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A class of finite symmetric graphs with 2-arc transitive quotients

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Abstract

Let Γ be a finite G-symmetric graph whose vertex set admits a non-trivial G-invariant partition $\mathscr{B}$ with block size v. A framework for studying such graphs Γ was developed by Gardiner and Praeger which involved an analysis of the quotient graph $\Gamma_{\mathscr{B}}$ relative to $\mathscr{B}$, the bipartite subgraph $\Gamma[B, C]$ of Γ induced by adjacent blocks B, C of $\Gamma_{\mathscr{B}}$ and a certain 1-design $\mathscr{D}(B)$ induced by a block $B \in \mathscr{B}$. The present paper studies the case where the size k of the blocks of $\mathscr{D}(B)$ satisfies $k = v - 1$. In the general case, where $k = v - 1 \geqslant 2$, the setwise stabilizer G_B is doubly transitive on B and G is faithful on $\mathscr{B}$. We prove that $\mathscr{D}(B)$ contains no repeated blocks if and only if $\Gamma_{\mathscr{B}}$ is $(G, 2)$-arc transitive and give a method for constructing such a graph from a 2-arc transitive graph with a self-paired orbit on 3-arcs. We show further that each such graph may be constructed by this method. In particular every 3-arc transitive graph, and every 2-arc transitive graph of even valency, may occur as $\Gamma_{\mathscr{B}}$ for some graph Γ with these properties. We prove further that $\Gamma[B, C] \cong K_{v-1,v-1}$ if and only if $\Gamma_{\mathscr{B}}$ is $(G, 3)$-arc transitive.

1. *Introduction*

A graph Γ admitting a group G of automorphisms is said to be *G-symmetric* if G acts transitively on the set of ordered pairs of adjacent vertices of Γ. In many cases, for example if Γ is connected, such a group must be transitive on the vertex set $V(\Gamma)$. We assume that this is the case and moreover that there is a *non-trivial G-invariant partition* $\mathscr{B}$ of $V(\Gamma)$, that is, the elements of G permute the blocks B of $\mathscr{B}$ blockwise, and $1 < |B| < |V(\Gamma)|$. Such a graph is said to be an *imprimitive G-symmetric* graph. A study of such graphs was initiated in [**5**] by Gardiner and Praeger. It was suggested there that three objects associated with $\mathscr{B}$ had a strong influence on the structure of Γ, namely the quotient graph $\Gamma_{\mathscr{B}}$, the bipartite subgraph $\Gamma[B, C]$ of Γ induced by two adjacent blocks B, C of $\mathscr{B}$, and a 1-design $\mathscr{D}(B)$ induced on a block $B \in \mathscr{B}$ (see Section 2 for the definitions). It was further suggested that these three geometric objects might provide a good framework for investigating imprimitive symmetric graphs. The paper [**5**] was written in the context of *G-locally primitive* graphs (that is, G_α induces a primitive group on the set $\Gamma(\alpha)$ of vertices adjacent to a vertex α) and the theory was extended to general symmetric graphs in the sequels [**6**, **7**]. In particular, the case where Γ is G-locally primitive and the size k of the blocks of

$\mathscr{D}(B)$ satisfies $k = v - 1$ was studied in [**5**, section 5]. In this paper we extend that investigation to the class of all imprimitive G-symmetric graphs with $k = v - 1$. The assumption $k = v - 1$ is equivalent to the following: for distinct blocks $B, C \in \mathscr{B}$, either there are no edges between B and C, or there is a unique vertex $\alpha \in B$ such that $\Gamma(\alpha) \cap C = \varnothing$.

In the special case where $k = v - 1 = 1$, we define two G-symmetric graphs $\Gamma^{\star}, \Gamma^{\#}$ with vertex set $V(\Gamma)$ which are covers of $\Gamma_{\mathscr{B}}$ (see Section 3). Our main focus, however, is the general case where $k = v - 1 \geqslant 2$. We investigate this case in Section 4 and prove in particular that G acts faithfully on $\mathscr{B}$ and G_B is doubly transitive on B (see Theorem 5). Thus the design $\mathscr{D}(B)$ is degenerate, with each k-element subset of B occurring as a (possibly repeated) block of $\mathscr{D}(B)$. In Sections 5 and 6 we continue this investigation in the special case where $\mathscr{D}(B)$ contains no repeated blocks. Not only is this a natural assumption geometrically, but also we prove that $\mathscr{D}(B)$ has no repeated blocks if and only if $\Gamma_{\mathscr{B}}$ is $(G, 2)$-arc transitive. (An *s-arc* is a sequence $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of vertices such that α_i, α_{i+1} are adjacent and $\alpha_{i-1} \neq \alpha_{i+1}$ for each i. The graph Γ is said to be (G, s)-*arc transitive* if G is transitive on the s-arcs of Γ.) In this case (see Proposition 7) $\Gamma_{\mathscr{B}}$ has valency v and we show that the vertices of Γ may be labelled by the arcs of $\Gamma_{\mathscr{B}}$. We continue the investigation of this case in Section 6 where we first give a construction of a family of graphs which satisfy these conditions. The construction requires a $(G, 2)$-arc transitive graph Σ of valency $v \geqslant 3$ and a self-paired G-orbit Δ of 3-arcs of Σ (where Δ is said to be *self-paired* if $(\alpha, \beta, \gamma, \delta) \in \Delta$ if and only if $(\delta, \gamma, \beta, \alpha) \in \Delta$). For such Σ and Δ, the graph $\mathrm{Arc}_{\Delta}(\Sigma)$ is defined to have vertices the arcs of Σ with (σ, τ) joined by an edge to (σ', τ') if and only if $(\tau, \sigma, \sigma', \tau') \in \Delta$. We show in Theorem 10 that $\mathrm{Arc}_{\Delta}(\Sigma)$ is an imprimitive G-symmetric graph relative to a certain partition $\mathscr{B}(\Sigma)$ of the arcs of Σ, that '$k = v - 1$' and that $\mathscr{D}(B)$ has no repeated blocks for $B \in \mathscr{B}(\Sigma)$. We further show in Theorem 11 that every graph Γ satisfying these conditions is isomorphic to $\mathrm{Arc}_{\Delta}(\Gamma_{\mathscr{B}})$ for some Δ. Thus we have the following result, which is the main theorem of this paper.

Theorem 1. *Let Γ be a finite G-symmetric graph and $\mathscr{B}$ a non-trivial G-invariant partition of $V(\Gamma)$ with block size $v \geqslant 3$ such that $\mathscr{D}(B)$ has block size $v - 1$. Then $\mathscr{D}(B)$ contains no repeated blocks if and only if $\Gamma_{\mathscr{B}}$ is $(G, 2)$-arc transitive. In this case $\Gamma \cong \mathrm{Arc}_{\Delta}(\Gamma_{\mathscr{B}})$ for some self-paired G-orbit Δ of 3-arcs of $\Gamma_{\mathscr{B}}$. Conversely, for any self-paired G-orbit Δ of 3-arcs of a $(G, 2)$-arc transitive graph Σ of valency $v \geqslant 3$, the graph $\Gamma = \mathrm{Arc}_{\Delta}(\Sigma)$, group G and partition $\mathscr{B}(\Sigma)$ (defined in Section 6) satisfy all the conditions above.*

We note that every 3-arc transitive graph, and every 2-arc transitive graph of even valency, may occur as the graph Σ (see Remark 4(*c*)). This theorem follows immediately from Theorems 8, 10 and 11. If the 3-arcs in Δ form 3-cycles then the possibilities for Γ are given more explicitly in Theorem 8(*b*). For the case where $\Gamma_{\mathscr{B}}$ is $(G, 3)$-arc transitive there is a unique graph $\mathrm{Arc}_{\Delta}(\Gamma_{\mathscr{B}})$, namely where Δ is the set of all 3-arcs of $\Gamma_{\mathscr{B}}$. In this case we have the following characterization.

Theorem 2. *Suppose that Γ, G and $\mathscr{B}$ are as in Theorem 1 and that $\mathscr{D}(B)$ contains no repeated blocks. Then the following conditions are equivalent*:

(*a*) *$\Gamma_{\mathscr{B}}$ is $(G, 3)$-arc transitive*;

(*b*) *$\Gamma[B, C] \cong K_{v-1,v-1}$*;

(*c*) $\Gamma \cong \mathrm{Arc}_\Delta(\Gamma_{\mathscr{B}})$ *with* Δ *the set of all* 3*-arcs of* $\Gamma_{\mathscr{B}}$.
Thus in this case Γ *is uniquely determined by* $\Gamma_{\mathscr{B}}$.

In Theorem 1, in the case where $\mathscr{D}(B)$ has no repeated blocks, we do not know very much about the structure of the graphs $\mathrm{Arc}_\Delta(\Gamma_{\mathscr{B}})$ in general. If however $\Gamma_{\mathscr{B}}$ has girth 3 then $\Gamma_{\mathscr{B}}$ is a vertex-disjoint union of complete graphs K_{v+1}, Δ consists of all the 3-cycles of $\Gamma_{\mathscr{B}}$ and Γ is disconnected with all connected components complete graphs K_v (see Theorem 8(*b*)). On the other hand, if girth $(\Gamma_{\mathscr{B}}) \geqslant 5$ then we derive some weak upper bounds on the valency of Γ (see Corollary 1 in Section 5).

2. *Definitions, notation and preliminaries*

Let Γ be a finite G-symmetric graph such that there is a non-trivial G-invariant partition $\mathscr{B}$ of $V(\Gamma)$. We define the *quotient graph* $\Gamma_{\mathscr{B}}$ to be the graph with vertex set $\mathscr{B}$ in which two blocks $B, C \in \mathscr{B}$ are adjacent if and only if there exist $\alpha \in B, \beta \in C$ such that α and β are adjacent in Γ. It is clear that G induces an action (possibly unfaithful) on $\mathscr{B}$ and under this action $\Gamma_{\mathscr{B}}$ is G-symmetric. We suppose throughout that $(\mathscr{B}, \Gamma_{\mathscr{B}})$ is *non-trivial* in the sense that $\mathscr{B}$ is a non-trivial partition and that $\Gamma_{\mathscr{B}}$ has at least one edge. Then it is easy to see (see for example [**5**, **8**]) that each block of $\mathscr{B}$ is an *independent set* of Γ (that is, a subset of $V(\Gamma)$ such that no two vertices are adjacent in Γ). For each $\alpha \in V(\Gamma)$, $B(\alpha)$ denotes the block of $\mathscr{B}$ containing α.

For any two *adjacent blocks* $B, C \in \mathscr{B}$, we denote by $\Gamma(B)$ (respectively $\Gamma(C)$) the set of vertices of Γ adjacent to at least one vertex in B (respectively C); let $\Gamma[B, C]$ be the induced bipartite subgraph of Γ with $\Gamma(C) \cap B$ and $\Gamma(B) \cap C$ as the parts of the bipartition. Then $\Gamma[B, C]$ is $(G_{B\cup C})$-symmetric, where $G_{B\cup C}$ is the setwise stabilizer of $B \cup C$ in G. In particular, if $\Gamma[B, C]$ is a perfect matching between the vertices of B and C, then Γ is said to be a *cover* of $\Gamma_{\mathscr{B}}$.

For each block B, we denote by $\Gamma_{\mathscr{B}}(B)$ the set of blocks of $\mathscr{B}$ that are adjacent to B in $\Gamma_{\mathscr{B}}$; and we define $\mathscr{D}(B)$ as the design with point set B and blocks $\Gamma(C) \cap B$ (with possible repetitions) for $C \in \Gamma_{\mathscr{B}}(B)$. We emphasize that $\mathscr{D}(B)$ may have repeated blocks since we may have $\Gamma(C_1) \cap B = \Gamma(C_2) \cap B$ for distinct $C_1, C_2 \in \Gamma_{\mathscr{B}}(B)$. Set $k := |\Gamma(B) \cap C|$ for adjacent blocks B, C and $r := |\Gamma_{\mathscr{B}}(\alpha)|$ for $\alpha \in V(\Gamma)$, where $\Gamma_{\mathscr{B}}(\alpha) := \{B \in \mathscr{B} \colon B \cap \Gamma(\alpha) \neq \varnothing\}$. Let $v := |B|$ be the size of the blocks in $\mathscr{B}$ and $b := \mathrm{val}\,(\Gamma_{\mathscr{B}}) = |\Gamma_{\mathscr{B}}(B)|$ be the valency of $\Gamma_{\mathscr{B}}$. Then $vr = bk$ and $\mathscr{D}(B)$ is a 1-(v, k, r) design with b blocks (see [**2**] for terminology on designs).

Since Γ is G-symmetric, the bipartite graph $\Gamma[B, C]$ and the 1-design $\mathscr{D}(B)$ are, up to isomorphism, independent of the choice of the adjacent blocks B, C and the block B, respectively. *Thus, with any imprimitive G-symmetric graph Γ and non-trivial G-invariant partition $\mathscr{B}$ of $V(\Gamma)$ we have associated a triple* $(\Gamma_{\mathscr{B}}, \Gamma[B, C], \mathscr{D}(B))$.

For blocks $B, C, D \in \mathscr{B}$, let G_B be the setwise stabilizer of B in G and similarly let $G_{B,C} = (G_B)_C = (G_C)_B$ and $G_{B,C,D} = (G_{B,C})_D$. For a vertex $\alpha \in V(\Gamma)$, let $G_{\alpha,B}$ denote the subgroup of G fixing α and B setwise. Let $G_{[B]}$ be the setwise stabilizer in G of B and each of the blocks in $\Gamma_{\mathscr{B}}(B)$. Let $G_{(B)}$ be the pointwise stabilizer of B in G. We say that Γ is *vertex-distinguishable* with respect to $\mathscr{B}$ if, for any two adjacent blocks B, C of $\mathscr{B}$ and distinct vertices $\alpha, \beta \in \Gamma(B) \cap C$, we have $\Gamma(\alpha) \cap B \neq \Gamma(\beta) \cap B$. The following lemma exemplifies some graphs of this kind.

Lemma 1. *Suppose* Γ *is a finite* G*-symmetric graph admitting a non-trivial* G-

invariant partition $\mathscr{B}$. Then Γ is vertex-distinguishable with respect to $\mathscr{B}$ if, for adjacent blocks B, C of $\mathscr{B}$, one of the following conditions holds:

(*a*) *$\Gamma[B, C]$ is a matching;*
(*b*) *$\Gamma[B, C]$ is a complete bipartite graph minus a perfect matching between the vertices of $\Gamma(C) \cap B$ and $\Gamma(B) \cap C$;*
(*c*) *$G_{B,C}$ acts primitively on $\Gamma(B) \cap C$ and $\Gamma[B, C] \ncong K_{k,k}$.*

Proof. Clearly, the result is true whenever (*a*) or (*b*) occurs. Suppose that the condition (*c*) is satisfied. If there exist distinct $\alpha, \beta \in \Gamma(B) \cap C$ such that $\Gamma(\alpha) \cap B = \Gamma(\beta) \cap B$, then $\{\gamma \in C\colon \Gamma(\gamma) \cap B = \Gamma(\alpha) \cap B\}$ is a block of imprimitivity for $G_{B,C}$ in $\Gamma(B) \cap C$ and has size at least 2. Since this action is primitive, it follows that $\Gamma(\gamma) \cap B = \Gamma(\alpha) \cap B$ for all $\gamma \in \Gamma(B) \cap C$. This implies that $\Gamma[B, C] \cong K_{k,k}$, a contradiction. Thus, Γ is vertex-distinguishable with respect to $\mathscr{B}$.

Let G_1, G_2 be groups acting on finite sets Δ_1, Δ_2, respectively. The action of G_1 on Δ_1 is said to be *permutationally isomorphic* [**4**, pp. 17] to the action of G_2 on Δ_2 if there exists a bijection $\lambda\colon \Delta_1 \to \Delta_2$ and a group isomorphism $\psi\colon G_1 \to G_2$ such that $\lambda(\alpha^g) = (\lambda(\alpha))^{\psi(g)}$ for all $\alpha \in \Delta_1$ and $g \in G_1$. We refer to [**4**, **11**] for other terminology for permutation groups used in the paper.

Since the main concern of this paper is the case where $k = v - 1$, we introduce some special notation for this case. Suppose $k = v - 1$. Let $\alpha \in V(\Gamma)$, set $B = B(\alpha)$ and let $\mathscr{B}(\alpha) := \{C \in \mathscr{B}\colon \Gamma(C) \cap B = B \setminus \{\alpha\}\}$. Thus $\mathscr{B}(\alpha)$ is the set of blocks adjacent to $B(\alpha)$, but containing no vertex adjacent to α. Let $A(\alpha) := \{(B, C)\colon C \in \mathscr{B}(\alpha)\}$, the set of arcs of $\Gamma_{\mathscr{B}}$ from B to a block of $\Gamma_{\mathscr{B}}(B)$ containing no vertices adjacent to α. We will show that the vertices of Γ can be labelled with the sets $A(\alpha)$. Let $\mathbf{A}(B) := \{A(\alpha)\colon \alpha \in B\}$ for a block $B \in \mathscr{B}$ and $\mathbf{A} := \{A(\alpha)\colon \alpha \in V(\Gamma)\}$.

Lemma 2. *Suppose Γ is a finite G-symmetric graph and $\mathscr{B}$ is a non-trivial G-invariant partition of $V(\Gamma)$ with $k = v - 1 \geqslant 1$. Then the map $(A(\alpha))^g = A(\alpha^g)$ for $\alpha \in V(\Gamma), g \in G$, defines an action of G on $\mathbf{A}$ and the actions of G on $V(\Gamma)$ and $\mathbf{A}$ are permutationally isomorphic with respect to the bijection $\lambda\colon \alpha \mapsto A(\alpha)$.*

Proof. It is straightforward to check that $(A(\alpha))^g = A(\alpha^g)$ defines an action. Let α, β be distinct vertices of Γ. If $B(\alpha) \neq B(\beta)$, then the arcs in $A(\alpha)$ and $A(\beta)$ have different initial vertices; if $B(\alpha) = B(\beta)$, then $\mathscr{B}(\alpha) \cap \mathscr{B}(\beta) = \varnothing$ as $k = v - 1$. In both cases, we get $A(\alpha) \neq A(\beta)$ and hence $\lambda\colon \alpha \mapsto A(\alpha)$ is a bijection from $V(\Gamma)$ to $\mathbf{A}$. For any $\alpha \in V(\Gamma)$ and $g \in G$, we have $\lambda(\alpha^g) = A(\alpha^g) = (A(\alpha))^g = (\lambda(\alpha))^g$. So the actions of G on $V(\Gamma)$ and $\mathbf{A}$ are permutationally isomorphic.

Next we define a graph Γ' associated with Γ in the case $k = v - 1$.

Definition 1. Let Γ' be the graph with vertex set $V(\Gamma)$ in which two vertices α, β are adjacent if and only if $B(\beta) \in \mathscr{B}(\alpha)$ and $B(\alpha) \in \mathscr{B}(\beta)$ (see Fig. 1). In other words, α, β are adjacent in Γ' if and only if $B(\alpha), B(\beta)$ are adjacent in $\Gamma_{\mathscr{B}}$, α is the only vertex in $B(\alpha)$ not adjacent to any vertex in $B(\beta)$ and β is the only vertex in $B(\beta)$ not adjacent to any vertex in $B(\alpha)$.

Note that $(\alpha, \beta) \mapsto (B(\alpha), B(\beta))$ establishes a bijection from the set of arcs of Γ' to the set of arcs of $\Gamma_{\mathscr{B}}$.

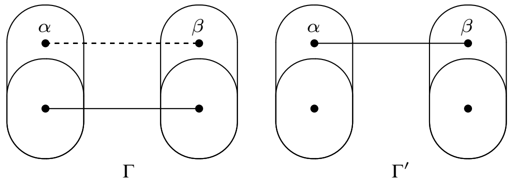

Fig. 1. The definition of Γ'.

PROPOSITION 3. *Suppose* Γ *is a finite* G*-symmetric graph and* $V(\Gamma)$ *admits a non-trivial* G*-invariant partition* $\mathscr{B}$ *with* $k = v - 1 \geqslant 1$. *Then* Γ' *is a* G*-symmetric graph.*

Proof. Let $(\alpha, \beta), (\gamma, \delta)$ be distinct arcs of Γ'. Then $(B(\alpha), B(\beta)), (B(\gamma), B(\delta))$ are distinct arcs of $\Gamma_{\mathscr{B}}$. Since $\Gamma_{\mathscr{B}}$ is G-symmetric, there exists $g \in G$ such that $(B(\alpha), B(\beta))^g = (B(\gamma), B(\delta))$, that is, $(B(\alpha^g), B(\beta^g)) = (B(\gamma), B(\delta))$. Since α is the only vertex in $B(\alpha)$ not adjacent to any vertex in $B(\beta)$, we know that α^g is the only vertex in $B(\alpha^g) = B(\gamma)$ not adjacent to any vertex in $B(\beta^g) = B(\delta)$ and γ is the only vertex in $B(\gamma)$ not adjacent to any vertex in $B(\delta)$. So we must have $\alpha^g = \gamma$. Similarly, $\beta^g = \delta$. Hence $(\alpha, \beta)^g = (\gamma, \delta)$ and Γ' is a G-symmetric graph.

We use K_n and $K_{n,n}$ to denote, respectively, the complete graph with n vertices and the complete bipartite graph with n vertices in each part of its bipartition. For an integer $n \geqslant 1$ and a graph Γ, $n \cdot \Gamma$ denotes the vertex-disjoint union of n copies of Γ. Thus, a matching with n edges is the graph $n \cdot K_2$. We use $\overline{\Gamma}$ to denote the *complement* graph of Γ with respect to the complete graph. The *girth* of Γ, denoted by girth (Γ), is the length of a shortest cycle in Γ if the graph Γ contains cycles and is defined to be ∞ otherwise. A cycle (path, respectively) of length n is called an *n-cycle* (*n-path*, respectively) and is denoted by C_n (P_n, respectively). A *clique* of Γ is a set of vertices of Γ which induces a complete subgraph. A clique with n vertices is called an *n-clique*. The distance in Γ between two vertices $\alpha, \beta \in V(\Gamma)$ is denoted by $d_\Gamma(\alpha, \beta)$. For a G-vertex-transitive graph Γ, it is easy to show that Γ is $(G, 2)$-arc transitive if and only if G_α is doubly transitive on $\Gamma(\alpha)$ for some $\alpha \in V(\Gamma)$.

3. *The case where* $k = 1$ *and* $v = 2$

In the remainder of the paper we will assume that Γ is a G-symmetric graph and $\mathscr{B}$ is a G-invariant partition of $V(\Gamma)$ such that $(\mathscr{B}, \Gamma_{\mathscr{B}})$ is non-trivial (that is, $\mathscr{B}$ is non-trivial and $\Gamma_{\mathscr{B}}$ has at least one edge) and $k = v - 1$. We distinguish the following two cases:

(I) $k = v - 1 = 1$; and

(II) $k = v - 1 \geqslant 2$.

In this section we discuss Case (I), which can occur in a non-trivial way (see the examples in [**5**, section 5] and see also Theorem 9 and the remarks following it). The characterization of Γ in Case (I) varies in difficulty according to the nature of $\Gamma_{\mathscr{B}}$. For example, if $\Gamma_{\mathscr{B}} = C_n$, then $r = 1$ and Γ is uniquely determined (see [**5**, theorem 4·1(*a*)]), namely $\Gamma = n \cdot K_2$, while if $\Gamma_{\mathscr{B}}$ is a complete graph, then it seems rather difficult to determine or describe Γ (see [**5**, section 4]).

Suppose then that $k = v - 1 = 1$. For each vertex α, let $B(\alpha) = \{\alpha, \alpha'\}$ denote the block of $\mathscr{B}$ containing α, so $B(\alpha) = B(\alpha')$. The adjacency relation for the graph Γ' defined in Definition 1 becomes: α and β are adjacent in Γ' if and only if α' and β'

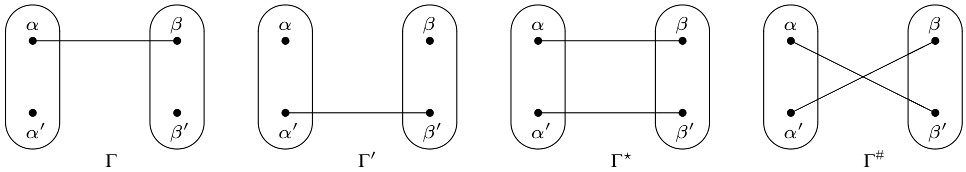

Fig. 2. The definitions of $\Gamma', \Gamma^\star$ and $\Gamma^\#$.

are adjacent in Γ. Besides Γ', we can associate with Γ two other graphs $\Gamma^\star$ and $\Gamma^\#$ (see Fig. 2) defined as follows.

Definition 2. (*a*) Let $\Gamma^\star$ be the graph with vertex set $V(\Gamma)$ in which $\{\alpha, \beta\}$ is an edge if and only if either $\{\alpha, \beta\}$ or $\{\alpha', \beta'\}$ is an edge of Γ.

(*b*) Let $\Gamma^\#$ be the graph with vertex set $V(\Gamma)$ such that $\{\alpha, \beta'\}$ and $\{\alpha', \beta\}$ are edges of $\Gamma^\#$ if and only if either $\{\alpha, \beta\}$ or $\{\alpha', \beta'\}$ is an edge of Γ.

The following result is analogous to [**5**, lemma 5·1] without assuming G-local primitivity. It shows that the quotient graph $\Gamma_{\mathscr{B}}$ may be covered by two (possibly non-isomorphic) symmetric graphs each with block size two. Let z be the involution which interchanges the two vertices in each block of $\mathscr{B}$.

Theorem 4. *Suppose that Γ is a finite G-symmetric graph and $\mathscr{B}$ is a non-trivial G-invariant partition of $V(\Gamma)$ with block size $v = k + 1 = 2$. Then G is faithful on $\mathscr{B}$. Furthermore,*

(*a*) *$\Gamma' \cong \Gamma$, and Γ' is G-symmetric;*

(*b*) *both $\Gamma^\star$ and $\Gamma^\#$ are $(G \times \langle z \rangle)$-symmetric and $\mathscr{B}$ is a $(G \times \langle z \rangle)$-invariant partition of $V(\Gamma)$. Also, $\Gamma^\star_{\mathscr{B}} = \Gamma^\#_{\mathscr{B}} = \Gamma_{\mathscr{B}}$ and both $\Gamma^\star$ and $\Gamma^\#$ are covers of $\Gamma_{\mathscr{B}}$.*

Proof. Let $B(\alpha) = \{\alpha, \alpha'\}$ be a block of $\mathscr{B}$. If $g \in G$ is any element which maps α to α', then g interchanges α and α'. Hence g interchanges $\Gamma_{\mathscr{B}}(\alpha)$ and $\Gamma_{\mathscr{B}}(\alpha')$. Note that $\Gamma_{\mathscr{B}}(\alpha)$ and $\Gamma_{\mathscr{B}}(\alpha')$ are disjoint since $k = 1$. Thus, g acts non-trivially on $\mathscr{B}$, and it follows that G is faithful on $\mathscr{B}$. By Proposition 3, Γ' is G-symmetric and the mapping $z\colon \alpha \mapsto \alpha'$, for $\alpha \in V(\Gamma)$, is an isomorphism from Γ to Γ'.

Clearly, $\langle G, z \rangle \cong G \times \mathbb{Z}_2$. Since the edge set of $\Gamma^\star$ is the union of the sets of edges of Γ and Γ' it follows from (*a*) that $G \times \langle z \rangle \leqslant \mathrm{Aut}(\Gamma^\star)$ and that $G \times \langle z \rangle$ is transitive on the arcs of $\Gamma^\star$. Also, $\mathscr{B}$ is a $(G \times \langle z \rangle)$-invariant partition of $V(\Gamma)$ and $\Gamma^\star$ is a cover of $\Gamma^\star_{\mathscr{B}} = \Gamma_{\mathscr{B}}$. Moreover, $\Gamma_{\mathscr{B}} = \Gamma^\#_{\mathscr{B}}$ and $\Gamma^\#$ is a cover of $\Gamma_{\mathscr{B}}$. For two adjacent blocks $B = \{\alpha, \alpha'\}$ and $C = \{\beta, \beta'\}$ of $\Gamma_{\mathscr{B}}$, suppose that (α, β) is an arc of Γ. Then (α, β') and (β, α') are arcs of $\Gamma^\#$ which are interchanged by z. It is also easy to check that G preserves the edge set of $\Gamma^\#$. It follows that $G \times \langle z \rangle$ is transitive on the arcs of $\Gamma^\#$.

Remark 1. The graphs $\Gamma^\star$, $\Gamma^\#$ defined in Definition 2 may, or may not, be isomorphic to each other. For example, if $\Gamma_{\mathscr{B}} = C_4$, then both $\Gamma^\star$ and $\Gamma^\#$ are $2 \cdot C_4$; while if $\Gamma_{\mathscr{B}} = C_3$, then $\Gamma^\star = C_6$ whilst $\Gamma^\# = 2 \cdot C_3$. So $\Gamma^\star$ and $\Gamma^\#$ may be non-isomorphic covers of $\Gamma_{\mathscr{B}}$.

4. *A general discussion*: $k = v - 1 \geqslant 2$

In the remaining sections of the paper we investigate the general case where $v = k + 1 \geqslant 3$. Note that if, in addition, Γ is G-locally primitive, then $\mathscr{D}(B)$ contains no repeated blocks (see [**5**, lemma 3·3]). This however is not true in general for symmetric graphs. We consider the general case in this section and the subsequent sections are devoted to studying the case where $\mathscr{D}(B)$ has no repeated blocks. In Lemma 2 we defined an action of G on $\mathbf{A}$ by $(A(\alpha))^g = A(\alpha^g)$ for $\alpha \in V(\Gamma), g \in G$, and proved that this action is permutationally isomorphic to the action of G on $V(\Gamma)$. Thus, G_B induces an action on $\mathbf{A}(B)$. Part (*b*) of the following theorem shows that, if $v = k + 1 \geqslant 3$, then this action of G_B is doubly transitive.

THEOREM 5. *Suppose that Γ is a finite G-symmetric graph and $\mathscr{B}$ is a non-trivial G-invariant partition of $V(\Gamma)$ with block size $v = k + 1 \geqslant 3$. Let B be a block of $\mathscr{B}$ and $\alpha \in B$ and set $m = |\mathscr{B}(\alpha)|$ (where $\mathscr{B}(\alpha) = \{C \in \mathscr{B}\colon \Gamma(C) \cap B = B \setminus \{\alpha\}\}$). Then the following hold.*

(*a*) *$\mathscr{D}(B)$ has v distinct blocks, each repeated exactly m times, so $b = mv$, $r = m(v-1)$ and $\mathscr{D}(B)$ is a 2-$(v, v-1, m(v-2))$-design.*

(*b*) *The actions of G_B on B and $\mathbf{A}(B)$ are permutationally isomorphic (with respect to the bijection $\alpha \mapsto A(\alpha)$, for $\alpha \in B$) and doubly transitive.*

(*c*) *G_α has two orbits on $\Gamma_{\mathscr{B}}(B)$, namely, $\mathscr{B}(\alpha)$ and $\Gamma_{\mathscr{B}}(B) \setminus \mathscr{B}(\alpha)$.*

(*d*) *G acts faithfully on $\mathscr{B}$. Moreover, $G_{[B]} \leqslant G_{(B)}$ and equality holds whenever $\mathscr{D}(B)$ contains no repeated blocks.*

(*e*) *If $\mathscr{D}(B)$ contains no repeated blocks and, if $\Gamma_{\mathscr{B}}$ is connected and Γ is vertex-distinguishable with respect to $\mathscr{B}$, then G_B acts faithfully on B and $\Gamma_{\mathscr{B}}(B)$.*

Proof. (*a*) Since G_B is transitive on B, each $(v-1)$-subset of B is a block of $\mathscr{D}(B)$ and hence $\mathscr{D}(B)$ has v distinct blocks each repeated m times. So we have $b = mv$. This, together with $vr = bk = b(v-1)$, gives $r = m(v-1)$. In particular, $\mathscr{D}(B)$ is a 2-$(v, v-1, m(v-2))$-design.

(*b*) It follows from Lemma 2 that the actions of G_B on B and $\mathbf{A}(B)$ are permutationally isomorphic. For pairwise distinct vertices $\alpha, \beta, \gamma \in B$ (note that $v \geqslant 3$), let $C \in \mathscr{B}(\beta)$ and $D \in \mathscr{B}(\gamma)$. Since $k = v - 1$, C contains a neighbour δ of α and D contains a neighbour ε of α. By the transitivity of G_α on $\Gamma(\alpha)$, there exists $g \in G_\alpha$ such that $\delta^g = \varepsilon$. Hence, $(B, C)^g = (B, D)$. Since $\mathbf{A}(B)$ is a (G_B)-invariant partition of the set $\{(B, E)\colon E \in \Gamma_{\mathscr{B}}(B)\}$, it follows from $(B, C)^g = (B, D)$ that $(A(\beta))^g = A(\gamma)$ and $(A(\alpha))^g = A(\alpha)$. Thus, G_α is transitive on $\mathbf{A}(B) \setminus \{A(\alpha)\}$. Since $G_\alpha = (G_B)_\alpha$ and G_B is transitive on $\mathbf{A}(B)$, it follows that G_B is doubly transitive on $\mathbf{A}(B)$ and hence doubly transitive on B as well.

(*c*) Clearly, $\mathscr{B}(\alpha)$ is (G_α)-invariant. Let $C, D \in \mathscr{B}(\alpha)$. Since $\Gamma_{\mathscr{B}}$ is G-symmetric, there exists $g \in G$ with $B^g = B, C^g = D$. Now $\alpha^g = \alpha$ for otherwise α is adjacent to no vertex in C but α^g is adjacent to at least one vertex in $C^g = D$. Thus, $g \in G_\alpha$ and so G_α is transitive on $\mathscr{B}(\alpha)$. Now let $C, D \in \Gamma_{\mathscr{B}}(B) \setminus \mathscr{B}(\alpha)$. Then $\alpha \in \Gamma(C) \cap \Gamma(D) \cap B$. So there exist $\beta \in C, \gamma \in D$ which are adjacent to α. Since Γ is G-symmetric, there exists $g \in G$ with $(\alpha, \beta)^g = (\alpha, \gamma)$. Thus, $g \in G_\alpha$ and $C^g = D$. So $\Gamma_{\mathscr{B}}(B) \setminus \mathscr{B}(\alpha)$ is a (G_α)-orbit.

(*d*) If $g \in G_{[B]}$ then, for each $\beta \in B$, g fixes setwise each block $C \in \mathscr{B}(\beta)$ and hence fixes setwise $\Gamma(C) \cap B$. Therefore, g fixes $B \setminus (\Gamma(C) \cap B) = \{\beta\}$. Thus, $G_{[B]} \leqslant G_{(B)}$. Moreover, if $g \in G$ fixes setwise each block of $\mathscr{B}$, then it lies in $G_{[B]}$ for each B and hence fixes each vertex of Γ; this implies that $g = 1$. So G is faithful on $\mathscr{B}$. Suppose that $\mathscr{D}(B)$ contains no repeated blocks and $g \in G_{(B)}$. Then for each $\alpha \in B$, g fixes the unique block in $\mathscr{B}(\alpha)$ and hence g fixes each block of $\Gamma_{\mathscr{B}}(B)$ setwise, so $g \in G_{[B]}$.

(*e*) Let $g \in G_{(B)} = G_{[B]}$. Then for $C \in \Gamma_{\mathscr{B}}(B)$, g fixes the unique vertex in $C\setminus(\Gamma(B)\cap C)$ and, for each $\beta \in \Gamma(B)\cap C$, we have $\beta^g \in \Gamma(B)\cap C$ and $\Gamma(\beta)\cap B = \Gamma(\beta^g)\cap B$ (since g fixes B pointwise). Since Γ is vertex-distinguishable with respect to $\mathscr{B}$, we get $\beta^g = \beta$. Thus $g \in G_{(C)}$ and hence $G_{(B)} \leqslant G_{(C)}$. By a similar argument $G_{(C)} \leqslant G_{(B)}$, so $G_{(B)} = G_{(C)}$. Since $\Gamma_{\mathscr{B}}$ is connected, this equality is true for any two blocks B, C (not necessarily adjacent) and hence $G_{(B)} = 1 = G_{[B]}$. Thus, G_B is faithful on B and on $\Gamma_{\mathscr{B}}(B)$.

Note that if in addition Γ is G-locally primitive then (i) $\Gamma[B, C]$ is a matching [**5**] and (ii) $\mathscr{D}(B)$ contains no repeated blocks [**5**, lemma 3·3]. From (i) and Lemma 1 we know that Γ is vertex-distinguishable with respect to $\mathscr{B}$ and hence G_B is faithful on B (Theorem 5(*e*)) if $\Gamma_{\mathscr{B}}$ is connected. Also from (i) and (ii), we know that, for each $\alpha \in B$, there exists a bijection from $B\setminus\{\alpha\}$ to $\Gamma(\alpha)$, namely each $\beta \in B\setminus\{\alpha\}$ corresponds to the unique neighbour of α in the unique block of $\mathscr{B}(\beta)$. So G_α is primitive on $B\setminus\{\alpha\}$ (that is, G_B is *doubly primitive* on B) as G_α is primitive on $\Gamma(\alpha)$. Combining these with Theorem 5(*a*)(*b*)(*e*), we deduce theorem 5·3 of [**5**], one of the results which motivated this investigation.

Now we consider the graph Γ' defined in Definition 1. Each maximal clique of Γ' has at most $m+1$ vertices since the valency of Γ' is m, where $m = |\mathscr{B}(\alpha)|$. The following result shows that if each maximal clique of Γ' does contain $m+1$ vertices, or equivalently if $\Gamma' \cong \ell \cdot K_{m+1}$ for some ℓ, then we obtain a second G-invariant partition of $V(\Gamma)$. This condition holds in particular when $m = 1$ and Proposition 6 will be used in this case in the next section.

Proposition 6. *Suppose that Γ is a finite G-symmetric graph with a non-trivial G-invariant partition $\mathscr{B}$ with blocks of size $v = k+1 \geqslant 3$. Let $\alpha \in V(\Gamma)$. Then $\mathscr{P} = \{(\{\alpha\}\cup\Gamma'(\alpha))^g : g \in G\}$ is a G-invariant partition of $V(\Gamma)$ if and only if $V(\Gamma)$ is a disjoint union of $(m+1)$-cliques of Γ', where $m = |\mathscr{B}(\alpha)|$.*

Proof. Set $\Gamma'(\alpha) = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and $B' = \{\alpha\}\cup\Gamma'(\alpha)$ and suppose that $V(\Gamma)$ is a disjoint union of $(m+1)$-cliques of Γ'. Then B' is the unique $(m+1)$-clique of Γ' containing α. Since G permutes the connected components of Γ', it follows that $\mathscr{P}$ is a G-invariant partition of $V(\Gamma)$.

Conversely, suppose $\mathscr{P}$ is a G-invariant partition of $V(\Gamma)$. For $i = 1, 2, \ldots, m$, let $g \in G$ be such that $\alpha^g = \alpha_i$. Then $B'^g = B'$ since α_i is in both B' and B'^g and hence $\Gamma'(\alpha_i) = \Gamma'(\alpha^g) = (\Gamma'(\alpha))^g = (B'\setminus\{\alpha\})^g = B'\setminus\{\alpha_i\}$. Therefore, B' is a clique of Γ' with the maximum possible size $m+1$. In other words, $V(\Gamma)$ is a disjoint union of $(m+1)$-cliques of Γ'.

5. *The case $\mathscr{D}(B)$ contains no repeated blocks*

From now on we focus on the case where $k = v-1 \geqslant 2$ and $\mathscr{D}(B)$ has no repeated blocks. We will prove in Theorem 8 below that this is true if and only if $k = v-1 \geqslant 2$ and $\Gamma_{\mathscr{B}}$ is $(G, 2)$-arc transitive. In this case, each vertex of Γ may be labelled by an arc of $\Gamma_{\mathscr{B}}$. In the main result of this section, Theorem 8, we classify all the graphs Γ for which adjacent vertices of Γ have labels involving at most three distinct blocks of $\mathscr{B}$. In the final section we consider the general case and complete the proofs of Theorems 1 and 2.

Thus we suppose that $k = v - 1 \geqslant 2$ and $\mathscr{D}(B)$ has no repeated blocks. Then the valency of Γ' is 1 and each vertex $\alpha \in V(\Gamma)$ has a unique *mate* α', namely the unique vertex adjacent to α in Γ'. Hence the partition $\mathscr{P}$ defined in Proposition 6 consists of the pairs $\{\alpha, \alpha'\}$ and the map $z\colon \alpha \mapsto \alpha'$ defines a G-invariant bijection on $V(\Gamma)$. So $A(\alpha)$ contains only one arc $(B(\alpha), B(\alpha'))$ and $B(\alpha')$ is the unique block in $\Gamma_{\mathscr{B}}(B)$ fixed setwise by G_α (see Theorem 5(c)). As in the G-locally primitive case [**5**], the mapping λ of Lemma 2 defines, for each $\alpha \in V(\Gamma)$, a unique label '$B(\alpha)B(\alpha')$' for α with the blocks of $\mathscr{B}$ containing α and α' as the first and the second coordinates, respectively. Set $B^* = B^z = \{\text{'}CB\text{'}\colon C \in \Gamma_{\mathscr{B}}(B)\}$ for $B \in \mathscr{B}$. Then $B^* \cap \Gamma(B) = \varnothing$, so no neighbour of $\alpha \in B$ has a label involving B as either coordinate.

Proposition 7. *Suppose that Γ is a finite G-symmetric graph, $\mathscr{B}$ is a non-trivial G-invariant partition of $V(\Gamma)$ with block size $v = k + 1 \geqslant 3$ such that $\mathscr{D}(B)$ contains no repeated blocks. Then $\Gamma_{\mathscr{B}}$ has valency $b = v$. Let $z\colon \alpha \mapsto \alpha'$, $\alpha \in V(\Gamma)$, as defined above. Then also*

(a) *the actions of G on $V(\Gamma)$ and on the set of arcs of $\Gamma_{\mathscr{B}}$ are permutationally isomorphic and each $\alpha \in V(\Gamma)$ can be uniquely labelled by a pair 'BB'' of adjacent blocks of $\mathscr{B}$, where $B = B(\alpha)$ and B' is the unique block in $\Gamma_{\mathscr{B}}(B)$ fixed setwise by G_α.*

(b) *z centralizes G and is an involution* (*that is, $z^2 = 1$*) *and $\mathscr{P} = \{\{\alpha, \alpha'\}\colon \alpha \in V(\Gamma)\}$ is a G-invariant partition of $V(\Gamma)$.*

(c) *$\mathscr{B}^* := \{(B^*)^g\colon g \in G\}$ is a G-invariant partition of $V(\Gamma)$ with blocks of size v; and $G_{B^*} = G_B$ is doubly transitive on B and B^*.*

Proof. Theorem 5(a) implies that $b = v$. Each $A(\alpha)$ can be identified with the arc $(B(\alpha), B(\alpha'))$ of $\Gamma_{\mathscr{B}}$ and each arc of $\Gamma_{\mathscr{B}}$ has this form. So from Lemma 2 the actions of G on $V(\Gamma)$ and on the set of arcs of $\Gamma_{\mathscr{B}}$ are permutationally isomorphic. Clearly, z is an involution and, from Proposition 6, $\mathscr{P}$ is a G-invariant partition of $V(\Gamma)$. For each $g \in G$ and 'BD' $\in V(\Gamma)$, we have 'BD'^{zg} = 'DB'^{g} = 'D^gB^g' = 'B^gD^g'^{z} = 'BD'^{gz} and hence z centralizes G. Since $(B^*)^g = \{\text{'}C^gB^g\text{'}\colon \text{'}CB\text{'} \in B^*\} = (B^g)^*$, it follows from $B^* \cap (B^*)^g \neq \varnothing$ that $g \in G_B$ and consequently $(B^*)^g = B^*$. Thus, $\mathscr{B}^*$ is a G-invariant partition of $V(\Gamma)$ with block size v. Clearly, $G_{B^*} = G_B$ and the actions of G_B on B and B^* are permutationally isomorphic with respect to $z\colon \alpha \mapsto \alpha'$. So by Theorem 5($b$), G_B is doubly transitive on both B and B^*.

The main result of this section is the following theorem.

Theorem 8. *Suppose that Γ is a finite G-symmetric graph and $\mathscr{B}$ is a non-trivial G-invariant partition of $V(\Gamma)$ with block size $v = k + 1 \geqslant 3$. Then $\mathscr{D}(B)$ contains no repeated blocks if and only if $\Gamma_{\mathscr{B}}$ is $(G, 2)$-arc transitive. Furthermore, in this case either*

(a) *adjacent vertices have labels involving four distinct blocks, or*

(b) *there exist two adjacent vertices of Γ which share the same second coordinate. In this case, $\Gamma[B, C]$ is a matching of $v - 1$ edges, $\Gamma \cong n(v + 1) \cdot K_v$ and $\Gamma_{\mathscr{B}} \cong n \cdot K_{v+1}$ for some integer $n \geqslant 1$ and the group induced on the connected component $\{B\} \cup \Gamma_{\mathscr{B}}(B)$ of $\Gamma_{\mathscr{B}}$ is 3-transitive. In particular, if $\Gamma_{\mathscr{B}}$ is connected, then $\Gamma \cong (v + 1) \cdot K_v$, $\Gamma_{\mathscr{B}} \cong K_{v+1}$ and G acts faithfully on $\mathscr{B}$ as a 3-transitive permutation group of degree $v + 1$.*

Proof. Suppose $\mathscr{D}(B)$ has no repeated blocks. Then for each $\alpha \in B$, $A(\alpha)$ can be identified with the unique block in $\mathscr{B}(\alpha)$. So Theorem 5(*b*) implies that G_B is doubly transitive on $\Gamma_{\mathscr{B}}(B)$. Then since G is transitive on $\mathscr{B}$, it follows that $\Gamma_{\mathscr{B}}$ is $(G, 2)$-arc transitive. Conversely suppose that $\Gamma_{\mathscr{B}}$ is $(G, 2)$-arc transitive and let α, β, γ be pairwise distinct vertices of B. (Note that $v \geqslant 3$.) If $\mathscr{D}(B)$ contains repeated blocks (that is, $m \geqslant 2$), then there are distinct blocks $C_1, C_2 \in \mathscr{B}(\alpha)$. Let $D \in \mathscr{B}(\beta)$ and $E \in \mathscr{B}(\gamma)$. By the $(G, 2)$-arc transitivity of $\Gamma_{\mathscr{B}}$ there exists $g \in G_B$ with $(C_1, C_2)^g = (D, E)$. Note that $\{\mathscr{B}(\delta)\colon \delta \in B\}$ is a (G_B)-invariant partition of $\Gamma_{\mathscr{B}}(B)$ (by Lemma 2). So $C_1^g = D$ implies $(\mathscr{B}(\alpha))^g = \mathscr{B}(\beta)$, whilst $C_2^g = E$ implies $(\mathscr{B}(\alpha))^g = \mathscr{B}(\gamma)$. This contradiction shows that $\mathscr{D}(B)$ contains no repeated blocks. Thus the first assertion is proved.

For the rest of the proof we assume that $\mathscr{D}(B)$ has no repeated blocks. If adjacent vertices of Γ have different second coordinates, then it follows from the definition of the labels that two adjacent vertices of Γ have labels involving four distinct blocks. Suppose there exist two adjacent vertices whose second coordinates are the same. Since G acts transitively on $\mathscr{B}$, we may assume without loss of generality that there are two adjacent vertices in B^*. Since G_{B^*} is doubly transitive on B^*, it follows that B^* induces a complete graph K_v. Since Γ is G-symmetric and since $\mathscr{B}^*$ is G-invariant, it follows that each edge of Γ joins two vertices in the same block of $\mathscr{B}^*$. This means that each block of $\mathscr{B}^*$ induces a connected component K_v of Γ and hence $\Gamma = |\mathscr{B}^*|\cdot K_v$. This implies in particular that $\Gamma[B, C]$ is a matching of $v - 1$ edges. Note that any two blocks in $\Gamma_{\mathscr{B}}(B)$ are adjacent in $\Gamma_{\mathscr{B}}$ and hence $\{B\} \cup \Gamma_{\mathscr{B}}(B)$ induces a complete subgraph K_{v+1} of $\Gamma_{\mathscr{B}}$. Since the valency of $\Gamma_{\mathscr{B}}$ is $b = v$, the subgraph induced by $\{B\} \cup \Gamma_{\mathscr{B}}(B)$ is a connected component of $\Gamma_{\mathscr{B}}$. This implies (i) $\Gamma_{\mathscr{B}} = n \cdot K_{v+1}$ and hence $\Gamma = n(v + 1) \cdot K_v$, where n is the number of connected components of $\Gamma_{\mathscr{B}}$; and (ii) since G is transitive on $\mathscr{B}$ and G_B is doubly transitive on $\Gamma_{\mathscr{B}}(B)$, as shown above, it follows that the group induced on the connected component $\{B\} \cup \Gamma_{\mathscr{B}}(B)$ of $\Gamma_{\mathscr{B}}$ is 3-transitive. In particular, if $\Gamma_{\mathscr{B}}$ is connected, then $\Gamma_{\mathscr{B}} = K_{v+1}$, $\Gamma = (v + 1) \cdot K_v$ and G is 3-transitive on $\mathscr{B} = \{B\} \cup \Gamma_{\mathscr{B}}(B)$ with degree $|\mathscr{B}| = v + 1$. From Theorem 5(*d*), G is also faithful on $\mathscr{B}$.

Remark 2. It follows from the classification of finite multiply-transitive permutation groups (which relies on the finite simple group classification, see [**3**, p. 8]) that in Theorem 8(*b*), if $\Gamma_{\mathscr{B}}$ is connected and two adjacent vertices of Γ share the same second coordinate, then G is one of S_{v+1} $(v \geqslant 3)$, A_{v+1} $(v \geqslant 4)$, M_{v+1} $(v = 10, 11, 21, 22, 23)$, M_{11} $(v = 11)$, $PSL(2, v) \leqslant G \leqslant P\Gamma L(2, v)$ (v a prime power), $G = AGL(d, 2)$ $(v = 2^d - 1)$, or $\mathbb{Z}_2^4.A_7$ $(v = 15)$.

According to Theorem 8, under the assumption that $\mathscr{D}(B)$ contains no repeated blocks, all possibilities for the graphs Γ, $\Gamma_{\mathscr{B}}$, $\Gamma[B, C]$ and the group G are known if there are two adjacent vertices of Γ sharing the same second coordinate. For the remaining case where the labels of any two adjacent vertices involve four distinct blocks, the following theorem gives some structural information about Γ and $\Gamma_{\mathscr{B}}$ provided the girth of $\Gamma_{\mathscr{B}}$ is sufficiently large. A mapping $\varphi\colon V(\Gamma) \to V(\Sigma)$ between the vertex sets of two graphs Γ and Σ is called a *graph homomorphism* if φ maps adjacent vertices of Γ to adjacent vertices of Σ; if in addition φ is one-to-one, then it is called a *graph monomorphism*.

THEOREM 9. *Suppose that* Γ *is a finite G-symmetric graph and* $\mathcal{B}$ *is a non-trivial G-invariant partition of* $V(\Gamma)$ *with block size* $v = k+1 \geqslant 3$ *such that* $\mathcal{D}(B)$ *contains no repeated blocks. Suppose further that* girth $(\Gamma_{\mathcal{B}}) \geqslant 5$. *Then*

(*a*) $\Gamma[\{\alpha,\alpha'\},\{\beta,\beta'\}] \cong K_2$ *for adjacent blocks* $\{\alpha,\alpha'\}$ *and* $\{\beta,\beta'\}$ *of* $\mathcal{P}$.

(*b*) $\Gamma[B^*,C^*]$ *is a matching for adjacent blocks* B^*, C^* *of* $\mathcal{B}^*$ *and if in addition* girth $(\Gamma_{\mathcal{B}}) \geqslant 7$ *then* $\Gamma[B^*,C^*] \cong K_2$.

(*c*) *The involution* $z\colon \alpha \mapsto \alpha'$ $(\alpha \in V(\Gamma))$ *defines a graph monomorphism from* Γ *to the complement* $\overline{\Gamma}$, *and* z *interchanges the two partitions* $\mathcal{B}$ *and* $\mathcal{B}^*$. *Moreover,* z *induces graph monomorphisms from* $\Gamma_{\mathcal{B}}$ *to* $\overline{\Gamma_{\mathcal{B}^*}}$ *and from* $\Gamma_{\mathcal{B}^*}$ *to* $\overline{\Gamma_{\mathcal{B}}}$, *defined by* $B \mapsto B^*$ *and* $B^* \mapsto B$, *respectively.*

Proof. The assumption girth $(\Gamma_{\mathcal{B}}) \geqslant 5$ implies that adjacent vertices of Γ have labels involving four distinct blocks. Suppose that $\{`BD', `DB'\}$ and $\{`CE', `EC'\}$ are blocks of $\mathcal{P}$ with $`DB'$ and $`EC'$ adjacent in Γ. (This is represented diagramatically in Fig. 3, where the two dashed boxes represent B^* and C^* respectively.) Then B, C, D, E are pairwise distinct blocks by our assumption about the labels. Note that $`BD'$ is not adjacent to $`EC'$ and $`DB'$ is not adjacent to $`CE'$ for otherwise (B,D,E,B) or (C,D,E,C) would be a triangle of $\Gamma_{\mathcal{B}}$, contradicting girth $(\Gamma_{\mathcal{B}}) \geqslant 5$. Similarly, $`BD' = `DB'^z$ is not adjacent to $`CE' = `EC'^z$, for otherwise (B,D,E,C,B) would be a 4-cycle of $\Gamma_{\mathcal{B}}$. Thus, $\Gamma[\{`BD', `DB'\},\{`CE', `EC'\}] \cong K_2$ and (*a*) holds.

In particular, the non-adjacency of $`BD'$ and $`CE'$ implies that z is a graph monomorphism from Γ to $\overline{\Gamma}$. By the definition of z, two vertices α, β lie in the same block B of $\mathcal{B}$ if and only if α^z, β^z lie in the same block B^* of $\mathcal{B}^*$. Hence z induces the bijection $B \mapsto B^*$ from $\mathcal{B}$ to $\mathcal{B}^*$. Suppose B^*, C^* are adjacent blocks of $\mathcal{B}^*$, say $`DB', `EC'$ are adjacent vertices of Γ, where $D \in \Gamma_{\mathcal{B}}(B), E \in \Gamma_{\mathcal{B}}(C)$ (see Fig. 3). If B and C were adjacent in $\Gamma_{\mathcal{B}}$ then (B,D,E,C,B) would be a 4-cycle in $\Gamma_{\mathcal{B}}$, which is not the case. Thus B, C are not adjacent in $\Gamma_{\mathcal{B}}$, that is to say, if B, C are adjacent in $\Gamma_{\mathcal{B}}$, then B^*, C^* are not adjacent in $\Gamma_{\mathcal{B}^*}$. Therefore, the bijection $B \mapsto B^*$ induced by z is a graph monomorphism from $\Gamma_{\mathcal{B}}$ to $\overline{\Gamma_{\mathcal{B}^*}}$ and similarly the bijection $B^* \mapsto B$ is a graph monomorphism from $\Gamma_{\mathcal{B}^*}$ to $\overline{\Gamma_{\mathcal{B}}}$.

If $`DB'$ were adjacent to a second vertex, say $`E_1C'$, in C^*, then (D,E,C,E_1,D) would be a 4-cycle of $\Gamma_{\mathcal{B}}$, contradicting the assumption that girth $(\Gamma_{\mathcal{B}}) \geqslant 5$. Therefore, $\Gamma[B^*,C^*]$ is a matching. Now suppose girth $(\Gamma_{\mathcal{B}}) \geqslant 7$ and suppose that there is an edge $\{`D_1B', `E_1C'\}$ connecting B^* and C^*, distinct from $\{`DB', `EC'\}$. If $D_1 = D$ then $E_1 \neq E$ and (D,E,C,E_1,D) is a 4-cycle and similarly if $E_1 = E$ then $D_1 \neq D$ and (E,D_1,B,D,E) is a 4-cycle. Hence $\{D,E\} \cap \{D_1,E_1\} = \emptyset$, but in this case (B,D,E,C,E_1,D_1,B) is a 6-cycle. Hence $\Gamma[B^*,C^*] \cong K_2$.

It is worth noticing that, under the assumptions of Theorem 9, the G-invariant partition $\mathcal{P}$ satisfies all the assumptions of Section 3. Thus, from Theorem 4, we know that the graphs $\Gamma^{\star} = \Gamma \cup \Gamma'$ and $\Gamma^{\#}$ defined in Definition 2 with respect to $\mathcal{P}$ are both covers of $\Gamma_{\mathcal{P}}$.

Remark 3. From the group theoretical point of view (see, for example, [**9**, theorem 2·1(*b*)]), Theorem 9(*c*) shows that z carries the arc set $A(\Gamma)$ of Γ to a self-paired G-orbital on $V(\Gamma)$ disjoint from Γ_1 and hence $z(A(\Gamma)) \subseteq \Gamma_i$ for some $i \geqslant 2$, where $\Gamma_i := \{(\alpha,\beta)\colon d_{\Gamma}(\alpha,\beta) = i\}$. This parameter i might have a strong influence on the structure of Γ. Essentially the same argument as that used in the proof of Theorem

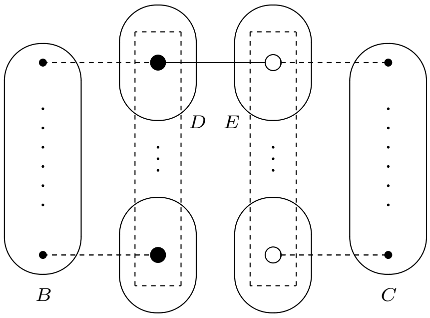

Fig. 3. Blocks of $\mathscr{B}$, $\mathscr{B}^*$ and $\mathscr{P}$.

9 shows that $i \geqslant \text{girth}(\Gamma_{\mathscr{B}}) - 3$ (so in particular $i \geqslant 2$ if $\text{girth}(\Gamma_{\mathscr{B}}) \geqslant 5$). However, we have been unable to determine the exact value of i.

One consequence of Theorem 9 is that the valencies of Γ and $\Gamma_{\mathscr{B}^*}$ are bounded as shown below. We denote by $\text{val}(\Gamma)$ the valency of a graph Γ.

Corollary 1. *Under the assumptions of Theorem* 9, $\text{val}(\Gamma) \leqslant (|V(\Gamma)| - 2)/4$ *and* $\Gamma_{\mathscr{B}^*}$ *has valency at most* $(|V(\Gamma)|/v) - v - 1$. *If in addition* $\text{girth}(\Gamma_{\mathscr{B}}) \geqslant 7$, *then* $\text{val}(\Gamma) \leqslant (|V(\Gamma)|/v^2) - (1/v) - 1$.

Proof. By Theorem 9, each edge of Γ joining α and β corresponds to a unique 3-path $\alpha, \beta', \alpha', \beta$ of $\overline{\Gamma}$ and conversely each 3-path of $\overline{\Gamma}$ of this form corresponds to a unique edge of Γ. One can see that the 3-paths of $\overline{\Gamma}$ with this form corresponding to distinct edges of Γ are pairwise edge-disjoint and that they have no common edges with Γ' (the latter being contained in $\overline{\Gamma}$). So $|E(\overline{\Gamma})| \geqslant 3|E(\Gamma)| + |V(\Gamma)|/2$, that is, $\text{val}(\overline{\Gamma}) \geqslant 3 \cdot \text{val}(\Gamma) + 1$. Thus, we have $\text{val}(\Gamma) \leqslant (|V(\Gamma)| - 2)/4$. Now by Theorem 9($c$), we have $\text{val}(\Gamma_{\mathscr{B}}) + \text{val}(\Gamma_{\mathscr{B}^*}) \leqslant |\mathscr{B}| - 1 = (|V(\Gamma)|/v) - 1$, which yields the second inequality since $\text{val}(\Gamma_{\mathscr{B}}) = v$. Note that by Theorem 9(b), $\text{val}(\Gamma_{\mathscr{B}^*}) = v \cdot \text{val}(\Gamma)$ if $\text{girth}(\Gamma_{\mathscr{B}}) \geqslant 7$, which implies the last inequality.

6. *Construction of the graphs* $\text{Arc}_\Delta(\Sigma)$ *and proofs of Theorems* 1 *and* 2

A fundamental problem arising from the approach used in the paper is that of reconstructing Γ from the triple $(\Gamma_{\mathscr{B}}, \Gamma[B, C], \mathscr{D}(B))$. In this section we study this problem for the case where $k = v - 1 \geqslant 2$ and $\mathscr{D}(B)$ contains no repeated blocks. Recall that, in this case, $\Gamma_{\mathscr{B}}$ is $(G, 2)$-arc transitive by Theorem 8. We will give an explicit construction for such graphs Γ from $(G, 2)$-arc transitive graphs Σ of valency $v \geqslant 3$. In particular if Σ is $(G, 3)$-arc transitive then the construction yields a unique graph Γ with the above properties and it has $\Gamma[B, C] \cong K_{v-1,v-1}$. Our proof of Theorem 2 shows that Γ is $(G, 3)$-arc transitive if and only if it is isomorphic to the graph obtained from a $(G, 3)$-arc transitive graph $\Gamma_{\mathscr{B}}$ by this construction.

We present the construction in a general setting, starting with a regular graph Σ of valency $v \geqslant 3$. (A graph is *regular* if its vertices have the same valency.) Let $A_i(\Sigma)$ denote the set of i-arcs of Σ, for i a positive integer, so $A(\Sigma) = A_1(\Sigma)$. For a subset Δ of $A_i(\Sigma)$ the *paired subset* of Δ is defined by

$$\Delta^\circ := \{(\sigma_i, \sigma_{i-1}, \dots, \sigma_1, \sigma_0) \colon (\sigma_0, \sigma_1, \dots, \sigma_{i-1}, \sigma_i) \in \Delta\}$$

and Δ is said to be *self-paired* if $\Delta = \Delta^\circ$. The data needed for our construction are a regular graph Σ and a self-paired subset of $A_3(\Sigma)$.

Definition 3. Let Σ be a finite regular graph of valency $v \geqslant 3$ and let Δ be a non-empty self-paired subset of $A_3(\Sigma)$. Define $\mathrm{Arc}_\Delta(\Sigma)$ to be the graph with vertex set $A(\Sigma)$ such that $(\sigma,\tau),(\sigma',\tau') \in A(\Sigma)$ are joined by an edge in $\mathrm{Arc}_\Delta(\Sigma)$ if and only if $(\tau,\sigma,\sigma',\tau') \in \Delta$. We call $\mathrm{Arc}_\Delta(\Sigma)$ the 3-arc graph of Σ corresponding to Δ.

The requirement that Δ is self-paired ensures that adjacency in $\mathrm{Arc}_\Delta(\Sigma)$ is well-defined (in the sense that (σ,τ) is joined to (σ',τ') if and only if (σ',τ') is joined to (σ,τ)). There are several natural partitions of the vertex set of $\mathrm{Arc}_\Delta(\Sigma)$, namely

(i) $\mathscr{P}(\Sigma) := \{\{(\sigma,\tau),(\tau,\sigma)\}\colon (\sigma,\tau) \in A(\Sigma)\}$;

(ii) $\mathscr{B}(\Sigma) := \{B(\sigma)\colon \sigma \in V(\Sigma)\}$, where $B(\sigma) := \{(\sigma,\tau)\colon \tau \in \Sigma(\sigma)\}$;

(iii) $\mathscr{B}^*(\Sigma) := \{B^*(\sigma)\colon \sigma \in V(\Sigma)\}$, where $B^*(\sigma) := \{(\tau,\sigma)\colon \tau \in \Sigma(\sigma)\}$.

Each subgroup $G \leqslant \mathrm{Aut}\,(\Sigma)$ induces natural actions on $A(\Sigma)$ and $A_3(\Sigma)$ and, provided G leaves Δ invariant, G will preserve the adjacency relation for $\mathrm{Arc}_\Delta(\Sigma)$ and hence will induce a (faithful) action as a group of automorphisms of $\mathrm{Arc}_\Delta(\Sigma)$. Moreover, the three partitions $\mathscr{P}(\Sigma)$, $\mathscr{B}(\Sigma)$ and $\mathscr{B}^*(\Sigma)$ are all G-invariant. We note the following relations between the G-actions on Σ and $\mathrm{Arc}_\Delta(\Sigma)$: the proofs are straightforward and are omitted.

Lemma 3. *Let* Σ, Δ *be as in Definition* 3 *and let* $G \leqslant \mathrm{Aut}\,(\Sigma)$ *leave* Δ *invariant. Then*

(*a*) $\mathrm{Arc}_\Delta(\Sigma)$ *is* G*-vertex-transitive if and only if* Σ *is* G*-symmetric.*

(*b*) $\mathrm{Arc}_\Delta(\Sigma)$ *is* G*-symmetric if and only if* G *is transitive on* Δ.

(*c*) *For* $\sigma \in V(\Sigma)$, $G_\sigma = G_{B(\sigma)} = G_{B^*(\sigma)}$ *and the actions of* G_σ *on* $\Sigma(\sigma), B(\sigma)$ *and* $B^*(\sigma)$ *are permutationally isomorphic.*

Thus if $G \leqslant \mathrm{Aut}\,(\Sigma)$, G is transitive on Δ and Σ is G-symmetric, then $\mathrm{Arc}_\Delta(\Sigma)$ is an imprimitive G-symmetric graph relative to each of the partitions above. Because of our remarks at the beginning of this section we will explore further the case where Σ is $(G,2)$-arc transitive, with particular attention to the partition $\mathscr{B}(\Sigma)$. Moreover in the case where Δ consists of *proper 3-arcs* (that is, for $(\tau,\sigma,\sigma',\tau') \in \Delta$ we have $\tau \neq \tau'$), for adjacent vertices (σ,τ) and (σ',τ') of $\mathrm{Arc}_\Delta(\Sigma)$, the four labels $\sigma,\tau,\sigma',\tau'$ are pairwise distinct.

Theorem 10. *Suppose* Σ *is a finite* $(G,2)$*-arc transitive graph with valency* $v \geqslant 3$ *and* $G \leqslant \mathrm{Aut}\,(\Sigma)$. *Suppose* Δ *is a self-paired* G*-orbit of* 3*-arcs of* Σ. *Set* $\Gamma = \mathrm{Arc}_\Delta(\Sigma)$. *Then*

(*a*) *for adjacent blocks* $B(\sigma), B(\sigma')$ *of* $\Gamma_{\mathscr{B}(\Sigma)}$, (σ,σ') *is the unique element of* $B(\sigma)$ *which is not adjacent to an element of* $B(\sigma')$ (*that is, '*$k = v-1$*'*).

(*b*) $\Gamma_{\mathscr{B}(\Sigma)} \cong \Sigma$ *and* $\mathscr{D}(B(\Sigma))$ *has no repeated blocks.*

(*c*) *If* Δ *contains a* 3*-cycle then* Δ *consists of all the* 3*-cycles of* Σ *and both* $\mathrm{Arc}_\Delta(\Sigma)$ *and* Σ *are vertex disjoint unions of complete graphs, as specified in Theorem* 8(*b*). *The connected components of* $\mathrm{Arc}_\Delta(\Sigma)$ *are the induced subgraphs on the blocks of* $\mathscr{B}^*(\Sigma)$.

(*d*) *On the other hand if* Δ *consists of proper* 3*-arcs then adjacent vertices of* $\mathrm{Arc}_\Delta(\Sigma)$ *involve four distinct vertices of* Σ.

Proof. Since $B(\sigma), B(\sigma')$ are adjacent in $\Gamma_{\mathscr{B}(\Sigma)}$, there exist $(\sigma,\tau),(\sigma',\tau') \in A(\Sigma)$ such that $(\tau,\sigma,\sigma',\tau') \in \Delta$. In particular $(\sigma,\sigma') \in A(\Sigma)$. Conversely, if $(\sigma,\sigma') \in A(\Sigma)$

then, since $\Delta \neq \varnothing$ and Σ is $(G,2)$-arc transitive it follows that there exist τ, τ' such that $(\tau,\sigma,\sigma',\tau') \in \Delta$ and hence such that (σ,τ) and (σ',τ') are adjacent in Γ. Thus $B(\sigma)$ is adjacent to $B(\sigma')$ in $\Gamma_{\mathscr{B}(\Sigma)}$. This proves that $\Gamma_{\mathscr{B}(\Sigma)} \cong \Sigma$.

It follows from the definition of a 3-arc that (σ,σ') is not adjacent to any vertex of $B(\sigma')$. Let $(\sigma,\rho) \in B(\sigma)$ with $\rho \neq \sigma'$. Then some $g \in G$ maps the 2-arc (τ,σ,σ') to the 2-arc (ρ,σ,σ') of Σ and hence g maps the edge $\{(\sigma,\tau),(\sigma',\tau')\}$ of Γ to $\{(\sigma,\rho),(\sigma',(\tau')^g)\}$. Thus (σ,ρ) is joined to some vertex of $B(\sigma')\setminus\{(\sigma',\sigma)\}$. It is now clear that the set of points of $\mathscr{D}(B(\sigma))$ incident with the block $B(\sigma')$ is $B(\sigma)\setminus\{(\sigma,\sigma')\}$. So $\mathscr{D}(B(\sigma))$ has no repeated blocks.

If Δ contains a 3-cycle then, since Σ is $(G,2)$-arc transitive, the end points of every 2-arc of Σ are adjacent vertices of Σ, so Σ is a disjoint union of complete graphs. From the previous paragraph it follows that Δ contains all the 3-cycles of Σ and that (σ,τ) is adjacent to (σ',τ') in $\mathrm{Arc}_\Delta(\Sigma)$ if and only if (σ,σ') is an arc of Σ and $\tau=\tau'$. Thus the connected components of $\mathrm{Arc}_\Delta(\Sigma)$ are the blocks $B^*(\tau)$ of $\mathscr{B}^*(\Sigma)$ and each is a complete graph. By Lemma 3, the conditions of Theorem 8(b) hold, so $\Gamma \cong \mathrm{Arc}_\Delta(\Sigma)$ and $\Gamma_{\mathscr{B}(\Sigma)} \cong \Sigma$ are as given there. On the other hand, if Δ consists of proper 3-arcs then adjacent vertices (σ,τ) and (σ',τ') of $\mathrm{Arc}_\Delta(\Sigma)$ involve four distinct vertices of Σ.

Thus under the assumptions of Theorem 10, we see that the graph $\mathrm{Arc}_\Delta(\Sigma)$ satisfies all the hypotheses of Theorem 1. We now show that every graph satisfying the hypotheses of Theorem 1 is isomorphic to $\mathrm{Arc}_\Delta(\Gamma_{\mathscr{B}})$ for some Δ. Theorems 10 and 11 together yield a proof of Theorem 1 stated in the introduction.

Theorem 11. *Suppose that Γ is a finite G-symmetric graph admitting a non-trivial G-invariant partition $\mathscr{B}$ of block size $v = k+1 \geqslant 3$ such that $\Gamma_{\mathscr{B}}$ is $(G,2)$-arc transitive, so the vertices of Γ are labelled with the arcs of $\Gamma_{\mathscr{B}}$. Then $\Gamma \cong \mathrm{Arc}_\Delta(\Gamma_{\mathscr{B}})$ for Δ the (self-paired) G-orbit in $A_3(\Gamma_{\mathscr{B}})$ containing the 3-arc (C,B,D,E), where (‘BC’, ‘DE’) is an arc of Γ. In particular, Δ contains a 3-cycle if and only if $\Gamma, \Gamma_{\mathscr{B}}$ are as in Theorem 8(b).*

Proof. Let (‘BC’, ‘DE’) be an arc of Γ. Then by the labelling defined before Proposition 7, it is clear that (C,B,D,E) is a 3-arc of $\Gamma_{\mathscr{B}}$. Let Δ be the G-orbit containing it. Since G is transitive on $A(\Gamma)$, Δ is independent of the choice of arc and Δ is self-paired. Since every arc of Γ is of the form (‘B^gC^g’, ‘D^gE^g’) for some $g \in G$, and since $(C^g,B^g,D^g,E^g) = (C,B,D,E)^g \in \Delta$, it follows from Definition 3 that $\Gamma \cong \mathrm{Arc}_\Delta(\Gamma_{\mathscr{B}})$. Finally, by Theorem 10(c) and (d), Δ contains a 3-cycle if and only if the second coordinates of labels for adjacent vertices of Γ are equal and hence $\Gamma, \Gamma_{\mathscr{B}}$ are as in Theorem 8(b).

In the case where $\Gamma_{\mathscr{B}}$ is 3-arc transitive, we have $\Delta = A_3(\Gamma_{\mathscr{B}})$ in Theorem 11 and hence there is a unique graph Γ. Theorem 2 gives a characterization of this case and we prove this now.

Proof of Theorem 2. Since $\mathscr{D}(B)$ has no repeated blocks, $\Gamma_{\mathscr{B}}$ is $(G,2)$-arc transitive, by Theorem 1. Suppose that (‘BC’, ‘DE’) is an arc of Γ and let Δ be the G-orbit in $A_3(\Gamma_{\mathscr{B}})$ containing the 3-arc (C,B,D,E). By Theorem 11, $\Gamma \cong \mathrm{Arc}_\Delta(\Gamma_{\mathscr{B}})$. Now each 3-arc (C_1,B,D,E_1) of $\Gamma_{\mathscr{B}}$ corresponds to a unique ordered pair ‘BC_1’, ‘DE_1’ of adjacent vertices of Γ and vice versa, where $C_1 \in \Gamma_{\mathscr{B}}(B)\setminus\{D\}$ and $E_1 \in \Gamma_{\mathscr{B}}(D)\setminus\{B\}$. Thus we have the following: $\Gamma[B,D] \cong K_{v-1,v-1} \Leftrightarrow$ for any such C_1, E_1, ‘BC_1’, ‘DE_1’

are adjacent in $\Gamma \Leftrightarrow$ for any such C_1, E_1, there exists $g \in G$ with $(`BC', `DE')^g = (`BC_1', `DE_1') \Leftrightarrow$ for any such C_1, E_1, there exists $g \in G$ with $(C, B, D, E)^g = (C_1, B, D, E_1) \Leftrightarrow$ for any such C_1, E_1, the 3-arc (C_1, B, D, E_1) is in $\Delta \Leftrightarrow \Delta = A_3(\Gamma_{\mathscr{B}})$ $\Leftrightarrow \Gamma_{\mathscr{B}}$ is $(G, 3)$-arc transitive.

Remark 4. (*a*) The structure of $\mathrm{Arc}_\Delta(\Sigma)$ for $(G, 2)$-arc transitive graphs Σ is of considerable interest. Zhou [**12**] has explored the family of these graphs for which Σ is a near-polygonal graph and Δ is the set of 3-arcs occurring in the distinguished 'polygons' of Σ. This case is of particular interest in connection with section 5 of [**5**].

(*b*) The construction of the graphs $\mathrm{Arc}_\Delta(\Sigma)$ bears some similarity with the covering graph construction of Biggs [**1**, pp. 149–154]. The graphs $\mathrm{Arc}_\Delta(\Sigma)$ are 'almost multicovers' of the 2-arc transitive graph Σ.

(*c*) Let Σ be a $(G, 2)$-arc transitive (and G-vertex-transitive) graph and let σ, σ' be a pair of adjacent vertices of Σ. Then G contains an element g which interchanges σ and σ'. Let $\tau \in \Sigma(\sigma) \setminus \{\sigma'\}$. Then $\tau' := \tau^g \in \Sigma(\sigma') \setminus \{\sigma\}$ and $(\tau, \sigma, \sigma', \tau')$ is a 3-arc of Σ. Also $\tau^{g^2} \in \Sigma(\sigma) \setminus \{\sigma'\}$. If it is possible to choose g and τ such that $\tau^{g^2} = \tau$, then g maps the 3-arc $(\tau, \sigma, \sigma', \tau')$ to its reverse $(\tau', \sigma', \sigma, \tau)$ and hence the G-orbit Δ containing $(\tau, \sigma, \sigma', \tau')$ is self-paired. This is certainly possible if any one of the following conditions holds:

(i) σ and σ' are interchanged by an element g of order 2;

(ii) the valency $|\Sigma(\sigma)|$ of Σ is even (since we may take g to be a 2-element and $g^2 \in G_{\sigma\sigma'}$);

(iii) Σ is $(G, 3)$-arc transitive;

(iv) the actions of $G_{\sigma\sigma'}$ on $\Sigma(\sigma) \setminus \{\sigma'\}$ and $\Sigma(\sigma') \setminus \{\sigma\}$ are permutationally isomorphic, in the sense that $G_{\sigma\sigma'\tau}$ fixes a point $\rho \in \Sigma(\sigma') \setminus \{\sigma\}$, and σ', τ are the only points of $\Sigma(\sigma)$ fixed by $G_{\sigma\sigma'\tau}$. (For if $h \in G_{\sigma\sigma'}$ maps τ' to ρ, then gh interchanges σ and σ', and maps τ to ρ and hence normalizes $G_{\sigma\sigma'\tau} = G_{\sigma\sigma'\rho}$. Therefore gh interchanges τ and ρ and hence reverses the 3-arc $(\tau, \sigma, \sigma', \rho)$.)

If any of these conditions holds, then Σ will occur as the quotient graph $\Gamma_{\mathscr{B}}$ for a graph Γ satisfying the hypotheses of Theorem 1.

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