L}(\alpha)} = G_\alpha$.*

(*c*) *$G_{B,\mathscr{L}(\alpha)}$ is transitive on $\Gamma_{\mathscr{B}}(\alpha)$, for $\alpha \in B$.*

Proof. Since G is transitive on $V(\Gamma)$, it is easy to see that $\Omega(\Gamma, \mathscr{B})$ is a G-orbit on the flags of $\mathscr{D}(\Gamma, \mathscr{B})$. Clearly, the flags ($B(\alpha)$, $\mathscr{L}(\alpha)$), ($B(\beta)$, $\mathscr{L}(\beta)$) in $\Omega(\Gamma, \mathscr{B})$ corresponding to two distinct vertices α, β are identical if and only if α, β are in the same block of $\mathscr{B}$ and $\Gamma_{\mathscr{B}}(\alpha) = \Gamma_{\mathscr{B}}(\beta)$. In other words, the flags ($B(\alpha)$, $\mathscr{L}(\alpha)$) for $\alpha \in V(\Gamma)$ are pairwise distinct if and only if $\mathscr{D}^*(B)$ contains no repeated blocks. In this case it is easy to see that the mapping $\rho : \alpha \mapsto (B(\alpha), \mathscr{L}(\alpha))$ (for $\alpha \in V(\Gamma)$) is bijective, and that the actions of G on $V(\Gamma)$ and on $\Omega(\Gamma, \mathscr{B})$ are permutationally equivalent with respect to ρ. The truth of (b) follows from a routine argument. For two blocks $C, D \in \Gamma_{\mathscr{B}}(\alpha)$, there exist $\beta \in C$ and $\gamma \in D$ which are adjacent to α. So, by the G-symmetry of Γ, there exists $g \in G$ such that $(\alpha, \beta)^g = (\alpha, \gamma)$. This implies $g \in G_\alpha$ and $C^g = D$. Hence G_α is transitive on $\Gamma_{\mathscr{B}}(\alpha)$, that is, $G_{B,\mathscr{L}(\alpha)}$ is transitive on $\Gamma_{\mathscr{B}}(\alpha)$ and (c) is proved. □

Now we can proceed to the proof of the main result.

Proof of Theorem 1.1. Suppose that Γ, G, $\mathscr{B}$ and r are as in the first part of Theorem 1.1. Then by Lemma 3.1, $\mathscr{D} := \mathscr{D}(\Gamma, \mathscr{B})$ is a G-point-transitive and G-block-transitive 1-design with block size $r+1$. From Lemma 3.2, $\Omega := \Omega(\Gamma, \mathscr{B})$ is a G-orbit on the flags of $\mathscr{D}$, and the mapping $\rho : \gamma \mapsto (B(\gamma), \mathscr{L}(\gamma))$, for $\gamma \in V(\Gamma)$, is a bijection from $V(\Gamma)$ to Ω. In particular, we have $|\Omega(B)| = |B| \geqslant 2$. For $(B, \mathscr{L}) \in \Omega(B)$, say $\mathscr{L} = \mathscr{L}(\alpha)$ for some $\alpha \in B$, we have $\mathscr{L} \setminus \{B\} = \Gamma_{\mathscr{B}}(\alpha)$. So it follows from Lemma 3.2(c) that $G_{B,\mathscr{L}}$ is transitive on $\mathscr{L} \setminus \{B\}$. Therefore, Ω is a feasible G-orbit on the flags of $\mathscr{D}$.

Clearly, for each arc (α, β) of Γ, we have $B(\alpha) \neq B(\beta)$ and $B(\alpha)$, $B(\beta) \in \mathscr{L}(\alpha) \cap \mathscr{L}(\beta)$. Therefore, setting

$$\Psi := \{((B(\alpha), \mathscr{L}(\alpha)), (B(\beta), \mathscr{L}(\beta))) : (\alpha, \beta) \in \mathrm{Arc}(\Gamma)\},$$

we have $\Psi \subseteq \Omega^{(2)}$, and Ψ is self-paired and compatible with Ω. By Lemma 3.2(a), the actions of G on $V(\Gamma)$ and Ω are permutationally equivalent with respect to the bijection ρ defined above. Since Γ is G-symmetric, this implies that $\Psi = ((B(\alpha), \mathscr{L}(\alpha)), (B(\beta), \mathscr{L}(\beta)))^G$, for a fixed arc (α, β) of Γ. Hence Ψ is a self-paired G-orbit on $\Omega^{(2)}$ compatible with Ω. One can easily check that the bijection ρ defines an isomorphism from Γ to the G-flag graph $\Gamma(\mathscr{D}, \Omega, \Psi)$, and hence the first part of Theorem 1.1 is proved.

Suppose conversely that $\mathscr{D}$, G, Ω, Ψ and r are as in the second part of Theorem 1.1. Let $\Gamma := \Gamma(\mathscr{D}, \Omega, \Psi)$, and let $\mathscr{B} := \mathscr{B}(\Omega)$ be as defined in (1). Then it follows from the definition that Γ is a G-symmetric graph with vertex set Ω, and that $\mathscr{B}$ is a nontrivial G-invariant partition of Ω with block size $|\Omega(\sigma)| \geqslant 2$, where σ is a point of $\mathscr{D}$. To complete the proof, we need to show that the block size of $\mathscr{D}^*(\Omega(\sigma))$ is equal to r and that $\mathscr{D}^*(\Omega(\sigma))$ contains no repeated blocks.

Let $\Omega(\sigma)$, $\Omega(\tau)$ be adjacent blocks of $\mathscr{B}$. Then there exist $(\sigma, L) \in \Omega(\sigma)$ and $(\tau, N) \in \Omega(\tau)$ such that (σ, L), (τ, N) are adjacent in Γ, that is, $((\sigma, L), (\tau, N)) \in \Psi$. So we have $\sigma \neq \tau$ and σ, $\tau \in L \cap N$ by the compatibility of Ψ with Ω. Since Ω is feasible, it follows from (b) in Definition 2.1 that, for any $\delta \in L \setminus \{\sigma\}$, there exists $g \in G_{\sigma,L}$ such that $\tau^g = \delta$. Setting $M := N^g$, then we have $(\delta, M) = (\tau, N)^g \in \Omega$. Since g fixed σ, it fixes $\Omega(\sigma)$ setwise, and moreover $\sigma \in N$ implies $\sigma \in M$. Also, $\sigma \neq \tau$ implies that $\sigma = \sigma^g \neq \tau^g = \delta$. Thus we have $((\sigma, L), (\delta, M)) = ((\sigma, L), (\tau, N))^g \in \Psi$, that is, (σ, L) and (δ, M) are adjacent in Γ. Hence $\Omega(\delta) \in \Gamma_{\mathscr{B}}((\sigma, L))$. Conversely, suppose that $\Omega(\delta) \in \Gamma_{\mathscr{B}}((\sigma, L))$ for some point δ of $\mathscr{D}$. Then there exists $(\delta, M) \in \Omega(\delta)$ such that (σ, L) and (δ, M) are adjacent in Γ. So $((\sigma, L), (\delta, M)) \in \Psi$ and hence there exists $h \in G$ such that $((\sigma, L), (\tau, N))^h = ((\sigma, L), (\delta, M))$. Thus we have $h \in G_{\sigma,L}$, $\tau^h = \delta$ and $N^h = M$. Since h fixes σ and fixes L setwise, and since $\tau \in L \setminus \{\sigma\}$, we have $\delta = \tau^h \in L \setminus \{\sigma\}$. Therefore, we have proved that $\Gamma_{\mathscr{B}}((\sigma, L)) = \{\Omega(\delta) : \delta \in L \setminus \{\sigma\}\}$, and hence $\mathscr{D}^*(\Omega(\sigma))$ has block size $|L \setminus \{\sigma\}| = r$. Moreover, since $\mathscr{D}$ contains no repeated blocks, we have $L \neq L'$ for distinct (σ, L), $(\sigma, L') \in \Omega(\sigma)$. This together with the argument above implies that $\Gamma_{\mathscr{B}}((\sigma, L)) \neq \Gamma_{\mathscr{B}}((\sigma, L'))$, and hence $\mathscr{D}^*(\Omega(\sigma))$ contains no repeated blocks. □

The special case where in addition $\Gamma_{\mathscr{B}}$ is a complete graph is particularly interesting. In this case, we have $\Gamma_{\mathscr{B}} \cong K_{b+1}$, where $b := |\Gamma_{\mathscr{B}}(B)|$ is the valency of $\Gamma_{\mathscr{B}}$. Since $\Gamma_{\mathscr{B}}$ is G-symmetric, this occurs if and only if G is doubly transitive on

$\mathscr{B}$. So in this case $\mathscr{D}(\Gamma, \mathscr{B}) = (\mathscr{B}, \mathbf{L})$ is a G-doubly transitive and G-block-transitive 2-$(b+1, r+1, \lambda)$ design, for some integer $\lambda \geqslant 1$, and is an extension of $\mathscr{D}^*(B)$. (As usual in the literature, when we say a design $\mathscr{D}$ is G-doubly transitive, we mean G is doubly transitive on the points of $\mathscr{D}$). Conversely, if $\mathscr{D}$ is a G-doubly transitive and G-block-transitive 2-$(b+1, r+1, \lambda)$ design, then for any G-flag graph $\Gamma = \Gamma(\mathscr{D}, \Omega, \Psi)$ of $\mathscr{D}$, we have $\Gamma_{\mathscr{B}(\Omega)} \cong K_{b+1}$. So Theorem 1.1 has the following consequence.

Corollary 3.3. *Let $b \geqslant 2$ and $r \geqslant 1$ be integers, and let G be a group. Then the following two assertions (a) and (b) are equivalent.*

(*a*) Γ *is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$ such that $\mathscr{D}^*(B)$ has block size r and contains no repeated blocks, and such that* $\Gamma_{\mathscr{B}} \cong K_{b+1}$.

(*b*) $\Gamma \cong \Gamma(\mathscr{D}, \Omega, \Psi)$, *for a G-doubly transitive and G-block-transitive* 2-$(b+1,\ r+1, \lambda)$ *design $\mathscr{D}$, a feasible G-orbit Ω on the flags of $\mathscr{D}$, and a self-paired G-orbit Ψ on $\Omega^{(2)}$ compatible with Ω.*

A lot of work on doubly transitive 2-designs has been done in the literature (see e.g. [1]). This would be very helpful in studying some particular families of G-symmetric graphs Γ such that $\Gamma_{\mathscr{B}}$ is a complete graph and $\mathscr{D}^*(B)$ contains no repeated blocks. Partial results towards such a study existed in the literature, as exemplified in the following example.

Example 3.4. Suppose Γ is a G-symmetric graph such that $k < v$ and G_B is doubly transitive on B. Then $\mathscr{D}^*(B)$ contains no repeated blocks. (Suppose otherwise, then since G_B is doubly transitive on the blocks of $\mathscr{D}^*(B)$, we would have $\Gamma_{\mathscr{B}}(\alpha) = \Gamma_{\mathscr{B}}(\beta)$ for all $\alpha, \beta \in B$. This implies $k = v$ and thus contradicts with our assumption.) Such graphs Γ with the additional properties that $\Gamma_{\mathscr{B}} \cong K_{b+1}$ and $v \geqslant b$ were classified in [5, Theorems 1.1 and 1.2]. Some of them were defined in terms of certain 2-designs. This is not a mere coincidence: From Corollary 3.3 all such graphs are G-flag graphs of some G-doubly transitive and G-block-transitive 2-designs.

4. Symmetric Graphs with $k = 1$

In this section we study G-symmetric graphs Γ admitting a nontrivial G-invariant partition $\mathscr{B}$ such that $k = 1$, that is, $\Gamma[B, C] \cong K_2$ for adjacent blocks B, C of $\mathscr{B}$. In this case $\mathscr{D}^*(B)$ contains no repeated blocks (see Lemma 4.1(a) below). Hence, from Theorem 1.1, Γ is isomorphic to a G-flag graph of $\mathscr{D} := \mathscr{D}(\Gamma, \mathscr{B})$. We will further characterize such a graph Γ as a G-flag graph $\Gamma(\mathscr{D}, \Omega, \Psi)$ with Ω satisfying some additional condition. Note that in this case $\mathscr{D}^*(B)$ has "blocks" $\Gamma_{\mathscr{B}}(\alpha) = \{C \in \mathscr{B} : \Gamma(C) \cap B(\alpha) = \{\alpha\}\}$, for $\alpha \in B$. For a regular graph Σ, we use val(Σ) to denote the valency of Σ.

Lemma 4.1. *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$ such that $\Gamma[B, C] \cong K_2$ for adjacent blocks B, C of $\mathscr{B}$. Then the following hold.*

(*a*) *$\mathscr{D}^*(B)$ contains no repeated blocks. Moreover, the set $\mathbf{E}(B) = \{\Gamma_{\mathscr{B}}(\alpha) : \alpha \in B\}$ of "blocks" of $\mathscr{D}^*(B)$ is a G_B-invariant partition of $\Gamma_{\mathscr{B}}(B)$.*

(*b*) *The block size r of $\mathscr{D}^*(B)$ is equal to* val(Γ), *and further* val($\Gamma_{\mathscr{B}}$) $= vr$.

(*c*) *If G is faithful on $V(\Gamma)$, then the induced action of G on $\mathscr{B}$ is faithful.*

(*d*) *If $\mathscr{L}(\alpha) = \mathscr{L}(\beta)$ for some pair of distinct vertices α, β of Γ, then $r+1$ divides $v|\mathscr{B}|$, $\Gamma \cong (v|\mathscr{B}|/(r+1)) \cdot K_{r+1}$, and $\mathscr{L}(\gamma) = \mathscr{L}(\delta)$ for all vertices γ, δ in the same component of Γ.*

Proof. (a) Since there is only one edge of Γ between two adjacent blocks of $\mathscr{B}$, we have $\Gamma_{\mathscr{B}}(\alpha) \cap \Gamma_{\mathscr{B}}(\beta) = \emptyset$ for distinct vertices α, β in the same block B of $\mathscr{B}$. Thus, $\mathbf{E}(B)$ is a partition of $\Gamma_{\mathscr{B}}(B)$, and in particular $\mathscr{D}^*(B)$ contains no repeated blocks. Suppose $(\Gamma_{\mathscr{B}}(\alpha))^g \cap \Gamma_{\mathscr{B}}(\beta) \neq \emptyset$ for some $\alpha, \beta \in B$ and $g \in G_B$, say $C^g = D$ for some $C \in \Gamma_{\mathscr{B}}(\alpha)$ and $D \in \Gamma_{\mathscr{B}}(\beta)$. Since α is the unique vertex in B adjacent to a vertex in C and since β is the unique vertex in B adjacent to a vertex in D, $C^g = D$ implies $\alpha^g = \beta$ and hence $(\Gamma_{\mathscr{B}}(\alpha))^g = \Gamma_{\mathscr{B}}(\beta)$. Therefore, $\mathbf{E}(B)$ is a G_B-invariant partition of $\Gamma_{\mathscr{B}}(B)$.

(b) This follows immediately from (a) and our assumption on $\Gamma[B, C]$.

(c) Suppose that $g \in G$ fixes setwise each block of $\mathscr{B}$. Then, for each $B \in \mathscr{B}$ and $\alpha \in B$, g fixes in particular each of the blocks in $\Gamma_{\mathscr{B}}(\alpha)$. So it follows from (a) that g fixes each vertex in B. Since this holds for each $B \in \mathscr{B}$, g fixes each vertex of Γ. So, if G is faithful on $V(\Gamma)$, then $g = 1$ and hence G is faithful on $\mathscr{B}$.

(d) Suppose that $\mathscr{L}(\alpha) = \mathscr{L}(\beta)$ for two distinct vertices $\alpha \in B$ and $\beta \in C$. Then $B \neq C$, $C \in \Gamma_{\mathscr{B}}(\alpha)$, $B \in \Gamma_{\mathscr{B}}(\beta)$, and in particular B, C are adjacent blocks. Moreover, since there is only one edge between B and C, the vertices α, β must be adjacent in Γ. So the transitivity of G_α on $\Gamma(\alpha)$ implies that, for each $\gamma \in \Gamma(\alpha)$, there exists $g \in G_\alpha$ such that $\beta^g = \gamma$. Since $\mathscr{L}(\alpha) = \mathscr{L}(\beta)$ we then have $\mathscr{L}(\alpha) = (\mathscr{L}(\alpha))^g = (\mathscr{L}(\beta))^g = \mathscr{L}(\gamma)$. In particular, this implies that each block in $\mathscr{L}(\alpha)$ other than $B(\gamma)$ contains a (unique) neighbour of γ, and so any two blocks in $\mathscr{L}(\alpha)$ are adjacent. For distinct vertices γ, $\delta \in \Gamma(\alpha)$, say $\delta \in D$, let δ' be the neighbour of γ in the block D. Then by the G-symmetry of Γ there exists $h \in G$ such that $(\alpha, \delta)^h = (\gamma, \delta')$. This implies $(\mathscr{L}(\alpha))^h = \mathscr{L}(\gamma)$ and $(\mathscr{L}(\delta))^h = \mathscr{L}(\delta')$. Since $\mathscr{L}(\alpha) = \mathscr{L}(\delta)$ as shown above, we have $\mathscr{L}(\delta') = \mathscr{L}(\gamma) = \mathscr{L}(\alpha)$. Thus δ' is adjacent to a vertex in B. However, our assumption on $\Gamma[B, D]$ implies that δ is the unique vertex in D adjacent to a vertex in B. So we must have $\delta' = \delta$. Thus we have shown that any two vertices in $\Gamma(\alpha)$ are adjacent. Hence $\{\alpha\} \cup \Gamma(\alpha)$ induces the complete graph K_{r+1}, which must be a connected component of Γ since $\mathrm{val}(\Gamma) = r$ by (b). Therefore, Γ is a union of vertex disjoint copies of K_{r+1}, that is, $\Gamma \cong n \cdot K_{r+1}$ for an integer n. Since $|V(\Gamma)| = n(r+1) = v|\mathscr{B}|$, we have $n = v|\mathscr{B}|/(r+1)$ and $r+1$ is a divisor of $v|\mathscr{B}|$. Obviously, in this case $\mathscr{L}(\gamma) = \mathscr{L}(\delta)$ holds for all vertices γ, δ in the same component of Γ. □

Part (d) of Lemma 4.1 implies that, if Γ satisfies $k = 1$ and is not a union of complete graphs, then the sets $\mathscr{L}(\alpha)$ (for $\alpha \in V(\Gamma)$) of blocks of $\mathscr{B}$ are pairwise distinct and thus $\mathscr{D}(\Gamma, \mathscr{B}) = (\mathscr{B}, \{\mathscr{L}(\alpha) : \alpha \in V(\Gamma)\})$. On the other hand, we will see in Example 4.5 that the opposite case can occur, that is, it may happen that $\mathscr{L}(\alpha) = \mathscr{L}(\beta)$ for some pair of distinct vertices α, β of Γ. In view of part (a) of Lemma 4.1, we give the following definition.

Definition 4.2. Let $\mathscr{D}$ and G be as in Definition 2.1. A G-orbit Ω on the flags of $\mathscr{D}$ is said to be a 1-*feasible* G-orbit if it is feasible and $L \cap N = \{\sigma\}$ holds for distinct $(\sigma, L), (\sigma, N) \in \Omega(\sigma)$, where σ is a point of $\mathscr{D}$.

Now we prove that, up to isomorphism, the class of G-symmetric graphs with $k = 1$ is precisely the class of G-flag graphs $\Gamma(\mathscr{D}, \Omega, \Psi)$ such that Ω is 1-feasible.

Theorem 4.3. *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathscr{B}$ such that $\Gamma[B, C] \cong K_2$ for adjacent blocks B, C of $\mathscr{B}$, and let r be the valency of Γ. Then $\Gamma \cong \Gamma(\mathscr{D}, \Omega, \Psi)$ for a certain G-point-transitive and G-block-transitive 1-design $\mathscr{D}$ with block size $r+1$, a certain 1-feasible G-orbit Ω on the flags of $\mathscr{D}$, and a certain self-paired G-orbit Ψ on $\Omega^{(2)}$ compatible with Ω. Moreover, either $\mathscr{D}$ is a 1-$(|\mathscr{B}|, r+1, v)$ design and $\Gamma \cong (v|\mathscr{B}|/(r+1)) \cdot K_{r+1}$, or $\mathscr{D}$ is a 1-$(|\mathscr{B}|, r+1, v(r+1))$ design.*

Conversely, for any G-point-transitive and G-block-transitive 1-design $\mathscr{D}$ with block size $r+1$, any 1-feasible G-orbit Ω on the flags of $\mathscr{D}$, and any self-paired G-orbit Ψ on $\Omega^{(2)}$ compatible with Ω, the graph $\Gamma = \Gamma(\mathscr{D}, \Omega, \Psi)$, group G, partition $\mathscr{B} = \mathscr{B}(\Omega)$ and integer r satisfy all the conditions above.

We will show further that, in both parts of this theorem, G is faithful on the vertices of Γ if and only if it is faithful on the points on $\mathscr{D}$.

Proof. For the first part, we have shown in Lemma 4.1(a), (b) that $\mathscr{D}^*(B)$ contains no repeated blocks and that the block size of $\mathscr{D}^*(B)$ is equal to the valency r of Γ. So by Lemma 3.1, $\mathscr{D} := \mathscr{D}(\Gamma, \mathscr{B})$ is a G-point-transitive and G-block-transitive 1-design with block size $r+1$. We have shown in the proof of Theorem 1.1 that $\Omega := \Omega(\Gamma, \mathscr{B})$ is a feasible G-orbit on the flags of $\mathscr{D}$, that $\Psi := \{((B(\alpha), \mathscr{L}(\alpha)), (B(\beta), \mathscr{L}(\beta))) : (\alpha, \beta) \in \mathrm{Arc}(\Gamma)\}$ is a self-paired G-orbit on $\Omega^{(2)}$ compatible with Ω, and that $\Gamma \cong \Gamma(\mathscr{D}, \Omega, \Psi)$. From Lemma 4.1(a), we have $\mathscr{L} \cap \mathscr{N} = \{B\}$ for distinct $(B, \mathscr{L}), (B, \mathscr{N}) \in \Omega(B)$. Hence Ω is 1-feasible. If there exist distinct vertices α, $\beta \in V(\Gamma)$ such that $\mathscr{L}(\alpha) = \mathscr{L}(\beta)$, then by Lemma 4.1(d), $\Gamma \cong (v|\mathscr{B}|/(r+1)) \cdot K_{r+1}$ and $\mathscr{D}$ has $v|\mathscr{B}|/(r+1)$ blocks. So $\mathscr{D}$ is a 1-$(|\mathscr{B}|, r+1, v)$ design. In the remaining case, $\mathscr{D}$ has $|\mathscr{B}|$ points, $v|\mathscr{B}|$ blocks and hence is a 1-$(|\mathscr{B}|, r+1, v(r+1))$ design. This proves the first part of the theorem. In addition, by Lemma 4.1(c), if G is faithful on $V(\Gamma)$, then it is also faithful on the point set $\mathscr{B}$ of $\mathscr{D}$.

Suppose conversely that $\mathscr{D}$, G, Ω, Ψ and r are as in the second part of the theorem. Then from Theorem 1.1, $\Gamma := \Gamma(\mathscr{D}, \Omega, \Psi)$ is a G-symmetric graph admitting $\mathscr{B} := \mathscr{B}(\Omega)$ as a nontrivial G-invariant partition such that $\mathscr{D}^*(\Omega(\sigma))$ has block size r and contains no repeated blocks. Let $\Omega(\sigma)$, $\Omega(\tau)$ be adjacent blocks of $\mathscr{B}$. Then there exist $(\sigma, L) \in \Omega(\sigma)$ and $(\tau, N) \in \Omega(\tau)$ such that $((\sigma, L), (\tau, N)) \in \Psi$. So we have $\sigma \neq \tau$ and σ, $\tau \in L \cap N$ by the compatibility of Ψ with Ω. Since Ω is 1-feasible this implies that, for any $(\sigma, L') \in \Omega(\sigma) \setminus \{(\sigma, L)\}$ and $(\tau, N') \in \Omega(\tau) \setminus \{(\tau, N)\}$, we have $\sigma \notin N'$ and $\tau \notin L'$. Thus none of $((\sigma, L), (\tau, N')), ((\sigma, L'), (\tau, N))$ and $((\sigma, L'), (\tau, N'))$ belongs to Ψ. In other words, the edge of Γ joining (σ, L) and (τ, N) is the only edge between $\Omega(\sigma)$ and $\Omega(\tau)$. Hence we have $\Gamma[\Omega(\sigma), \Omega(\tau)] \cong K_2$, and consequently the valency of Γ is equal to the block size r of $\mathscr{D}^*(\Omega(\sigma))$. If an element of G fixes each flag in Ω, then it must fix each point of $\mathscr{D}$. So if G is faithful on the points of $\mathscr{D}$, then it must be faithful on Ω, the vertex set of Γ. This completes the proof of Theorem 4.3, and that of the statement immediately following it. □

We pause here to point out that graphs satisfying the condition of Theorem 4.3 arise naturally from our study [6, 10, 12] of imprimitive G-symmetric graphs $(\Gamma, \mathscr{B})$ with $k = v - 1 \geqslant 2$. In fact, for such a graph Γ and any two adjacent blocks B, C of $\mathscr{B}$, there is a unique vertex α in B which is not adjacent to any vertex in C, and similarly there is a unique vertex β in C not adjacent to any vertex in B. By [6, Proposition 3], the graph Γ' with vertex set $V(\Gamma)$ and edges of the form $\{\alpha, \beta\}$ is G-symmetric and admits the same G-invariant partition $\mathscr{B}$ such that $\Gamma'[B, C] \cong K_2$. So, by Theorem 4.3, Γ' is isomorphic to a G-flag graph $\Gamma(\mathscr{D}, \Omega, \Psi)$. Moreover, as shown in [12, Lemma 2.2(d)], the 1-feasible G-orbit Ω satisfies the extra condition that, for any $(\sigma, L) \in \Omega$ and $\tau \in L \setminus \{\sigma\}$, $G_{\sigma\tau}$ is transitive on $\Omega(\sigma) \setminus \{(\sigma, L)\}$. This additional feature enables us to construct the graph Γ by using a similar flag graph construction, see [12] for details. (Note that the meanings of feasibility and compatibility in [12] are different from that used in this paper. In [12] a 1-feasible Ω satisfying the extra condition above is said to be feasible, and the compatibility condition is meant for constructing Γ).

Now let us return to our study of G-symmetric graphs $(\Gamma, \mathscr{B})$ with $k = 1$. Analogous to Corollary 3.3, we have the following consequence of Theorem 4.3. Note that, by Lemma 4.1(b), we have $\mathrm{val}(\Gamma_{\mathscr{B}}) = vr$ for the graph Γ in Theorem 4.3. So $\Gamma_{\mathscr{B}}$ is a complete graph if and only if $\Gamma_{\mathscr{B}} \cong K_{vr+1}$, and in this case G is doubly transitive on $\mathscr{B}$ and hence $\mathscr{D}(\Gamma, \mathscr{B})$ is a 2-$(vr+1, r+1, \lambda)$ design for some integer $\lambda \geqslant 1$. Moreover, since $\mathbf{E}(B)$ is a G_B-invariant partition of $\Gamma_{\mathscr{B}}(B) = \mathscr{B} \setminus \{B\}$ (Lemma 4.1(a)) with block size r, in the general case where $r \geqslant 2$, G_B is imprimitive on $\mathscr{B} \setminus \{B\}$ and in particular G is not triply transitive on $\mathscr{B}$.

Corollary 4.4. *Let $v \geqslant 2$ and $r \geqslant 1$ be integers, and let G be a group. Then the following two assertions* (*a*) *and* (*b*) *are equivalent.*

(*a*) *Γ is a G-symmetric graph of valency r which admits a nontrivial G-invariant partition $\mathscr{B}$ of block size v such that $\Gamma[B, C] \cong K_2$ for any two blocks B, C of $\mathscr{B}$* (*so $\Gamma_{\mathscr{B}} \cong K_{vr+1}$*).

(*b*) $\Gamma \cong \Gamma(\mathscr{D}, \Omega, \Psi)$, *for a G-doubly transitive and G-block-transitive* 2-$(vr+1, r+1, \lambda)$ *design $\mathscr{D}$, a* 1*-feasible G-orbit Ω on the flags of $\mathscr{D}$, and a self-paired G-orbit Ψ on $\Omega^{(2)}$ compatible with Ω.*

Moreover, for any point σ of *$\mathscr{D}$, the set of points of $\mathscr{D}$ other than σ admits a G_σ-invariant partition of block size r. Hence $\mathscr{D}$ is not G-triply transitive when $r \geqslant 2$. Furthermore, either $\lambda = r + 1$, or $\lambda = 1$ and $\Gamma \cong (v(vr+1)/(r+1)) \cdot K_{r+1}$.*

The value of λ above was determined by using known relations (see e.g. [1, 2.10, Chapter I]) among the parameters of a 2-design. In the case where $\lambda = 1$, the design $\mathscr{D}$ is a linear space, and the graph Γ is constructed from $\mathscr{D}$ by using the method given in part (b) of the following example. In general, a *linear space* [1] is an incidence structure of points and blocks (called *lines*) in which any two distinct points are incident with exactly one line, any point is incident with at least two lines, and any line with at least two points. We use $L_{\sigma\tau}$ to denote the unique line through two given points σ, τ in a linear space.

Example 4.5. (a) If $\mathscr{D}$ is a G-flag-transitive linear space, then the flag set Ω of $\mathscr{D}$ is the only G-orbit on the flags of $\mathscr{D}$. Clearly, Ω satisfies (a) in Definition 2.1 and the condition in Definition 4.2. So Ω is feasible if and only if it satisfies (b) in Definition 2.1, and in this case Ω is 1-feasible. For such an Ω, a G-orbit on $\Omega^{(2)}$ is self-paired and compatible with Ω if and only if it has the form

$$\Psi = \{((\sigma, L_{\sigma\tau}), (\tau, L_{\sigma\tau})) : (\sigma, \tau) \in P\},$$

for some self-paired G-orbit P on ordered pairs of distinct points of $\mathscr{D}$. For such a Ψ, set $\Gamma := \Gamma(\mathscr{D}, \Omega, \Psi)$ and $L := L_{\sigma\tau}$ for a fixed $(\sigma, \tau) \in P$. From (b) in Definition 2.1, for any $\delta \in L \setminus \{\sigma\}$ there exists $g \in G_{\sigma,L}$ such that $\tau^g = \delta$. So $(\sigma, \delta) = (\sigma, \tau)^g \in P$ and $((\sigma, L), (\delta, L)) = ((\sigma, L), (\tau, L))^g \in \Psi$. It follows that (σ, L) is adjacent in Γ to any $(\delta, L) \in \Omega$ with $\delta \in L \setminus \{\sigma\}$. Therefore, each connected component of Γ is a complete graph induced by a line $L_{\sigma\tau}$, for $(\sigma, \tau) \in P$. Such a graph Γ satisfies the condition in Lemma 4.1(d).

(b) In particular, if $\mathscr{D}$ is a G-doubly transitive linear space with $vr + 1$ points and with block size $r + 1$, then $\mathscr{D}$ is G-flag-transitive and its flag set Ω is 1-feasible. In this case the only self-paired G-orbit on $\Omega^{(2)}$ compatible with Ω is

$$\Psi := \{((\sigma, L), (\tau, L)) : L \text{ is a line of } \mathscr{D},\ \sigma,\ \tau \text{ are distinct points on } L\}.$$

Hence $\mathscr{D}$ has a unique G-flag graph $\Gamma(\mathscr{D}, \Omega, \Psi)$, of which each connected component is a complete graph induced by a line of $\mathscr{D}$.

A 1-design $\mathscr{D}$ with block size 2 can be viewed as a regular graph Σ, and vice versa, if we identify the blocks of $\mathscr{D}$ with the edges of Σ. The automorphism groups of the design $\mathscr{D}$ and the graph Σ are the same. Moreover, under this identification each flag (σ, L) of $\mathscr{D}$, say $L = \{\sigma, \tau\}$, is the arc (σ, τ) of Σ. Hence $\mathscr{D}$ is G-flag-transitive if and only if Σ is G-symmetric. We conclude this paper by giving the following somewhat trivial example, which is mainly for illustrative purpose.

Example 4.6. A G-flag-transitive 1-design $\mathscr{D}$ with block size $r + 1 := 2$ such that each point is incident with $d \geqslant 2$ blocks can be identified with a G-symmetric graph Σ of valency d. Since $\mathscr{D}$ is G-flag-transitive, the only G-orbit on the flags of $\mathscr{D}$ is the set Ω of all flags of $\mathscr{D}$, that is, the arc set Arc(Σ) of Σ. It is clear that Ω is 1-feasible, and that the only self-paired G-orbit on $\Omega^{(2)}$ compatible with Ω is $\Psi := \{((\sigma, \tau), (\tau, \sigma)) : (\sigma, \tau) \in \mathrm{Arc}(\Sigma)\}$. So we get a unique G-flag graph $\Pi := \Gamma(\mathscr{D}, \Omega, \Psi)$, which has vertex set Arc(Σ) and edges joining (σ, τ) and (τ, σ), for all $(\sigma, \tau) \in \mathrm{Arc}(\Sigma)$. Clearly, we have $\Pi \cong n \cdot K_2$ and $\Pi_{\mathscr{B}(\Omega)} \cong \Sigma$, where n is the number of edges of Σ. From Theorem 4.3, these graphs Π can represent all G-symmetric graphs Γ of valency $r = 1$ such that $V(\Gamma)$ admits a nontrivial G-invariant partition $\mathscr{B}$ with $\Gamma[B, C] \cong K_2$, for adjacent blocks B, C of $\mathscr{B}$. Moreover, any G-symmetric graph Σ with valency $d \geqslant 2$ can appear as the quotient $\Gamma_{\mathscr{B}}$ of such a graph Γ.

A linear space is *trivial* if each line contains exactly two points. The graph Γ in Corollary 4.4(a) with additional property $r = 1$ is precisely the G-flag graph (give in Example 4.5(b)) of a trivial G-doubly transitive linear space $\mathscr{D}$ with $v + 1$ points. Corollary 4.4 and Examples 4.5(b), 4.6 together imply the characterization of such graphs given in [4, Theorem 4.2].

Acknowledgement. Part of the work was done while the author was at the University of Western Australia.

References

[1] Beth T, Jungnickel D, Lenz H (1986) Design Theory. Cambridge: Univ Press
[2] Biggs NL (1993) Algebraic Graph Theory, 2nd edn. Cambridge: Univ Press
[3] Dixon JD, Mortimer B (1996) Permutation Groups. New York: Springer
[4] Gardiner A, Praeger CE (1995) A geometrical approach to imprimitive graphs. Proc London Math Soc **71**(3): 524–546
[5] Gardiner A, Praeger CE (2003) Symmetric graphs with complete quotients (Preprint)
[6] Li CH, Praeger CE, Zhou S (2000) A class of finite symmetric graphs with 2-arc transitive quotients. Math Proc Cambridge Philos Soc **129**: 19–34
[7] Li CH, Praeger CE, Venkatesh A, Zhou S (2002) Finite locally quasiprimitive graphs. Discrete Math **246**: 197–218
[8] Iranmanesh MA, Praeger CE, Zhou S (2003) Finite symmetric graphs with two-arc transitive quotients (submitted)
[9] Praeger CE (1985) Imprimitive symmetric graphs. Ars Combinatoria **19A**: 149–163
[10] Zhou S (2003) Almost covers of 2-arc transitive graphs. Combinatorica (to appear)
[11] Zhou S (2002) Imprimitive symmetric graphs, 3-arc graphs and 1-designs. Discrete Math **244**: 521–537
[12] Zhou S (2002) Constructing a class of symmetric graphs. European J Combinatorics **23**: 741–760

Author's address: Department of Mathematics and Statistics, The University of Melbourne, Parkville, VIC 3010, Australia, e-mail: smzhou@ms.unimelb.edu.au